Robust Inflation-Forecast-Based Rules to Shield against Indeterminacy*

Nicoletta Batini
International Monetary Fund

Alejandro Justiniano
International Monetary Fund

Paul Levine
University of Surrey

Joseph Pearlman
London Metropolitan University

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Abstract

We estimate several variants of a linearized form of a New Keynesian model using quarterly US data. Using these rival models and the estimated posterior probabilities we then design rules that are robust in two senses: '*weakly robust*' rules are guaranteed to be stable and determinate in all the possible variants of the model, whereas '*strongly robust*' rules, in addition, use the probabilities to minimize an expected loss function of the central bank subject to this model uncertainty. We find three main results. First, in our two model variants with the highest posterior model probabilities there are substantial stabilization gains from commitment. Second, an optimized inflation targeting rule feeding back on current inflation will result in a unique stable equilibrium and realize at least three-quarters of these potential gains, even if it is used in a variant of the model that is not the one for which it was designed. Third, the performance of optimimized inflation targeting rules perform increasing less well as the forward horizon increases from $j = 0$ to $j = 1, 2$ quarters. For j=2, only a rule designed for our most indeterminacy-prone model is weakly robust and yields determinacy across all models. A strongly robust rule can be designed that sacrifices performance in the least probable models for better performance in the most probable models.

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Contents

1 Introduction

"Uncertainty is not just an important feature of the monetary policy landscape; it is the defining characteristic of that landscape." Alan Greenspan[1]

This paper adopts a consistently Bayesian approach to the measurement of uncertainty and the design of robust rules for the conduct of monetary policy. Employing a New Keynesian model, the source of uncertainty in our paper concerns the structural parameters and the volatility of the white noise disturbances. We estimate several variants of a linearized form of the model using quarterly US data. From these competing specifications we obtain estimates for posterior model probabilities. Using these rival models and the estimated probabilities we then design rules that are robust in two senses: '*weakly robust*' rules are guaranteed to be stable and determinate in all the possible variants of the model whereas '*strongly robust*' rules, also guarantee stable and unique equilibria and, in addition, use the probabilities to minimize an expected loss function of the central bank subject to this model uncertainty.

Our approach thus differs from existing work on the design of robust policy rules in two important respects. First, existing work typically posits uncertainty by arbitrarily calibrating the relative probability of alternative models being true representations of the economy (see for example Angeloni *et al.* (2003); Coenen (2003)). This paper provides a first attempt to quantify and at the same time utilize estimated measures of uncertainty for the design of robust rules. Second we examine robust policy in a unified framework that compares different simple rules with each other, and with their optimal counterparts.

Throughout we focus on Taylor-type rules, and in particular on inflation-forecast-based (IFB) rules. These are 'simple' rules as in Taylor (1993), but where the policy instrument responds to deviations of expected, rather than current inflation from target. In most applications, the inflation forecasts underlying IFB rules are taken to be the endogenous rational-expectations forecasts conditional on an intertemporal equilibrium of the model. These rules are of specific interest because similar reaction functions are used in the Quarterly Projection Model of the Bank of Canada (see Coletti *et al.* (1996)), and in the Forecasting and Policy System of the Reserve Bank of New Zealand (see Black *et al.* (1997)) – two prominent inflation targeting central banks. As shown in Clarida *et al.*

[1] Federal Reserve Bank of Kansas (2003), Opening Remarks.

(2000) and Castelnuovo (2003), estimates of IFB-type rules appear to be a good fit to the actual monetary policy in the US and Europe of recent years.) However, with IFB rules indeterminacy can be particularly severe and can take two forms: if the response of interest rates to a rise in expected inflation is insufficient, then real interest rates fall thus raising demand and confirming any exogenous expected inflation. But indeterminacy is also possible if the rule is overly aggressive (Bernanke and Woodford (1997); Batini and Pearlman (2002); Giannoni and Woodford (2002); Batini *et al.* (2004), BLP hereafter).

We find three main results. First, in our two model variants with the highest posterior model probabilities there are substantial stabilization gains from commitment. This is measured for each model by comparing the expected loss from the optimal policies with and without commitment. We assess this gain to be between a 3.6% to 9.4% equivalent permanent increase in output. Second, an optimized inflation targeting rule feeding back on current inflation will result in a unique stable equilibrium and realize at least three-quarters of these potential gains even if the implemented rule is designed for the wrong model. Current inflation rules, in other words, are robust in both the weak and strong sense. Third, the optimized inflation targeting rules perform increasing less well as the forward horizon increases from $j = 0$ (the current inflation rule) to $j = 1, 2$ quarters. Denoting such a rule by IFBj, we find a qualitative difference between IFB1 and IFB2 rules. For IFB1 optimal rules, a rule designed for the wrong model is still weakly robust, but for IFB2 optimal rules this is no longer the case. Then only a rule designed for our most indeterminacy-prone model is weakly robust and yields determinacy across all models. In both cases a strongly robust rule can be designed that sacrifices performance in the least probable models for better performance in the most probable models.

The rest of the paper is organized as follows. Section 2 sets out a New Keynesian model which, in its most general form, exhibits persistence in both inflation and output. Sections 4 examines the indeterminacy problem of IFB rules using the root locus method employed by Batini and Pearlman (2002) and BLP. This analysis indicates which features of the model make it indeterminacy-prone. Section 5 first focuses on IFB and optimal rules without uncertainty before we turn to the robust policy problem in section 6. A final section 7 provides conclusions. The general solution solution procedures for computing the optimal rules are set out in an Appendix.

2 Recent Related Literature

Not surprisingly other approaches to policy design with uncertainty are found in a rapidly growing literature. Hansen and Sargent (2002) (henceforth H&S) adopt a minmax framework with three key ingredients that distinguishes it from alternatives. First, it conducts 'local analysis' in the sense that it assumes that the true model is known only up to some local neighborhood of models that surround the 'approximating' or 'core' model. Second, it uses a minmax criterion without priors in model space. Third, the type of uncertainty is both unstructured and additive being reflected in additive shock processes that are 'chosen' by malevolent nature to feed back on state variables so has to maximize the loss function the policy-maker is trying to minimize. Another strand retains the minmax framework but assumes bounded uncertainty about the values of certain parameters in the model (Giannoni (2002), Gaspar and Smets (2002), Angeloni *et al.* (2003)). Tetlow and von zur Muehlen (2002) provides a comparison of the H&S unstructured model uncertainty and the latter structured approaches. Walsh (2003) provides a useful overview of the literature and and carries out a number of policy exercises using a similar New Keynesian model to that in our paper. First, he assesses Taylor rules and first difference rules when target variables in these rules can only be measured imperfectly. Second, he examines 'robust, optimal, explicit instrument rules' proposed by Giannoni and Woodford (2002) and Svensson and Woodford (1999) that are robust in the sense of being independent of both the variance-covariance structure of the white-noise disturbances and the serial correlation of the disturbances. Third, he implements the H&S robust control procedure. The conclusions most relevant to our paper are that uncertainty about the output gap suggests using a first difference rule, and that parameter uncertainty has no general implications for the size of the feedback coefficients in the optimized simple rules.

3 The Model

Our model is the closed economy version of BLP. There is one traded risk-free nominal bond. A final homogeneous good is produced competitively using a CES technology consisting of a continuum of differentiated non-traded goods. Intermediate goods producers and household suppliers of labor have monopolistic power. Nominal prices of intermediate goods, are sticky. We incorporate a bias for consumption of home-produced goods, habit formation in consumption, and Calvo price setting with indexing of prices for those firms who, in a particular period, do not re-optimize their prices. The latter two aspects of the model follow Christiano *et al.* (2001) and, as with these authors, our motivation is an empirical one: to generate sufficient inertia in the model so as to enable it, in calibrated form, to reproduce commonly observed output, inflation and nominal interest rate responses to exogenous shocks.

Our model is stochastic with two exogenous AR(1) stochastic processes for total factor productivity in the intermediate goods sector and government spending.

3.1 Households

A representative household r maximizes

$$\mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t(r) - H_t)^{1-\sigma}}{1-\sigma} + \chi \frac{\left(\frac{M_t(r)}{P_t}\right)^{1-\varphi}}{1-\varphi} - \kappa \frac{N_t(r)^{1+\phi}}{1+\phi} + u(G_t) \right] \tag{1}$$

where $\mathcal{E}_t$ is the expectations operator indicating expectations formed at time t, $C_t(r)$ is an index of consumption, $N_t(r)$ are hours worked, H_t represents the habit, or desire not to differ too much from other consumers, and we choose it as $H_t = hC_{t-1}$, where C_t is the average consumption index and $h \in [0, 1)$. When $h = 0$, $\sigma > 1$ is the risk aversion parameter (or the inverse of the intertemporal elasticity of substitution)[2]. $M_t(r)$ are end-of-period nominal money balances and $u(G_t)$ is the utility from exogenous real government spending G_t.

The representative household r must obey a budget constraint:

$$P_t C_t(r) + D_t(r) + M_t(r) = W_t(r)N_t(r) + (1 + i_{t-1})D_{t-1}(r) + M_{t-1}(r) + \Gamma_t(r) - P_t\tau_t \tag{2}$$

where P_t is a price index, $D_t(r)$ are end-of-period holdings of riskless nominal bonds with nominal interest rate i_t over the interval $[t, t+1]$. $W_t(r)$ is the wage, $\Gamma_t(r)$ are dividends from ownership of firms and τ_t are lump-sum real taxes. In addition, if we assume that households' labour supply is differentiated with elasticity of supply η, then (as we shall see below) the demand for each consumer's labor is given by

$$N_t(r) = \left(\frac{W_t(r)}{W_t}\right)^{-\eta} N_t \tag{3}$$

[2]When $h \neq 0$, σ is merely an index of the curvature of the utility function.

where $W_t = \left[\int_0^1 W_t(r)^{1-\eta} dr\right]^{\frac{1}{1-\eta}}$ is an average wage index and $N_t = \int_0^1 N_t(r)dr$ is aggregate employment.

Maximizing (1) subject to (2) and (3) and imposing symmetry on households (so that $C_t(r) = C_t$, etc) yields standard results:

$$1 = \beta(1+i_t)\mathcal{E}_t\left[\left(\frac{C_{t+1}-H_{t+1}}{C_t-H_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \tag{4}$$

$$\left(\frac{M_t}{P_t}\right)^{-\varphi} = \frac{(C_t-H_t)^{-\sigma}}{\chi P_t}\left[\frac{i_t}{1+i_t}\right] \tag{5}$$

$$\frac{W_t}{P_t} = \frac{\kappa}{(1-\frac{1}{\eta})}N_t^{\phi}(C_t-H_t)^{\sigma} \tag{6}$$

(4) is the familiar Keynes-Ramsey rule adapted to take into account of the consumption habit. In (5), the demand for money balances depends positively on consumption relative to habit and negatively on the nominal interest rate. Given the central bank's setting of the latter, (5) is completely recursive to the rest of the system describing our macro-model and will be ignored in the rest of the paper. (6) reflects the market power of households arising from their monopolistic supply of a differentiated factor input with elasticity η.

3.2 Firms

Competitive final goods firms use a continuum of non-traded intermediate goods according to a constant returns CES technology to produce aggregate output

$$Y_t = \left(\int_0^1 Y_t(m)^{(\zeta-1)/\zeta} dm\right)^{\zeta/(\zeta-1)} \tag{7}$$

where ζ is the elasticity of substitution. This implies a set of demand equations for each intermediate good m with price $P_t(m)$ of the form

$$Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t \tag{8}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm\right]^{\frac{1}{1-\zeta}}$. P_t is an aggregate intermediate price index, but since final goods firms are competitive and the only inputs are intermediate goods, it is also the domestic price level.

In the intermediate goods sector each good m is produced by a single firm m using only differentiated labour with another constant returns CES technology:

$$Y_t(m) = A_t\left(\int_0^1 N_{tm}(r)^{(\eta-1)/\eta} dr\right)^{\eta/(\eta-1)} \tag{9}$$

where $N_{tm}(r)$ is the labour input of type r by firm m and A_t is an exogenous shock capturing shifts to trend total factor productivity (TFP) in this sector. Minimizing costs $\int_0^1 W_t(r)N_{tm}(r)dr$ and aggregating over firms leads to the demand for labor as shown in (3). In a equilibrium of equal households and firms, all wages adjust to the same level W_t and it follows that $Y_t = A_t N_t$.

For later analysis it is useful to define the real marginal cost as the wage relative to domestic producer price. Using (6) and $Y_t = A_t N_t$ this can be written as

$$MC_t \equiv \frac{W_t}{A_t P_t} \quad = \quad \frac{\kappa}{(1-\frac{1}{\eta})A_t}\left(\frac{Y_t}{A_t}\right)^{\phi}(C_t - H_t)^{\sigma} \tag{10}$$

Now we assume that there is a probability of $1-\xi$ at each period that the price of each intermediate good m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is indexed to last period's aggregate producer price inflation.[3] With indexation parameter $\gamma \geq 0$, this implies that successive prices with no reoptimization are given by $P_t^0(m)$, $P_t^0(m)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma}$, $P_t^0(m)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma}, \ldots$. For each intermediate producer m the objective is at time t to choose $\{P_t(m)\}$ to maximize discounted profits

$$\mathcal{E}_t \sum_{k=0}^{\infty}\left(\frac{\xi}{1+i_t}\right)^k Y_{t+k}(m)\left[P_t^0(m)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma} - \frac{W_{t+k}}{A_t}\right] \tag{11}$$

given i_t (since firms are atomistic), subject to (8). The solution to this is

$$\mathcal{E}_t \sum_{k=0}^{\infty}\left(\frac{\xi}{1+i_t}\right)^k Y_{t+k}(m)\left[P_t^0(m)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma} - \frac{1}{(1-1/\zeta)}\frac{W_{t+k}}{A_t}\right] = 0 \tag{12}$$

and by the law of large numbers the evolution of the price index is given by

$$P_{t+1}^{1-\zeta} = \xi\left(P_t\left(\frac{P_t}{P_{t-1}}\right)^{\gamma}\right)^{1-\zeta} + (1-\xi)(P_{t+1}^0)^{1-\zeta} \tag{13}$$

3.3 Equilibrium

In equilibrium, goods markets, money markets and the bond market all clear. Equating the supply and demand of the consumer good we obtain

$$Y_t = AN_t = C_t + G_t \tag{14}$$

[3]Thus we can interpret $\frac{1}{1-\xi}$ as the average duration for which prices are left unchanged.

A balanced budget government budget constraint

$$G_t = \tau_t + \frac{M_t - M_{t-1}}{P_t} \tag{15}$$

completes the model. Given interest rates i (expressed later in terms of an optimal or IFB rule) the money supply is fixed by the central banks to accommodate money demand. By Walras' Law we can dispense with the bond market equilibrium condition and therefore the government budget constraint that determines taxes τ_t. Then the equilibrium defined at $t = 0$ as stochastic processes C_t, D_t, P_t, M_t, W_t, Y_t, N_t, given past price indices and exogenous TFP and government spending processes.

3.4 Linearization and State Space Representation

We now linearize about the deterministic zero-inflation steady state. Output is then at its sticky-price, imperfectly competitive natural rate and from the Keynes-Ramsey condition (4) the nominal rate of interest is given by $\bar{\imath} = \frac{1}{\beta} - 1$. Now define all lower case variables as proportional deviations from this baseline steady state.[4]

Then the linearization takes the form:

$$\begin{aligned}
\pi_t &= \frac{\beta}{1+\beta\gamma}\mathcal{E}_t\pi_{t+1} + \frac{\gamma}{1+\beta\gamma}\pi_{t-1} + \frac{(1-\beta\xi)(1-\xi)}{(1+\beta\gamma)\xi}mc_t && (16)\\
mc_t &= -(1+\phi)a_t + \frac{\sigma}{1-h}(c_t - hc_{t-1}) + \phi y_t && (17)\\
c_t &= \frac{h}{1+h}c_{t-1} + \frac{1}{1+h}\mathcal{E}_t c_{t+1} - \frac{1-h}{(1+h)\sigma}(i_t - \mathcal{E}_t\pi_{t+1}) && (18)\\
y_t &= \frac{\bar{C}}{\bar{Y}}c_t + \frac{\bar{G}}{\bar{Y}}g_t && (19)\\
g_t &= \rho_g g_{t-1} + \epsilon_{gt} && (20)\\
a_t &= \rho_a a_{t-1} + \epsilon_{at} && (21)
\end{aligned}$$

Variables y_t, c_t, mc_t, a_t, g_t are proportional deviations about the steady state. π_t and i_t are absolute deviations about the steady state.[5] For later use we require the *output gap* the difference between output for the sticky price model obtained above and output when

[4]That is, for a typical variable X_t, $x_t = \frac{X_t - \bar{X}}{\bar{X}} \simeq \log\left(\frac{X_t}{\bar{X}}\right)$ where $\bar{X}$ is the baseline steady state. The interest rate however is now expressed as an absolute deviation about $\bar{\imath}$.

[5]Note that (20) and (18) imply an IS curve given by $y_t - \frac{h}{1+h}y_{t-1} - \frac{1}{1+h}\mathcal{E}_t y_{t+1} - s_g(g_t - \frac{h}{1+h}g_{t-1} - \frac{1}{1+h}\mathcal{E}_t g_{t+1}) + \frac{(1-h)(1-s_g)}{(1+h)\sigma}(i_t - \mathcal{E}_t\pi_{t+1}) = 0$ where $s_g = \frac{\bar{G}}{\bar{Y}}$.

prices are flexible, y_{nt} say. The latter, obtained by putting $\xi = 0$ into (16) to (19), is in deviation form given by[6]:

$$\frac{\sigma}{1-h}(c_{nt} - hc_{n,t-1}) + \phi y_{nt} = (1+\phi)a_t \tag{22}$$

$$y_{nt} = \frac{\bar{C}}{\bar{Y}}c_{nt} + \frac{\bar{G}}{\bar{Y}}g_t \tag{23}$$

We can write this system in state space form as

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathcal{E}_t \mathsf{x}_{t+1} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + Bi_t + C \begin{bmatrix} \epsilon_{gt+1} \\ \epsilon_{at+1} \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} y_t \\ y_{nt} \end{bmatrix} = E \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{25}$$

where $\mathsf{z}_t = [a_t, g_t, c_{t-1}, c_{n,t-1}, \pi_{t-1}]$ is a vector of predetermined variables at time t and $\mathsf{x}_t = [c_t, \pi_t]$ are non-predetermined variables. Rational expectations are formed assuming an information set $\{z_s, x_s\}$, $s \leq t$, the model and the monetary rule.

3.5 Estimation

3.5.1 Overview

In this section we estimate four main variants of model (16)-(21) using Bayesian methods. In particular, we estimate: the most general specification of the model with both inflation and habit persistence(we label this variant 'Z'); a version of the model without inflation persistence but with persistence in habits ($\gamma = 0$, we label this variant 'G'); a version without habit persistence but with persistence in inflation ($h = 0$, we label this variant 'H'); and finally a version with neither inflation nor habit persistence ($\gamma = h = 0$, we label this variant 'GH'). We close the model with a 1-quarter ahead IFB rule of the form (29) that is the subject of the next section.

Bayesian estimation of the model has the specific advantage that it provides a posterior distribution of the parameter values that allows us to make probabilistic statements about the functionals of the model(s)' parameters. Furthermore, it provides us with the odds on models that allow us to quantify how likely itis the data would have come from a model with both habit and inflation persistence as opposed to a framework with just one

[6]Note that the zero-inflation steady states of the sticky and flexi-price steady states are the same.

of these mechanisms or neither. In this sense the estimation method *per se* supplies us with a consistent measure of both parameter (posterior distribution of the parameters) and model (posterior odds) uncertainty.[7]

The idea here is to utilize both measures of uncertainty in the construction of a policy rule that, in the presence of such uncertainty, is robust in both the weak and strong senses.[8] The derivation of robust rules using consistent measures of parameter and model uncertainty directly from estimation of the model thus advances upon existing studies on the design of robust rules that instead 'calibrated' uncertainty in an ad hoc fashion (see, for example, Levin *et al.* (2001); Rudebusch (2002); Angeloni *et al.* (2003); Coenen (2003).

The sub-sections below offer: a brief sketch of the methods used in estimation (Subsection 3.5.2); a discussion of the specification of the prior distributions (Sub-section 3.5.3); the results from the estimation of our four model specifications (Sub-section 3.5.4); and a formal comparison of models (Sub-section 3.5.5). This sub-section shows how we obtain the posterior model probabilities that we use as weights for the competing model specifications in the analysis of robust IFB rules under uncertainty.

3.5.2 Methodology

Each model indexed by k and denoted m_k, has an associated set of unknown parameters $\omega_k \in \Omega_k$. Following a Bayesian approach, our aim is to characterize the posterior distribution of the models' parameters, $p\left(\omega_k|Y^T, m_k\right)$,where Y^T stands for the full sample of observed data (T denotes the number of observations). Having specified a (perhaps model specific) prior density, $p(\omega_k|m_k)$, the posterior of the parameters is given by

$$p\left(\omega_k|Y^T, m_k\right) = \frac{\mathcal{L}\left(\omega_k|Y^T, m_k\right) p(\omega_k|m_k)}{\int \mathcal{L}\left(\omega_k|Y^T, m_k\right) p(\omega_k|m_k) d\omega_k} \tag{26}$$

where $\mathcal{L}\left(\omega_k|Y^T, m_k\right)$ is the likelihood obtained under the assumption of normally distributed disturbances from the state-space representation implied by the solution of the

[7]Justiniano and Preston (2004) discuss the many additional advantages of using Bayesian methods to estimate dynamic stochastic general equilibrium models. These include overcoming convergence problems with numerical routines to maximize the likelihood as well as providing measures of uncertainty that need not assume a symmetric distribution.

[8]This paper for now only provides results for robustness with respect to model uncertainty. Current research is investigating robustness with respect to parameter uncertainty.

linear rational expectations model. The denominator in equation (26) corresponds to the marginal likelihood (also known as the 'marginal data density') and, as explained later, plays a key role in model comparisons.

The solution of the model is a non-linear function of the parameters which does not allow for any closed-form expression for the posterior density. Furthermore, the high-dimensionality of the parameters space renders numerical integration inefficient. Markov Chain Monte Carlo (MCMC) methods, however, provide a feasible and accurate approximation to this density.

Following Schorfheide (2000) the estimation follows a two step approach. In the first step, a numerical algorithm is used to approximate the posterior mode by combining the likelihood $\mathcal{L}(Y^T|\omega_k, m_k)$ with the prior. In the second step, the obtained posterior mode is then used as starting value (ω_k^0) for a Random Walk Metropolis algorithm that generates draws from the posterior $p(\omega_k|Y^T, m_k)$. At each step i of the Markov Chain, the proposal density used to draw a new candidate parameter ω_k^* is a normal centered at the current state of the chain, $N(\omega_k^i, c\Sigma_k)$. A new draw is then accepted with probability

$$\alpha = \min(1, \frac{\mathcal{L}(Y^T|\omega_k^*, m_k)p(\omega_k^*|m_k)}{\mathcal{L}(Y^T|\omega_k^i, m_k)p(\omega_k^i|m_k)})$$

If accepted, $\omega_k^{i+1} = \omega_k^*$; otherwise, $\omega_k^{i+1} = \omega_k^i$. We generate chains of 130,000 draws in this manner discarding the first 30,000 iterations.[9]

Point estimates of the parameters ω_k can be obtained from the generated values by using various location measures, such as mean or, as in this paper, medians. Similarly, measures of uncertainty follow from computing the percentiles of the draws.

3.5.3 Data and Priors

We estimate the model(s) using quarterly US data on real GDP (detrended–as standard in the literature we detrend this using a Hodrick-Prescott filter, see Lubik and Schorfheide (2003), Juillard *et al.* (2004)), the Federal Funds rate (annualized, in percentage points), and the annualized log difference of the consumer price index (CPI) for the sample 1984:I-

[9]This initial burn-in phase is intended to remove any dependence of the chain from its starting values.

2003:IV.[10] All series were obtained from DataStream International.

Following Lubik and Schorfheide (2004) rather than de-meaning the series, we estimate the mean of inflation and the (unobservable) real interest rate, π^* and r^* respectively, together with the model(s) parameters. In turn, this gives the following mapping between observables (superscript *obs*) and the variables following the solution of the model.

$$\begin{pmatrix} \pi_t^{obs} \\ y_t^{obs} \\ i_t^{obs} \end{pmatrix} = \begin{pmatrix} \pi^* \\ 0 \\ \pi^* + r^* \end{pmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} \pi_t \\ y_t \\ i_t \end{pmatrix}$$

In addition, the mean of the real rate gives us an estimate of the discount factor $\beta = 1/\sqrt[0.25]{1 + \frac{r^*}{100}}$.

To proceed with the Bayesian estimation we need a prior distribution for the parameters. Details on our priors are presented in **Table 1** in Appendix B reporting the type of density, mean and standard deviation for each coefficient.[11] The last two columns also provide the 1% and 99% percentiles of the prior ordinates. In choosing these densities we considered the entire spectrum of prior existing empirical estimates or calibrations. As a result, some of our priors are more widely dispersed, and therefore less tight than those chosen by other authors. [12]

The degree of habit formation (h), price indexation (γ) and interest smoothing in the IFB-type rule (ρ), as well as the autoregressive coefficients of the shocks (ρ_g and ρ_a) are all constrained to the unit interval, motivating our choice of Beta densities for these priors. The priors for h and γ are centered at 0.7, on the assumption that output and inflation are considerably inertial, in line with findings by Fuhrer and Moore (1995), Fuhrer (2000), Banerjee and Batini (2003, 2004) and Smets and Wouters (2004)(SW, 2004), among others. Likewise, our prior for the mean of ρ is rather high and close to the estimates from Clarida *et al.* (2000) (CGG,2000).

[10]Eight observations, corresponding to the perior 1982:I - 1983:IV are used to intialize the Kalman filter.

[11]In principle, is would be possible to specify flat or non-informative priors for estimating θ_k. However, in addition to being able to choose priors based on coefficients values available in the literature, flat priors are not well suited for model comparisons.

[12]Throughout the estimation of different models, the share of government expenditures in output is calibrated at 0.22,which represents the sample average of this coefficient for our sample.

Priors for σ and ϕ are shaped in the form of a Gamma density and are chosen to be fairly flat, reflecting the wide dispersion of existing empirical estimates and calibrations of these parameters in the literature. (see Nelson and Nikolov (2002)).

The slope of the Phillips' curve, $\lambda = \frac{(1-\beta\xi)(1-\xi)}{(1+\beta\gamma)\xi}$ is a function of the degree of price stickiness in the economy, ξ, and the discount factor. So we selected the prior for λ in line with the assumption that the quarterly discount factor is equal to 0.99 and prices are sticky for three quarters, as suggested by survey evidence on the average duration of US price contracts (see, for example, Blinder *et al.* (1998).[13]

Finally, the prior for θ accounts for the breadth of the spectrum of estimated responses to expected inflation by the US Federal Reserve. More specifically, our specification contains the 90% posterior intervals of Lubik and Schorfheide (2004)[14] and it is fairly looser than the prior specified by SW for the same parameter.[15]

3.5.4 Estimation Results

Table 2, Appendix B, summarizes the results of estimating the four model variants (G,H,GH and Z). The three columns for each specification report the median, 1st and 9th decile of the 100,000 draws generated using the Random Walk-Metropolis algorithm used to approximate the posterior densities.

A few important things emerge from the table. First, estimates of the policy coefficients are fairly robust across specifications. Posterior estimates of ρ are tightly concentrated on values that suggest a substantial degree of interest smoothing, in accordance with results reported by CGG amongst other authors. Meanwhile, the posterior density for θ is remarkably similar (that is both in medians and percentiles) across the first three specifications, implying a very aggressive response by the US Federal Reserve to expected

[13] It is worth noting that the results of the estimation from assuming a prior directly on the Calvo coefficient ξ are somewhat different. This may be because with a prior on lambda, as we have used now, the link between ξ and the discount factor in determining the slope of the PC is not imposed. We plan to re-run the estimation with this alternative prior as a robustness check.

[14] Note that in contrast to these authors however we constrain the estimation to the region of determinacy and therefore truncate the prior for θ. The results of their paper suggest, however, that at least for a Taylor rule on current inflation, indeterminacy has not been an issue for our sample. In light of the results in BLP, exploring whether their results extend to the estimation of IFB is left for a future project.

[15] In their paper, however, SW include the output gap in the Taylor rule.

inflation, in line with findings by CGG for a similar rule and sample.

The median estimates for r^* translate into a median value of 0.995 for the stochastic discount factor which, in turn, implies plausible estimates for the degree of price stickiness based on the inferred values for λ. The implied point estimates of ξ range from 0.36 up to 0.67, decreasing, as expected, depending on whether or not price indexation is allowed for.[16] These higher values are in accordance with Blinder *et al.* (1998) and Rotemberg and Woodford (1998), but contrast the high degree of price rigidity estimated by SW (2004).

Our estimates of σ are rather large. With no habits, these estimates map directly with the intertemporal elasticity of substitution and suggest that this may be quite small.[17] A common theme in papers estimating DSGE models is the difficulty in pinning down ϕ. Therefore, it is not surprising that, inference on the inverse Frisch elasticity of labor supply is susceptible to the specification of the model, and exhibits wide posterior probability intervals.

Turning to the coefficient governing habit formation, h is tightly estimated and suggests rather inertial consumption and output processes. Reported posterior intervals for h are almost identical to the ones obtained by Juillard *et al.* (2004) and higher than the estimates by SW. By contrast, the posterior density of γ lies to the left of our chosen prior, suggesting, in contrast to studies mentioned earlier, that inflation is intrinsically not very persistent – a result that accords with findings in Erceg and Levin (2001), Taylor (2000) and Cogley and Sargent (2001).

Estimates of the shock processes reveal that both the technology and the government expenditure shock are highly persistent, and this holds true regardless of the exact model specification. Posterior estimates clearly attribute greater volatilty of shocks to the government expenditure component rather than to disturbances in technology.[18]

[16] Using $\lambda \equiv \frac{(1-\beta\xi)(1-\xi)}{(1+\beta\gamma)\xi}$ we obtain $\xi = 0.67, 0.36, 0, 60, 0.53$ corresponding to contract lengths, $\frac{1}{1-\xi}$, of 3.06, 1.57, 2.50 and 2.13 quarters for models G, H, GH and Z respectively.

[17] This result is attributable to a prior density centered on high values for σ. Redoing the estimation using the SW priors leads to point estimates far closer to one, clearly revealing that inference on this parameters is sensitive to the choice of priors.

[18] Indeed, the 1st posterior decile of the former exceeds the 9th decile of the latter, for all models, despite similar prior densities for the innovation standard deviations. As usual, exogenous disturbances to the monetary policy equation appear much less important than technology and government expenditure shocks in driving inflation, consumption and output processes.[19]

3.5.5 Model Comparison

Since the goal of this paper is to characterize the design of robust rules under uncertainty, it is important to investigate which specification seems to be best supported by the data. In doing so we do not intend to select any particular model as being the 'true' one but rather wish to compute posterior probabilities to place odds on the different models.

Bayesian methods for model comparisons allows us to obtain these posterior model probabilities in order to discriminate or aggregate across competing specifications, thereafter providing coefficient estimates that explicitly account for model uncertainty. Let us define m_k to be one possible element from the (discrete) set of competing models $\mu = \{G, H, GH, Z\}$. The posterior model probability for $p(m_k|Y^T)$ summarizes the evidence provided by the data in favor of m_k and is then given by

$$p(m_k|Y^T) = f(Y^T|m_k)p(m_k)/f(Y^T) \tag{27}$$

where $p(m_k)$ stands for the prior probability assigned to model k, that in our case equals $\frac{1}{4}$ since we treat each model as equiprobable a-priori. The first expression in the numerator is known as the marginal likelihood (or marginal data density) and was previously presented as the denominator in equation (26)

$$f(Y^T|m_k) = \int \mathcal{L}\left(\omega_k|Y^T, m_k\right) p(\omega_k|m_k)d\omega_k \tag{28}$$

Equation (27) can be conveniently re-written as

$$p(m_k|Y^T) = \frac{f(Y^T|m_k)p(m_k)}{\sum_{i\in\mu} f(Y^T|m_i)p(m_i)}$$

which emphasizes the key role the marginal likelihood plays in constructing odds on models.[20] Computing the marginal likelihood usually requires simulation methods and a number of proposals in this vein are now available from the statistics literature.[21]

In this paper, instead we estimate the model probabilities by relying on the Reversible Jump MCMC algorithm (RJMCMC) of Dellaportas *et al.* (2002)). This method belongs

[20]Notice that posterior model probabilities lead directly to the posterior odds that can be used to compare two models, say m_k and m_h by updating the prior odds with the Bayes Factor.

[21] Chib (2000) provides an excellent survey of simulation methods in general and discusses methods for model comparison

to the popular class of product space search algorithms, widely used in the statistic literature,[22] that allow for the joint estimation of the model indicator, m_k and parameters ω_k and which do not requiring therefor evaluating the marginal likelihood.[23]

Estimates of $p(m_k|Y^T)$ obtained with the RJMCMC for our four model variants are presented in **Table 3**, Appendix B. In line with results discussed above, the specification with habit persistence and no price indexation (G) attains highest posterior probability. Model Z that allows for both of these intrinsic mechanisms follows in probability ranking. In contrast, a model with no habit persistence is 9 times less likely than those specifications (Z and G) with endogenous persistence in consumption. Finally, the most restrictive model., GH attains the lowest posterior model probability further providing evidence of the need to incorporate at least one of the two intrinsic mechanisms imparting greater inertia to the model. Therefore, these results can be interpreted as suggesting that the addition of endogenous mechanisms of persistence, particularly habit in consumption, improve the fit of the model. These posterior odds will be used to weight the models for our analysis of uncertainty on the robustness of policy rules.

4 The Stability and Determinacy of IFB Rules

4.1 Theory

This section studies an IFB rule of the form

$$i_t = \rho i_{t-1} + \theta(1-\rho)\mathcal{E}_t \pi_{t+j} \tag{29}$$

[22]The interest reader is referred to Carlin and Han (2001) for an overview of these methods.

[23]The RJMCMC requires a set of preliminary runs in order to generate a proposal density, $V(\omega_k|m_k)$ for each model's parameters To this end we use the 100,000 draws generated for the estimation of the parameters discussed above and choose $V(\omega_k|m_k)$ to be a normal density, centered at the mean of those draws and with fatter tails than the posterior. The algorithm then operates over the product space $\mu \times \Pi_{k=1}\Omega_k$ by drawing , at each step, a candidate model m_k^* from a proposal density J -which in our case assigns equals probability to all models- drawing ω_k from $V(\omega_k|m_k)$ and then accepting or not the jump to m_k^* with a Metropolis type probability. We generated 200,000 draws in this manner, discard the first 20,000 and then compute the proportion of draws the sampler spent in each model to directly obtain the model probabilities.

where $j \geq 0$ is the forecast horizon, which is a feedback on single-period inflation over the period $[t+j-1, t+j]$.[24]

With rule (29), policymakers set the nominal interest rate so as to respond to deviations of the inflation term from target. In addition, policymakers smooth rates, in line with the idea that central banks adjust the short-term nominal interest rate only partially towards the long-run inflation target, which is set to zero for simplicity in our set-up.[25] The parameter $\rho \in [0, 1)$ measures the degree of interest rate smoothing. j is the feedback horizon of the central bank. When $j = 0$, the central bank feeds back from current dated variables only. When $j > 0$, the central bank feeds back instead from deviations of forecasts of variables from target. Finally, $\theta > 0$ is the feedback parameter: the larger is θ, the faster is the pace at which the central bank acts to eliminate the gap between expected inflation and its target value. We now show that, for given degrees of interest rate smoothing ρ, the stabilizing characteristics of these rules depend both on the magnitude of θ and the length of the feedback horizon j.

To understand better how the precise combination of the pair (j, θ), IFB rules can lead the economy into instability or indeterminacy consider the deterministic model economy (24) and (25) with interest rate rules of the form (29). g_t and a_t are exogenous stable processes and play no part in the stability analysis. For convenience, we therefore set them to zero.

Let z be the forward operator. Taking z-transforms of (16), (17), (18) and (29), the characteristic equation for the system is given by:

$$
\begin{aligned}
& (z-\rho)[(z-1)(z-h)(\beta z-1)(z-\gamma) - \frac{\lambda}{\mu} z^2 (\tilde{\phi} z + \mu(z-h))] \\
+ \quad & \frac{\lambda\theta}{\mu}(1-\rho)(\tilde{\phi} z + \mu(z-h)) z^{j+2} = 0 \qquad (30)
\end{aligned}
$$

where we have defined $\lambda \equiv \frac{(1-\beta\xi)(1-\xi)}{\xi}$, $\tilde{\phi} \equiv \frac{\bar{C}}{\bar{Y}}\phi$ and $\mu \equiv \frac{\sigma}{1-h}$. Equation (30) shows that the minimal state-space form of the system has dimension $\max(5, j+3)$. Since there are

[24] To set the model up with this rule in state-space form for $j \leq 1$ we simply need to augment the state vector with a lagged term i_{t-1}. For $j = 2$ replace t with $t+1$ in (16)-(17) and take expectations at time t. Then the state-space presentations remains of the same dimension. For $j > 2$ replace t with $t+j-1$ in (16)-(17) and take expectations at time t. The state vector must then be augmented with $\mathcal{E}_t\pi_{t+1} \cdots \mathcal{E}_t\pi_{t+j-2}$.

[25] For instance (29) can be written as $\Delta i_t = \frac{1-\rho}{\rho}[\theta\mathcal{E}_t\pi_{t+j} - i_t]$ which is a partial adjustment to a static IFB rule $i_t = \theta\mathcal{E}_t\pi_{t+j}$.

3 predetermined variables in the system, it follows that the saddle-path condition for a unique stable rational expectations solution is that the number of roots inside the unit circle of the complex plane is 3 and the number of outside the unit circle is $\max(2, j)$.

To identify values of (j, θ) that involve exactly three roots of equation (30) we graph the root locus of (θ, z) pairs that traces how the roots change as θ varies between 0 and ∞. All the graphs can be drawn by following the rules set out in Appendix A of BLP. Other parameters in the system, including the feedback horizon parameter j in the IFB rule, are kept constant. We generate separate charts, each conditioning on a different horizon assumption. Each chart shows the complex plane (indicated by the solid thin line),[26] the unit circle (indicated by the dashed line), and the root locus tracking zeroes of equation (30) as θ varies between 0 and ∞ (indicated by the solid bold line). The arrows indicate the direction of the arms of the root locus as θ increases. Throughout we experiment with both a 'high' and a 'low' $\frac{\lambda}{\mu}$, as defined after (30). The economic interpretation of these cases is that the high $\frac{\lambda}{\mu}$ case corresponds to low ξ (i.e., more flexible prices) and low $\frac{\sigma}{1-h}$ (low risk aversion and habit formation).

The term inside the square brackets in equation (30) corresponds to no nominal interest rate feedback rule (i.e., an open-loop interest rate policy). Then rule (29) is switched off and so the lagged term i_{t-1} disappears from our model; the system now requires exactly *two* stable roots for determinacy. Figure 1 plots the root locus in this case. Since with no policy θ is set to 0, the root locus is just a set of dots: namely, the roots of equation (30) when $\theta = 0$. Note that depending on the value of λ/μ, the position of these roots varies, and in the flexible price, low interdependence case where $\frac{\lambda}{\mu}$ is high, there are complex roots indicating oscillatory dynamics.[27] The diagram shows that there are *too many* stable roots in both cases (i.e. 3 instead of 2), which implies that with no interest rate feedback rule, there will always be indeterminacy in the system.

If the nominal interest rate rule is switched on and now feeds back on current rather than expected inflation, i.e. $j = 0$, then the root locus technique yields a pattern of zeros as depicted in Figure 2. Interest rate smoothing brings about a lag in the short-term

[26] In this plane, the horizontal axis depicts real numbers, and the vertical axis depicts imaginary numbers. If a root is complex, i.e. $z = x + iy$, then its complex conjugate $x - iy$ is also a root. Thus the root locus is symmetric about the real axis.

[27] How we find the position of these zeros is the main example of Appendix A.

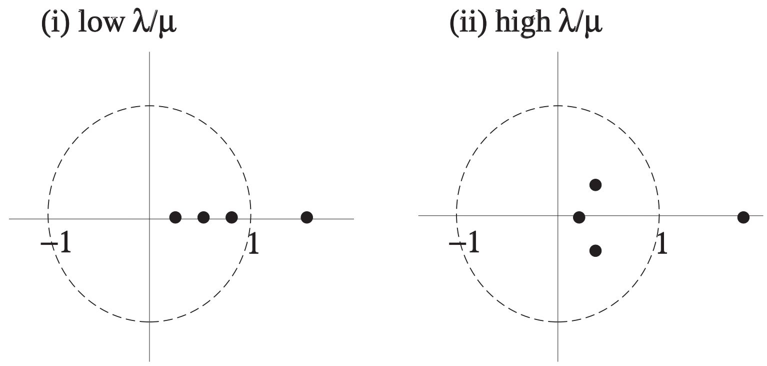

Figure 1: **Possible position of zeroes when $\theta = 0$**

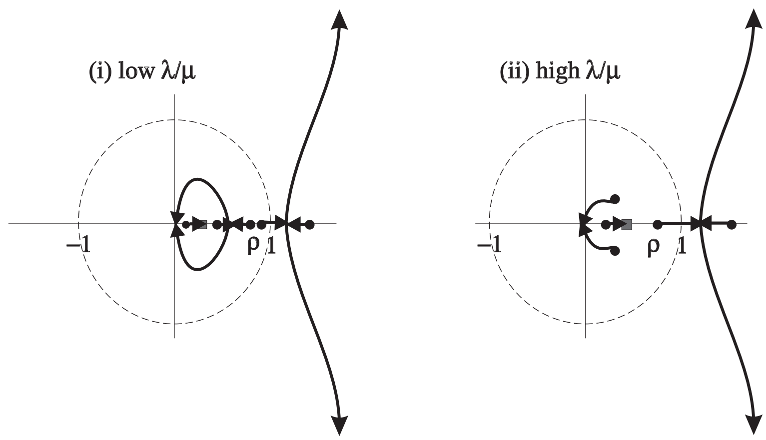

Figure 2: **Position of zeroes as θ changes using current inflation**

nominal interest rate and the system is now stable if it has exactly *three* stable roots (as we now have three predetermined variables in the system). The figure demonstrates that if θ is sufficiently large, one arm of the root locus starting originally at ρ exits the unit circle, turning one root from stable to unstable so that there are now three – as required – instead of four stable roots and the system has a determinate equilibrium. As $\theta \to \infty$, there are roots at $\pm i\infty$, two roots at 0, and one at $\mu h/(\tilde{\phi} + \mu)$, the latter shown as a square.

Note that when $\theta = z = 1$, the characteristic equation has the value 0, confirming that the branch of the root locus moving away from $z = \rho$ crosses the unit circle at a value $\theta = 1$. Thus we conclude that for a rule feeding back on current inflation, the system exhibits determinacy if and only if $\theta > 1$. For higher values of $j \geq 1$ we can draw the sequence of root locus diagrams shown in Figures 3-6, and so confirm the well-known 'Taylor

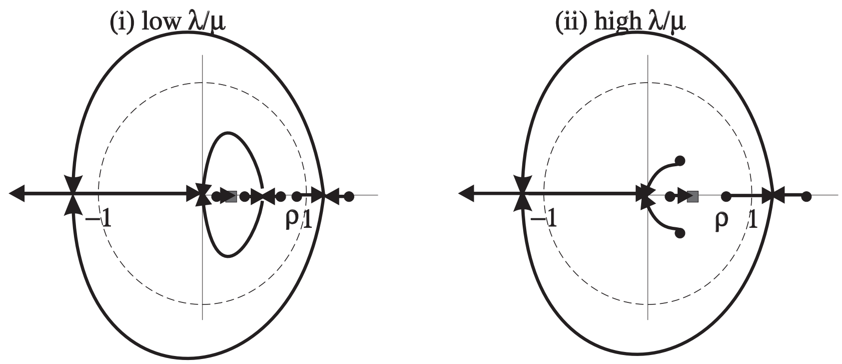

Figure 3: **Position of zeroes as θ changes: 1-period ahead expected inflation**

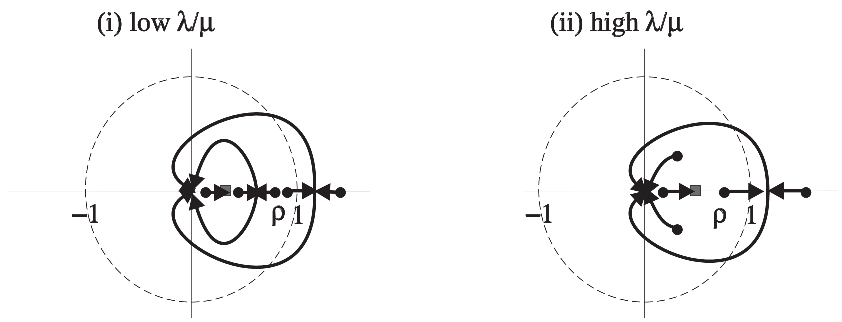

Figure 4: **Position of zeroes as θ changes: 2-period ahead expected inflation**

Principle' that interest rates need to react to inflation with a feedback greater than unity. However for $j \geq 1$ our diagrams show that an arm of the root locus re-enters the unit circle for some high $\theta > 1$ and indeterminacy re-emerges. Therefore $\theta > 1$ is necessary but not sufficient for stability and determinacy. Our results up to this point are summarized in proposition 1:

***Proposition 1*: For a rule feeding back on current inflation ($j = 0$), $\theta > 1$ is a necessary and sufficient condition for stability and determinacy. For higher feedback horizons ($j \geq 1$), $\theta > 1$ is a necessary but not sufficient condition for stability and determinacy.**

Now let $\bar{\theta}(j)$ be the upper critical value of θ for the system for a feedback horizon j. Figure 3 shows that for the case $j = 1$, i.e. one-quarter ahead forecasts which corresponds to a case studied by CGG (2000), indeterminacy occurs when this portion of the root locus enters the unit circle at $z = -1$.[28] The critical upper value for $\theta = \bar{\theta}(1)$ when this occurs is obtained by substituting $z = -1$ and $j = 1$ into the characteristic equation (30) to obtain:

$$\bar{\theta}(1) = \frac{1+\rho}{1-\rho}\left[1 + \frac{2(1+h)(1+\beta)(1+\gamma)\mu}{\lambda(\tilde{\phi} + \mu(1+h))}\right] \tag{31}$$

One important thing to note looking at this expression for a 1-period ahead IFB rule is that the greater is the degree of smoothing captured by the parameter ρ in the interest rate rule, the larger the maximum permissible value of θ before indeterminacy sets in. In this sense indeterminacy is less of a potential problem for high ρ. Similarly from (31) the problem of indeterminacy lessens for high h, γ and σ, and low λ and $\tilde{\phi}$. Notice from the definition of λ after (30) that low λ is associated with a high degree of price stickiness. With this is mind we can summarize these results as:

***Proposition 2*: For a 1-period ahead IFB rule, indeterminacy is less of a problem if there is a high degree of interest rate smoothing, habit persistence, price indexing, price stickiness and household risk aversion, and a low elasticity of disutility with respect to hours worked.**

Proceeding on to j-period ahead IFB rules for $j \geq 2$ the analysis is more difficult. For $j = 2$, Figures 4 shows that indeterminacy occurs when the root locus enters the unit circle at $z = cos(\psi) + isin(\psi)$ for some $\psi \in (0, \frac{\pi}{2})$. All our results up to this point are analytical using topological reasoning, but now the threshold $\bar{\theta}(j)$ for $j \geq 2$ must be found numerically. Given j, write the characteristic equation as

$$\sum_{k=1}^{max(5,j+3)} a_k(\theta) z^k = 0 \tag{32}$$

noting that some of the a_k are dependent on θ. The root locus meets the unit circle at $z = cos(\psi) + isin(\psi)$. Using De Moivre's theorem $z^k = cos(k\psi) + isin(k\psi)$ and equating real and imaginary parts we arrive at two equations which can be solved numerically for $\bar{\theta}$ and ψ.

As well as locating an upper threshold $\bar{\theta}(j)$, an even more significant result concerning indeterminacy emerges from Figure 4. This have been drawn in for values of ρ such that

[28] Thus Figure 3 portrays diagrammatically the result shown analytically by Woodford (2003), chapter 4, that there is a value of $\theta = \theta^S$ say, beyond which there is indeterminacy.

the two rightmost poles of the root locus are joined by straight lines that meet *outside* the unit circle. The implication is that for some values of $\theta > 1$, these yield unstable roots of the system, and therefore the system will have exactly three stable roots which is what is required for determinacy. (Note that if the arms of the root locus from ∞ cross the unit circle before these latter meet, then there may anyway be too many stable roots). However, for a lower value of ρ it could happen that rather than meeting to the right of $z = 1$, the two arms instead meet to the left of $z = 1$, that is *inside* the unit circle and then remain within it, as in figure 5. This would imply that for all θ there are always more than three stable roots, which would entail, in turn, indeterminacy for all values of θ. We therefore conclude that there is determinacy for θ slightly greater than 1 if the root locus passes through $z = 1$ from the left, as in figures 3 and 4. Conversely, Figure 5 for the left and middle examples show indeterminacy for all θ if the root locus passes through $z = 1$ from the right. However, to be certain that this result is true for all θ, we need to be able to show that once this arm of the root locus enters the unit circle it never leaves it, as is the case in the right hand example of Figure 5. The simplest case for which this 'pathological' behaviour cannot happen is when $h = \gamma = 0$. We can now show:

***Proposition 3*: For the general model there is always some lead J^S such that for**

$$j > J^S = \frac{1}{1-\rho} + \frac{(1-\beta)(1-\gamma)\sigma}{\lambda(\tilde{\phi}+\sigma)} \tag{33}$$

there is indeterminacy for all values of θ, provided that that the arm of the root locus from the right is 'non-pathological' in the sense that it enters the unit circle only once. If $h = \gamma = 0$ this is true if $\beta > \rho > \sqrt{2} - 1$ and $\frac{\lambda(\tilde{\phi}+\sigma)}{\sigma} > \frac{(1-\beta)(1+\rho)(1-\rho)^2}{\rho^2+2\rho-1}$.

***Proof*:** See BLP, Appendix B.

For $h, \gamma > 0$ the derivation of sufficient conditions that rule out pathological behaviour has proved elusive. However for small values of h, γ, the root locus diagrams correspond to the 'low λ/μ' ones of Figures 2-4, with the inner arms that lie off the real axis becoming vanishingly small as h, γ tend to 0. By a continuity argument therefore, it follows that the sufficient conditions of Proposition 3 apply in this case as well for small h, γ. Numerical experiments indicate that pathological behaviour does not occur for all realistic values of the parameters. Indeed it is extremely difficult to numerically produce diagrams such as

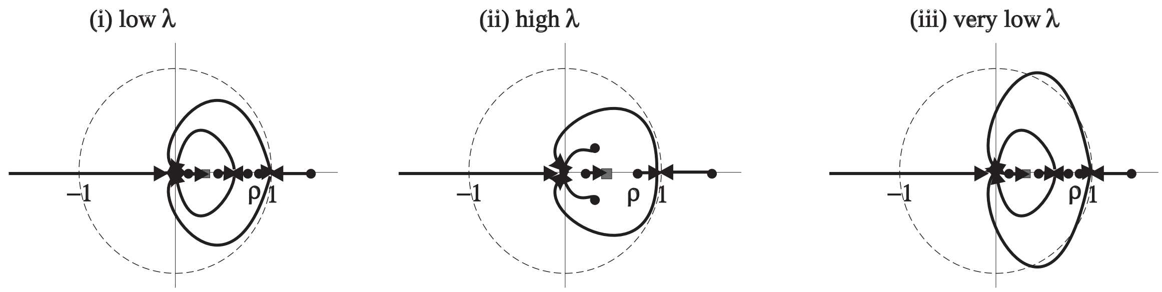

Figure 5: **Position of zeros as θ changes: 3-period ahead expected inflation, and low ρ**

that on the right-hand-side of figure 5. For example with other parameters set at central values the parameter ξ must exceed 0.9, corresponding the price contracts of 10 quarters to generate such an example. In addition our calibrated values indicate that the sufficient conditions in proposition 3 are easily satisfied.

Propositions 1 and 3 confirm, in a rigorous setting, the possibility of real indeterminacy for any IFB rule with lead $j \geq 1$ when the feedback inflation is below unity (the Taylor principle) and above a threshold $\bar{\theta}(j)$. The root locus diagrams in figures 3 and 4 show that $\theta(\bar{1}) > \theta(\bar{2})$, so that indeterminacy becomes more of a problem as j increases from $j = 1$ to $j = 2$. Tables 1a-1f below shows that this deterioration continues for higher j and eventually, from proposition 2, for high j no IFB rule of the form (29) results in a unique stable equilibrium. The value of ρ is crucial in determining the critical value of the lead j beyond which indeterminacy sets in. The lower ρ, the lower the maximum-permitted inflation horizon the central bank can respond to, and hence, the larger the region of indeterminacy under IFB rules. Thus the absence of interest rate smoothing has the same indeterminacy-inducing effect as high j.

In tables 1a parameter values are set as for model G (apart from θ which is calculated to be the threshold value). Then the numerical calculations for alternative values of parameters h, γ, λ, σ and ϕ are repeated in tables 1b-1f. The results show that proposition 2 which applies to 1-period ahead IFB rules only may well carry over to j-period ahead rules as well for changes to h, λ and γ, but not to σ and ϕ.

j	j=1	j=2	j=3	j=4	$j \geq 5$
$\bar{\theta}(j)$	222	23.4	5.2	1.74	indeterminacy

Table 1a. Critical upper bounds for $\bar{\theta}(j)$ for Model G. Parameter values: $h = 0.85$, $\gamma = 0$, $\lambda = 0.16$, $\sigma = 3.29$, $\phi = 1.46$.

j	j=1	j=2	j=3	j=4	$j \geq 5$
$\bar{\theta}(j)$	171	17.5	2.8	1.72	indeterminacy

Table 1b. Critical upper bounds for $\bar{\theta}(j)$ for Model G but with $h = 0$.

j	j=1	j=2	j=3	j=4	$j \geq 4$
$\bar{\theta}(j)$	328	48.6	6.7	1.84	indeterminacy

Table 1c. Critical upper bounds for $\bar{\theta}(j)$ for Model G but with $\gamma = 0.5$.

j	j=1	j=2	j=3	j=4	$j \geq 5$
$\bar{\theta}(j)$	43	6.9	2.6	1.47	indeterminacy

Table 1d. Critical upper bounds for $\bar{\theta}(j)$ for Model G but with $\lambda = 1$.

j	j=1	j=2	j=3	j=4	$j \geq 5$
$\bar{\theta}(j)$	209	25.4	5.9	2.07	indeterminacy

Table 1e. Critical upper bounds for $\bar{\theta}(j)$ for Model G but with $\sigma = 1$.

j	j=1	j=2	j=3	j=4	$j \geq 4$
$\bar{\theta}(j)$	215	24.5	5.6	2.04	indeterminacy

Table 1c. Critical upper bounds for $\bar{\theta}(j)$ for Model G but with $\phi = 3$.

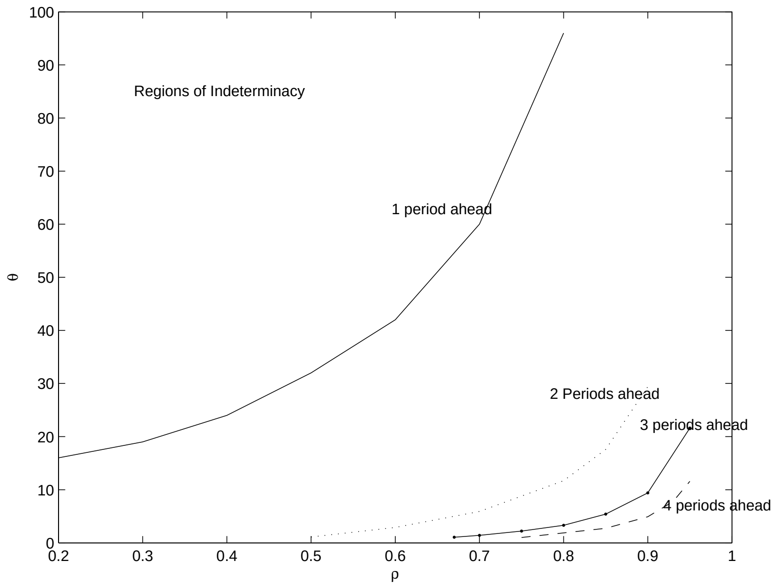

Figure 6: **Regions of Indeterminacy for single period inflation rate targets.**

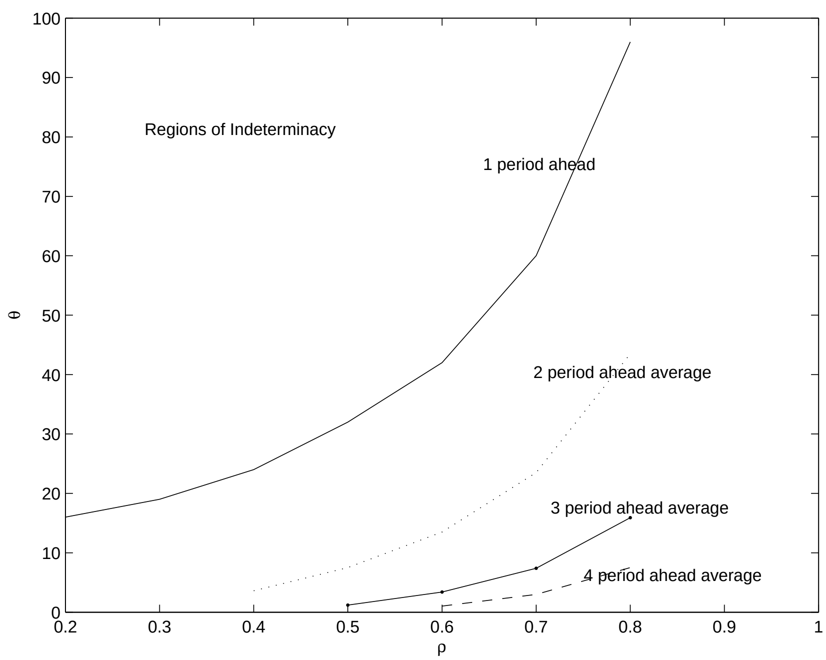

Figure 7: **Regions of Indeterminacy for average inflation rate targets.**

4.2 The Likelihood of Indeterminacy

Figures 6 and 7 show the indeterminacy for parameters ρ and θ based on model G. These figures are based on 1000 draws of parameter combinations using the estimated parameter distributions of section 2.5. Regions to the south-west of each contour then represent 100% confidence regions of determinacy. Figure 6 is drawn for single period IFB rules as in (29) for $j = 1, 2, 3, 4$. In figure 7, this is compared with an average IFB rule of the form

$$i_t = \rho i_{t-1} + \theta(1-\rho)\mathcal{E}_t \sum_{r=0}^{j} \pi_{t+r} \tag{34}$$

For both single-period ahead and average period rules, regions of determinacy indicate combinations of ρ and θ that are weakly robust for all possible non-policy parameter combinations in model G. The declining size of this region as the forward horizon j increases confirms the earlier theoretical results that show that IFB rules with unique and stable equilibria are increasingly constrained in the choice of (ρ, θ) with a qualitative change taking place between $j = 1$ and $j = 2$.[29]

5 Optimal Policy and Optimized IFB Rules without Model Uncertainty

Without model uncertainty, the policy problem of the central bank at time $t = 1$ is to choose in each period $t = 1, 2, 3, \cdots$ an interest rate i_t so as to minimize a loss function that depends on the variation of the output relative to an an output target $o_t = y_{nt} + k$, inflation and the change in the nominal interest rate[30]:

$$\Omega_0 = \mathcal{E}_0 \left[\frac{1}{2} \sum_{t=0}^{\infty} \beta_c^t \left[(y_t - o_t)^2 + b\pi_t^2 + c(i_t - i_{t-1})^2 \right] \right] \tag{35}$$

where β_c is the discount factor of the central bank. The term k is ambitious target for output that exceeds the natural level of output. It arises because natural level of output is not efficient (owing to mark-up pricing in a monopolistically competitive intermediate

[29]Ideally we would like to locate the 100%, 95% etc confidence regions of weak robustness across all possible non-policy parameter combinations *across all model variants*. This will feature in a future version of this paper.

[30]Notice this is a central bankers' loss function, not a welfare function. It describes the actual policy objectives banks have (or are instructed to have) rather than what they should have.

goods, market power in the labour market and habit persistence). This inefficiency can be seen from the model set out in BLP. Consider the steady state where $C_t = C_{t-1} = C$ and $Y = C + G$. The utility of the representative household is found by choosing Y to maximize

$$U = \frac{[(1-h)(Y-G)]^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi}\left(\frac{Y}{A}\right)^{1+\phi} \tag{36}$$

After some rearrangement the first order condition for the efficient level of output, $Y = Y^*$, is

$$(Y^*)^{\phi}(Y^* - G)^{\sigma} = \frac{(1-h)^{1-\sigma}A^{1+\phi}}{\kappa} \tag{37}$$

From BLP in the sticky-price model, the real marginal cost is given by

$$MC = \frac{\kappa}{(1-\frac{1}{\eta})A}\left(\frac{Y}{A}\right)^{\phi}(C(1-h))^{\sigma} = 1 - \frac{1}{\zeta} \tag{38}$$

where η and ζ are the elasticities for demand for differentiated labour and the differentiated intermediate good respectively. Rearranging this gives the following expression for the natural level of output Y^n

$$(Y^n)^{\phi}(Y^n - G)^{\sigma} = \frac{\left(1-\frac{1}{\zeta}\right)\left(1-\frac{1}{\eta}\right)A^{1+\phi}}{\kappa(1-h)^{\sigma}} \tag{39}$$

It follows that $Y^n < Y^*$ iff

$$\left(1-\frac{1}{\zeta}\right)\left(1-\frac{1}{\eta}\right) < 1-h \tag{40}$$

In the absence of habit persistence ($h = 0$) this will also hold, but with habit persistence it is possible that the steady state natural rate of output is *too large, not too small*. The intuition here is that with habit persistence each household derive utility from consumption at time t relative to hC_{t-1} where C_t is aggregate consumption considered exogenous by each household. In equilibrium other households behave similarly and the consumption-leisure choice of each household is distorted in favour of too much consumption and too little leisure (therefore too much labour supply and output) compared with to the efficient choice of the central planner. Thus for high h and high values of elasticities ζ and η (which reduce the efficiency in the output and labour markets) it is possible that $k < 0$ which will lead to a negative inflation bias! However for elasticities sufficiently close to unity (which is consistent with empirical estimates) (40) will hold and $k > 0$. In fact in our simulations we will set $k = \frac{Y^* - Y^n}{Y^n} > 0$ thus implying combinations of these elasticities consistent with this chosen value.

5.1 Optimal Policy with and without Commitment

Before turning to IFB rules, we compute the optimal policies where the policy maker can commit, and the optimal discretionary policy where no commitment mechanism is in place. We compare the performance of these policies with that of an estimated one-period-ahead IFB rule. All parameter values apart from those defining the central bank's loss function are based on model G as reported in section 2.5.[31]

In our linear-quadratic framework optimal policies (including those for optimal IFB rules) conveniently decompose into deterministic and stochastic components. Let target variables in (35) be written as sums of a deterministic stochastic components such as $y_t = \bar{y}_t + \tilde{y}_t$ where all variables are expressed in deviation form about the baseline zero-inflation deterministic steady state of the model. Then the loss function decomposes as

$$\begin{aligned}\Omega_0 &= \frac{1}{2}\sum_{t=0}^{\infty}\beta_c^t\left[(\bar{y}_t-\bar{o}_t)^2+b\bar{\pi}_t^2+c(\bar{i}_t-\bar{i}_{t-1})^2+\mathcal{E}_0\left[(\tilde{y}_t-\tilde{o}_t)^2+b\tilde{\pi}_t^2+c(\tilde{i}_t-\tilde{i}_{t-1})^2\right]\right]\\ &= \bar{\Omega}_0+\tilde{\Omega}_0 \qquad (41)\end{aligned}$$

say. The policymaker can then design an optimal policy consisting of an open-loop trajectory that minimizes $\bar{\Omega}_0$ subject to the deterministic model plus a feedback rule that minimizes $\tilde{\Omega}_0$ subject to a stochastic model expressing stochastic deviations about the open-loop trajectory. By the property of *certainty equivalence* for full optimal policies, but *not* optimized simple rules, the feedback rule is independent of both the initial values of the predetermined variables and the variance-covariance matrix of the white-noise disturbances.

Figures 8-11 show the deterministic component of inflation and the output gap under optimal policies compared with the trajectories under the estimated one-period ahead IFB rule. The optimal policy under commitment provides a benchmark with which to compare the loss in other policy rules. Comparing the optimal discretionary policy with the latter gives an empirical assessment of the potential gains from commitment. In these simulations we have set $c = 1$ in (35), $k = 5\%$ and calibrated b to result in an annual inflationary bias (the long-run inflation rate) under discretion of 5%. This gave $b = 2.5, 1.5, 0.85$ for models G, GH and Z respectively. The discount factor of the central bank was set at $\beta_c = 0.988$ which corresponds to an annual discount rate of 5%.

[31]Details of solution procedures are provide in the Appendix.

In figures 8 and 9 the central bank responds only to an ambitious output target $k = 5\%$ with all predetermined variables at the beginning of period 1 at the baseline steady. In figure 6 with commitment the central bank engages in a bout of inflation engineered by a drop in the interest rate, but this quickly falls to close to zero. Under discretion however we have the familiar inflationary bias of 1.3% per quarter used to calibrate b. Corresponding to these inflation rates the output gap $y_t - y_t^n$ in figure 7 rises by 0.22% moving a little way towards its target of $k = 5\%$, and then falls towards zero. Under discretion there is a smaller rise in the output gap and in the long-run there is a permanent increase arising from the small output-inflation trade-off in the model.

In figures 10 and 11 we suppress the ambitious output target by putting $k = 0$ and allow the policymaker to engage in a deterministic stabilization exercise in response to a shock to TFP of 1% at the beginning of period $t = 1$. We can now compare the stabilization performance of the optimal rules with the estimated IFB rule. In figures 10 and 11 we can see that the commitment policy stabilizes inflation and the output gap somewhat better than the discretionary policy and that both optimal policies are far superior in this respect to the estimated actual rule. In figures 12 and 13 we repeat this exercise for a shock to government spending. From figure 12, this increases inflation and the central bank responds by raising the real interest rate and moderating the increase in demand. The flexible price level of output rate y_{nt} rises with government spending because the latter crowds out consumption and households respond by supplying more labour. Aggregate demand then rises, but by less than y_{nt} and the net effect of these changes in figure 11 is to see the output gap initially fall before gradually returning to its steady state. In figure 13 changes in the natural level of output, y_{nt} dominate the output gap $y_t - y_{nt}$, so there are imperceptible differences between the policy rules.

5.2 Optimized IFB Rules

We now turn to optimized IFB rules and optimal Taylor-type rules feeding back on either current inflation alone or on inflation and the output gap. The latter is expressed as

$$i_t = \rho i_{t-1} + (1 - \rho)[\theta \pi_t + \theta_y (y_t - y_t^n)] \tag{42}$$

Starting at the steady state rules of the form (29) or (42) are stabilization rules responding only to displacements of a_t and g_t. In all the results from this point onwards

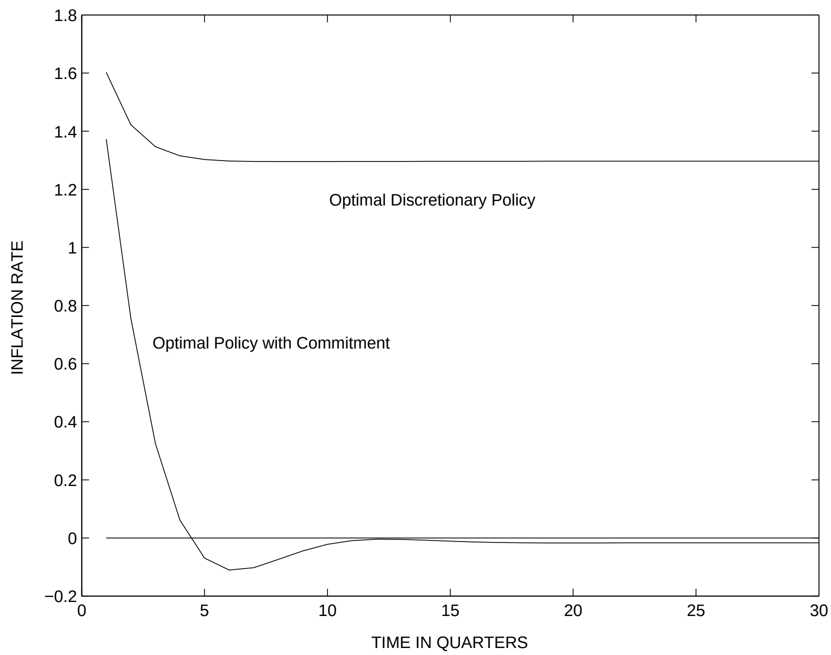

Figure 8: **Quarterly Inflation Rate (%) for Deterministic Optimal Policy.** $k =$ 5%**,** $a(0) = g(0) = 0$**.**

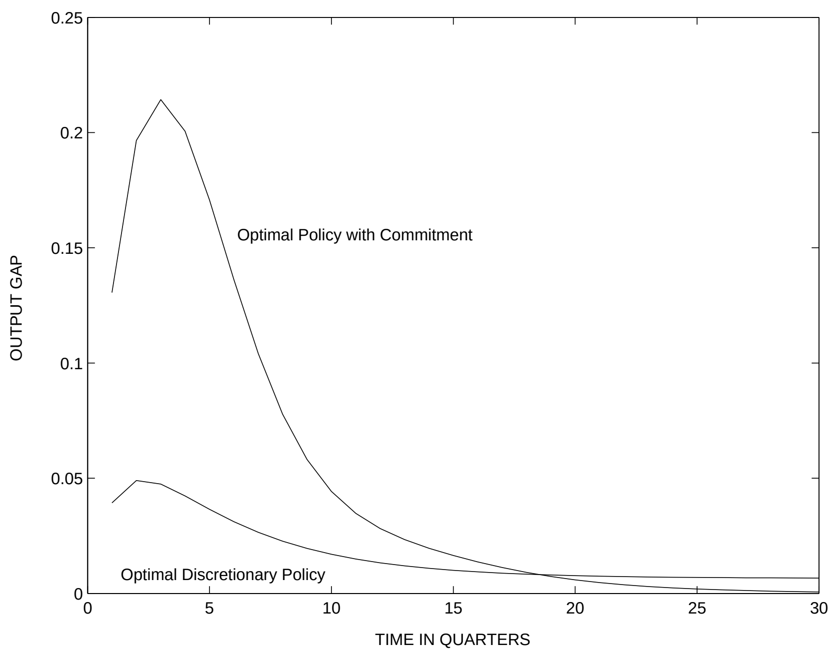

Figure 9: **Output Gap (% deviation from baseline) for Deterministic Optimal Policy.** $k = 5\%$**,** $a(0) = g(0) = 0$**.**

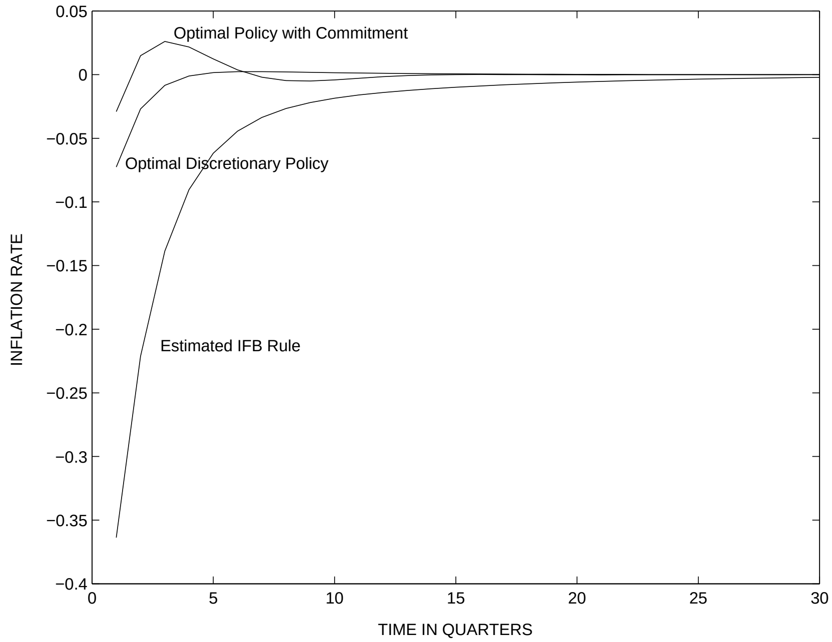

Figure 10: **Quarterly Inflation Rate (%) for Stabilization Policy: Supply Shock,** $k = 0\%$**,** $a(0) = 1$**;** $g(0) = 0$**.**

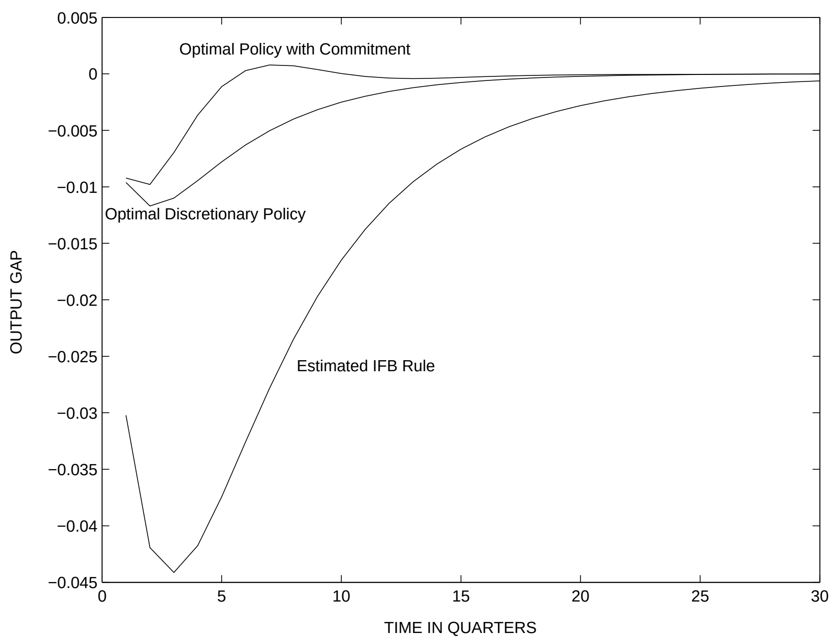

Figure 11: **Output Gap (% deviation from baseline)for Stabilization Policy: Supply Shock,** $k = 0\%$**,** $a(0) = 1$**;** $g(0) = 0$**.**

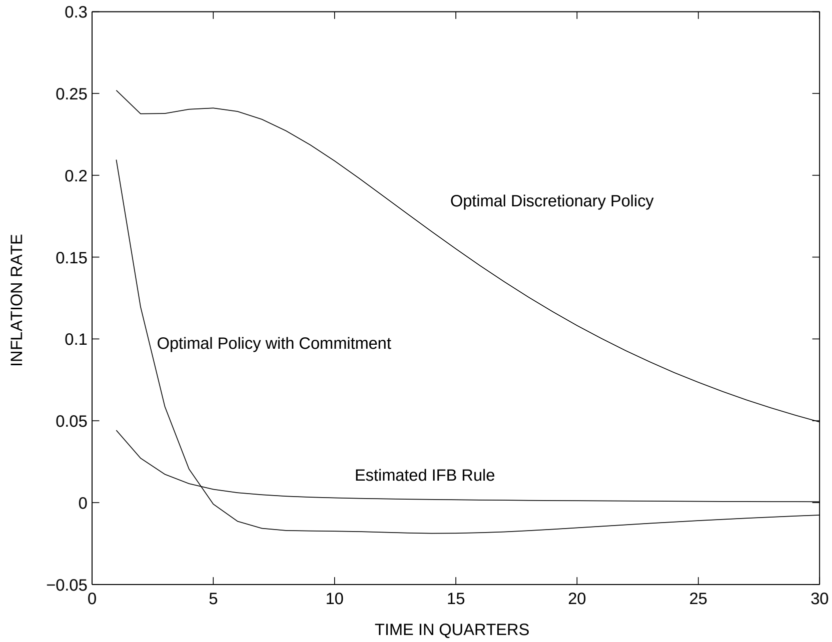

Figure 12: **Quarterly Inflation Rate (%) for Stabilization Policy: Demand Shock, $k = 0\%$, $a(0) = 0$; $g(0) = 1$.**

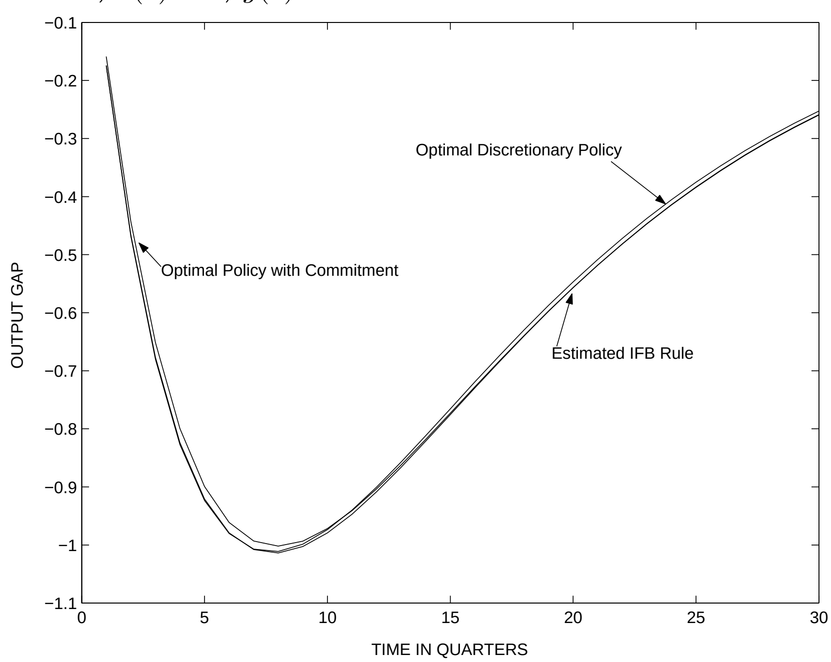

Figure 13: **Output Gap (% deviation from baseline)for Stabilization Policy: Demand Shock, $k = 0\%$, $a(0) = 0$; $g(0) = 1$.**

we focus exclusively on stabilization policy by putting $a_0 = g_0 = k = 0$ so there is no deterministic component of policy in response to an ambitious output target.[32] Given the estimated variance-covariance matrix of the white noise disturbances, an optimal combination (θ, ρ) can be found for each rule defined by the time horizon $j \geq 0$, and for the Taylor rule, and optimal combination (θ, θ_y, ρ). The results are shown in tables 2 to 5 for the estimated models G, GH, H and Z of section 2.5.

Rule	ρ	θ	θ_y	Loss Function	% Output Equivalent
Estimated 1-period ahead IFB	0.80	2.56	0	2711	1.27
Feedback on π_t	0.94	5.00	0	2696	0.90
Taylor Rule	0.97	4.03	0.09	2686	0.66
1-period ahead IFB	0.83	4.66	0	2708	1.20
2-period ahead IFB	0.58	2.65	0	2749	2.20
Optimal Commitment	n.a.	n.a.	n.a	2659	0
Optimal Discretion	n.a.	n.a.	n.a	3045	9.42

Table 2. Model G: Optimal Rules, Optimized Simple rules and the Estimated 1-period ahead IFB Rule Compared.[33]

[32]Since the IFB rule assumes a commitment mechanism, the policymaker in principle should be able to implement a policy $i_t = \bar{i}_t$ plus a feedback component such as (29) or (42) relative to $\bar{i}_t$, where the latter is the optimal deterministic trajectory found in the previous section.

[33]Let Ω=loss from rule, Ω^{OPT} =loss from optimal rule with commitment. A 1% permanent fall in the output gap leads to a reduction in the loss function of $\frac{1}{2(1-\beta^{CB})} = 41$ in our calibration. The % output equivalent loss is then a measure of the degree of sub-optimality of the IFB or Taylor Rule and is defined as $\frac{\Omega-\Omega^{OPT}}{41} \times 100$ where $\Omega^{OPT} = 125$. In each model the weight b on inflation in (35) was chosen to give an annual inflationary bias of 5%. Denoting this coefficient by $b_s, s = G, GH, H, Z$ this gave calibrated values $b_G = 2.5$, $b_{GH} = 1.5$, $b_H = 2.5$ and $b_Z = 0.85$. The central banker's quarterly discount factor was set at $\beta_c = 0.988$ corresponding to an annual discount rate of 5%. Optimized simple rules were restricted to the ranges $\rho \in [0, 1]$ and $\theta \in [1, 5]$.

Rule	ρ	θ	θ_y	Loss Function	% Output Equivalent
Estimated 1-period ahead IFB	0.72	2.64	0	58.4	1.38
Feedback on π_t	0.67	5.00	0	5.14	0.12
Taylor Rule	0.74	5.00	1.0	4.75	0.11
1-period ahead IFB	0.60	2.09	0	56.5	1.37
2-period ahead IFB	0.57	2.20	0	198	4.82
Optimal Commitment	n.a.	n.a.	n.a	0.31	0
Optimal Discretion	n.a.	n.a.	n.a	4.58	0.10

Table 3. Model GH: As for table 2.

Rule	ρ	θ	θ_y	Loss Function	% Output Equivalent
Estimated 1-period ahead IFB	0.67	2.68	0	302	7.36
Feedback on π_t	0.68	4.98	0	8.14	0.19
Taylor Rule	0.76	5	0.55	7.92	0.19
1-period ahead IFB	0.61	1.55	0	232	5.65
2-period ahead IFB	0.67	3.65	0	3421	83
Optimal Commitment	n.a.	n.a.	n.a	0.19	0
Optimal Discretion	n.a.	n.a.	n.a	7.93	0.19

Table 4. Model H: As for table 2.

Rule	ρ	θ	θ_y	Loss Function	% Output Equivalent
Estimated 1-period ahead IFB	0.77	2.25	0	3812	10.8
Feedback on π_t	0.92	4.64	0	3380	0.24
Taylor Rule	0.95	3.37	0.035	3380	0.20
1-period ahead IFB	0.83	2.82	0	3416	1.12
2-period ahead IFB	0.49	1.93	0	3708	8.2
Optimal Commitment	n.a.	n.a.	n.a	3370	0
Optimal Discretion	n.a.	n.a.	n.a	3519	3.63

Table 5. Model Z: As for table 2.

A number of interesting observations emerge from table 2. First, for the most probable model G, comparing the optimal policies with commitment and with discretion in table 2, the stabilization gain from commitment is equivalent to a 9.42% permanent increase in output as seen by the last column. If the policymaker can commit using a simple rule, the best one in this respect is a Taylor rule, and this realizes a large part of the potential stabilization gains from commitment. Second, the estimated 1-period ahead IFB rule is sub-optimal in its class by an output equivalent of only 0.07% increase in output. Third, the determinacy conditions on ρ and θ severely constrain the range of possible stabilizing rules and as a result compared with the Taylor rule, IFB rules *perform very badly, especially as j increases.* This is what one would expect from proposition 2.

Comparing results across models, models GH and H where there is no habit persistence exhibit quite different results from models G and Z with habit persistence. First habit persistence considerably increases the potential stabilization gains from commitment. In its absence these gains drop to only 0.1% and 0.19% in models GH and H respectively. In such a world there is less of a rationale for simple rules as commitment mechanisms and indeed they fail to perform better than optimal discretion in terms of the policymaker's loss function. If there is a rationale it lies with possibility that the optimal discretionary policy set out in Appendix A.2 gives indeterminacy when implemented as a rule. Although the procedure set out computes a unique time consistent solution and policy rule of the form $i_t = Dz_t$, we find for all models that if we plug the rule into the model we get do indeterminacy. This raises a problem of how such a the rule can be implemented and leaves us with a role for simple rules. Second, the poor performance of IFB rules carries over for remaining models, especially for model H which combines an absence of habit persistence with the indexing of Calvo contracts. Now a 2-period IFB rule results in an output equivalent loss of 83%. The reason for this can be seen from proposition 2. Model H, as well as exhibiting no habit persistence has a high degree of price flexibility, a low degree of household risk aversion and a high elasticity of disutility of work, all factors that encourage indeterminacy according to proposition 2 and the numerical simulations in tables 1a-1f. Since in a quarterly model, a 2-period is well within the practices of inflation-targeting central banks, we regard model H as implausible.[34] We therefore confine ourselves to only three models, G, GH and Z in what follows.

6 Robust Rules with Model Uncertainty

6.1 Theory

In this section we consider model uncertainty in the form of uncertain estimates of the non-policy parameters of the model, $\Theta = (\beta, \gamma, \xi, \phi, \sigma, h, \rho_a, \rho_b, \zeta, \eta, \kappa, \sigma_{at}^2, \sigma_{gt}^2)$. Suppose the state of the world s is described by a model with $\Theta = \Theta^s$ expressed in state-space form as

$$\begin{bmatrix} \mathsf{z}_{t+1}^s \\ \mathcal{E}_t \mathsf{x}_{t+1}^s \end{bmatrix} = A^s \begin{bmatrix} \mathsf{z}_t^s \\ \mathsf{x}_t^s \end{bmatrix} + B^s i_t^s + C^s \begin{bmatrix} \epsilon_{gt+1} \\ \epsilon_{at+1} \end{bmatrix} \quad (43)$$

$$o_i^s = E^s \begin{bmatrix} \mathsf{z}_t^s \\ \mathsf{x}_t^s \end{bmatrix} \quad (44)$$

where $\mathsf{z}_{t-1}^s = [a_t^s, g_t^s, c_{t-1}^s, c_{n,t-1}^s, \pi_{t-1}^s]$ is a vector of predetermined variables at time t and $\mathsf{x}_t = [c_t^s, \pi_t^s]$ are non-predetermined variables in state s of the world. In (43) and (44) it is important to stress that variables are in deviation form about a zero-inflation steady state of the model in state s. For example output in deviation form is given by $y_t^s = \frac{Y_t^s - \bar{Y}^s}{\bar{Y}_s}$ where $\bar{Y}^s$ is the steady state of the model in state s defined by parameters Θ^s and $i_t^s = i_t - \bar{i}^s$ where the natural rate of interest in model s, $\bar{i}_s = \frac{1}{\beta^s} - 1$.

Our approach to robust policy design is to set up a *composite model* of outputs from each of the states $s = 1, 2, \cdots, n$ and to minimize the expected loss across these states using the posterior probabilities obtained in section 2.5. To do so we must set up the model in state s in terms of the *actual* interest rate, not the deviation about the steady state. Then augmenting the state vector to become $\mathsf{z}_t^s = [1, a_t^s, g_t^s, c_{t-1}^s, c_{n,t-1}^s, \pi_{t-1}^s]$ we can rewrite (43) as

$$\begin{bmatrix} \mathsf{z}_{t+1}^s \\ \mathcal{E}_t \mathsf{x}_{t+1}^s \end{bmatrix} = A^s \begin{bmatrix} \mathsf{z}_t^s \\ \mathsf{x}_t^s \end{bmatrix} + B^s i_t + C^s \begin{bmatrix} \epsilon_{gt+1} \\ \epsilon_{at+1} \end{bmatrix}; \;\; s = 1, 2, \cdots, n \quad (45)$$

[34]This is also borne out by an alternative Modified Harmonic Mean estimator proposed by Geweke (1999) which also found that the H-model posterior probability was very low, but not that of the GH model.

$$\Omega_0 = \frac{1}{2}\sum_{t=0}^{\infty}\beta_c^t\sum_{s=1}^{n}p_s\Big[(\bar{y}_t^s-\bar{o}_t^s)^2+b(\bar{\pi}_t^s)^2+c(\bar{i}_t-\bar{i}_{t-1})^2$$
$$+ \ \mathcal{E}_0\left[(\tilde{y}_t^s-\tilde{o}_t^s)^2+b(\tilde{\pi}_t^s)^2+c(\tilde{i}_t-\tilde{i}_{t-1})^2\right]\Big] \quad (46)$$

In (46) the output target in state s of the world is given by $o_t^s = y_{nt} + k^s$ where the ambitious output target $k^s = \frac{(Y^*)^s - Y_n^s}{Y_n^s}$ depends on s. In fact we will continue to assume that the central bank has no ambitious output targets and set $k^s = 0$ in its loss function. However with model uncertainty there is still a deterministic component of policy arising from differences in the natural rate of interest compatible with zero inflation in the steady state, $\bar{i}^s = \frac{1}{\beta^s} - 1$.[35] A rule specifying $i_t = \bar{i}^s$ in the long-run will only result in zero inflation in model s. From the consumers' Euler equation (4) in model r with $\beta^r > \beta^s$, implementing the rule designed for model s gives a steady state inflation rate $\bar{\pi}^r$ given by

$$\frac{\beta^r(1+\bar{i}^s)}{(1+\bar{\pi}^r)} = \frac{\beta^r}{\beta^s(1+\bar{\pi}^r)} = 1 \quad \text{i.e., } \bar{\pi}^r = \frac{\beta^r}{\beta^s} - 1 > 0 \quad (47)$$

Our robust rule designed for any model specifies a natural zero inflation rate of interest $\bar{i}_R$, corresponding to a discount factor $\beta_R = \frac{1}{1+\bar{i}_R}$ to result in an expected long-run inflation rate across models of zero. This implies β_R is determined by

$$\sum_{s=1}^{n} p_s\left[\frac{\beta_s}{\beta_R} - 1\right] \Rightarrow \beta_R = \sum_{s=1}^{n} p_s\beta_s \quad (48)$$

That is, β_R is the expected value of β_s across the model variants.

There is one final consideration first raised by Levine (1986) that is usually ignored in the literature. Up to now we have assumed that private sector expectations $\mathcal{E}_t \mathrm{x}_{t+1}^s$ are state s model-consistent expectations. In other worlds in each state of the world the private sector knows the state and faces no model uncertainty. In a more general formulation of the problem we can relax this assumption and assume that both the policymaker and the private sector faces model uncertainty. Suppose that in state s of the world the latter believes model s' with probability $q_{ss'}$. Then $\mathcal{E}_t \mathrm{x}_{t+1}^s$ must be replaced by the composite expectation $\sum_{s'=1}^{n} q_{ss'}\mathcal{E}_t \mathrm{x}_{t+1}^{s'}$ and the model no longer decomposes into independent systems. In the results that follow we bypass this complication and confine ourselves to model-consistent expectations in each state of the world.

[35] In fact estimated differences in β^s between models are not great, so the point we make here is only potentially important.

6.2 Robust Rules Across Three Rival Models

We now report results for IFB rules with horizon $j = 0, 1, 2$ for three rival models G, GH and Z. The diagonal elements of table 6 gives the policymaker's losses obtained previously in tables 2 to 5 when the optimal rule designed for model s=G, GH, Z is implemented in that model. Figures in brackets refer to output equivalent % losses. We refer to these rules as IFBj(s) for horizon $j = 0, 1, 2$. The off-diagonal entries show the loss outcome when the rule designed for model s is implemented on model $r \neq s$. A striking pattern emerges from this table: whereas the current inflation rules IFB0 are remarkably robust across models, this is no longer true for IFB1 and IFB2 rules. An IFB1 rule designed for the wrong model perform particularly badly in model GH with losses increasing to an equivalent of a permanent decrease in output of around 10%. Matters become worse for the IFB2 rule. Optimal rules designed for models G and H give indeterminacy in model GH.

Rule	Model G	Model GH	Model Z
IFB0(G)	2696 (0.90)	7.48 (0.18)	3380 (0.24)
IFB0(GH)	2698 (0.95)	5.14 (0.12)	3381 (0.27)
IFB0(Z)	2698 (0.95)	6.90 (0.16)	3380 (0.20)
IFB1(G)	2708 (1.20)	397 (9.67)	3429 (1.44)
IFB1(GH)	2716 (1.39)	56.5 (1.39)	3430 (1.46)
IFB1(Z)	2712 (1.29)	441 (10.8)	3416 (1.12)
IFB1(Robust)	2709 (1.22)	64.1 (1.56)	3418 (1.17)
IFB2(G)	2749 (2.20)	indeterminacy	3745 (9.2)
IFB2(GH)	2752 (2.27)	198 (4.82)	3732 (8.83)
IFB2(Z)	2759 (2.44)	indeterminacy	3708 (8.2)
IFB2(Robust)	2751 (2.24)	189.4 (4.83)	3734 (8.88)

Table 6. Value of Loss Function for Different Rules with Model Uncertainty[36]

[36]Current Inflation (s) denotes the optimal rule feeding back on π_t designed for model $s = G, GH, Z$. Similarly IFBj(s) denotes the outcome from the j-horizon IFB rule designed for model s. Each row then gives the value of the loss function for models $s = G, GH, Z$. The % output equivalent losses are in brackets. Diagonal elements correspond to losses in tables 2 to 5.

A simple solution to the lack of robustness of rules IFB1 and IFB2 is to design them on the assumption that the world is characterized by zero habit persistence and price indexing, i.e., by model GH. However by using the procedure set out in the previous subsection we can do better and design a robust rules that shields against indeterminacy in all states and performs better on average across models. Using the probabilities $p_G = 0.56$, $p_{GH} = 0.12$ and $p_Z = 0.32$ for models G, GH and Z respectively, table 7 reports the optimal robust rule. The relevant rows of table 6 show that, compared with the rule designed for model GH, our robust rule sacrifices performance in model GH (which only occurs with low probability) for better performance in the more probable states G and H.

Robust Rule	ρ	θ
1-period ahead IFB	0.830	3.442
2-period ahead IFB	0.574	2.261

Table 7. Robust IFB Rules.

7 Conclusions

Both our theoretical results on IFB rules in section 3 and our numerical results of that and later sections indicate that they become increasing prone to the problem of indeterminacy as the forward horizon increases from $j = 0$ to $j = 2$. As a consequence optimized rules of this type perform increasing worse too. For $j = 2$ we found that the strongly robust rule offers little improvement over the weakly robust rule that only shielded the economy against indeterminacy.

In view of this result the question arises of why do central banks pursue forward-looking targeting rules in the first place? Two main reasons for favouring such rules are commonly cited. First, the delayed response of inflation to interest rate changes obliges monetary authorities to react in a pre-emptive fashion to expected inflation in the future. Second, by targeting inflation in the future in a simple and accountable fashion, the central bank can respond to shocks whilst at the same time providing the private sector with assurances that inflation will eventually return to its long-run target of zero inflation, in our set-up. Of these two reasons only the second is compatible with our analysis. Central banks can only target *forecasts* of future inflation and these can only be conditional on information available at the time the interest rate is set, i.e., the state vector at time t in (24). By committing to a rule that feeds back on inflation $j \geq 1$ periods ahead, since this forecast can be expressed as a linear combination of these state variables, the authority is severely constraining how the interest rate should in effect respond to this information, and it is this constraint that lies at the heart of the poor performance of these rules. It may well be the case that a long forward horizon is necessary to establish the commitment to a low inflation target, but this credibility argument needs to be formalized.

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A Computation of Policy Rules

The general model in deterministic form takes the form

$$\begin{bmatrix} z_{t+1} \\ x^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} z_t \\ x_t \end{bmatrix} + Bw_t \tag{A.1}$$

where z_t is an $(n-m) \times 1$ vector of predetermined variables including non-stationary processed, z_0 is given, w_t is a vector of policy variables, x_t is an $m \times 1$ vector of non-predetermined variables and $x^e_{t+1,t}$ denotes rational (model consistent) expectations of x_{t+1} formed at time t. Then $x^e_{t+1,t} = x_{t+1}$ and letting $y_t^T = \begin{bmatrix} z_t^T & x_t^T \end{bmatrix}$ (A.1) becomes

$$y_{t+1} = Ay_t + Bw_t \tag{A.2}$$

Define target variables s_t by

$$s_t = My_t + Hw_t \tag{A.3}$$

and the policy-maker's loss function at time t by

$$\Omega_t = \frac{1}{2} \sum_{i=0}^{\infty} \lambda^t [s^T_{t+i} Q_1 s_{t+i} + w^T_{t+i} Q_2 w_{t+i}] \tag{A.4}$$

which we rewrite as

$$\Omega_t = \frac{1}{2} \sum_{i=0}^{\infty} \lambda^t [y^T_{t+i} Q y_{t+i} Q y_{t+i} + 2y^T_{t+i} U w_{t+i} + w^T_{t+i} R w_{t+i}] \tag{A.5}$$

where $Q = M^T Q_1 M$, $U = M^T Q_1 H$, $R = Q_2 + H^T Q_1 H$, Q_1 and Q_2 are symmetric and non-negative definite R is required to be positive definite and $\lambda \in (0,1)$ is discount factor. The procedures for evaluating the three policy rules are outlined in the rest of this appendix (or Currie and Levine (1993) for a more detailed treatment).

A.1 The Optimal Policy with Commitment

Consider the policy-maker's *ex-ante* optimal policy at $t = 0$. This is found by minimizing Ω_0 given by (A.5) subject to (A.2) and (A.3) and given z_0. We proceed by defining the Hamiltonian

$$H_t(y_t, y_{t+1}, \mu_{t+1}) = \frac{1}{2}\lambda^t (y_t^T Q y_t + 2y_t^T U w_t + w_t^T R w_t) + \mu_{t+1}(Ay_t + Bw_t - y_{t+1}) \tag{A.6}$$

where μ_t is a row vector of costate variables. By standard Lagrange multiplier theory we minimize

$$L_0(y_0, y_1, \ldots, w_0, w_1, \ldots, \mu_1, \mu_2, \ldots) = \sum_{t=0}^{\infty} H_t \tag{A.7}$$

with respect to the arguments of L_0 (except z_0 which is given). Then at the optimum, $L_0 = \Omega_0$.

Redefining a new costate vector $p_t = \lambda^{-1}\mu_t^T$, the first-order conditions lead to

$$w_t = -R^{-1}(\lambda B^T p_{t+1} + U^T y_t) \tag{A.8}$$

$$\lambda A^T p_{t+1} - p_t = -(Q y_t + U w_t) \tag{A.9}$$

Substituting (A.8) into (A.2)) we arrive at the following system under control

$$\begin{bmatrix} I & \lambda B R^{-1} B^T \\ 0 & \lambda(A^T - U R^{-1} B^T) \end{bmatrix} \begin{bmatrix} y_{t+1} \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} A - B R^{-1} U^T & 0 \\ -(Q - U R^{-1} U^T & I \end{bmatrix} \begin{bmatrix} y_t \\ p_t \end{bmatrix} \tag{A.10}$$

To complete the solution we require $2n$ boundary conditions for (A.10). Specifying z_0 gives us $n-m$ of these conditions. The remaining condition is the 'transversality condition'

$$\lim_{t\to\infty} \mu_t^T = \lim_{t\to\infty} \lambda^t p_t = 0 \tag{A.11}$$

and the initial condition

$$p_{20} = 0 \tag{A.12}$$

where $p_t^T = \begin{bmatrix} p_{1t}^T & p_{2t}^T \end{bmatrix}$ is partitioned so that p_{1t} is of dimension $(n-m) \times 1$. Equation (A.3), (A.8), (A.10) together with the $2n$ boundary conditions constitute the system under optimal control.

Solving the system under control leads to the following rule

$$w_t = -F \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix} \tag{A.13}$$

$$\begin{bmatrix} z_{t+1} \\ p_{2t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ S_{21} & S_{22} \end{bmatrix} G \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix} \tag{A.14}$$

$$N = \begin{bmatrix} S_{11} - S_{12} S_{22}^{-1} S_{21} & S_{12} S_{22}^{-1} \\ -S_{22}^{-1} S_{21} & S_{22}^{-1} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{A.15}$$

$$x_t = -\begin{bmatrix} N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix} \tag{A.16}$$

where $F = -(R + B^T S B)^{-1}(B^T S A + U^T), G = A - BF$ and

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{A.17}$$

partitioned so that S_{11} is $(n-m) \times (n-m)$ and S_{22} is $m \times m$ is the solution to the steady-state Ricatti equation

$$S = Q - UF - F^TU^T + F^TRF + \lambda(A - BF)^TS(A - BF) \tag{A.18}$$

The cost-to-go for the optimal policy (OP) at time t is

$$\Omega_t^{OP} = -\frac{1}{2}(\mathrm{tr}(N_{11}Z_t) + \mathrm{tr}(N_{22}p_{2t}p_{2t}^T)) \tag{A.19}$$

where $Z_t = z_t z_t^T$. To achieve optimality the policy-maker sets $p_{20} = 0$ at time $t = 0$. At time $t > 0$ there exists a gain from reneging by resetting $p_{2t} = 0$. It can be shown that $N_{22} < 0$, so the incentive to renege exists at all points along the trajectory of the optimal policy. This is the time-inconsistency problem.

A.2 The Dynamic Programming Discretionary Policy

The evaluate the discretionary (time-consistent) policy we rewrite the cost-to-go Ω_t given by (A.5) as

$$\Omega_t = \frac{1}{2}[y_t^TQy_t + 2y_t^TUw_t + w_t^TRw_t + \lambda\Omega_{t+1}] \tag{A.20}$$

The dynamic programming solution then seeks a stationary solution of the form $w_t = -Fz_t$ in which Ω_t is minimized at time t subject to (1) in the knowledge that a similar procedure will be used to minimize Ω_{t+1} at time $t + 1$.

Suppose that the policy-maker at time t expects a private-sector response from $t + 1$ onwards, determined by subsequent reoptimisation, of the form

$$x_{t+\tau} = -N_{t+1}z_{t+\tau},\ \tau \geq 1 \tag{A.21}$$

The loss at time t for the *ex ante* optimal policy was from (A.8) found to be a quadratic function of x_t and p_{2t}. We have seen that the inclusion of p_{2t} was the source of the time inconsistency in that case. We therefore seek a lower-order controller $w_t = -Fz_t$ with the cost-to-go quadratic in z_t only. We then write $\Omega_{t+1} = \frac{1}{2}Z_{t+1}^TS_{t+1}Z_{t+1}$ in (A.20). This leads to the following iterative process for F_t

$$w_t = -F_tz_t \tag{A.22}$$

where

$$\begin{aligned}
F_t &= (\overline{R}_t + \lambda\overline{B}_t^TS_{t+1}\overline{B}_t)^{-1}(\overline{U}_t^T + \lambda\overline{B}_t^TS_{t+1}\overline{A}_t) \\
\overline{R}_t &= R + K_t^TQ_{22}K_t + U^{2T}K_t + K_t^TU^2 \\
K_t &= -(A_{22} + N_{t+1}A_{12})^{-1}(N_{t+1}B^1 + B^2) \\
\overline{B}_t &= B^1 + A_{12}K_t \\
\overline{U}_t &= U^1 + Q_{12}K_t + J_t^TU^2 + J_t^TQ_{22}J_t \\
\overline{J}_t &= -(A_{22} + N_{t+1}A12)^{-1}(N_{t+1}A_{11} + A_{12})
\end{aligned}$$

$$
\begin{aligned}
\overline{A}_t &= A_{11} + A_{12}J_t \\
S_t &= \overline{Q}_t - \overline{U}_t F_t - F_t^T \overline{U}^T + \overline{F}_t^T \overline{R}_t F_t + \lambda(\overline{A}_t - \overline{B}_t F_t)^T S_{t+1}(\overline{A}_t - \overline{B}_t \overline{F}_t) \\
\overline{Q}_t &= Q_{11} + J_t^T Q_{21} + Q_{12}J_t + J_t^T Q_{22}J_t \\
N_t &= -J_t + K_t F_t
\end{aligned}
$$

where $B = \begin{bmatrix} B^1 \\ B^2 \end{bmatrix}$, $U = \begin{bmatrix} U^1 \\ U^2 \end{bmatrix}$, $A = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix}$, and Q similarly are partitioned conformably with the predetermined and non-predetermined components of the state vector.

The sequence above describes an iterative process for F_t, N_t, and S_t starting with some initial values for N_t and S_t. If the process converges to stationary values, F, N and S say, then the time-consistent feedback rule is $w_t = -Fz_t$ with loss at time t given by

$$\Omega_t^{TC} = \frac{1}{2} z_t^T S z_t = \frac{1}{2} \text{tr}(SZ_t) \tag{A.23}$$

A.3 Optimized Simple Rules

We now consider simple sub-optimal rules of the form

$$w_t = Dy_t = D \begin{bmatrix} z_t \\ x_t \end{bmatrix} \tag{A.24}$$

where D is constrained to be sparse in some specified way. Rule can be quite general. By augmenting the state vector in an appropriate way it can represent a PID (proportional-integral-derivative)controller (though the paper is restricted to a simple proportional controller only).

Substituting (A.3) into (A.5) gives

$$\Omega_t = \frac{1}{2} \sum_{i=0}^{\infty} \lambda_t y_{t+i}^T P_{t+i} y_{t+i} \tag{A.25}$$

where $P = Q + UD + D^T U^T + D^T RD$. The system under control (A.1), with w_t given by (A.3), has a rational expectations solution with $x_t = -Nz_t$ where $N = N(D)$. Hence

$$y_t^T P y_t = z_t^T T z_t \tag{A.26}$$

where $T = P_{11} - N^T P_{21} - P_{12}N + N^T P_{22} N$, P is partitioned as for S in (A.17) onwards and

$$z_{t+1} = (G_{11} - G_{12}N) z_t \tag{A.27}$$

where $G = A + BD$ is partitioned as for P. Solving (A.27) we have

$$z_t = (G_{11} - G_{12}N)^t z_0 \tag{A.28}$$

Hence from (A.29), (A.26) and (A.28) we may write at time t

$$\Omega_t^{SIM} = \frac{1}{2} z_t^T V z_t = \frac{1}{2} \text{tr}(V Z_t) \tag{A.29}$$

where $Z_t = z_t z_t^T$ and V satisfies the *Lyapunov* equation

$$V = T + H^T V H \tag{A.30}$$

where $H = G_{11} - G_{12}N$. At time $t = 0$ the optimized simple rule is then found by minimizing Ω_0 given by (A.29) with respect to the non-zero elements of D given z_0 using a standard numerical technique. An important feature of the result is that unlike the previous solution the optimal value of D, D^* say, is not independent of z_0. That is to say

$$D^* = D^*(z_0)$$

A.4 The Stochastic Case

Consider the stochastic generalization of (A.1)

$$\begin{bmatrix} z_{t+1} \\ x^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} z_t \\ x_t \end{bmatrix} + B w_t + \begin{bmatrix} u_t \\ 0 \end{bmatrix} \tag{A.31}$$

where u_t is an $n \times 1$ vector of white noise disturbances independently distributed with $\text{cov}(u_t) = \Sigma$. Then, it can be shown that certainty equivalence applies to all the policy rules apart from the simple rules (see Currie and Levine (1993)). The expected loss at time t is as before with quadratic terms of the form $z_t^T X z_t = \text{tr}(X z_t, Z_t^T)$ replaced with

$$\mathcal{E}_t \left(\text{tr} \left[X \left(z_t z_t^T + \sum_{i=1}^{\infty} \lambda^t u_{t+i} u_{t+i}^T \right) \right] \right) = \text{tr} \left[X \left(z_t^T z_t + \frac{\lambda}{1-\lambda} \Sigma \right) \right] \tag{A.32}$$

where $\mathcal{E}_t$ is the expectations operator with expectations formed at time t.

Thus for the optimal policy with commitment (A.19) becomes in the stochastic case

$$\Omega_t^{OP} = -\frac{1}{2} \text{tr} \left(N_{11} \left(Z_t + \frac{\lambda}{1-\lambda} \Sigma \right) + N_{22} p_{2t} p_{2t}^T \right) \tag{A.33}$$

For the time-consistent policy (A.23) becomes

$$\Omega_t^{TC} = -\frac{1}{2} \text{tr} \left(S \left(Z_t + \frac{\lambda}{1-\lambda} \Sigma \right) \right) \tag{A.34}$$

and for the simple rule, generalizing (A.29)

$$\Omega_t^{SIM} = -\frac{1}{2} \text{tr} \left(V \left(Z_t + \frac{\lambda}{1-\lambda} \Sigma \right) \right) \tag{A.35}$$

The optimimized simple rule is found at time $t = 0$ by minimizing Ω_0^{SIM} given by (A.35). Now we find that

$$D^* = D^* \left(z_0 + \frac{\lambda}{1-\lambda} \Sigma \right) \tag{A.36}$$

or, in other words, the optimimized rule depends both on the initial displacement z_0 and on the covariance matrix of disturbances Σ.

APPENDIX B: ESTIMATION RESULTS

Table1 : Priors for Baseline model

	Distribution	Mean	Standard Deviation	Percentiles 1%	99%
ρ_i	B	0.75	0.15	0.538	0.981
θ	G	1.7	0.5	1.099	3.074
λ	G	0.15	0.1	0.044	0.473
φ	G	1.75	0.5	1.148	3.118
σ	G	1.5	0.8	0.609	3.952
γ	B	0.7	0.1	0.566	0.897
h	B	0.7	0.1	0.566	0.897
ρ_a	B	0.7	0.15	0.492	0.959
ρ_g	B	0.7	0.15	0.492	0.959
π^*	G	4	2	1.745	10.045
r*	G	2	1	0.872	5.023
sd_g	IG1	1.7	inf	0.635	9.260
sd_e	IG1	1	inf	0.372	5.699
sd_a	IG1	1.7	inf	0.635	9.260

For all models, g is calibrated to 0.22. Distributions: G (Gamma), B (Beta), and), IG1 (Inverse Gamma-1). ρ corresponds to the autoregressive coefficient of an AR(1) process. sd stands for the standard deviation of the shocks. .Last two columns, report the inverse cumulative distribution function of each prior ordinate for thepercentiles 0.01 and 0.99.

APPENDIX B: ESTIMATION RESULTS

Table 2: Parameter estimates for j=1

Model	Model G ($\gamma = 0$)			Model H ($h = 0$)			Model GH ($\gamma = h = 0$)			Model Z		
	Posterior Distribution			Posterior Distribution			Posterior Distribution			Posterior Distribution		
Coefficient	Median	10%	90%	Median	10%	90%	Median	10%	90%	Median	10%	90%
ρ_i	0.80	[0.73	0.85]	0.67	[0.55	0.75]	0.72	[0.64	0.79]	0.77	[0.71	0.83]
θ	2.55	[2.15	2.99]	2.68	[2.18	3.29]	2.64	[2.14	3.20]	2.25	[1.86	2.58]
λ	0.16	[0.08	0.30]	0.71	[0.49	0.98]	0.27	[0.14	0.44]	0.47	[0.25	0.74]
φ	1.46	[1.06	1.94]	2.40	[1.70	3.15]	2.16	[1.47	2.77]	1.32	[0.99	1.81]
σ	3.29	[2.28	4.98]	2.50	[1.47	4.29]	3.91	[2.86	5.76]	3.23	[2.55	4.21]
γ				0.59	[0.45	0.72]				0.54	[0.40	0.68]
h	0.85	0.74	0.91							0.85	[0.73	0.92]
ρ_a	0.91	[0.85	0.94]	0.91	[0.87	0.94]	0.90	[0.85	0.93]	0.94	[0.89	0.96]
ρ_g	0.92	[0.88	0.95]	0.91	[0.88	0.93]	0.90	[0.87	0.93]	0.93	[0.89	0.96]
π^*	3.00	[2.43	3.60]	2.72	[2.14	3.32]	2.96	[2.40	3.51]	2.88	[1.75	3.55]
r^*	1.86	[1.38	2.49]	1.82	[1.26	2.48]	1.90	[1.33	2.46]	1.80	[1.25	2.38]
sd_g	2.19	[1.99	2.39]	3.23	[2.53	4.17]	2.75	[2.35	3.11]	2.23	[2.05	2.47]
sd_e	0.17	[0.15	0.19]	0.20	[0.18	0.23]	0.17	[0.16	0.19]	0.19	[0.17	0.21]
sd_a	0.78	[0.61	1.01]	0.51	[0.42	0.64]	0.59	[0.48	0.74]	0.72	[0.57	0.93]

Median and posterior deciles of the draws generated with a Random Walk Metropolis algorithm. Discarded the first 30,000 draws, retained the remaining 100,000 values

APPENDIX B: ESTIMATION RESULTS

Table 3: Model Comparisons

Reversible Jump MCMC

Model	*Posterior Odds, P(m | data)*
G (γ = 0)	0.56
H (h= 0)	0.09
GH (γ =h= 0)	0.03
Z	0.32

Reversible MCMC of Dellaportas et al. (2002). 100,000 draws to obtain the proposal densities. For the Metropolis step, Discarded the first 20,000 values and retained the remaining 180,000. Posterior odds P(m|data) based on assigning each model equal prior probability. Model proposal density assigns equal probability to the jump to any of four possible models, regardless of the current model in the chain.