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# A Consistent Nonparametric Test for Causality in Quantile

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# A Consistent Nonparametric Test for Causality in Quantile

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**Abstact**

This paper proposes a nonparametric test of causality in quantile. Zheng (1998) has proposed an idea to reduce the problem of testing a quantile restriction to a problem of testing a particular type of mean restriction in independent data. We extend Zheng's approach to the case of dependent data, particularly to the test of Granger causality in quantile. The proposed test statistic is shown to have a second-order degenerate U-statistic as a leading term under the null hypothesis. Using the result on the asymptotic normal distribution for a general second order degenerate U-statistics with weakly dependent data of Fan and Li (1996), we establish the asymptotic distribution of the test statistic for causality in quantile under β-mixing (absolutely regular) process.

Key Words: Granger Causality, Quantile, Nonparametric Test

JEL classification: C14, C52

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## 1. Introduction

Whether movements in one economic variable cause reactions in another variable is an important issue in economic policy and also for the financial investment decisions. A framework for investigating causality has been developed by Granger (1969). Testing for Granger causality between economic time series has been studied intensively in empirical macroeconomics and empirical finance. The majority of research results have been obtained in the context of Granger causality in the conditional mean. The conditional mean, though, is a questionable element of analysis if the distributions of the variables involved are non-elliptic or fat tailed as to be expected with financial returns. The fixation of causality analysis on the mean might result in many unclear results on Granger causality. Also, the conditional mean targets on an overall summary for the conditional distribution. A tail area causal relation may be quite different to that of the center of the distribution. Lee and Yang (2007) explore money-income Granger causality in the conditional quantile by using parametric quantile regression and find that Granger causality is significant in tail quantiles, while it is not significant in the center of the distribution.

This paper investigates Granger causality in the conditional quantile. It is well known that the conditional quantile is insensitive to outlying observations and a collection of conditional quantiles can characterize the entire conditional distribution. Based on the kernel method, we propose a nonparametric test for Granger causality in quantile. Testing conditional quantile restrictions by nonparametric estimation techniques in dependent data situations has not been considered in the literature before. This paper therefore intends to fill this literature gap.

Recently, the problem of testing the conditional mean restrictions using nonparametric estimation techniques has been actively extended from independent data to dependent data. Among the related work, only the testing procedures of Fan and Li (1999) and Li (1999) are consistent and have the standard asymptotic distributions of the test statistics. For the general hypothesis testing problem of the form $E(\varepsilon \mid z) = 0$ a.e., where $\varepsilon$ and $z$ are the regression error term and the vector of regressors respectively, Fan and Li (1999) and Li (1999) all consider the distance measure of $J = E[\varepsilon E(\varepsilon \mid z) f(z)]$ to construct kernel-based consistent test procedures. For the advantages of using distance measure $J$ in kernel-based testing procedures, see Li and Wang (1998) and Hsiao and Li (2001). A feasible test statistic based on the measure $J$ has a second order degenerate U-statistics as the leading term under the null hypothesis. Generalizing Hall's (1984) result for independent data, Fan and Li (1999) establish the asymptotic normal distribution for a general second order degenerate U-statistics with dependent data.

All the results stated above on testing mean restrictions are however irrelevant when testing quantile restrictions. Zheng (1998) proposed an idea to transform quantile restrictions to mean restrictions in independent data. Following his idea, one can use the existing technical results on testing mean restrictions in testing quantile restrictions. In this paper, by combining the Zheng's idea and the results of Fan and Li (1999) and Li (1999), we derive a test statistic for Granger causality in quantile and establish the asymptotic normal distribution of the proposed test statistic under the beta-mixing process. Our testing procedure can be extended to several hypotheses testing problems with conditional quantile in dependent data; for example, testing a parametric regression functional form, testing the insignificance of a subset of regressors, and testing semiparametric versus nonparametric regression models.

The paper is organized as follows. Section 2 presents the test statistic. Section 3 establishes the asymptotic normal distribution under the null hypothesis of no causalty in quantile. Technical proofs are given in Appendix.

## 2. Nonparametric Test for Granger-Causality in Quantile

To simplify the exposition, we assume a bivariate case, or only $\{y_t, w_t\}$ are observable. Denote $U_{t-1} = \{y_{t-1}, \cdots, y_{t-p}, w_{t-1}, \cdots, w_{t-q}\}$ and $W_{t-1} = \{w_{t-1}, \cdots, w_{t-q}\}$. Granger causality in mean (Granger, 1988) is defined as

(i) $w_t$ does not cause $y_t$ *in mean* with respect to $U_{t-1}$ if

$$E(y_t \mid U_{t-1}) = E(y_t \mid U_{t-1} - W_{t-1}) \text{ and}$$

(ii) $w_t$ is a prima facie cause *in mean* of $y_t$ with respect to $U_{t-1}$ if

$$E(y_t \mid U_{t-1}) \neq E(y_t \mid U_{t-1} - W_{t-1}),$$

Motivated by the definition of Granger-causality in mean, we define Granger causality in quantile as

(1) $w_t$ does not cause $y_t$ *in quantile* with respect to $U_{t-1}$ if

$$Q_\theta(y_t \mid U_{t-1}) = Q_\theta(y_t \mid U_{t-1} - W_{t-1}) \text{ and} \tag{1}$$

(2) $w_t$ is a prima facie cause *in quantile* of $y_t$ with respect to $U_{t-1}$ if

$$Q_\theta(y_t \mid U_{t-1}) \neq Q_\theta(y_t \mid U_{t-1} - W_{t-1}), \tag{2}$$

where $Q_\theta(y_t \mid \cdot) \equiv \inf\{y_t \mid F(y_t \mid \cdot) \geq \theta\}$ is the $\theta$th($0 < \theta < 1$) conditional quantile of $y_t$. Denote $x_t \equiv (y_{t-1}, \cdots, y_{t-p})$, $z_t \equiv (y_{t-1}, \cdots, y_{t-p}, w_{t-1}, \cdots, w_{t-q})$, and the conditional distribution function $y$ given $v$ by $F_{y|v}(y \mid v)$, $v = (x, z)$. Denote $Q_\theta(v_t) \equiv Q_\theta(y_t \mid v_t)$. In this paper, $F_{y|v}(y \mid v)$ is assumed to be absolutely continuous in $y$ for almost all $v = (x, z)$. Then we have

$$F_{y|v}\{Q_\theta(v_t) \mid v_t\} = \theta, \quad v = (x, z)$$

and from the definitions (1) and (2), the hypotheses to be tested are

$$\mathrm{H}_0: \quad \Pr\{F_{y|z}(Q_\theta(x_t) \mid z_t) = \theta\} = 1 \tag{3}$$

$$\mathrm{H}_1: \quad \Pr\{F_{y|z}(Q_\theta(x_t) \mid z_t) = \theta\} < 1. \tag{4}$$

Zheng (1998) proposed an idea to reduce the problem of testing a quantile restriction to a problem of testing a particular type of mean restriction. The null hypothesis (3) is true if and only if $E\left[I\{y_t \leq Q_\theta(x_t) \mid z_t\}\right] = \theta$ or $I\{y_t \leq Q_\theta(x_t)\} = \theta + \varepsilon_t$ where $E(\varepsilon_t \mid z_t) = 0$ and $I(\cdot)$ is the indicator function. There is a rich literature on constructing nonparametric tests for conditional mean restrictions. Refer to Li (1998) and Zheng (1998) for the list of related works. While various distance measures can be used to consistently test the hypothesis (3), we consider the following distance measure,

$$J \equiv \mathrm{E}\left[\{F_{y|z}(Q_\theta(x_t) \mid z_t) - \theta\}^2 f_z(z_t)\right], \tag{5}$$

where $f_z(z)$ be the marginal density function of $z$. Note that $J \geq 0$ and the equality holds if and only if $\mathrm{H}_0$ is true, with strict inequality holding under $\mathrm{H}_1$. Thus $J$ can be used as a proper candidate for consistent testing $\mathrm{H}_0$ (Li, 1999, p. 104). Since $\mathrm{E}(\varepsilon_t \mid z_t) = F_{y|z}\left\{Q_\theta(x_t) \mid z_t\right\} - \theta$, we have

$$J = \mathrm{E}\left\{\varepsilon_t \ \mathrm{E}(\varepsilon_t \mid z_t) f_z(z_t)\right\}. \tag{6}$$

The test is based on a sample analog of $\mathrm{E}\{\varepsilon \ \mathrm{E}(\varepsilon \mid z) f_z(z)\}$. Including the density function $f_z(z)$ is to avoid the problem of trimming on the boundary of the density function, see Powell, Stock, and Stoker (1989) for an analogue approach. The density weighted conditional expectation $\mathrm{E}(\varepsilon \mid z) f_z(z)$ can be estimated by kernel methods

$$\hat{E}(\varepsilon_t \mid z_t) \hat{f}_z(z_t) = \frac{1}{(T-1)h^m} \sum_{s \neq t}^{T} K_{ts} \varepsilon_s, \tag{7}$$

where $m = p + q$ is the dimension of $z$, $K_{ts} = K\left\{(z_t - z_s)/h\right\}$ is the kernel function and $h$ is a bandwidth. Then we have a sample analog of $J$ as

$$\begin{aligned} J_T &\equiv \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \neq t}^{T} K_{ts} \varepsilon_t \varepsilon_s \\ &= \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \neq t}^{T} K_{ts} \left[ I\left\{y_t \leq Q_\theta(x_t)\right\} - \theta \right] \left[ I\left\{y_s \leq Q_\theta(x_s)\right\} - \theta \right] \end{aligned} \tag{8}$$

The $\theta$-th conditional quantile of $y_t$ given $x_t$, $Q_\theta(x_t)$, can also be estimated by the nonparametric kernel method

$$\hat{Q}_\theta(x_t) = \hat{F}_{y|x}^{-1}(\theta \mid x_t), \tag{9}$$

where

$$\hat{F}_{y|x}(y_t \mid x_t) = \frac{\sum_{s \neq t} L_{ts} I(y_s \leq y_t)}{\sum_{s \neq t} L_{ts}} \tag{10}$$

is the Nadaraya-Watson kernel estimator of $F_{y|x}(y_t \mid x_t)$ with the kernel function of $L_{ts} = L\left(\frac{x_t - x_s}{a}\right)$ and the bandwidth parameter of $a$. The unknown error $\varepsilon$ can be estimated as:

$$\hat{\varepsilon}_t \equiv I\left\{y_t \le \hat{Q}_\theta(x_t)\right\} - \theta . \tag{11}$$

Replacing $\varepsilon$ by $\hat{\varepsilon}$, we have a kernel-based feasible test statistic of $J$,

$$\hat{J}_T \equiv \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \ne t}^{T} K_{ts} \hat{\varepsilon}_t \hat{\varepsilon}_s$$

$$= \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \ne t}^{T} K_{ts} \left[ I\left\{y_t \le \hat{Q}_\theta(x_t)\right\} - \theta \right] \left[ I\left\{y_s \le \hat{Q}_\theta(x_s)\right\} - \theta \right] \tag{12}$$

## 3. The Limiting Distributions of the Test Statistic

Two existing works are useful in deriving the limiting distribution of the test statistic; one is Theorem 2.3 of Franke and Mwita (2003) on the uniform convergence rate of the nonparametric kernel estimator of conditional quantile; another is Lemma 2.1 of Li (1999) on the asymptotic distribution of a second-order degenerate U-statistic, which is derived from Theorem 2.1 of Fan and Li (1999). We restate these results in lemmas below for ease of reference.

**Lemma 1 (Franke and Mwita)** *Suppose Conditions (A1)(v)-(vii) and (A2)(iii) of Appendix hold. The bandwidth sequence is such that* $a = o(1)$ *and* $\tilde{S}_T = Ta^p (s_T \log T)^{-1} \to \infty$ *for some* $s_T \to \infty$. *Let* $S_T = a^2 + \tilde{S}_T^{-1/2}$. *Then for the nonparametric kernel estimator of conditional quantile of* $\hat{Q}_\theta(x_t)$ *of equation (9), we have*

$$\sup_{\|x\| \in G} \left| \hat{Q}_\theta(x) - Q_\theta(x) \right| = O\left(S_T\right) + O\left(\frac{1}{Ta^p}\right) \quad a.s. \tag{13}$$

**Lemma 2 (Li / Fan and Li)** *Let* $L_t = (\varepsilon_t, z_t)^{\mathrm{T}}$ *be a strictly stationary process that satisfies the condition (A1)(i)-(iv) of Appendix,* $\varepsilon_t \in R$ *and* $z_t \in R^m$, $K(\cdot)$ *be the kernel function with h being the smoothing parameter that satisfies the condition (A2)(i)-(ii) of Appendix. Define*

$$\sigma_\varepsilon^2(z) = E[\varepsilon_t^2 \mid z_t = z] \quad and \tag{14}$$

$$J_T \equiv \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \neq t}^{T} K_{ts} \varepsilon_t \varepsilon_s \tag{15}$$

*Then*

$$Th^{m/2} J_T \;\rightarrow\; \mathrm{N}(0, \sigma_0^2) \;\; \textit{in distribution,} \tag{16}$$

*where* $\sigma_0^2 = 2\mathrm{E}\left\{\sigma_\varepsilon^4(z_t) f_z(z_t)\right\}\left\{\int K^2(u)du\right\}$ *and* $f_z(\cdot)$ *is the marginal density function of* $z_t$.

Technical conditions required to derive the asymptotic distribution of $\hat{J}_T$ are given in Appendix, which are adopted from Li (1999) and Franke and Mwita (2003). In the assumptions we use the definitions of Robinson (1988) for the class of kernel functions $\Upsilon_v$ and the class of functions $\mathsf{A}_v^\infty$, defined in Appendix.

Conditions (A1)(i)-(iv) and (A2)(i)-(ii) are adopted from condition (A1) and (A2) of Li (1999), which are used to derive the asymptotic normal distribution of a second-order degenerate U-statistic. Conditions (A1)(v)-(vii) and (A2)(iii) are conditions (A1), (A2), (B1), (B2), (C1) and (C2) of Franke and Mwita (2003), which are required to get the uniform convergence rate of nonparametric kernel estimator of conditional quantile with mixing data. Finally Conditions (A2)(iv)-(v) are adopted from conditions of Lemma 2 of Yoshihara (1976), which are required to get the asymptotic equivalence of nondegenerate U-statistic and its projection under the $\beta$-mixing process.

We consider testing for local departures from the null that converge to the null at the rate $T^{-1/2}h^{-m/4}$. More precisely we consider the sequence of local alternatives:

$$\mathrm{H}_{1T}: \quad F_{y|z}\left\{Q_\theta(x_t) + d_T l(z_t) \mid z_t\right\} = \theta, \tag{17}$$

where $d_T = T^{-1/2}h^{-m/4}$ and the function $l(\cdot)$ and its first-derivatives are bounded.

**Theorem 1**. *Assume the conditions (A1) and (A2). Then*

*(i) Under the null hypothesis (3),* $Th^{m/2}\hat{J}_T \rightarrow \mathrm{N}(0, \sigma_0^2)$ *in distribution, where*

$\sigma_0^2 = 2\mathrm{E}\left\{\sigma_\varepsilon^4(z_t) f_z(z_t)\right\}\int K^2(u)du$ *and* $\sigma_\varepsilon^2(z_t) = E(\varepsilon_t^2 \mid z_t) = \theta(1-\theta)$.

*(ii) under the null hypothesis (3),* $\hat{\sigma}_0^2 \equiv 2\theta^2(1-\theta)^2 \dfrac{1}{T(T-1)h^m}\sum_{s\neq t} K_{ts}^2$ *is a consistent estimator of* $\sigma_0^2 = 2\mathrm{E}\left\{\sigma_\varepsilon^4(z_t) f_z(z_t)\right\}\int K^2(u)du$.

*(iii) under the alternative hypothesis (4),* $\hat{J}_T \to_p E\{[F_{y|z}(Q_\theta(x_t)\mid z_t)-\theta]^2 f_z(z_t)\} > 0$.

(iv) *under the local alternatives (17),* $Th^{m/2}\hat{J}_T \to \mathrm{N}(\mu,\sigma_0^2)$ *in distribution, where* $\mu = E\left[ f_{y|z}^2\left\{Q_\theta(z_t)\mid z_t\right\} l^2(z_t) f_z(z_t)\right]$.

Theorem 1 generalizes the results of Zheng (1998) of independent data to the weakly dependent data case. A detailed proof of Theorem is given in the Appendix. The main difficulty in deriving the asymptotic distribution of the statistic defined in equation (12) is that a nonparametric quantile estimator is included in the indicator function which is not differentiable with respect to the quantile argument and thus prevents taking a Taylor expansion around the true conditional quantile $Q_\theta(x_t)$. To circumvent the problem, Zheng (1989) appealed to the work of Sherman (1994) on uniform convergence of U-statistics indexed by parameters. In this paper, we bound the test statistic by the statistics in which the nonparametric quantile estimator in the indicator function is replaced with sums of the true conditional quantile and upper and lower bounds consistent with uniform convergence rate of the nonparametric quantile estimator, $1(y_t \le Q_\theta(x_t) - C_T)$ and $1(y_t \le Q_\theta(x_t) + C_T)$.

An important further step is to show that the differences of the ideal test statistic $J_T$ given in equation (8) and the statistics having the indicator functions obtained from the first step stated above is asymptotically negligible. We may directly show that the second moments of the differences are asymptotically negligible by using the result of Yoshihara (1976) on the bound of moments of U-statistics for absolutely regular processes. However, it is tedious to get bounds on the second moments with dependent data. In the proof we instead use the fact that differences are second-order degenerate U-statistics. Thus by using the result on the asymptotic normal distribution of the second-order degenerate U-statistic of Fan and Li

(1999), we can derive the asymptotic variance which is based on the i.i.d. sequence having the same marginal distributions as weakly dependent variables in the test statistic. With this little trick we only need to show that the asymptotic variance is $o(1)$ in an i.i.d. situation. For details refer to the Appendix.

## 4. Conclusion

This paper has provided a consistent test for Granger-causality in quantile. The test can be extended to testing conditional quantile restrictions with dependent data; for example, testing misspecification test, testing the insignificance of a subset of regressors, testing some semiparametric versus nonparametric models, all in quantile regression models.

## Appendix

Here we collect all required assumptions to establish the results of Theorem 1.

**(A1)** (i) $\{y_t, w_t\}$ is strictly stationary and absolutely regular with mixing coefficients $\beta(\tau) = O(\rho^\tau)$ for some $0 < \rho < 1$.

(ii) For some integer $v \geq 2$, $f_y$, $f_z$, and $f_x$ all are bounded and belong to $\mathrm{A}_v^\infty$ (see D2).

(iii) with probability one, $E[\varepsilon_t \mid \mu_{-\infty}^t(z), \mu_{-\infty}^{t-1}(z)] = 0$ . $E\left[\left|\varepsilon_t^{4+\eta}\right|\right] < \infty$ and $E\left[\left|\varepsilon_{t_1}^{i_1}\varepsilon_{t_2}^{i_2}\cdots\varepsilon_{t_l}^{i_l}\right|^{1+\xi}\right] < \infty$ for some arbitrarily small $\eta > 0$ and $\xi > 0$, where $2 \leq l \leq 4$ is an integer, $0 \leq i_j \leq 4$ and $\sum_{j=1}^{l} i_j \leq 8$ . $\sigma_\varepsilon^2(z) = E(\varepsilon^2 \mid z)$ , $\mu_{\varepsilon 4}(z) = E\left[\varepsilon_t^4 \mid z_t = z\right]$ all satisfy some Lipschitz conditions: $\left|a(u+v) - a(u)\right| \leq D(u)\left\|v\right\|$ with $E\left[\left|D(z)\right|^{2+\eta'}\right] < \infty$ for some small $\eta' > 0$, where $a(\cdot) = \sigma_\varepsilon^2(\cdot),\ \mu_{\varepsilon 4}(\cdot)$.

(iv) Let $f_{\tau_1,\ldots,\tau_l}(\ )$ be the joint probability density function of $\left(z_{\tau_1},\ldots,z_{\tau_l}\right)$ $\left(1 \leq l \leq 3\right)$. Then $f_{\tau_1,\ldots,\tau_l}(\ )$ is bounded and satisfies a Lipschitz condition: $\left|f_{\tau_1,\ldots,\tau_l}\left(z_1+u_1, z_2+u_2, \ldots z_l+u_l\right) - f_{\tau_1,\ldots,\tau_l}\left(z_1, z_2, \ldots z_l\right)\right| \leq D_{\tau_1,\ldots,\tau_l}\left(z_1,\ldots,z_l\right)\left\|u\right\|$ , where $D_{\tau_1,\ldots,\tau_l}(\ )$ is integrable and satisfies the condition that $\int D_{\tau_1,\ldots,\tau_l}\left(z_1,\ldots,z_l\right)\left\|z\right\|^{2\xi} < M < \infty$, $\int D_{\tau_1,\ldots,\tau_l}\left(z_1,\ldots,z_l\right) f_{\tau_1,\ldots,\tau_l}\left(z_1,\ldots,z_l\right) dz < M < \infty$ for some $\xi > 1$.

(v) For any $y, x$ satisfying $0 < F_{y|x}(y \mid x) < 1$ and $f_x(x) > 0$ ; for fixed $y$ , the conditional distribution function $F_{y|x}$ and the conditional density function $f_{y|x}$ belong to $\mathrm{A}_3^\infty$; $f_{y|x}(Q_\theta(x) \mid x) > 0$ for all $x$; $f_{y|x}$ is uniformly bounded in $x$ and $y$ by $c_f$, say.

(vi) For some compact set $G$, there are $\varepsilon > 0, \gamma > 0$ such that $f_x \geq \gamma$ for all $x$ in the $\varepsilon$ -neighborhood $\left\{x \mid \left\|x-u\right\| < \varepsilon,\ u \in G\right\}$ of $G$ ; For the compact set $G$ and some compact neighborhood $\Theta_0$ of 0, the set $\Theta = \{v = Q_\theta(x) + \mu \mid x \in G, \mu \in \Theta_0\}$ is compact and for some constant $c_0 > 0$, $f_{y|x}(v \mid x) \geq c_0$ for all $x \in G, v \in \Theta$. (vii) There is an increasing sequence $s_T$ of positive integers such that for some finite $A$,

$$\frac{T}{s_T}\beta^{2s_T/(3T)}(s_T) \leq A,\ 1 \leq s_T \leq \frac{T}{2} \text{ for all } T \geq 1.$$

**(A2)** (i) we use product kernels for both $L(\cdot)$ and $K(\cdot)$, let $l$ and $k$ be their corresponding univariate kernel which is bounded and symmetric, then $l(\cdot)$ is non-negative, $l(\cdot) \in \Upsilon_v$, $k(\cdot)$ is non-negative and $k(\cdot) \in \Upsilon_2$.

(ii) $h = O(T^{-\alpha'})$ for some $0 < \alpha' < (7/8)m$.

(iii) $a = o(1)$ and $\tilde{S}_T = Ta^p(s_T \log T)^{-1} \to \infty$ for some $s_T \to \infty$

(iv) there exists a positive number $\delta$ such that for $r = 2 + \delta$ and a generic number $M_0$

$$\int\int \left| \frac{1}{h^m} K\left(\frac{z_1 - z_2}{h}\right) \right|^r dF_z(z_1)dF_z(z_2) \leq M_0 < \infty \text{ and}$$

$$E\left| \frac{1}{h^m} K\left(\frac{z_1 - z_2}{h}\right) \right|^r \leq M_0 < \infty$$

(v) for some $\delta'$ $(0 < \delta' < \delta)$, $\beta(T) = O(T^{-(2+\delta')/\delta'})$.

The following definitions are due to Robinson (1988).

**Definition (D1)** $\Upsilon_\lambda$, $\lambda \geq 1$ is the class of even functions $k : R \to R$ satisfying

$$\int_R u^i k(u)du = \delta_{i0} \qquad (i = 0,1,\ldots,\lambda - 1),$$

$$k(u) = O\left((1 + |u|^{\lambda+1+\varepsilon})^{-1}\right), \text{ for some } \varepsilon > 0,$$

where $\delta_{ij}$ is the Kronecker's delta.

**Definition (D2)** $\mathrm{A}_\mu^\alpha$, $\alpha > 0$, $\mu > 0$ is the class of functions $g : R^m \to R$ satisfying that

$g$ is $(d-1)$ -times partially differentiable for $d-1 \le \mu \le d$ ; for some $\rho > 0$ , $\sup_{y \in \phi_{z\rho}} \left| g(y) - g(z) - G_g(y,z) \right| / \left| y - z \right|^{\mu} \le D_g(z)$ for all $z$, where $\phi_{z\rho} = \{ y \mid |y - z| < \rho \}$ ; $G_g = 0$ when $d = 1$; $G_g$ is a $(d-1)$ th degree homogeneous polynomial in $y - z$ with coefficients the partial derivatives of $g$ at $z$ of orders 1 through $d-1$ when $d > 1$; and $g(z)$, its partial derivatives of order $d-1$ and less, and $D_g(z)$, has finite $\alpha$ th moments.

**Proof of Theorem (i)**

In the proof, we use several approximations to $\hat{J}_T$. We define them now and recall a few already defined statistics for convenience of reference.

$$\hat{J}_T \equiv \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \neq t}^{T} K_{ts} \hat{\varepsilon}_t \hat{\varepsilon}_s \tag{A.1}$$

$$J_T \equiv \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \neq t}^{T} K_{ts} \varepsilon_t \varepsilon_s \tag{A.2}$$

$$J_{TU} \equiv \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \neq t}^{T} K_{ts} \varepsilon_{tU} \varepsilon_{sU} \tag{A.3}$$

$$J_{TL} \equiv \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \neq t}^{T} K_{ts} \varepsilon_{tL} \varepsilon_{sL} \tag{A.4}$$

where $\hat{\varepsilon}_t = I\{ y_t \le \hat{Q}_\theta(x_t) \} - \theta$,

$\varepsilon_t = I\{ y_t \le Q_\theta(x_t) \} - \theta$,

$\varepsilon_{tU} = I\{ y_t + C_T \le Q_\theta(x_t) \} - \theta$,

$\varepsilon_{tL} = I\{ y_t - C_T \le Q_\theta(x_t) \} - \theta$ and

$C_T$ is an upper bound consistent with the uniform convergence rate of the nonparametric estimator of conditional quantile given in equation (13). The proof of Theorem 1 (i) consists of three steps.

Step 1. Asymptotic normality:

$$Th^{m/2} J_T \to \mathrm{N}(0, \sigma_0^2), \tag{A.5}$$

where $\sigma_0^2 = 2\mathrm{E}\left\{\theta^2(1-\theta)^2 f(z_t)\right\}\left\{\int K^2(u)du\right\}$ under the null.

Step 2. Conditional asymptotic equivalence:

Suppose that both $Th^{m/2}(J_T - J_{TU}) = o_p(1)$ and $Th^{m/2}(J_T - J_{TU}) = o_p(1)$.

Then $Th^{m/2}(\hat{J}_T - J_T) = o_p(1)$. (A.6)

Step 3. Asymptotic equivalence:

$Th^{m/2}(J_T - J_{TU}) = o_p(1)$ and $Th^{m/2}(J_T - J_{TL}) = o_p(1)$. (A.7)

The combination of Steps 1-3 yields Theorem 1 (i).

*Step 1: Asymptotic normality.*

Since $J_T$ is a degenerate U-statistic of order 2, the result follows from Lemma 2.

□

*Step 2: Conditional asymptotic equivalence.*

The proof of Step 2 is motivated by the technique of Härdle and Stoker (1989) which was used in treating trimming indicator function asymptotically. Suppose that the following two statements hold.

$$Th^{m/2}(J_T - J_{TU}) = o_p(1) \text{ and} \tag{A.8}$$

$$Th^{m/2}(J_T - J_{TL}) = o_p(1) \tag{A.9}$$

Denote $C_T$ as an upper bound consistent with the uniform convergence rate of the nonparametric estimator of conditional quantile given in equation (13). Suppose that

$$\sup|\hat{Q}_\theta(x) - Q_\theta(x)| \le C_T. \tag{A.10}$$

If inequality (A.3) holds, then the following statements also hold:

$$\{\ Q_\theta(x) > y_t + C_T\ \} \subset \{\ \hat{Q}_\theta(x) > y_t\ \} \subset \{\ Q_\theta(x) > y_t - C_T\ \}, \tag{A.11-1}$$

$$1(\ Q_\theta(x) > y_t + C_T\ ) \le 1(\ \hat{Q}_\theta(x) > y_t\ ) \le 1(\ Q_\theta(x) > y_t - C_T\ ), \tag{A.11-2}$$

$$J_{TU} \le \hat{J}_T \le J_{TL}, \text{ and} \tag{A.11-3}$$

$$|Th^{m/2}(J_T - \hat{J}_T)| \le \max\ \{\ |Th^{m/2}(J_T - J_{TU})|\ ,\ |Th^{m/2}(J_T - J_{TL})|\ \} \tag{A.11-4}$$

Using (A.10) and (A.11-4), we have the following inequality;

$$\Pr\left\{ | Th^{m/2}(J_T - \hat{J}_T) | > \delta \;\middle|\; \sup | \hat{Q}_\theta(x) - Q_\theta(x) | \le C_T \right\}$$

$$\le \Pr\left\{ \max \{ |Th^{m/2}(J_T - J_{TU})| \;,\; |Th^{m/2}(J_T - J_{TL})| \} > \delta \;\middle|\; \sup | \hat{Q}_\theta(x) - Q_\theta(x) | \le C_T \right\}$$

, for all $\delta > 0$. (A.12)

Invoking Lemma 1 and condition A2(iii), we have

$$\Pr\left\{ \sup | \hat{Q}_\theta(x) - Q_\theta(x) | \le C_T \right\} \to 1 \text{ as } T \to \infty . \tag{A.13}$$

By (A.8) and (A.9), as $T \to \infty$, we have

$$\Pr\left\{ \max \{ |Th^{m/2}(J_T - J_{TU})| \;,\; |Th^{m/2}(J_T - J_{TL})| \} > \delta \right\} \to 0, \text{ for all } \delta > 0 .$$

(A.14)

Therefore, as $T \to \infty$,

the L.H.S. of the inequality (A.12) $-$ $\Pr\left\{ | Th^{m/2}(J_T - \hat{J}_T) | > \delta \right\} \to 0$ and

the L.H.S. of the inequality (A.12) $\to 0$.

In summary, we have that if both $Th^{m/2}(J_T - J_{TU}) = o_p(1)$ and $Th^{m/2}(J_T - J_{TU}) = o_p(1)$, then $Th^{m/2}(\hat{J}_T - J_T) = o_p(1)$. □

*Step 3: Asymptotic equivalence.*

In the remaining proof, we focus on showing that $Th^{m/2}(J_T - J_{TU}) = o_p(1)$, with the proof of $Th^{m/2}(J_T - J_{TL}) = o_p(1)$ being treated similarly. The proof of Step 3 is close in lines with the proof in Zheng (1998). Denote

$$H_T(s,t,g) \equiv K_{ts} \{1(y_t \le g(x_t)) - \theta\} \{1(y_s \le g(x_s)) - \theta\} \text{ and} \tag{A.15}$$

$$J[g] \equiv \frac{1}{T(T-1)h^m} \sum_{t=1}^{T} \sum_{s \ne t}^{T} H_T(s,t,g) . \tag{A.16}$$

Then we have $J_T \equiv J[Q_\theta]$ and $J_{TU} \equiv J[Q_\theta - C_T]$. We decompose $H_T(s,t,g)$ into three parts;

$$H_T(s,t,g) = K_{ts} \{1(y_t \le g(x_t)) - F(g(x_t) | z_t)\} \{1(y_s \le g(x_s)) - F(g(x_s) | z_s)\}$$

$$+\ 2\times K_{ts}\{1(y_t \le g(x_t)) - F(g(x_t)\,|\,z_t)\}\{F(g(x_s)\,|\,z_s)-\theta\}$$

$$+\ K_{ts}\{F(g(x_t)\,|\,z_t)-\theta\}\{F(g(x_s)\,|\,z_s)-\theta\}$$

$$=\ H_{1T}(s,t,g)+2H_{2T}(s,t,g)+H_{3T}(s,t,g) \qquad \text{(A.17)}$$

Then let $J_j[g]=\frac{1}{T(T-1)h^m}\sum_{t=1}^{T}\sum_{s\neq t}^{T} H_{jT}(s,t,g)$ , $i=1,2,3$ . We will treat $J_j[Q_\theta]-J_j[Q_\theta - C_T]$ $for\ j=1,2,3$ separately.

[1] $Th^{m/2}\left[J_1(Q_\theta)-J_1(Q_\theta-C_T)\right]=o_p(1)$:

By simple manipulation, we have

$$J_1(Q_\theta)-J_1(Q_\theta-C_T)$$

$$=\frac{1}{T(T-1)h^m}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\left[H_{1T}(s,t,Q_\theta)-H_{1T}(s,t,Q_\theta-C_T)\right]$$

$$=\frac{1}{T(T-1)h^m}\sum_{t=1}^{T}\sum_{s\neq t}^{T} K_{ts}\Big\{\left[1_t(Q_\theta)-F_t(Q_\theta)\right]\left[1_s(Q_\theta)-F_s(Q_\theta)\right]$$

$$-\left[1_t(Q_\theta-C_T)-F_t(Q_\theta-C_T)\right]\left[1_s(Q_\theta-C_T)-F_s(Q_\theta-C_T)\right]\Big\} \qquad \text{(A.18)}$$

To avoid tedious works to get bounds on the second moment of $J_1(Q_\theta)-J_1(Q_\theta-C_T)$ with dependent data, we note that the R.H.S. of (A.18) is a degenerate U-statistic of order 2. Thus we can apply Lemma 2 and have

$$Th^{m/2}\left[J_1(Q_\theta)-J_1(Q_\theta-C_T)\right]\ \to\ \mathrm{N}(0,\sigma_1^2)\ \text{ in distribution,} \qquad \text{(A.19)}$$

where the definition of the asymptotic variance $\sigma_1^2$ is based on the i.i.d. sequence having the same marginal distributions as weakly dependent variables in (A.18). That is,

$$\sigma_1^2=\tilde{E}\left[H_{1T}(s,t,Q_\theta)-H_{1T}(s,t,Q_\theta-C_T)\right],$$

where the notation $\tilde{E}$ is expectation evaluated at an i.i.d. sequence having the same marginal distribution as the mixing sequences in (A.18) (Fan and Li (1999), p. 248). Now, to show that $Th^{m/2}\left[J_1(Q_\theta)-J_1(Q_\theta-C_T)\right]=o_p(1)$, we only need to show that the asymptotic variance $\sigma_1^2(z)$ is $o(1)$ with i.i.d data. We have

$$\tilde{E}\ \left[H_{1T}(s,t,Q_\theta)-H_{1T}(s,t,Q_\theta-C_T)\right]^2$$

$$\leq\ \Lambda\ \tilde{\mathrm{E}}\Big\{\ \left[1_t(Q_\theta)-F_t(Q_\theta)\right]\left[1_s(Q_\theta)-F_s(Q_\theta)\right]$$

$$-\left[1_t(Q_\theta-C_T)-F_t(Q_\theta-C_T)\right]\left[1_s(Q_\theta-C_T)-F_s(Q_\theta-C_T)\right]\ \Big\}^2$$

$$\leq\Lambda\ \tilde{E}\left\{F_t(Q_\theta)\left[1-F_t(Q_\theta)\right]F_s(Q_\theta)\left[1-F_s(Q_\theta)\right]\right\}$$

$$+\tilde{E}\left\{F_t(Q_\theta-C_T)\left[1-F_t(Q_\theta-C_T)\right]F_s(Q_\theta-C_T)\left[1-F_s(Q_\theta-C_T)\right]\right\}$$

$$-2E\Big\{\ \left[F_t(\min(Q_\theta,Q_\theta-C_T)-F_t(Q_\theta)F_t(Q_\theta-C_T)\right]$$

$$\times\left[F_s(\min(Q_\theta,Q_\theta-C_T)-F_s(Q_\theta)F_s(Q_\theta-C_T)\right]\ \Big\}$$

$$=\Lambda\ \tilde{E}\left\{\left[F_t(Q_\theta)-F_t(Q_\theta)F_t(Q_\theta)\right]\left[F_s(Q_\theta)-F_s(Q_\theta)F_s(Q_\theta)\right]\right\}$$

$$-\Lambda\ \tilde{E}\Big\{\ \left[F_t(\min(Q_\theta,Q_\theta-C_T)-F_t(Q_\theta)F_t(Q_\theta-C_T)\right]$$

$$\times\left[F_s(\min(Q_\theta,Q_\theta-C_T)-F_s(Q_\theta)F_s(Q_\theta-C_T)\right]\ \Big\}$$

$$+\Lambda\ \tilde{E}\Big\{\ \left[F_t(Q_\theta-C_T)-F_t(Q_\theta-C_T)F_t(Q_\theta-C_T)\right]$$

$$\times\left[F_s(Q_\theta-C_T)-F_s(Q_\theta-C_T)F_s(Q_\theta-C_T)\right]\Big\}$$

$$-\Lambda\ \tilde{E}\Big\{\ \left[F_t(\min(Q_\theta,Q_\theta-C_T)-F_t(Q_\theta)F_t(Q_\theta-C_T)\right]$$

$$\times\left[F_s(\min(Q_\theta,Q_\theta-C_T)-F_s(Q_\theta)F_s(Q_\theta-C_T)\right]\ \Big\}$$

$$\leq\Lambda\left|C_T\right|=o(1). \tag{A.20}$$

where the last equality holds by the smoothness of conditional distribution function and its bounded first derivative due to Assumption (A.8). Thus we have

$$Th^{m/2}\left[J_1(Q_\theta)-J_1(Q_\theta-C_T)\right]=o_p(1) \tag{A.21}$$

[2] $Th^{m/2}\left[J_2(Q_\theta)-J_2(Q_\theta-C_T)\right]=o_p(1)$:

Noting that $H_{2T}(s,t,Q_\theta)=0$ because of $F_{y|z}(Q_\theta(x_s)\mid z_s)-\theta=0$, we have

$$J_2(Q_\theta)-J_2(Q_\theta-C_T)$$

$$= -J_2(Q_\theta - C_T)$$

$$= -\frac{1}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t}^{T} \frac{1}{h^m} K\left(\frac{z_t - z_s}{h}\right)$$

$$\times \{1(y_t \le Q_\theta(x_t) - C_T) - F_{y|z}(Q_\theta(x_t) - C_T \mid z_t)\} \{F_{y|z}(Q_\theta(x_s) - C_T \mid z_s) - \theta\} \quad \text{(A.22)}$$

Denote $S(g) \equiv \partial F[g]/\partial g$. By taking a Taylor expansion of $F_{y|z}(Q_\theta(x_s) - C_T \mid z_s)$ around $Q_\theta(x_s)$, we have

$$J_2(Q_\theta) - J_2(Q_\theta - C_T)$$

$$= -\frac{1}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t}^{T} \frac{1}{h^m} K\left(\frac{z_t - z_s}{h}\right) \{1(y_t \le Q_\theta(x_t) - C_T) - F_{y|z}(Q_\theta(x_t) - C_T \mid z_t)\}$$

$$\times (-C_T) S(\bar{Q}_\theta(x_s))$$

$$= C_T \frac{1}{T} \sum_{t=1}^{T} \{1(y_t \le Q_\theta(x_t) - C_T) - F_{y|z}(Q_\theta(x_t) - C_T)\} S(\bar{Q}_\theta(x_s)) \hat{f}_z(z_t)$$

$$\equiv C_T \frac{1}{T} \sum_{t=1}^{T} u_t S(\bar{Q}_\theta(x_s)) \hat{f}_z(z_t), \quad \text{(A.23)}$$

where $\bar{Q}_\theta$ is between $Q_\theta$ and $Q_\theta - C_T$. Thus we have

$$E\left|J_2(Q_\theta) - J_2(Q_\theta - C_T)\right|$$

$$\le \Lambda C_T \frac{1}{T} \sum_{t=1}^{T} E\left|u_t \hat{f}_z(z_t)\right|$$

$$\le \Lambda C_T \frac{1}{T} \sum_{t=1}^{T} E\left\{u_t^2 \hat{f}_z^2(z_t)\right\}$$

$$= O\left(C_T (Th^m)^{-1}\right), \quad \text{(A.24)}$$

where the first inequality holds due to Assumption (1)(v) and the last equality is derived by using Lemma C.3(iii) of Li (1999) that is proved in the proof of Lemma A.4(i) of Fan and Li (1996c).

Thus, we have

$$Th^{m/2}\left[J_2(Q_\theta)-J_2(Q_\theta-C_T)\right]$$

$$=O_p\left(C_T h^{-m/2}\right)$$

$$=o_p(1)\,. \tag{A.25}$$

[3] $Th^{m/2}\left[J_3(Q_\theta)-J_3(Q_\theta-C_T)\right]=o_p(1)$:

Noting that $H_{3T}(s,t,Q_\theta)=0$ because of $F(Q_\theta(x_j)\mid z_j)-\theta=0$ for $j=t,s$, we have

$$J_3(Q_\theta)-J_3(Q_\theta-C_T)$$

$$=-\frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\frac{1}{h^m}K\left(\frac{z_t-z_s}{h}\right)$$

$$\times\{F(Q_\theta(x_t)-C_T\mid z_t)-\theta\}\{F(Q_\theta(x_s)-C_T\mid z_s)-\theta\}$$

$$=\frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\frac{1}{h^m}K\left(\frac{z_t-z_s}{h}\right)C_T^2 S(\bar{Q}_\theta(x_t))S(\bar{Q}_\theta(x_s))$$

$$=C_T^2\frac{1}{T}\sum_{t=1}^{T}S(\bar{Q}_\theta(x_t))S(\bar{Q}_\theta(x_s))\,\hat{f}_z(z_t) \tag{A.26}$$

Thus, we have

$$E\left|J_3(Q_\theta)-J_3(Q_\theta-C_T)\right|$$

$$\leq\ \Lambda C_T^2\frac{1}{T}\sum_{t=1}^{T}E\left|\hat{f}_z(z_t)\right|$$

$$\leq\ \Lambda C_T^2\frac{1}{T}\sum_{t=1}^{T}E\left|f_z(z_t)\right|+\Lambda C_T^2\frac{1}{T}\sum_{t=1}^{T}E\left|\hat{f}_z(z_t)-f_z(z_t)\right|$$

$$\leq\ \Lambda C_T^2\frac{1}{T}\sum_{t=1}^{T}Ef_z(z_t)+\Lambda C_T^2\frac{1}{T}\sum_{t=1}^{T}E\left\{\hat{f}_z(z_t)-f_z(z_t)\right\}^2$$

$$=O\left(C_T^2\right) \tag{A.27}$$

Finally, we have

$$Th^{m/2}\left[J_3(Q_\theta)-J_3(Q_\theta-C_T)\right]$$

$$=O_p\left(Th^{m/2}C_T^2\right)$$

$$= o_p(1) . \qquad (A.28)$$

By combining (A.21), (A.25) and (A.28), we have the result of Step 3 □

**Proof of Theorem 1 (ii)**

Since

$$\sigma_0^2 = 2\theta^2(1-\theta)^2 \mathrm{E}\left\{ f_z\ (z_t) \right\} \int K^2(u)du \quad \text{and}$$

$$\hat{\sigma}_0^2 \equiv 2\theta^2(1-\theta)^2 \frac{1}{T(T-1)h^m} \sum_{s \neq t} K_{ts}^2 ,$$

it is enough to show that

$$\sigma_T^2 \equiv \frac{1}{T(T-1)h^m} \sum_{s \neq t} K_{ts}^2$$

$$= \mathrm{E}\left\{ f_z\ (z_t) \right\} \int K^2(u)du + o_p(1) \qquad (A.29)$$

Note that $\sigma_T^2$ is a nondegenerate U-statistic of order 2 with kernel

$$H_T(z_t, z_s) = \frac{1}{h^m} K^2\left( \frac{z_t - z_s}{h} \right). \qquad (A.30)$$

Since Assumption (A2)(iv)-(v) satisfy the conditions of Lemma 2 of Yoshihara (1976) on the asymptotic equivalence of U-statistic and its projection under $\beta$-mixing, we have for $\gamma = 2(\delta - \delta')/\delta'(2+\delta) > 0$

$$\sigma_T^2 \equiv \frac{1}{T(T-1)} \sum_{s \neq t} H_T(z_t, z_s)$$

$$= \int\int H_T(z_1, z_2) dF_z(z_1) dF_z(z_2)$$

$$+2T^{-1} \sum_{t=1}^{T} \left[ \int H_T(z_t, z_2) dF_z(z_2) - \int\int H_T(z_1, z_2) dF_z(z_1) dF_z(z_2) \right] + O_p(T^{-1-\gamma})$$

$$= \int\int H_T(z_1, z_2) dF_z(z_1) dF_z(z_2) + o_p(1)$$

$$= \int\int \frac{1}{h^m} K^2\left( \frac{z_1 - z_2}{h} \right) dF_z(z_1) dF_z(z_2) + o_p(1)$$

$$= \int K^2(u)\,du \int f_z^2(z)dz + o_p(1) \qquad \text{(A.31)}$$

The result of Theorem (ii) follows from (A.31). □

**Proof of Theorem 1 (iii)**

The proof of Theorem (iii) consists of the two steps.

Step 1. Show that $\hat{J}_T = J_T + o_p(1)$ under the alternative hypothesis (4).

Step 2. Show that $J_T = J + o_p(1)$ under the alternative hypothesis (4),

where $J = E\{[F_{y|z}(Q_\theta(x_t) \mid z_t) - \theta]^2 f_z(z_t)\}$. The combination of Steps 1 and 2 yields Theorem (iii).

*Step 1: Show that* $\hat{J}_T = J_T + o_p(1)$ *under the alternative hypothesis.*

We need to show that the results of Step 2 and Step 3 in the proof of Theorem (i) hold under the alternative hypothesis. First, we show that the result of Step 2 in the proof of Theorem (i) still holds under the alternative hypothesis. We can show that $J_2(Q_\theta - C_T) = o_p(1)$ by the same procedures as in (A.24). Thus we focus on showing that $J_2(Q_\theta) = o_p(1)$. As in the proof of Theorem (i), denote $S(g) \equiv \partial F[g]/\partial g$. By taking a Taylor expansion of $F_{y|z}(Q_\theta(x_s) \mid z_s)$ around $Q_\theta(z_s)$, we have

$$J_2(Q_\theta) = -\frac{1}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t}^{T} \frac{1}{h^m} K\left(\frac{z_t - z_s}{h}\right) \{1(y_t \le Q_\theta(x_t)) - F_{y|z}(Q_\theta(x_t) \mid z_t)\}$$

$$\times S(\bar{Q}_\theta(x_s, z_s))$$

$$= \frac{1}{T} \sum_{t=1}^{T} \{1(y_t \le Q_\theta(x_t)) - F_{y|z}(Q_\theta(x_t))\} S(\bar{Q}_\theta(x_s, z_s)) \hat{f}_z(z_t)$$

$$\equiv \frac{1}{T} \sum_{t=1}^{T} u_t S(\bar{Q}_\theta(x_s, z_s)) \hat{f}_z(z_t), \qquad \text{(A.32)}$$

where $\bar{Q}_\theta(x_s, z_s)$ is between $Q_\theta(x_s)$ and $Q_\theta(z_s)$. By using the same procedures as in

(A.24), we have

$$J_2(Q_\theta) = O\left(T^{-1}h^{-m}\right). \tag{A.33}$$

Next, we show that the result of Step 3 in the proof of Theorem (i) holds under the alternative hypothesis. Since $F(Q_\theta(x_j) \mid z_j) - \theta \neq 0$ for $j = t, s$ under the alternative hypothesis, we have

$$\begin{aligned}
& J_3(Q_\theta) - J_3(Q_\theta - C_T) \\
&= \frac{1}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t}^{T} \frac{1}{h^m} K\left(\frac{z_t - z_s}{h}\right) \times \{F(Q_\theta(x_t) \mid z_t) - \theta\}\{F(Q_\theta(x_s) \mid z_s) - \theta\} \\
&\quad - \frac{1}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t}^{T} \frac{1}{h^m} K\left(\frac{z_t - z_s}{h}\right) \\
&\qquad \times \{F(Q_\theta(x_t) - C_T \mid z_t) - \theta\}\{F(Q_\theta(x_s) - C_T \mid z_s) - \theta\} \\
&= \frac{1}{T} \sum_{t=1}^{T} \{F(Q_\theta(x_t) \mid z_t) - \theta\}\{F(Q_\theta(x_s) \mid z_s) - \theta\} \hat{f}_z(z_t) \\
&\quad - \frac{1}{T} \sum_{t=1}^{T} \{F(Q_\theta(x_t) - C_T \mid z_t) - \theta\}\{F(Q_\theta(x_s) - C_T \mid z_s) - \theta\} \hat{f}_z(z_t).
\end{aligned} \tag{A.34}$$

By taking a Taylor expansion of $F_{y|z}(Q_\theta(x_j) - C_T \mid z_j)$ around $Q_\theta(z_j)$ for $j = t, s$, we have

$$\begin{aligned}
& J_3(Q_\theta) - J_3(Q_\theta - C_T) \\
&= \frac{1}{T} \sum_{t=1}^{T} \{F(Q_\theta(x_t) \mid z_t) - \theta\} C_T S(\bar{Q}_\theta(x_s)) \hat{f}_z(z_t) \\
&\quad + \frac{1}{T} \sum_{t=1}^{T} C_T S(\bar{Q}_\theta(x_t))\{F(Q_\theta(x_s) \mid z_s) - \theta\} \hat{f}_z(z_t) \\
&\quad - \frac{1}{T} \sum_{t=1}^{T} C_T^2 S(\bar{Q}_\theta(x_t)) S(\bar{Q}_\theta(x_s)) \hat{f}_z(z_t).
\end{aligned} \tag{A.35}$$

We further take Taylor expansion of $F_{y|z}(Q_\theta(x_j) \mid z_j)$ around $Q_\theta(z_j)$ for $j = t, s$ and have

$$J_3(Q_\theta) - J_3(Q_\theta - C_T)$$

$$=\frac{1}{T}\sum_{t=1}^{T} S(\bar{Q}_\theta(x_t,z_t))C_T S(\bar{Q}_\theta(x_s))\hat{f}_z(z_t)$$

$$+\frac{1}{T}\sum_{t=1}^{T} C_T S(\bar{Q}_\theta(x_t))S(\bar{Q}_\theta(x_s,z_s))\hat{f}_z(z_t)$$

$$-\frac{1}{T}\sum_{t=1}^{T} C_T^2 S(\bar{Q}_\theta(x_t))S(\bar{Q}_\theta(x_s))\hat{f}_z(z_t), \quad \text{(A.36)}$$

where $\bar{Q}_\theta(x_s,z_s)$ is between $Q_\theta(x_s)$ and $Q_\theta(z_s)$. Then by using the same procedures as in (A.27), we have

$$J_3(Q_\theta)-J_3(Q_\theta-C_T)=O\left(C_T\right). \quad \text{(A.37)}$$

Now we have the result of Step 1 for the proof of Theorem (iii). □

*Step 2: Show that* $J_T = J + o_p(1)$ *under the alternative hypothesis.*

Using (7) and uniform convergence rate of kernel regression estimator under $\beta$-mixing process, we have

$$J_T=\frac{1}{T(T-1)h^m}\sum_{t=1}^{T}\sum_{s\neq t}^{T} K_{ts}\varepsilon_t\varepsilon_s$$

$$=\frac{1}{T}\sum_{t=1} \hat{E}(\varepsilon_t\mid z_t)\hat{f}_z(z_t)\varepsilon_t$$

$$=\frac{1}{T}\sum_{t=1} E(\varepsilon_t\mid z_t)f_z(z_t)\varepsilon_t$$

$$+\frac{1}{T}\sum_{t=1} \left\{\hat{E}(\varepsilon_t\mid z_t)\hat{f}_z(z_t)-E(\varepsilon_t\mid z_t)f_z(z_t)\right\}\varepsilon_t$$

$$=\frac{1}{T}\sum_{t=1} E(\varepsilon_t\mid z_t)f_z(z_t)\varepsilon_t+o_p(1)$$

$$=E\left[E(\varepsilon_t\mid z_t)f_z(z_t)\varepsilon_t\right]+o_p(1)$$

$$=J+o_p(1) \quad \text{(A.38)}$$

□

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