Extending the CAPM model

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June 25, 2004

Preliminary version 0.5b

Abstract

This (preliminary version of the) paper extends the well known Capital Asset Pricing Model by Sharpe and Lintner to a multi-period context with possibly price dependent preferences. The model is built from individual forward looking agents adopting a portfolio selection scheme similar to the portfolio selection theory devised by Markowitz. We allow agents to use past and present price information to forecast both the expected return and the variance of asset returns, but with possibly different econometric forecasting techniques. Since the effects of price dependent preferences of agents are complicated, we use Microscopic Simulations to investigate the effects on equilibrium asset prices and on returns over an extended time period in a temporary equilibrium context. We also test whether the assumption of rational expectations makes sense.

Comments welcome

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1 Introduction

The benchmark equilibrium model in finance is the single-period Capital Asset Pricing Model (CAPM, see Sharpe (1964), or Lintner (1965)) in which investors are mean-variance efficient, and where the mean and variance are assumed to be known (rational expectations).

The aim of this paper is to investigate economies in which economic agents (investors) choose their portfolios, similar as in the CAPM, as a trade-off between mean and variance of the wealth at their time horizon, but where the mean and variance are not known in advance, but have to be estimated. We assume that the economic agents use econometric techniques to calculate these estimates. Different agents will find different estimates, for instance, due to different sample sizes or sampling frequencies, or due to different econometric estimation techniques employed. We consider single-period time horizons, as in the benchmark CAPM, and multi-period time horizons where investors might have different time horizons. In the latter case we assume that the investors use the closed-form solution for the mean-variance problem as derived by Li and Ng (2000) in order to calculate their portfolio holding demand.

Since every time period new information becomes available, the mean and variance are re-estimated every period. So, the estimates will be time-varying and investor-specific. As a consequence, the equilibrium concept to be used cannot be full rational equilibrium where present and future –correctly anticipating– prices are set such that the present and future markets clear. Instead, we use the concept of temporary equilibrium: each period that period's prices are set such that the markets clear in that period.

Since we cannot solve the equilibrium prices analytically, we use microscopic simulation to simulate the economy over time. From that point of view, the set-up chosen in this paper can be seen as an alternative to the work by, for instance, Levy et al. (2000), Arthur et al. (1996), LeBaron (1999), LeBaron et al. (1999), or LeBaron (2001). In these microscopic simulation models, economic agents are usually subdivided in various types, such as technical analysts or believers in the expectation hypothesis. In our economy, the agents are all of the same type, i.e., rational in the sense of utility maximization, but different in the way the parameters characterizing the future are estimated, due to different sample sizes, different econometric techniques employed, etcetera.

The resulting asset prices and returns generated by the microscopic simulations are subsequently investigated empirically. For instance, we consider asset return predictability, we model the volatility of the returns, and we test the random walk hypothesis. In this way we can find out whether the economic agents do estimate the mean and variance in an (econometrically) rational way. Moreover, assuming that the investors perform such an econometric analysis themselves, we can use the outcomes to improve the investors' estimation procedures.

The remainder of the current preliminary version of the paper is organized as follows. In the next section we present the economy and introduce notation. Then we formulate a single period problem for investors and describe the equilibrium concept. Section 5 contains some simulation results for a single period model. In section 6, we extend the problem for investors to a multi-period setting and show the effects of this extension on the simulated processes. In section 7 we describe an empirical analysis of simulation results and we conclude in section 8 with a summary and some ideas for future research.

2 Set-up economy and notation

The time span of interest is $\{0, 1, 2, \ldots, T\}$ ($T \in \mathbb{N}_{++}$). A particular period will be denoted by t. There are $J \in \mathbb{N}_{++}$ risky assets in the market and 1 riskless asset (asset 0). A particular asset will be denoted by j. The assets have return $r_t = (r_t^0, r_t^1, \ldots, r_t^J)'$ (where r_t^j denotes the return of asset j at time t) and is defined by

$$r_t^j \equiv \frac{p_t^j(1 + d_t^j)}{p_{t-1}^j}. \tag{1}$$

where p_t^j denotes the price of asset j at time t and d_t^j denotes the dividend paid out during the period between time $t-1$ and t in numbers of assets. Hence the value of the dividend of asset j at time t is $p_t^j d_t^j$. We assume that the riskless asset does not pay out dividends, i.e., $d_t^0 = 0$ for every t.

At time $t = 0$, $I \in \mathbb{N}_{++}$ mean-variance investors (where an individual investor will be denoted by i) will enter the market. These investors are characterized by their asset holdings h_{it} (in number of assets) at time t, the risk-aversion parameter $\gamma_i \in \mathbb{R}_{++}$, the memory M_i and the forecast horizon T_i. We model agents with a rolling horizon from time t to time $t + T_i$ at every t. Investors also have ideas about the assets that are available in the market. These ideas will be summarized by

$$r_{it\tau}^j = \mathrm{E}_i\left(r_\tau^j | I_{it}\right)$$

and

$$\Sigma_{it\tau} = \mathrm{Cov}_i\left(r_\tau | I_{it}\right)$$

where $\mathrm{E}_i(\cdot|\cdot)$ and $\mathrm{Cov}_i(\cdot|\cdot)$ denote the individual i specific time t expectation and covariance operator respectively. I_{it} denotes the information set of individual i at time t. This information set includes the last M_i returns. It is assumed throughout this paper that $\Sigma_{it\tau}$ is positive semi-definite. Furthermore, $r_{it\tau}$ and $\Sigma_{it\tau}$ are functions of current price information p_t unless specified otherwise. This reflects price dependent preferences as discussed by Balasko (2003b).

This notation implies that investor i can have a different model for the returns on assets than investor $i' \neq i$. Define the number of assets that investor i gets when asset j pays out a dividend by

$$\overline{h}_{it}^j \equiv d_t^j h_{i,t-1}^j.$$

This dividend payment in terms of assets is proportional to the current holdings of an agent. We define the portfolio selection of investor i at time t by θ_{it}. The amount invested in asset j by a particular agent i at time t will be denoted by θ_{it}^j and is defined by $\theta_{it}^j \equiv p_t^j h_{it}^j$. Define the total wealth of agent i at time t by

$$w_{it} \equiv p_t' h_{it} = \sum_{j=0}^{J} \theta_{it}^j.$$

For any vector $q \in \mathbb{R}^{J+1}$, we adopt the notation

$$q = \begin{pmatrix} q^0 \\ \tilde{q} \end{pmatrix}$$

to separately denote the first component, $q^0 \in \mathbb{R}$, of the vector and the vector containing the remaining components, $\tilde{q} \in \mathbb{R}^J$.

Define furthermore

$$R_t \equiv \left((r_t^1 - r_t^0), (r_t^2 - r_t^0), \ldots, (r_t^J - r_t^0)\right)'$$

which is the excess return of an asset over the riskfree asset. $R_{it\tau}$ is defined similarly as

$$R_{it\tau} \equiv \left((r_{it\tau}^1 - r_{it\tau}^0), (r_{it\tau}^2 - r_{it\tau}^0), \ldots, (r_{it\tau}^J - r_{it\tau}^0)\right)'.$$

Shorthand notation will be used for one-period ahead expectations, namely, r_{it} and R_{it}. Finally, the symbol '$\odot$' will be used to denote the Hadamard-Schur product and the symbol '$\oslash$' will denote the Hadamard-Schur quotient.

3 Single period Mean-Variance analysis with heterogeneous expectations

Let us start by formulating the standard CAPM model in terms of wealth of investors and heterogeneous expectations. We formulate a repeated single period CAPM model in which each agent solves at each time t

$$\begin{aligned} \max_{h_{it}} \quad & \mathrm{E}_t\left(w_{i,t+1}|I_{it}\right) - \gamma_i \mathrm{Var}_i\left(w_{i,t+1}|I_{it}\right) \\ \text{subject to} \quad & p'_{t+1} h_{i,t+1} = p'_{t+1}(h_{i,t} + \overline{h}_{i,t+1}) \end{aligned} \tag{2}$$

We rewrite the budget restriction as follows

$$\begin{aligned} w_{i,t+1} &\equiv p'_{t+1} h_{i,t+1} = p'_{t+1}(h_{it} + \overline{h}_{i,t+1}) \\ &= \sum_{j=0}^{J} p_{t+1}^j (h_{it}^j + h_{i,t+1}^j) = \sum_{j=0}^{J} p_{t+1}^j (1 + d_{t+1}^j) h_{it}^j \\ &= \sum_{j=0}^{J} r_{t+1}^j \theta_{it}^j = \left(w_{it} - \sum_{j=1}^{J} \theta_{it}^j\right) r_{t+1}^0 + \sum_{j=1}^{J} \theta_{it}^j r_{t+1}^j \\ &= r_{t+1}^0 w_{it} + R'_{t+1} \tilde{\theta}_{it} \end{aligned}$$

and hence the problem becomes

$$\begin{aligned} \max_{h_{it}} \quad & \mathrm{E}_t\left(w_{i,t+1}|I_{it}\right) - \gamma_i \mathrm{Var}_i\left(w_{i,t+1}|I_{it}\right) \\ \text{subject to} \quad & w_{i,t+1} = r_{t+1}^0 w_{it} + R'_{t+1} \tilde{\theta}_{it} \end{aligned} \tag{2'}$$

The solution to this problem, as can easily be verified, is given by

$$\begin{aligned}\tilde{\theta}_{it}^{*} &= \frac{1}{2\gamma_i}\Sigma_{it,t+1}^{-1}R_{it,t+1}\\ \theta_{it}^{0*} &= w_{it} - \sum_{j=1}^{J}\theta_{it}^{j*}\end{aligned} \tag{3}$$

4 Equilibrium concept

In the standard CAPM model, the important assumption of rational expectations is made to derive a closed form expression for equilibrium risk premia. However, we explicitly did not assume that here. The fact that all investors in the market know the exact distribution of asset returns is very hard to defend. Furthermore, the setup suggests that the temporary equilibrium concept is the only natural one. Hence we drop rational expectations and step to the concept of temporary equilibria in which the market is in equilibrium at every single time period, but where expectations over future returns are not required to actually be fulfilled in future periods.

This equilibrium concept has first been described by Grandmont (1988) and the set-up used here is inspired by Balasko (2003a). We use the word economy for a collection of agents that use problem (2') to determine their behavior with respect to buying and selling the available assets. This economy is a CAPM-like economy in the sense that agents are mean-variance optimizers.

To be able to define a temporary equilibrium in our economy, we first define the net demand function.
We define the (market) net demand function $Z_t : \mathbb{R}^{J+1} \to \mathbb{R}^{J+1}$ of the economy described in the preceding sections by

$$Z_t(p_t) \equiv \sum_{i=1}^{I}\left(h_{it}^{*} - \left(h_{i,t-1} + \overline{h}_{it}\right)\right) \tag{4}$$

and define $\tilde{Z}_t : \mathbb{R}^{J+1} \to \mathbb{R}^{J}$ in the obvious way

$$\tilde{Z}_t(p_t) \equiv \sum_{i=1}^{I}\left(\tilde{h}_{it}^{*} - \left(\tilde{h}_{i,t-1} + \tilde{\overline{h}}_{it}\right)\right) \tag{5}$$

and

$$Z_t^j(p_t) \equiv \sum_{i=1}^{I}\left(\frac{\theta_{it}^{j*}}{p_t^j} - \left(\tilde{h}_{i,t-1}^{j} + \tilde{\overline{h}}_{it}^{j}\right)\right) \tag{6}$$

for $\in \{0, 1, \ldots, J\}$. Notice that the net demand function is measured in numbers of assets. Next, we define an equilibrium of this economy at time t as a price vector p_t^* such that $Z_t(p_t^*) = 0$. It can trivially be seen from the budget restrictions of investors that Walras' law holds in the described economy for all t. It follows that we have to apply a normalization to asset prices. We normalize $p_t^0 = 1$ for all $t \in \{0, 1, 2, \ldots, T\}$. Besides that, when numerically computing equilibrium prices, we can forget about the market for the riskless asset since that will clear simultaneously when all markets for the risky assets clear.

Notice that the equilibrium price of asset j must satisfy

$$Z_t^j(p_t^*) \equiv \sum_{i=1}^{I} \left(\frac{\theta_{it}^{j*}}{p_t^{j*}} - \left(\tilde{h}_{i,t-1}^j + \tilde{\tilde{h}}_{it}^j \right) \right) = 0. \tag{7}$$

5 Simulating the one period CAPM model

We start the exploration of the CAPM model by simulating the one period version as described in section 3. We setup 10 agents with characteristics as listed in table 1. The number of agents is kept low to keep computations practically feasible. The agents use the following very simple model to predict future

Table 1: Starting values for agents.

Characteristic	Value
Memory M_i	from uniform distribution $[100, 200]$
Risk aversion γ_i	from uniform distribution $[1, 2]$
Endowments $h_{i,0}$	$(0, 1)'$ for all i

returns and the variance-covariance matrix of returns.

$$r_{it,t+1} = \frac{1}{M_i} \sum_{m=1}^{M_i} r_{t-m}$$

and

$$\Sigma_{it,t+1} = \frac{1}{M_i} \sum_{m=1}^{M_i} (r_{t-m} - r_{it,t+1})'(r_{t-m} - r_{it,t+1})$$

for every i, t and τ. Notice that this does not include current price information, p_t. As a deviation from rationality, noise is added to predictions of expected return in the following way.

$$r_{it,t+1} := r_{it,t+1}\varepsilon_{it}, \qquad \text{with } \varepsilon_{it} \sim N(1, \sigma^2)$$

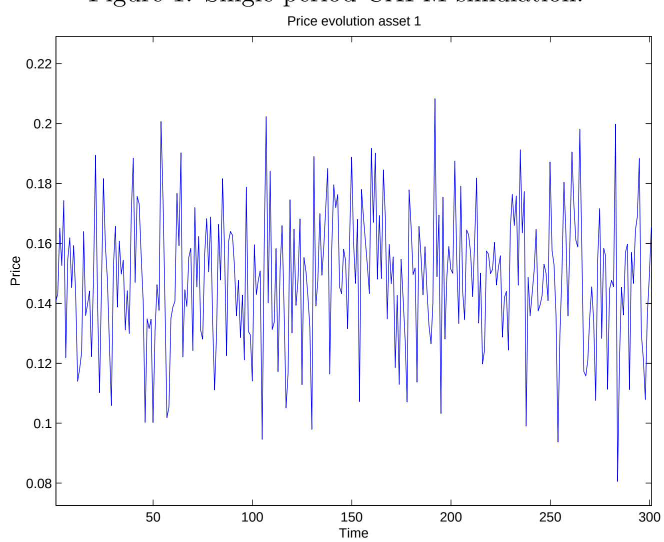
and we set $\sigma = 0.01$. We force a no short-sell restriction to avoid prices driving negative and we ignore dividends for this moment. This means that $d_t^j = 0$ for every j and t.

In the following simulations, we consider a single asset as is custom in literature. Especially in simple economies in which agents follow simple learning rules and are quite alike, they tend to learn which asset is the best very rapidly, resulting in prices going to zero for other assets. However, the setup will be kept general to be able to extend the simulations later on.

We start by generating a random history for agents (which will be reused to keep results comparable). We do this by simulating the standard Black-Scholes model for a considerable period. Next, we startup the economy using this historical information and let it run for at least 5000 periods to wash out the effects that the random historical information could have through the memory of agents. The new historical information that is created this way is used in subsequent simulations.

In figure 1 we see a typical price pattern resulting from the simulation as described above.

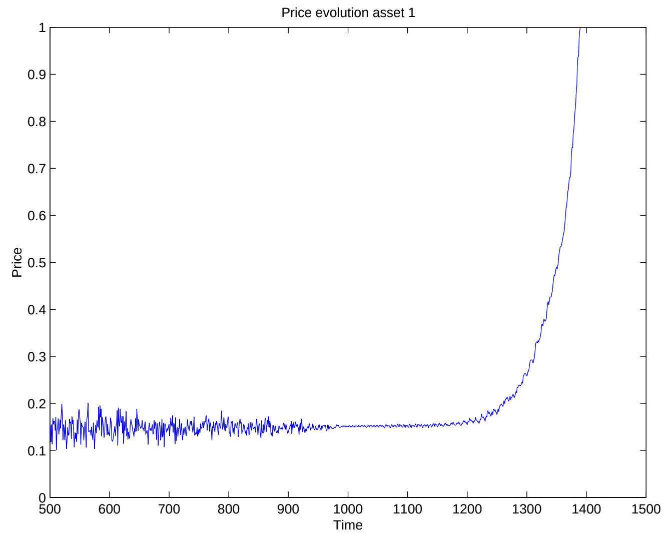
Figure 1: Single period CAPM simulation.



We clearly see the absence of structural growth in prices. This might be resolved by introducing a dividend process. However, we will not do that yet. This will complicate matters and distract attention from the basic characteristics of the economy.

As experiments, we consider two situations. First, we are interested what happens to the economy when we reduce the noise that makes agents deviate from rationality. In the exemplary experiment, we reduce the variance of ε_{it} between periods 500 and 1000 to zero. An typical simulation result can be found in figure 2. The result can easily be interpreted from the fact that a convergence

Figure 2: Single period CAPM simulation, reducing noise.



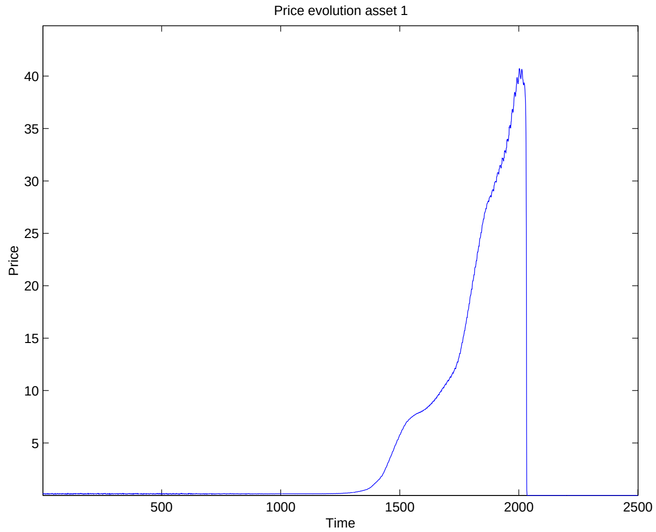
to rationality, reduces the variability in demand, which is reflected in prices. As a result, variance of returns decrease and the stock becomes more attractive. This, in its turn, will lead to increasing stock prices, increasing returns, etcetera. This is a self-fulfilling prophecy. At a certain moment, the stock can become to volatile and the market crashes. We can view an example of that in figure 3.

Another interesting experiment is to see what happens to this economy when a shock is applied to prices. Since the price level itself is irrelevant in the economy at hand, the effect will work through the returns and the memory of agents. In the following example, we apply a multiplicative shock to the price of the risky asset of magnitude 5. A resulting simulated price process can be found in figure 4.

Directly after the positive shock, the agents react to the increased volatility by reducing demand and hence prices decrease. However, gradually, the event starts to play a smaller role in the economy because of the limited memory of agents and we observe that prices go back to the approximate same level as before applying the shock.

This benchmark economy has shown to be a very interesting starting point for extensions to the economy. One such extension will be discussed in the next section and will allow investors to optimize their investment over multiple future periods.

Figure 3: Single period CAPM simulation, crashing market.



6 Multi-period CAPM

In this section, we formulate a CAPM-like model with multi-period forward looking agents. Investors maximize their expected final wealth w_{iT_i}. But they do not want to bear too much risk. The risk-aversion parameter takes care of a weighting of expected return and variance. The problem at time $t \in \{0, 1, \ldots, T\}$ can be stated as

$$\begin{aligned} \max_{\theta_{it}} \quad & \mathrm{E}_{it}\left(w_{iT_i}\right) - \gamma_i \mathrm{Var}_{it}\left(w_{iT_i}\right) \\ \text{subject to} \quad & p'_{\tau+1} h_{i,\tau+1} = p'_{\tau+1}(h_{i\tau} + \overline{h}_{i,\tau+1}) \\ & \text{for all } \tau \in \{t, t+1, \ldots, t+T_i-1\} \end{aligned} \tag{8}$$

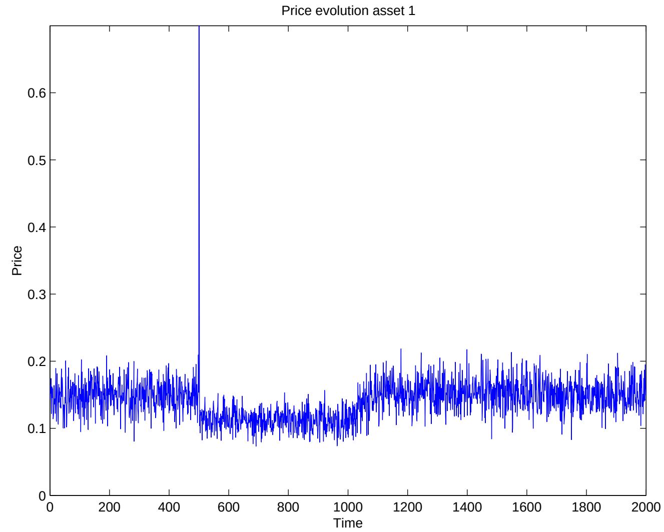
Recall that w_{it} denotes the wealth of investor i at time t, γ_i its risk aversion, h_{it} the endowments in numbers of assets and θ_{it} the optimal portfolio measured in wealth.

Notice that we can rewrite this problem in a similar fashion as done in section 3. This results in

$$\begin{aligned} \max_{\theta_{it}} \quad & \mathrm{E}_{it}\left(w_{iT_i}\right) - \gamma_i \mathrm{Var}_{it}\left(w_{iT_i}\right) \\ \text{subject to} \quad & w_{i,\tau+1} = r^0_{\tau+1} w_{i\tau} + R'_{\tau+1} \tilde{\theta}_{i\tau} \\ & \text{for all } \tau \in \{t, t+1, \ldots, t+T_i-1\} \end{aligned} \tag{8'}$$

When we additionally assume that the vectors $r_{it\tau}$ for a specific investor i and time t and for all $\tau \in \{t, t+1, \ldots, t+T_i-1\}$ are considered to be statistically

Figure 4: Single period CAPM simulation, impulse response.



independent, then the problem formulated above is a special case of the model discussed in Li and Ng (2000). The authors have solved problem (8') for one investor, fixed (implicit) prices and known beliefs about the parameters of the assets using Dynamic Programming. The optimal portfolio path for $\tau \in \{t, t+1, \ldots, t+T_i-1\}$ is given by

$$\begin{aligned}\tilde{\theta}^*_{it\tau} = &-r^0_{it,\tau+1}\,\mathrm{E}_i^{-1}\left(R_{\tau+1}R'_{\tau+1}|I_{it}\right)R_{it,\tau+1}w_{i\tau} \\ &+\left(\prod_{k=t+1}^{t+T_i} r^0_{itk}w_{it} + \frac{1}{2\gamma_i\left(\prod_{k=t+1}^{t+T_i}(1-B_{itk})\right)}\right)\left(\prod_{k=\tau+2}^{t+T_i}\frac{1}{r^0_{itk}}\right) \\ &\times \mathrm{E}_i^{-1}\left(R_{\tau+1}R'_{\tau+1}|I_{it}\right)R_{it,\tau+1}\end{aligned} \tag{9}$$

where

$$B_{it,\tau+1} = R'_{it,\tau+1}\,\mathrm{E}_i^{-1}\left(R_{\tau+1}R'_{\tau+1}|I_{it}\right)R_{it,\tau+1}.$$

Notice that we extended notation a bit. $\theta_{it\tau}$ denotes the optimal portfolio for a future time $\tau \geq t$ for agent i at time t. The latter indicates that this agent will be using the information set I_{it}. Under the assumption $d^0_t = 0$ and the normalization $p^0_t = 1$ for every t, equation (9) simplifies to

$$\tilde{\theta}^*_{it\tau} = \left(w_{it} - w_{i\tau} + \frac{1}{2\gamma_i\left(\prod_{k=t+1}^{t+T_i}(1-B_{itk})\right)}\right)\mathrm{E}_i^{-1}\left(R_{\tau+1}R'_{\tau+1}|I_{it}\right)R_{it,\tau+1} \tag{10}$$

Agents have a rolling horizon from time t to $t+T_i$. This implies that they solve problem 8' at every t using the latest available information in the market. With this in mind, we can simplify the solution even further. For brevity, θ^*_{itt} will be denoted by θ^*_{it} and

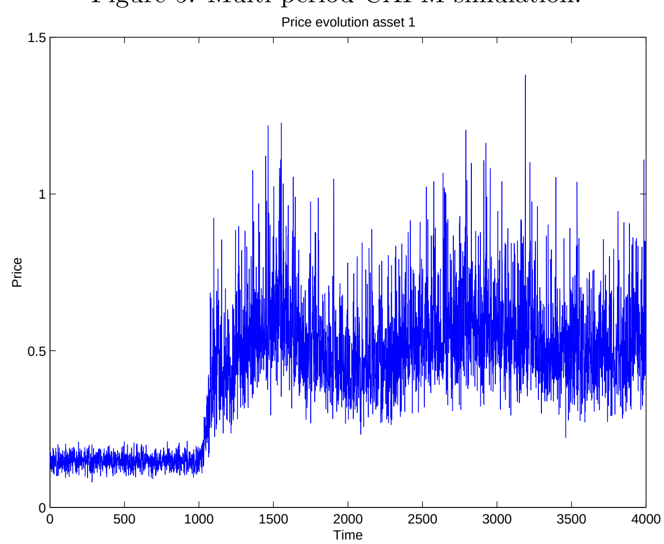
$$\tilde{\theta}^*_{it} = \frac{1}{2\gamma_i \left(\prod_{k=t+1}^{t+T_i}(1-B_{itk})\right)} \mathrm{E}_i^{-1}\left(R_{t+1}R'_{t+1}|I_{it}\right) R_{it,t+1}. \tag{11}$$

The amount $\theta^{0*}_{it} = w_{it} - \sum_{j=1}^{J} \tilde{\theta}^{j*}_{it}$ will be invested in the riskless asset. Notice that the optimal portfolio will be measured in wealth that will be invested in the different assets. Contrary to that, the holdings h_{it} of an investor will be measured in numbers of assets. The demand function in numbers of assets is denoted by $h^*_{it} = \theta^*_{it} \oslash p_t$, where the indices of h^*_{it} have the same meaning as those of θ^*_{it} and where h^*_{it} will be the shorthand notation for h^*_{itt}. We show in appendix A that the optimal portfolio path in (11) reduces to the optimal portfolio in (3) when $T_i = 1$ for investor i.

To study the effect of multi period optimizations on prices, we start a simulation as described in section 5. Parameters are again set as described in table 1. From time 1000 to time 1100, we increase the planning horizon of agents each two periods by one period.

From equation (11), we can see that, since $0 \leq B_{itk} \leq 1$, the effect of an increasing time horizon will be an increase in demand at every t. When inspecting figure 5, we see that this also realizes in simulation. Notice that, as a

Figure 5: Multi period CAPM simulation.



result, also the noise over the price process has increased. This is caused by the multiplicative specification of noise over expectations.

7 Empirical analysis

The interesting exercise is to see whether an assumption of rational expectations would make sense in the economies described in the preceding sections. We can verify this is in a number of ways as suggested in literature.

We decompose the simulated returns of an asset in the following way.

$$r_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim F, \quad t = 1, 2, \ldots, T-1$$

where $\mu_t = \mathrm{E}\left(r_{t+1}|I_t\right)$ and $\sigma_t = \mathrm{Var}\left(r_{t+1}|I_t\right)$. Predictability of asset returns and variance can then for instance be tested with the following hypotheses.

$$\begin{array}{ll} H_0: \ \mu_t = \mu & H_0: \ \sigma_t = \sigma \\ H_1: \ \mu_t = \mu(x_t) & H_1: \ \sigma_t = \sigma(x_t) \end{array}$$

where $\mu(x_t)$ denotes some function of x_t which contains the inputs to the simulation at time t. A possible alternative volatility model could be ARCH or GARCH. It would also be interesting to test whether ε_t is normally distributed, or

$$\begin{array}{l} H_0: \ F = \Phi \\ H_1: \ F \neq \Phi \end{array}$$

where Φ denotes the standard normal distribution function.

Predictability can also be tested in the context of the Random Walk hypothesis. In this case, we postulate the following model for simulated returns

$$r_{t+1} = c + \rho r_t + \varepsilon_{t+1}, \quad \mathrm{E}\left(\varepsilon_{t+1}\right) = 0, \quad t = 1, 2, \ldots, T-1 \tag{12}$$

and test whether ρ, the correlation coefficient, is significantly different from 0.

We haven't had the opportunity yet to perform all shortly mentioned test. However, we estimated the model in equation 12 using the data generated by the simulation which resulted in figure 1. The sample size is 300 observations. The results are as follows (standard errors in parentheses).

$$\begin{array}{ll} \hat{c} = 1.4703 & (0.0297) \\ \hat{\rho} = -0.4381 & (0.0284) \end{array}$$

showing that in this particular simulation, there is significant negative autocorrelation present. This indicates that an assumption of rational expectations does not seem to be justified in the economy that was on the basis of the simulation used.

8 Summary and future research

So far, in this preliminary version of the paper, we have concentrated on setting up a Microscopic Simulations model based on the well known CAPM model. The CAPM model has shown to be an interesting starting point for a simulation model for financial markets and can be extended in several ways. The effect of most of these extensions cannot be computed analytically and hence we used the Microscopic Simulations technique. The simulations that have been performed already indicate that an assumption of rational expectations which is normally made within the CAPM model, does not seem to be justified in the economies that we studied.

Future research will mainly focus on extending the presented model with price dependent preferences as discussed in, for instance, Balasko (2003b). In general, equilibrium prices will be defined by the equilibrium condition

$$Z_t^j(p_t^*) \equiv \sum_{i=1}^{I} \left(\frac{\theta_{it}^{j*}(p_t^*)}{p_t^{j*}} - \left(\tilde{h}_{i,t-1}^j + \tilde{\tilde{h}}_{it}^j \right) \right) = 0. \tag{13}$$

Since $\theta_{it}^j(p_t)$ is now a function of J asset prices, finding equilibrium prices becomes much more complicated.

Furthermore, we have a choice in the specification of price dependent preferences. In the current context, two approaches seem natural. The first approach is by making investor's risk aversion a function of current price information, $\gamma_i(p_t)$. When integrating current price information through risk aversion, the risk aversion could be modeled as an increasing (decreasing) function of the difference between hypothetical prices and previous period prices for risk averse (risk loving) investors.

The big disadvantage of this method is that there is not a clearly best strategy for modeling $\gamma_i(p_t)$. This gives a lot of freedom to the researcher. However, this freedom also makes it difficult making a decision between the many possible ways modeling risk aversion.

An alternative, with more clear paths to be followed, is offered when considering to integrate current price information through the predictions that investors make for return and risk of assets. Well known econometric models which are used in daily practice can be used to create predictions of agents.

As a first experiment with price dependent preferences, we considered forecasting functions of the form

$$r_{it,t+1}(p_t) = \frac{1}{M_i} \sum_{m=0}^{M_i-1} r_{t-m}(p_t)$$

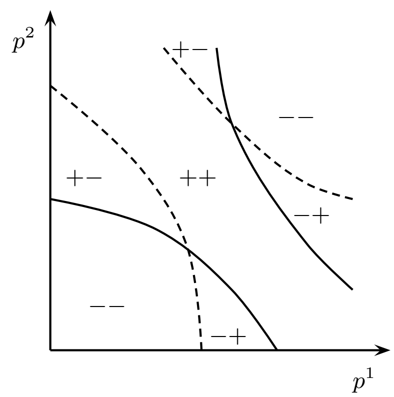
and

$$\Sigma_{it,t+1}(p_t) = \frac{1}{M_i} \sum_{m=0}^{M_i-1} \left(r_{t-m}(p_t) - r_{it,t+1}(p_t)\right)' \left(r_{t-m}(p_t) - r_{it,t+1}(p_t)\right)$$

This has major effects to the demand function of an individual agent, mostly through the forecasted variance $\Sigma_{it\tau}$. Not only will the demand curves change due to this extension, they might even give rise to multiple equilibria in the market.

A particular sign diagram of a market net demand function for two assets that results when modeling investors to use the forecasting functions above is depicted in figure 6. This diagram shows that there are two equilibria in the market.

Figure 6: Sign diagram of a three-dimensional net demand function (asset 1 dashed)



It is necessary to select one equilibrium. We will use a homotopy method to numerically locate this equilibrium. Effects on the economies resulting from price dependent preferences will be studied.
Future research will also include an extension of learning rules for investors. Investors will be able to use more advanced econometric techniques for predicting asset prices or returns and will be able to use the information that they pick up from the market such as knowledge about predictability of asset returns and correlation.

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A Optimal portfolio paths

In this appendix, we show that the multi-period optimal portfolio path (11) reduces to the single period optimal portfolio (3) when $T_i = 1$. The multi-period optimal portfolio path is give by

$$\tilde{\theta}_{it}^* = \frac{1}{2\gamma_i \left(\prod_{k=t+1}^{t+T_i} (1 - B_{itk}) \right)} \mathrm{E}_i^{-1} \left(R_{t+1} R_{t+1}' | I_{it} \right) R_{it,t+1}.$$

We use

$$(A + uv')^{-1} = A^{-1} - \frac{(A^{-1}u)(v'A^{-1})}{1 + v'A^{-1}u}$$

for a matrix A and vector u and v such that dimensions match. Hence we can write for $T_i = 1$

$$\begin{aligned} B_{it,t+1} &= R_{it,t+1}' (\Sigma_{it,t+1} + R_{it,t+1} R_{it,t+1}')^{-1} R_{it} \\ &= R_{it,t+1}' \Sigma_{it,t+1}^{-1} R_{it,t+1} - \frac{(R_{it,t+1}' \Sigma_{it,t+1}^{-1} R_{it,t+1})^2}{1 + R_{it,t+1}' \Sigma_{it,t+1}^{-1} R_{it,t+1}} \end{aligned}$$

and hence it can be verified that

$$\begin{aligned} \tilde{\theta}_{it}^* = & \frac{1 + R_{it,t+1}' \Sigma_{it,t+1}^{-1} R_{it,t+1}}{2\gamma_i} \\ & \times \left(\Sigma_{it,t+1}^{-1} - \frac{(\Sigma_{it,t+1}^{-1} R_{it,t+1})(R_{it,t+1}' \Sigma_{it,t+1}^{-1})}{1 + R_{it,t+1}' \Sigma_{it,t+1}^{-1} R_{it,t+1}} \right) R_{it,t+1} \\ = & \frac{1}{2\gamma_i} \Sigma_{it,t+1}^{-1} R_{it,t+1} \end{aligned}$$

which is exactly the single period optimal portfolio of equation (3).

It is important to notice (when assuming that $\mathrm{E}_i \left(R_\tau^j | I_{it} \right) \geq 0$ for every j) that $0 \leq B_{it\tau} \leq 1$. The first inequality can easily be seen from the fact that $\mathrm{E}_i \left(R_{\tau+1} R_{\tau+1}' | I_{it} \right) = \mathrm{Cov}_i \left(R_\tau | I_{it} \right) + \mathrm{E}_i \left(R_\tau | I_{it} \right) \mathrm{E}_i \left(R_\tau | I_{it} \right)'$ is positive semi-definite. Furthermore, positive semi-definiteness of $\mathrm{Cov}_i \left(R_\tau | I_{it} \right)$ is defined by $y' \mathrm{Cov}_i \left(R_\tau | I_{it} \right) y \geq 0$ for any $y \in \mathbb{R}^J$. Taking $y = \mathrm{E}_i^{-1} \left(R_{\tau+1} R_{\tau+1}' | I_{it} \right) \mathrm{E}_i \left(R_\tau | I_{it} \right)$ results in $B_{it\tau} - B_{it\tau}^2 \geq 0$ which in turn gives the desired result.