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On the Robustness of Herds*

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ABSTRACT

Herd behavior is argued by many to be present in many markets. Existing models of such behavior have been subjected to two apparently devastating critiques. The continuous investment critique is that in the basic model herds disappear if simple zero-one investment decisions are replaced by the more appealing assumption that investment decisions are continuous. The price critique is that herds disappear if, as seems natural, other investors can observe asset market prices. We argue that neither critique is devastating. We show that once we replace the unappealing exogenous timing assumption of the early models that investors move in a pre-specified order by a more appealing endogenous timing assumption that investors can move whenever they choose then herds reappear.

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Herd behavior is widely thought of as being prevalent in many markets. For example, Calvo and Mendoza (1996)) and Chari and Kehoe (2002) argue that herd behavior accounts for a substantial fraction of the volatility in capital flows to emerging markets. More generally, Devenow and Welch (1996) report that participants in financial markets and financial economists believe that herd-like behavior is widespread.

A recent literature has developed models of herd behavior. The early models of herd behavior of Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992) have been criticized on the grounds that two unpalatable assumptions play a key role in generating herds. First, investment decisions are zero-one. Second, the models abstract from prices. Lee (1993) argued that if investment decisions are continuous then herds disappear. Avery and Zemsky (2000) argued once agents are allowed to trade in financial markets, prices reveal information and herds disappear. We label these critiques the *continuous investment critique* and the *price critique* respectively. The investment critique seems to quite strong because the scale of investments can often be changed easily. The price critique is also strong because it suggests that herd behavior will not be observed in financial markets. Taken together these critiques seem to be devastating to anyone who wants to use the herd framework to understand actual economies.

We argue that neither critique is devastating and that both can be overturned by replacing an unappealing simplifying assumption of the early herd models with a more natural assumption for applied situations. The unappealing assumption used in the early herd literature is an *exogenous timing* assumption, namely that investors move in a pre-specified order. We show that when we replace this assumption with the more natural and perhaps more realistic *endogenous timing* assumption, namely that investors can move at any time, then the two critiques are overturned and herds reappear.

We begin with a simple baseline model similar to the early models to set the stage for the critiques and our answers. In it investors move in a pre-specified order and must invest in either a risky or a safe project, so that the model has exogenous timing and discrete investments. Each investor receives a private signal indicating that the returns on the risky project are either high or low. In this model a small number of high signals leads all future investors to invest in the risky project regardless of their signals while a small number of low signals leads all future investors to forgo investing in the risky project. These outcomes are inefficient relative to a benchmark public signal game. We call these outcomes *herds* because they satisfy two criteria: investors make the same decisions regardless of their signals and the outcomes are inefficient.

We show that both the continuous investment critique and the price critique have merit with exogenous timing. To illustrate the continuous investment critique we take the baseline model with exogenous timing and allow investment to be continuous. We find that Lee's (1993) critique applies: herds of investment disappear. The basic idea is that if investment decisions are continuous then differences in signals always lead to at least small differences in investment. These differences in investment can be used to infer the underlying signal so that information never becomes trapped and there are no herds.

To illustrate the price critique we take the baseline model with exogenous timing and allow asset trade with market determined prices. We find that Avery and Zemsky's (1998) critique applies: herds of investment disappear. The basic idea is that prices reveal information. In particular, prices continuously adjust to trades and hence can be used to infer underlying signals so that information never becomes trapped and there are no herds.

We show that both critiques are overturned when we allow for endogenous timing. Under this timing there is a tradeoff between investing and waiting: waiting is potentially beneficial because investors can gain information but it is costly because of discounting. We find that even with continuous investment there can be herds of investment and that even with prices and asset trade there can be herds of investment. In both cases, the reason is that at some point the gains from waiting for more information are outweighed by the costs from waiting due to discounting and investors choose to invest with relatively little information. We show that these resulting herds of investment are inefficient relative to a natural benchmark game with public signals.

An interesting feature of our results is that in the baseline model with discrete investments and without asset trade, the investment outcomes with endogenous timing are essentially identical to those with exogenous timing. Thus, in that model, the exogenous timing assumption leads to a useful and innocuous simplification of the analysis. With either continuous investments or asset trading, the outcomes under the two timing assumptios can be quite different.

In the Appendix we take up a third critique, namely there are other ways to share information besides having investors infer information from investment decisions. In particular, if investors are allowed to communicate with each other, they have no incentive to misrepresent their information and there is no essential reason for information to get trapped. This *information-sharing critique* also suggests that herds are quite fragile.

To investigate this critique we amend the baseline model by allowing investors to send messages about their signal to other investors. We show that this critique has some merit when timing is exogenous by showing there is an equilibrium in which all investors truthfully reveal their signals and there are no herds. We then show this critique can be overturned by allowing endogenous timing and assuming that there is a small *early-mover advantage* in that the rate of return earned on the risky project is higher when there are fewer investors. We show that even with information sharing there can still be herds.

Our paper is related to an extensive literature on herds. (See the survey by Bikhchandani, Hirshleifer, and Welch (1998) for results from this literature and references.) Several papers are closely related. Caplin and Leahy (1993), Chamley and Gale (1994), Gul and Lindholm (1995) allow for private signals and endogenous timing of decisions, however, none of them is directed towards the critiques of the early herd literature. The most similar of these is Chamley and Gale (1994) who emphasize the same tradeoff between the cost of waiting due to discounting and the benefits to waiting from garnering information from the actions of others. They also share our emphasis, as does Vives (1997), that equilibria are socially inefficient due to infomation externalities.

Our paper is also related to those of Glosten and Milgrom (1985), Lee (1998) and Avery and Zemsky (1998). Glosten and Milgrom show that prices eventually reveal the true value of the asset as long as informed traders trade in all periods. In this sense, Glosten and Milgrom (1985) is a critique of herd models the predates the recent herd literature. Lee (1998) and Avery and Zemsky (1998) retain the asumption of exogenous timing and offer two possible answers to the price critique. Lee shows that fixed costs of trading can lead informed investors to stop trading after some point so that information becomes trapped. Avery and Zemsky show that with nonmonotonic signals even with exogenous timing, herds can appear. One feature of our formulation that differs from the market microstructure literature following Glosten and Milgrom (1985) is that we allow for both investment and asset trading decisions.

Allowing for both seems essential in developing models of herd behavior in some situations, such as those of capital flows into and out of emerging market economies.

Finally, this paper is also related to the large literature about prices revealing information following Grossman and Stiglitz (1980).

1. The Baseline Model of Herd Behavior

Consider an economy that lasts for T periods, from $t = 0, 1, \ldots, T$. There are $N < T$ risk neutral investors who must decide to invest in either a safe asset or a risky asset. The investors are ordered in a sequence, $t = 1, 2, \ldots, N$. Investor t starts in period t with one unit invested in the safe asset and must make a once-and-for-all decision in period t to switch this unit to the risky asset. This decision is irrevocable. Let $x_t = 1$ denote an investment in the risky project in period t and $x_t = 0$ denotes no investment in the risky project. For convenience we say that investor t *invests* whenever $x_t = 1$ and we say that investor t *does not invest* whenever $x_t = 0$. The state of the economy y is either high, denoted H, or low, denoted L. The state of the economy is unknown to investors and it becomes known in period T.

Investing in the risky asset in period t yields a return of $e^{R(T-t)}$ if the state is high and 0 from the risky asset if the state is low. We normalize the return in the safe asset to be 1. An investor who invests in the risky asset in period t gets an expected return of

$$e^{R(T-t)}\text{Prob}_t(\text{state is } H)$$

where $\text{Prob}_t(\text{state is } H)$ is the conditional probability that the investor assigns to the state being H.

At the beginning of period t, investor t privately observes one of two possible signals

$s \in \{H, L\}$: that the economy is in a high state or that the economy is in the low state. The signals are informative and symmetric in the sense that

$$\text{(1)} \qquad \Pr(s = H \mid y = H) = \Pr(s = L \mid y = L) = q > 1/2$$

as well as conditionally independent across investors.

The public history in any period t, $h_t = (x_1, \ldots, x_{t-1})$, is the sequence of investment decisions up through the beginning of period t where $x_t = 1$ denotes an investment in the risky project in period t and $x_t = 0$ denotes no investment in the risky project. We let $p_t(h_t)$ be the *public beliefs*, namely the probability that the state is high conditional on the public history h_t. Investor $t's$ history is the public history h_t plus this investor's private signal s_t. The investor *t's strategy* is a function $x_t(h_t, s_t)$. Investor $t's$ *beliefs* are $p_t(h_t, s_t)$, where $p_t(h_t, s_t)$ is the probability that the state is high conditional on the history h_t and the signal s_t.

A *perfect Bayesian equilibrium* is a collection of strategies and beliefs $\{x_t, p_t\}_{t=1}^{N}$ such that (i) for all t, histories h_t, and signals s_t, the strategy $x_t(h_t, s_t)$ solves

$$\text{(2)} \qquad V(p_t(h_t, s_t), t) = \max_{x \in X} e^{R(T-t)} p_t(h_t, s_t) x + (1 - x)$$

where $X = \{0, 1\}$ and (ii) the beliefs satisfy Bayes' rule wherever it applies.

In constructing an equilibrium, we find it useful to first define the public beliefs $p_t(h_t)$ given some public history h_t and then use them to construct each investor's strategies. For arbitrary beliefs p, we use Bayes' rule to define $P_H(p)$ and $P_L(p)$ as the updated beliefs that the state is high given that signals H and L were received:

$$\text{(3)} \qquad P_H(p) = \frac{pq}{pq + (1-p)(1-q)}$$

$$\text{(4)} \qquad P_L(p) = \frac{p(1-q)}{p(1-q) + (1-p)q},$$

where q is defined in (1). Let $P(0) = p_0$, $P(1) = P_H(P(0))$, $P(2) = P_H(P(1))$, and so on, and let $P(-1) = P_L(P(0))$, $P(-2) = P_L(P(-1))$, and so on. Thus, $P(k)$ for $k > 0$ is the prior probability that the state is high if k high signals have been received, and $P(k)$ for $k < 0$ is the prior probability that the state is high if k low signals have been received. Notice from the symmetry in (1) that

$$(5) \qquad P_H(P_L(p)) = P_L(P_H(p)) = p.$$

It follows from (5) that the effect on the prior of a given set of signals is summarized by the number of high signals minus the number of low signals in the set. Thus, for example, receiving two high signals and one low signal yields the same prior as receiving one high signal.

We focus on the region of the parameter space that satisfies the following:

$$(6) \qquad e^{R(T-N)}P(0) > 1$$

$$(7) \qquad e^{RT}P(-1) < 1$$

Note that because $R > 0$ these assumptions imply the following. Assumption (6) implies that in any period t, between the two options of investing in the risky asset given belief $P(0)$ or never investing in the risky asset, it is better to invest. Assumption (7) implies that at any date t, between the two options of investing in the risky asset given belief $P(-1)$ or never investing in the risky asset, it is better to not invest. Under assumptions (6) and (7), given beliefs p, each investor's choice is static and of the form

$$(8) \qquad x(p) = \left\{ \begin{array}{c} 1 \text{ if } p \geq P(0) \\ 0 \text{ if } p \leq P(-1) \end{array} \right\}$$

where we have made use of the feature that beliefs are always of the form $P(k)$ for some k.

Now we describe, informally, the outcomes of the equilibrium that we will construct. Investor 1 invests if the signal is high and does not if the signal is low. Investor $2's$ decision depends on the history from period 1. If there was an investment in period 1, investor 2 invests regardless of the signal that investor receives and so do all future investors. We say that this history starts a *cascade with investment.* If there was zero investment in period 1, then the investor 2 invests if and only if the signal received is high. At the beginning of period 3, if there has been no investment in either periods 1 or 2, then no investor invests in period 3 or in any subsequent period. We will say that this history starts a *cascade with no investment* . If there has been no investment in period 0 but an investment in period 1, then the investor in period 2 invests if and only if the signal received is high. (Note that we use the term cascade to mean an outcome in which after some point all investors take the same action regardless of their signal. We will define a herd to be a cascade that is inefficient relevant to a natural benchmark.)

More generally, histories of the form (1), $(0,1,1)$, $(0,1,0,1)$, $\ldots$, $(0,1,0,1,\ldots,0,1,1)$ start cascades with investment. Histories of the form $(0,0)$, $(0,1,0,0),\ldots$, $(0,1,0,1,\ldots,$ $0,1,0,0)$ start cascades with no investment.

More formally, we proceed as follows. The strategy for investor t is

$$(9) \qquad x_t(h_t, s_t) = \left\{ \begin{array}{c} 1 \text{ if } p_t(h_t, s_t) \geq P(0) \\ \\ 0 \text{ otherwise} \end{array} \right\}.$$

Define public beliefs $p_t(h_t)$ recursively as follows from the initial prior $P(0)$. For $t \geq 0$, given

x_t and $p_t(h_t)$ equal to either $P(-1)$ or $P(0)$,

$$(10) \quad p_{t+1}(h_{t+1}) = \left\{ \begin{array}{l} P_H(p_t(h_t)) \text{ if } x_t = 1 \\ \\ P_L(p_t(h_t)) \text{ if } x_t = 0 \end{array} \right\}.$$

For $p_t(h_t)$ greater than $P(0)$ or less than $P(-1)$, beliefs are unchanged, so that $p_{t+1}(h_{t+1}) = p_t(h_t)$.

Notice that, for simplicity, we have constructed an equilibrium in which strategies depend only on the probability that a state is high and not on time or on the number of investors who have previously invested.

Built into these beliefs is the idea that investors look at past investors' actions and try to infer their signals. On the equilibrium path, investors infer the following. If investor t takes an action which is consistent with that investor having received a high signal but not consistent with that investor having received a low signal, then all future investors infer that a high signal was received. Inferences are made similarly for actions consistent with a low signal but not with a high signal. If a investor takes an action which is consistent with both a high signal and a low signal, then future investors infer nothing. Finally, off the equilibrium path, if a investor takes an action which is consistent with neither a high signal nor a low signal, then future investors infer nothing.

It is easy to see that the constructed strategies and beliefs constitute a perfect Bayesian equilibrium. To see this note that the constructed beliefs satisfy Bayes' rule. We repeatedly use the observation that by construction, for any history h_t, $p_t(h_t) = P(k)$ for some integer k. Suppose that investor t faces a history (h_t, s_t) that has associated beliefs $p_t(h_t, s_t) \geq P(0)$. Under assumption (6), investing gives a strictly higher return than not investing. Suppose next that investor t faces a history (h_t, s_t) that has associated beliefs $p_t(h_t, s_t) \leq P(-1)$. Under assumption (7), not investing gives a strictly higher return than investing. We summarize this discussion with the following proposition:

Proposition 1. Under assumption (6), the constructed strategies and beliefs constitute a perfect Bayesian equilibrium.

Next, we are interested in exploring if there is any sense in which the outcomes in this equilibrium are inefficient relative to some benchmark. Our benchmark is a public signal game in which the signals are observed by all investors. In period t, given the history of signals $s^t = (s_0, s_1, \ldots, s_t)$, the history of investments is irrelevant. Thus, we define both investments z_t and beliefs p_t as functions of s^t alone. An equilibrium is defined as before. Let $z_t(s^t)$ denote the investment decision in period t with signal history s^t. In the private signal game, given a history of signals s^t, we can recursively substitute out for the investment strategies and write the investment outcome simply as a function of s^t. Let $x_t(s^t)$ denote such an investment outcome.

We say that the private signal game has a *herd at* s^t if (i) for all future histories s^r containing s^t, $z_r(s^r)$ is that same for all s^r and (ii) for some future history s^r containing s^t, $z_r(s^r) \neq x_r(s^r)$. We say that a herd at s^t is a *herd of investment* if $x_r(s^r) > 0$ for all future histories s^r containing s^t. Likewise, a herd at s^t is a *herd of no investment* if $x_r(s^r) = 0$ for all future histories s^r containing s^t.

The first clause in our definition of a herd requires that individuals make the same decisions regardless of their signals, namely that the outcome is a cascade. The second clause requires that these decisions are inefficient relative to the public signal game, so that a herd is an inefficient cascade. Several authors (including Banerjee 1992) have defined notions of herd-like behavior which only require the first clause, namely our notion of cascade. With only the first clause, if we started agents with prior below $P(-N)$ they would never invest in either game regardless of the signals. In a sense everyone is doing the same thing because they are all doing the right thing. The second clause ensures that when everyone is doing the same thing, relative to the public signal game, they are doing the wrong thing.

Proposition 2. *(Herd Behavior)* Under (6) and (7) the equilibrium has both herds of investment and herds of no investment.

Proof. To see that there is a herd of investment consider the history of signals $s^N = (H, L, L, \ldots, L)$. In the private signal game, $x_t = 1$ for all t. In the public signal game, using (6) and (7), $z_0 = z_1 = 1$, and $z_t = 0$ for $t \geq 2$. Likewise there are clearly herds of no investment. Consider the history of signals $s^N = (L, L, H, \ldots, H)$. In the private signal game $x_t = 0$ for all t, while in the public signal game for $t \geq 4$ the prior rises above $P(0)$ and z_t is 1 thereafter. Q.E.D.

Our demonstration that herds are robust depends crucially on allowing for endogenous timing, that is, allowing investors to invest at any time they choose. In Chari and Kehoe (2002) we analyse this baseline model allowing for endogenous timing. It turns out that the equilibrium in that model is essentially identical to the equilibrium here with exogenous timing under one additional assumption. This assumption is

$$(11) \quad e^{RT} P(0) < v_H(P(0)) e^{R(T-1)} P(1) + v_L(P(0))$$

where $v_H(p) = pq + (1-p)(1-q)$ is the probability that a high signal is received when the prior is p and $v_L(p) = p(1-q) + (1-p)q$ is the probability that a low signal is received when the prior is p.

Assumption (11) essentially says that discounting is small relative to the value of information. To interpret (11) consider an economy with endogenous timing. The left-side of (11) is the value of investing in period 0 at $P(0)$. Now, suppose that an investor knows that if he waits in period 0 , with probability $v_H(P(0))$ he will either be able to infer a high signal occured and with probability $v_L(P(0))$ he will be able to infer that a low signal occurred. Then (11) says that the investor is better off waiting to receive the information rather than investing immediately.

In a model with endogenous timing waiting and receiving information is beneficial because investors have the option of not investing if the signals are sufficiently low. We call this benefit the *no-investment option value.* The cost of waiting comes from a kind of discounting in that investors forgo the flow return from investing. Assumption (11) requires that the no investment option value be large relative to discounting.

To facilitate comparisons between the assumptions made here and the assumptions made in later sections it is useful to note that (6), (7) and (11) can be written as

(12) $\quad V(P(0), N) > 1$

(13) $\quad V(P(-1), 1) = 1$ and

(14) $\quad V(P(0), 0) < v_H(P(0))V(P(1), 1) + v_L(P(0)).$

2. Herds with Continuous Investment

One critique of the baseline herd model is that the results depend critically on the action space being coarse relative to the signal space. Here we show that this critique has some merit. We show that if the action space is continuous then there can be no herds of investment. We answer this critique by showing that once we drop the unappealing assumption of exogenous timing the critique is overturned.

A. The Continuous Investment Critique

Let us now suppose that the investment choices are continuous in that $X = [0, 1]$. We assume that an investment of x units in period t yields a return of $e^{R(T-t)}f(x)$ if the state is high and zero if the state is low where f is strictly concave, $f'(0)$ is finite, $f(0) = 0$ and $f(1) = 1$. The assumption that $f'(0)$ is finite is natural in any applied situation in which there is either a fixed cost or a minimum scale of production.

Consider the problem of investor t for some given beliefs p, given in (2), namely

$$(15) \quad V(p,t) = \max_{x \in X} e^{R(T-t)} p f(x) + (1-x)$$

The first order condition at an interior point is

$$(16) \quad f'(x) = e^{R(t-T)}/p.$$

Let $\underline{p}(t) = e^{R(t-T)}/f'(0)$. Clearly, investor t's optimal investment given beliefs p is

$$(17) \quad x(p,t) = \left\{ \begin{array}{c} (f')^{-1}(e^{R(t-T)}/p) \text{ if } p \geq \underline{p}(t) \\ \\ 0 \text{ otherwise} \end{array} \right\}.$$

The definition of a perfect Bayesian equilibrium here is the same as in the baseline model. We make the following assumptions,

$$(18) \quad V(P(0), N) > 1$$

$$(19) \quad V(P(-1), 1) = 1.$$

Assumptions (18) and (19) are identical to (12) and (13) except that here investment decisions are continuous.

We will show that, in contrast to baseline model with discrete investments, with continuous investments there can be no herds of investment. Intuitively, as long as the prior is above the cutoff level $\underline{p}(t)$,the investor t invests some positive amount, say x_t, given by (17). Using (16), the belief of the agent can be inferred to be

$$p_t = e^{R(t-T)}/f'(x_t)$$

and, given some public belief $p_{t-1,}$ the action x_t reveals investor t's signal. More formally we have that investor's beliefs are recursively defined as follows. On the equilibrium path, investors invert the invest decisions to infer the underlying signals and update their beliefs in the obvious manner. For any period t, with public beliefs $P(k)$ such that

$$(20) \quad x(P(k+1),t) \neq x(P(k-1),t)$$

public beliefs at $t+1$ are $P(k+1)$ if $x_t = x(P(k+1),t)$ and public beliefs at $t+1$ are $P(k-1)$ if $x_t = x(P(k-1),t)$. For any period t with public beliefs $P(k)$ such that

$$(21) \quad x(P(k+1),t) = x(P(k-1),t),$$

public beliefs at $t+1$ are simply $P(k)$. For investment decisions which are not on the equilibrium path we assume that public beliefs at $t+1$ are simply the public beliefs at t. Investor t updates the public beliefs using the signal and Bayes' rule. The investment strategies are given by substituting these beliefs into (17). Clearly, these strategies and beliefs constitute a perfect Bayesian equilibrium.

We then have the following proposition.

Proposition 3. *(The Continuous Investment Critique)* In the continuous investment model, under (18) and (19), there are herds of no investment but there are no herds of investment.

Proof. We first show that there is a herd of no investment. Consider the outcome path when the first two signals are (L, L). The investor at 0 has beliefs $P(-1)$ and under (19) does not invest. The public belief in period 1 is $P(-1)$ because (20) is satisfied in period 0. To see these note that if the investor at 0 would have received a H, under (18) this investor would have invested. The investor in period 1, having received a signal L, lowers his beliefs to $P(-2)$ and under (19) does not invest. The public belief in period 2 is $P(-2)$ because (20) is satisfied in period 1. In all subsequent periods, regardless of the signals, no investor invests and public beliefs stay at $P(-2)$. This is because (21) holds in all such periods. Thus the outcome path in the private signal game for any set of signals that begins with (L, L) is no investment in all periods. The outcome associated with the signals $(L, L, H, H, \ldots, H)$ is a herd of no investment. In the public signal game with these signals the beliefs rise to $P(0)$ in period 4 and thus investor 4 invests. So do all subsequent investors. In the private signal game these signals lead to no investment in all periods.

We prove that there are no herds of investment by showing that in any outcome path with positive investment in the tails, the public can infer private signals exactly so that the signals are effectively public. Suppose we have an outcome path where in all periods r following some period t investment is positive. Then, from (18) and (19), we have that investor $r's$ beliefs must be at least as large as $P(0)$ for all r. The public's beliefs must be at least $P(-1)$. At these beliefs, the investment strategies satisfy (20) and the public inverts the investment decision to infer the signal exactly in period t and all subsequent periods. Next we show that in all periods before t, the public can infer the signal exactly. To see this result note that if public beliefs reach $P(-2)$ in any period, no investor invests and public beliefs stay at $P(-2)$. Thus, in all periods before t public beliefs must have been at least $P(-1)$. But then, the investment strategies satisfy (20) and can be inverted to obtain the signal. Since the signals are effectively public, it follows that the outcomes of the private and public signal games must coincide. Q.E.D.

B. Answering the Continuous Investment Critique

Here we show that once we allow for endogenous timing herds reappear even though investment decisions are continuous.

The model with endogenous timing is identical to that with exogenous timing with several exceptions. Here we allow investors to invest at any time, although the decision must be made once-and-for-all. In each period $t = 0, \ldots, T_1$ with $T_1 < T$, one signal arrives to the economy and is randomly distributed to one and only one agent among the set of investors who have not already received a signal.[1] (Conceptually, it is easy to instead allow signals to arrive intermittently, say, according to a Poisson process. The results are similar, however, the resulting algebra is more complicated.)

We let the number of investors N be large relative to T_1. The state becomes known at $T > T_1$. All that happens from $T_1 + 1$ to T is that the investments grow, but during this time no signals arrive and no actions can be taken.

The only publicly observable event in any period t is the aggregate quantity of investment denoted by x_t. The public history $h_t = (x_0, x_1, , \ldots, x_{t-1})$ records the aggregate quantity of investments in each period up through the beginning of period t. Investors also record the signal they receive, if any, and the date they receive it. Thus, the history of an investor i at t who received a signal at r is $h_{it} = (h_t, s_r, r)$ and we let $(h_t, \emptyset, \emptyset)$ denote the history of an investor who has not received a signal. Notice that at each date t, given their histories, investors can be described as belonging to one of several groups. Any investor who has already invested is *inactive*. The active investors in period t consist of a *newly informed* investor who receives the signal at the beginning of period t, *previously informed* investors who received a signal at some date r before t and *uninformed* investors who have not yet received a signal.

An investor's strategy and beliefs are sequences of functions $x_t(h_{it})$ and $p_t(h_{it})$ that map their histories into actions and into the probability that the state is high. The payoffs are defined as follows. The payoff for an investor who makes an investment decision in period t with history h_{it} is

$$(22) \quad V_t(h_{it}) = \max_{x \in X} e^{R(T-t)} p_t(h_{it}) f(x) + 1 - x.$$

The payoff for an investor who waits at time t is given by

$$(23) \quad W_t(h_{it}) = \sum_{h_{it+1}} \mu_t(h_{it+1}|h_{it}) \max\{V_{t+1}(h_{it+1}), W_{t+1}(h_{it+1})\}),$$

where $\mu_t(h_{it+1}|h_{it})$ is the conditional distribution over history at $t+1$ given the history at t. Clearly, an investor invests at t if $V_t(h_{it}) \geq W_t(h_{it})$ and waits otherwise. Notice that the conditional distributions $\mu_t(h_{it+1}|h_{it})$ are induced from the strategies and the structure of exogenous uncertainty of the game in the obvious way. Notice also that we have imposed symmetry by supposing that all investors who have the same histories take the same actions and have the same beliefs. Here, a *perfect Bayesian equilibrium* is a set of strategies $x_t(h_{it})$, a set of conditional distributions $\mu_t(h_{it+1}|h_{it})$ and a set of beliefs $p(h_{it})$ such that i) for every history h_{it}, such that the investor has not invested before t, the waiting decision is optimal and, conditional on making an investment, the investment level $x_t(h_{it})$ is optimal, $ii)$ the conditional distributions $\mu_{it}(h_{it+1}|h_{it})$ and the beliefs $p_{it}(h_{it})$ are consistent with Bayes' rule wherever possible and arbitrary otherwise.

In addition to (18) and (19) we make the following assumptions,

$$(24) \quad V(P(0), 0) < v_H(P(0))V(P(1), 1) + v_L(P(0))$$

and

$$(25) \quad V(P(1), t) > v_H(P(1))V(P(2), t+1) + v_L(P(1))V(P(0), t+1)$$

$$(26) \quad V(P(2), t) > v_H(P(2))V(P(3), t+1) + v_L(P(2))V(P(1), t+1)$$

Note that (19) is identical to (14). Furthermore, (25) and (26) are automatically satisfied in the baseline model. For example, to see that (25) is satisfied, note that under (18), $V(p, t) = e^{R(T-t)}p$ for $p \geq P(0)$. From Bayes' rule it follows that $P(1) = v_H(P(1))P(2) + v_L(P(1)P(0)$ so that (25) reduces to $e^{R(T-t)}P(1) > e^{R(T-t-1)}P(1)$ which obviously holds.

In this model, waiting and receiving information has two benefits. The first is that investors have the option of never investing if the future information turns out to be sufficiently low. We have called this benefit the no investment option value. The second, which we call the *fine-tuning option value,* comes from better information allowing investors who have already decided to invest to adjust the size of their projects. The cost of waiting comes from a kind of discounting in that investors forgo the flow return from investing.

Assumption (24) requires that the no investment option value be large relative to discounting. We will use this assumption to ensure that in equilibrium an uninformed investor will find it optimal to wait at $P(0)$. The right-side of (24) will turn out to be the payoff to an uninformed investor who follows the following possibly suboptimal strategy at public prior of $P(0)$. Wait. In the next period invest if he infers that the informed investor received a high signal and never invest if he infers that the informed investor received a low signal.

Assumptions (25) and (26) require that the fine-tuning option value is small relative to discounting in the following sense. Assumption (25) requires that investing at a prior of $P(1)$ dominates waiting one period, receiving a signal and then investing the optimal higher amount if the signal is high and the optimal lower amount if the signal is low. Assumption (26) imposes a similar requirement at a prior $P(2)$. We use assumption (25) and (26) to show that starting at history in which uninformed investors' priors are $P(0)$, a newly informed investor who has received a high signal will invest immediately rather than attempting to learn from the investments of future newly informed investors and then optimally adjusting the size of his investment. Theses assumptions are satisfied if f is sufficiently concave at $x(P(1), t)$ and $x(P(2), t)$

The strategies and beliefs for our equilibrium are as follows. The strategy for all uninformed and previously informed investors is

$$(27) \quad x_t(h_{it}) = \left\{ \begin{array}{c} 1 \text{ if } p_t(h_{it}) \geq P(1) \\ \\ 0 \text{ otherwise} \end{array} \right\}$$

for $t \leq T_1 - 1$ and $x_{T_1}(h_{iT_1}) = 1$ if and only if $p_{T_1}(h_{iT_1}) \geq P(0)$. The strategy for newly informed investors is

$$(28) \quad x_t(h_{it}) = \left\{ \begin{array}{c} 1 \text{ if } p_t(h_{it}) \geq P(0) \\ \\ 0 \text{ otherwise} \end{array} \right\}$$

for $t \leq T_1$. Notice that the uninformed and previously informed investors need to be more optimistic than newly informed investors in order to invest before T_1. We refer to $P(1)$ and $P(0)$ as the *cutoff levels* for the uninformed and informed investors respectively.

The beliefs of uniformed investors at history $h_{it} = (h_t, \emptyset, \emptyset)$ are defined recursively. Given $p_{t-1}(h_{it-1})$ and a total investment of x_{t-1} at $t-1$, the beliefs at t are given as follows. For $p_{t-1}(h_{it-1})$ equal to either $P(-1)$ or $P(0)$

$$(29) \quad p_t(h_{it}) = \left\{ \begin{array}{l} P_L(p_{t-1}(h_{it-1})) \text{ if } x_{t-1} = 0, \\ \\ P_H(p_{t-1}(h_{it-1})) \text{ if } x_{t-1} > 0 \end{array} \right\}$$

where $p_0(h_{i0}) = P(0)$. For $p_{t-1}(h_{it-1})$ either greater than or equal to $P(1)$ or less than or equal to $P(-2)$, $p_t(h_{it}) = p_{t-1}(h_{it-1})$.

The beliefs of the newly informed investors at history $h_{it} = (h_t, s, t)$ are simply those of the uninformed investor, updated by the newly informed investor's signal, namely $p_t(h_t, s, t) = P_s(p_t(h_t, \emptyset, \emptyset))$ for $s = H, L$. The beliefs of the previously informed investor at t who received his signal in $t-1$ with history $h_{it} = (h_t, s, t-1)$ are defined as follows. If no other investor invested at $t-1$, this investor's beliefs are the same as they were in period $t-1$, namely $p_t(h_t, s, t) = P_s(p_{t-1}(h_{t-1}, \emptyset, \emptyset))$ for $s = H, L$. If some other investor invested at $t-1$, $p_t(h_t, s, t) = P(2)$. The beliefs of previously informed investors who received their signals before period $t-1$ are recursively defined using (29) except that the recursion starts at r, with the beliefs of the newly informed investor at r, namely $p_r(h_{ir}, s, r)$. These strategies and beliefs induce the conditional distributions $\mu_t(h_{it+1}|h_{it})$ in the obvious manner.

Built into our beliefs is the idea that investors look at previous investors' actions and try to infer their signals. On the equilibrium path and for deviations that they cannot detect investors infer the following. Consider the uninformed investors at public beliefs p equal to either $P(-1)$ or $P(0)$. If they see $x_t = x(P_H(p), t)$ they infer that the newly informed investor received a high signal. If they see $x_t = x(P_L(p), t) = 0$ they infer that the newly informed investor received a low signal. If public beliefs equal $P(-2)$, they expect to see no investment regardless of the newly informed investor's signal. We have also filled in beliefs off the equilibrium path in an intuitive manner. Our results are unaffected by these choices.

On the equilibrium path and for undetectable deviations the newly informed investors simply update the beliefs of the uninformed investors with their private signal. The previously informed investor that was newly informed at $t-1$, simply updates the beliefs of the newly informed investor at $t-1$ appropriately. The previously informed investor that was newly informed at $r < t-1$, simply updates the beliefs of the previously informed investor at $t-1$ appropriately.

An important feature of the strategies is that cutoff level for investment for the uninformed investors is higher than that for the newly informed investors. To understand why this is necessary suppose first that both types of investors invest if their beliefs are greater than or equal to $P(0)$. To see why this cannot be an equilibrium consider a deviation by an uninformed investor at $t = 0$ with beliefs $P(0)$ to waiting. Since the newly informed investor invests if and only if his signal is high, the deviating investor learns the value of the signal. By (24) this deviation increases payoffs.

Suppose next that the cutoff level for both types of investors is $P(1)$. Suppose the first signal is L. The newly informed at 0 does not invest and the other investors infer he got a low signal and their priors are $P(-1)$. The newly informed investor at date 1 is supposed to wait regardless of his signal. Thus the prior of the uninformed investor stays at $P(-1)$ and thus all newly informed investors at all future dates also wait. After a history of signals L, H, the newly informed at date 1 has a prior of $P(0)$. A deviation to investing, by (18), raises the payoffs.

These arguments help explain why the cutoff levels of the informed and uninformed investors must be different. We now show that when these cutoffs have the form in (27) and (28) the strategies and beliefs are an equilibrium. (The first part of the proof of this proposition is similar to that in Chari and Kehoe 2002.)

Proposition 5. Under assumptions $(18), (19)$ and $(24) - (26)$, the strategies and beliefs in (27)-(29) constitute a perfect Bayesian equilibrium.

Proof. By construction the beliefs in (29) satisfy Bayes' rule. We repeatedly use the observation that by construction for any history h_{it}, $p_t(h_{it}) = P(k)$ for some integer k .

Consider first optimality for histories with no detectable deviations. Consider the strategies of the uninformed investors. For a history of the uninformed investor with beliefs $P(1)$ the uninformed investor is supposed to invest. Suppose instead that the investor deviates and waits. If for all future histories the investor ends up investing then waiting merely reduces the length of time of investment in the high return project and, by discounting, yields a strictly lower payoff. Thus, the only way that this deviation can be profitable is that there are some future histories in which this investor never invests. Consider the most pessimistic information the investor could receive. Recall that for such a history all other active investors invest at t. Thus, by waiting the uninformed investor receives no new information from others. By waiting the uninformed investor could receive a signal in the future. But even if the future signal is $s = L$, this investor's belief will be $P(0)$ and by (18) he will invest. Thus, even under the most pessimistic information it is optimal to invest, and hence waiting is not profitable. Clearly, for histories with $p_t(h_{it}) \geq P(1)$ it is also optimal to invest. For a history of the uninformed investor with beliefs $P(0)$, (24) ensures that it is optimal to wait, while with beliefs $P(-1)$ and below (19) ensures that it is optimal to wait.

Consider next the strategies of the informed investors at some history h_{it}. The interesting histories are those in which the uninformed investors' beliefs are $P(0)$ or $P(-1)$ and the newly informed investor has just received a high signal. The strategy for the newly informed investor specifies invest and suppose this investor deviates and waits, presumably to garner information about the signals of subsequent informed investors. If for all future histories the investor ends up investing then waiting merely reduces the length of time of investment in the high return project. Thus, the only way that this deviation can be profitable is that there are some future histories in which this investor never invests. After such a deviation the beliefs of the newly informed investor are always 2 higher than that of the uninformed investors. The reason is that the newly informed investor′s private signal raised his beliefs by 1 and the deviation by the newly informed investor did not affect his own beliefs while it lowered the uninformed investors' beliefs by 1.

Consider first a history h_{it} in which the uniformed investors' beliefs are $P(-1)$ and suppose that the newly informed investor receives a high signal and hence has beliefs $P(0)$. If the newly informed investor deviates and waits then this deviation triggers a cascade with no investment. To see this note that the deviation causes the uninformed investors' beliefs to be $P(-2)$ permanently. Given these beliefs uninformed investors never invest. Future newly informed investors update their beliefs to at most $P(-1)$ and do not invest either. Thus this deviation garners no new information. The beliefs of the deviating investor remain at $P(0)$. Assumption (6) then implies that given these beliefs it is optimal for the newly informed investor to invest at t.

Next, consider a history h_{it} in which the uniformed investors' beliefs are $P(0)$ and the newly informed investor receives a high signal and hence has beliefs $P(1)$. We use assumption (25) and (26) to show that investing immediately dominates attempting to learn from the investments of future newly informed investors and then optimally adjusting the size of his investment. Suppose the newly informed investor deviates and waits. In all subsequent periods, the beliefs of this deviating investor are 2 higher than that of the uninformed investors in the sense that if public beliefs are at $P(k)$ the deviating investor's beliefs are at $P(k+2)$. The reason is that the newly informed investor's private signal raised his beliefs by 1 and the deviation by the newly informed investor lowered the uninformed investor's beliefs by 1 without affecting his own beliefs. Next, note that a herd with investment starts when uninformed investors' beliefs are $P(1)$ and a herd of no investment starts when uninformed investors' beliefs are $P(-2)$. Hence, in any period after the deviation, the deviating investor's beliefs can be one of four values, $P(0)$, $P(1)$, $P(2)$, or $P(3)$. Moreover, if this investor's beliefs reach $P(0)$ or $P(3)$ they stay there because the uninformed investors are then in either a cascade without investment or a cascade with investment.

We recursively solve for the strategy the yields the highest payoff following the deviation at t as follows. Suppose that the investor has not invested until T_1. For all four beliefs (18) implies that it is optimal to invest. Suppose that the investor has not invested until $T_1 - 1$. If the investor's beliefs are at either $P(0)$ or $P(3)$ they stay there at T_1 and discounting makes it optimal to invest at $T_1 - 1$. If the investor's beliefs are at $P(1)$ (25) makes it optimal to invest immediately while if the investor's beliefs are at $P(2)$ (26) makes it optimal to invest immediately. Repeating this argument, conditional on waiting at t, the best deviation is for the investor to invest at $t + 2$. Assumptions (25) and (26) then imply that investing at t, dominates waiting and then investing at $t+2$.

For histories in which the newly informed investor's beliefs are at or below $P(-1)$, (19) implies that deviating to investing is not optimal.

Finally, it is easy to show that for histories off the equilibrium path the strategies for all investors are optimal. *Q.E.D.*

To show that there are herds we need an appropriate benchmark. The public signal version of this game is not the appropriate benchmark. In the private signal game the uninformed agents can only react to the revealed information with a one period lag, while in the public signal game they can react immediately.

The benchmark we use is a public signal game which captures the informational lags built into the private signal game. Consider a game with public information lags in which the uninformed agents learn the realization of the date t signal after they have made their period t investment decisions. In period t, given the history of publicly observed signals $s^{t-1} = (s_0, s_1, \ldots, s_{t-1})$, the history of investments is irrelevant. Thus, for uninformed investors we define both investments z_t and beliefs p_t as functions of s^{t-1} alone. Investments and beliefs of informed investors are functions of s^t. An equilibrium is defined as before. In the private signal game, given a history of signals s^t, we can recursively substitute out for the investment strategies and write the investment outcome for the uninformed investors simply as a function of s^{t-1} and for the informed investors as a function of s^t. The definition of a herd of investment and a herd of no investment is the same as before.

Under the following assumption

$$(30) \quad V(P(1),0) < v_H(P(1))V(P(2),1) + v_L(P(1))\left[v_H(P(0))V(P(1),2) + v_L(P(0))\right].$$

we can show that there are herds. We think of (30) as strengthened version of (24), since, given (7), it is easy to see that (30) implies (24). Assumption (30) implies that, in the public signal game, in any period t investing at $P(1)$ dominated by the following strategy. Wait until $t+1$, if the signal is high invest, if the signal is low, wait until $t+2$ and invest if and only if the signal is high. The following proposition is immediate.

Proposition 6. *(Herd with Continuous Investment)* Under $(18), (19), (24) - (26)$, and (30) there are both herds of investment and herds of no investment in the model with endogenous timing.

3. Herds with Prices

Here we consider a variant of the baseline model in which investors trade investment projects. We add a market maker who sets prices in a competitive fashion together with some idiosyncratic traders. Otherwise the basic setup is quite similar.

There are two types of agents in our model: T risk neutral investors, who are ordered in a sequence $t = 1, \ldots, T$. and a large number of *market makers*. Investors can be of one of two types, idiosyncratic or informed. *Idiosyncratic investors* are imagined to have some idiosyncratic reason that makes them want to buy or sell with probability 1/2 each, in any period regardless of the price. In any given period the probability that the investor is idiosyncratic is $1-\alpha$ and the probability that the investor is informed is α. *Informed investors* receive a signal about the underlying state with distribution given by (19).

Each investor is endowed with a risky project. Each project requires an investment of one unit of effort to become viable. In terms of market trades, the investor may either sell this project to the market maker, buy a second project from the market maker or not trade.

In terms of investments we assume that if the investor buys a second project, he must invest in both. This assumption is innocuous since in equilibrium an investor who wants to buy will always want to invest as much as possible. The choices of investors in any period are then: sell a project, buy a second project and invest in both, invest in own project without trading, do nothing. We denote these choices by S, B, I, N respectively. We let Q_{St} denote the price at which the investor sells the project and Q_{Bt} denote the price at which the investor buys the project.

A. The Price Critique

We begin a version of the model with exogenous timing of investment decisions. With this timing, in equilibrium, the choice B dominates I and we suppress the option I.

The idiosyncratic agents trade in a predetermined fashion and we only need model the strategies of the informed agents and the market maker. Since each informed agent takes an action in only one period, we need only define each informed agent's strategy for only one period. In each period t, the only publicly observable events, denoted by $z_t = (x_t, Q_{Bt}, Q_{St})$ where $x_t \in \{S, B, N\}$ denotes the sell/buy/do nothing decisions. The *public history* at t , denoted $h_t = (z_0, z_1, \ldots, z_{t-1})$ is the history of investor decisions up through the beginning of period t. Let $p_t(h_t)$ denote the *public beliefs*, namely the posterior probability that the state is H given the history h_t.

An informed agent at t, in addition to the public history and his own signal confronts a vector of selling prices and a vector of buying prices. Clearly, only the highest selling price and the lowest buying price are relevant to this agent's decision. An informed agent's history at t is $h_{It} = (h_t, s_t, Q_{St}, Q_{Bt})$ where $s_t \in \{H, L\}$ denotes this agent's signal and we interpret

Q_{St} and Q_{Bt} to be the highest selling price and the lowest buying price respectively. The informed agent's beliefs are denoted by $p_t(h_{It})$. This agent's strategy at t, $\sigma_t(h_t) \in \{B, S, N\}$ specifies an action buy (B), sell (S), or do nothing (N) for each possible history. The informed investor's payoff from selling is Q_{St}, from buying is

$$(31) \quad [e^{R(T-t)} p_t(h_{It}) - 1] + [e^{R(T-t)} p_t(h_{It}) - 1 - Q_{Bt}],$$

and 0 from not trading.

In addition to the investors, there are a large number of risk neutral *market makers.* The market makers do not receive signals. In period t, each market maker posts prices at which the investor can buy or sell one risky project. Let Q'_{Bt} and Q'_{St} denote the prices of a particular market maker. If an investor buys a project from this market maker, the market maker receives a payoff of

$$(32) \quad Q'_{Bt} - \max\{e^{R(T-t)} p_B(h_t, Q'_{Bt}) - 1, 0\}$$

where $p_B(h_t, Q'_{Bt})$ is the posterior of this market maker. This posterior depends on the market maker's posted price because the mix of investors attracted to the market maker depends on the posted price. (Of course, the mix also depends on prices posted by other market makers $Q_{Bt}(h_t)$ and $Q_{St}(h_t)$, but these prices are known functions of the history h_t and hence are suppressed.) If an investor sells a project the market maker receives a payoff of

$$(33) \quad \max\{e^{R(T-t)} p_S(h_t, Q'_{St}) - 1, 0\} - Q'_{St}$$

where $p_S(h_t, Q'_{St})$ is the posterior of the market maker given Q'_{St}.

One interpretation of these payoffs is that each market maker is endowed with a project that he can run himself and that the market maker can buy a second project and operate it.

The opportunity cost of selling the project to the investor is $\max\{e^{R(T-t)}p_S(h_t, Q'_{Bt}) - 1, 0\}$. Thus, if the investor buys a project the profits of the market maker are given by (32). Clearly, if the investor sells a project the profits of the market maker are given by (33). Another interpretation is that market makers are intermediaries between the investors in this market and outside uninformed investors.

An equilibrium is a collection of strategies $\sigma_t(h_{It}, Q_{Bt}, Q_{St})$, prices $Q_{Bt}(h_t)$ and $Q_{St}(h_t)$ and beliefs $p_t(h_t), p_B(h_t, Q_{Bt}), p_S(h_t, Q_{St}), p_t(h_{It})$ such that *i*) the strategies $\sigma_t(h_{It})$ are optimal for the investors, *ii*) the prices set by market makers maximize profits given in (32) and (33), *iii*) the market maker's profits evaluated at $Q_{Bt}(h_t)$ and $Q_{St}(h_t)$ are equal to zero and *iv*) the beliefs satisfy Bayes' rule where possible.

In terms of characterizing the equilibrium it is easier to write the strategies as functions of the public belief $p = p_t(h_t)$ rather than the histories h_t directly. The investor's beliefs $p_t(h_{It})$ are obtained from public beliefs p by (3) if this investor's signal is H and (4) if this investor's signal is L. Clearly, given beliefs $p_t(h_{It})$, the informed investor's strategy has the following properties: buy if (31) is strictly positive and larger than Q_{St}, sell if Q_{St} is strictly larger than both (31) and 0, do nothing if both buying and selling give negative payoffs. We specify the actions taken when the investor is indifferent below.

Consider the strategies and beliefs of the market makers. Competition among market makers drives equilibrium profits to zero for any purchase or for any sale. In equilibrium, the market maker loses money trading with informed investors and makes money trading with idiosyncratic investors. Competition among the market makers leads them to value the project between the value implied by public beliefs and the value assigned by the investor given his private information.

In order to set up the problem of the market maker it is convenient to let

$$P_U(p) = \frac{p(\frac{1-\alpha}{2} + \alpha q)}{\frac{1-\alpha}{2} + \alpha[pq + (1-p)(1-q)]} \tag{34}$$

$$P_D(p) = \frac{p(\frac{1-\alpha}{2} + \alpha(1-q))}{\frac{1-\alpha}{2} + \alpha[p(1-q) + (1-p)q])}. \tag{35}$$

We will show that the equilibrium prices fall into two regions depending on the level of public beliefs p. In the high region, $1 < e^{R(T-t)}P_H(p)$ and the equilibrium prices are

$$Q_{Bt}(p) = \max\{e^{R(T-t)}P_U(p) - 1, 0\} \tag{36}$$

$$Q_{St}(p) = \max\{e^{R(T-t)}P_D(p) - 1, 0\}. \tag{37}$$

In the low region, $e^{R(T-t)}P_H(p) < 1$ and the equilibrium prices are both zero.

For both regions we show that profits are zero at the equilibrium prices and that no market maker can gain by deviating. We first show that profits are zero in the high region. To evaluate the expected profits of the market maker we need to form the posteriors of the market maker and hence we need the investor's optimal actions. It is immediate to check that the following choices are optimal at the equilibrium prices: investor t chooses B if $s_t = H$ and S if $s_t = L$. If the investor buys, the posterior of the market makers, conditional on a buy decision by the investor is $P_U(p)$ given in (34). This result follows from Bayes' rule. The probability that the investor is idiosyncratic is $(1-\alpha)/2$. The probability that the investor is informed and receives a high signal is αq. The prior probability that the state is H is p. Thus, the probability that the state is high and an investor buys from the market maker is given by the numerator of (34). The denominator is simply the probability that an investor buys from the market maker. Similar reasoning establishes that the posterior of the market makers to whom investors sell is given by $P_D(p)$. Given these posteriors it follows from substituting the posteriors and the equilibrium prices into (32) and (33) that the expected profits of the market maker are zero at these prices.

We next show that in the high region no market maker can gain by deviating. The interesting deviations are ones that raise lower the price at which can investors can buy and raise the price at which investors can sell. Clearly, in either case expected profits are negative.

Consider next the low region. It is immediate to check that doing nothing regardless of the signal is optimal for investor t. Since the informed investor does not trade, the market maker realizes that all trades are with idiosyncratic traders. Hence, the market makers' posterior stays at the public belief p. Substituting these posteriors and the equilibrium prices into (32) and (33) it follows that the expected profits of the market maker are zero at these prices. Clearly, any deviation by a market maker leads to negative profits.

Public beliefs evolve naturally from the strategies. If $1 < e^{R(T-t)}P_H(p)$ then

$$p_{t+1}(p) = \left\{ \begin{array}{l} P_U(p) \text{ if } z_t = B \\ \\ P_D(p) \text{ if } z_t = S \end{array} \right\}$$

while if $e^{R(T-t)}P_H(p) < 1$ then $p_{t+1}(p) = p$.

In equilibrium, public beliefs will always be equal to $P(k)$ for some integer k where $P(0) = p_0, P(1) = P_D(p_0)$, $P(-1) = P_U(p_0)$ and so on. This follows because P_U and P_D are symmetric, in the sense that $P_U(P_D(p)) = P_D(P_U(p)) = p$.

As before we define a herd relative to a public signal game. The public signal game which captures some of the information constraints of the private signal game is the following. In each period t, with probability α a signal H or L is drawn from the distribution in (1) and this signal is received by the informed investor at the beginning of the period. At the end of the period this signal becomes public. With probability $(1-\alpha)/2$, an idiosyncratic investor is instructed to buy and at the end of the period the public sees the signal H. With probability $(1-\alpha)/2$, an idiosyncratic investor is instructed to sell and at the end of the period the public sees the signal L.

In this equilibrium, under the analogs of (6) and (7), there are herds of no investment but no herds of investment. To see this result, assume

$$(38) \quad e^{R(T-N)}P(0) > 1$$

$$(39) \quad e^{RT}P(-1) < 1.$$

To see that there are herds of no investment, suppose that the first two investors sell. (One way this could happen is that the first two investors are informed investors who receive low signals, where we note that $P_H(P(-1)) > P(0)$. Other ways include the first two investors being idiosyncratic sellers or some combination of informed and idiosyncratic investors.) Then, public beliefs reach $P(-2)$. Thereafter, no informed investor trades and public beliefs stay at $P(-2)$. This outcome path is associated with a herd of no investment.

There are no herds of investment. To see this note that once public beliefs are at $P(-1)$ or higher, the informed investors take different actions for different signals. Subsequent investors then adjust their priors depending upon the observed action. In this sense, the proposition is a critique of herd behavior.

Proposition 8. *(The Asset Trading Critique)* The strategies, prices and beliefs described above constitute an equilibrium. Under (38) and (39), in this equilibrium there are herds of no investment but there are no herds of investment.

B. Answering the Price Critique

Next consider a version of the model which allows for endogenous timing of moves. In this version, we allow investors to invest at any time, although the decision must be made once-and-for-all. We also allow for the option I, namely, an investor can invest in his own project without trading.

In each period $t = 0, \ldots, T_1$ one signal arrives to the economy and is randomly distributed to one and only one agent among the set of investors who have not already received a signal. We let the number of investors N be large relative to T_1. The state is realized at 0 but only becomes known at $T > T_1$.

Let B_t denote the number of investors who buy and S_t denote the number who sell in period t. The publicly observable events in any period t are $n_t = (B_t, S_t)$ and the prices Q_{Bt} and Q_{St}. We let $z_t = (n_t, Q_{Bt}, Q_{St})$. The public history is $h_t = (z_0, z_1, , \ldots, z_{t-1})$. As before, we let h_{it} denote the history of investor i at t. This history records the signal they have received, if any, and the date they received it in addition to the public history. Here, among the active investors, in addition to the newly informed, the previously informed and the uninformed, we also have idiosyncratic investors. In what follows we do not need to define strategies for either inactive investors or idiosyncratic investors.

An investor's strategy and beliefs are sequences of functions $x_t(h_{it})$ and $p_t(h_{it})$ that map their histories into actions $\{B, S, I, N\}$ and into the probability that the state is high. The payoff for an investor who buys in period t with history h_{it} and current prices Q_{Bt} and Q_{St} is

$$(40) \quad V_{Bt}(h_{it}) = [e^{R(T-t)} p_t(h_{It}) - 1] + [e^{R(T-t)} p_t(h_{It}) - 1 - Q_{Bt}],$$

the payoff to an investor who sells is Q_{St},and the payoff to an investoe who chooses I is $V_{It}(h_{it}) = [e^{R(T-t)}p_t(h_{It}) - 1]$ The payoff for an investor who waits at time t is given by

$$(41) \quad W_t(h_{it}) = \sum_{h_{it+1}} \mu_t(h_{it+1}|h_{it}) \max\{V_{Bt+1}(h_{it+1}), Q_{St+1}(h_{t+1}), V_{It+1}(h_{it+1})W_{t+1}(h_{it+1})\}),$$

where $\mu_t(h_{it+1}|h_{it})$ is the conditional distribution over history at $t+1$ given the history at t. The future histories and the conditional distributions are induced from the strategies and the structure of exogenous uncertainty of the game in the obvious way. A *perfect Bayesian equilibrium* is defined in the obvious way.

We now show that now that we have allowed for endogenous timing there are both herds of investment as well as herds of no investment. To show this we assume (38) and (39) along with the analog of (24), namely,

$$(42) \quad e^{RT}P(0) - 1 < v_B(P(0))[e^{R(T-1)}P(1) - 1].$$

where $v_B(p) = (1-\alpha)/2 + \alpha[pq + (1-p)(1-q)]$ is the probability that investors will see a buy when the public beliefs are p.

Under these assumptions we show that the following strategies and beliefs constitute an equilibrium. It is convenient to define the strategies as functions of the public belief p. Let $k(t)$ be defined as the value of k that satisfies

$$e^{R(T-t)}P_H(P(-k-1)) < 1 < e^{R(T-t)}P_H(P(-k)).$$

At public beliefs $P(-k(t))$ the payoff from investing for a newly informed investor at t who receives a high signal is strictly positive, while at beliefs $P(-k(t)-1)$ this payoff is negative.

The newly informed investor's strategies at the equilibrium prices are as follows. In the region $p \geq P(-k(t))$, buy if the signal is high and sell if the signal is low. In the region

$p \leq P(-k(t)-1)$, do nothing. At the equilibrium prices the uninformed investors' strategies are as follows. For $t \leq T_1 - 1$, do nothing if $p \leq P(0)$ and invest if $p \geq P(1)$. At T_1, do nothing if $p \leq P(-1)$ and invest if $p \geq P(0)$. At prices other than the equilibrium prices, the strategies are obtained from (40) and (41). The equilibrium prices are (36) and (37) for $p \geq P(-k(t))$ and are both zero otherwise.

The public beliefs evolve as follows. For $p \geq P(-k(t))$, if a buy occurs then $p_{t+1} = P_U(p)$, if a sell occurs then $p_{t+1} = P_D(p)$ and for all events other than one buy or one sell, $p_{t+1} = P_D(p)$. For $p \leq P(-k(t)-1)$, $p_{t+1} = p$ for all occurrences at t.

The public signal game we use as a benchmark is the simply the one we defined with exogenous timing modified to allow endogenous timing. To ensure the inefficiency of the equilibrium cascades we use the following assumption,

$$(43) \quad e^{RT}P(1) - 1 < v_B(P(1))[e^{R(T-1)}P(2) - 1] + (1 - v_B(P(1))v_B(P(0))[e^{R(T-2)}P(1) - 1].$$

This assumption is the analog of (30).

Proposition 9. *(Herds with Asset Trading)* The above strategies of the investors, prices and beliefs constitute an equilibrium. In this equilibrium there are both herds of no investment and herds of investment.

Proof. The proof that these strategies of the investors are optimal and that the beliefs satisfy Bayes' rule is essentially identical to the one in Proposition 5. The proof that these prices are optimal and that there are herds of no investment is essentially identical to that in Proposition 8.

To see that there are herds of investment consider a period t in either the private or the public signal game in which public beliefs are $P(1)$. In the private signal game all investors invest. In the public signal game we show that (43) implies that the uninformed investor waits. We will show that waiting dominates investing. Consider the following (possibly suboptimal) strategy. Wait at t. If the end-of-period t public signal is H invest at $t+1$. If the end-of-period t public signal is L, wait at $t+1$. If the end-of-period $t+1$ signal is H invest at $t+2$, otherwise never invest. The payoff to this strategy is

$$(44) \quad v_B(P(1))[e^{R(T-t-1)}P(2)-1]+(1-v_B(P(1))v_B(P(0))[e^{R(T-t-2)}P(1)-1].$$

Assumption (43) implies that (44) is greater than the payoff to investing at t, namely $e^{R(T-t)}P(1)-1$. Q.E.D.

4. Conclusion

This paper has taken a step towards developing models of herd behavior that can be used in applied situations. We have done so by answering the two main critiques of the early models.

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Appendix

In this appendix we show that a third critique of herd behavior has merit with exogenous timing but that this critique is overturned under a mild additional assumption once we allow for endogenous timing of investment decisions.

The Information-Sharing Critique

Another critique of the baseline herd model is that agents have strong incentives to share information. Here we show that this critique also has merit. We show that if investors can share information by sending messages then there is an equilibrium of communication game which coincides with the equilibrium of the public signal game.

We modify our simple model with discrete investments as follows. At the beginning of each period t, investor t sends a message $m_t \in \{H, L, \emptyset\}$ about his signal in that period, where the empty set $\emptyset$ denotes an uninformative signal. The publicly observable events are the investment decisions in each period and the messages so that the public history $h_t = (x_0, x_1, \ldots, x_{t-1}, m_0, m_1, \ldots, m_{t-1})$. An perfect Bayesian equilibrium is defined as above.

Let us suppose that each investor t truthfully reveals the private signal at t. Since there are no costs for telling the truth, investors are indifferent about their messages, and it is optimal to tell the truth. Let the investment strategies be defined so that the outcomes coincide with those in the public signal game. The following proposition is immediate.

Proposition 4. *(The Information-Sharing Critique)* There is a truth-telling equilibrium of the communication game with outcomes that coincide with those of the public signal game.

Answering the Information-Sharing Critique

Here we allow investors to share information by sending messages. We show that if there is a flow benefit to being one of the early investors in the project then there are strong incentives to lie about private signals. In particular, we show that there does not exist a truth-telling equilibrium of the associated communication game.

Consider a version of the baseline model with $X = \{0, 1\}$ with several changes. Some of the changes are the same as those in the models with endogenous timing described in the text. Investors can invest at any time, although the decision must be made once-and-for-all. In each period $t = 0, \ldots, T_1$ with $T_1 < T$, one signal arrives to the economy and is randomly distributed to one and only one agent among the set of investors who have not already received a signal. We let the number of investors N be large relative to T_1. The state becomes known at $T > T_1$. .In addition, we make two assumptions. First, the rate of return earned in the risky project is $\bar{R}$ if there are fewer than T_1 investors and R otherwise, where $\bar{R} > R$. Second, at the beginning of each period t the informed investor sends a message $m_t \in \{H, L, \emptyset\}$ about his signal in that period. All other investors receive the message and then decide whether or not to invest in period t.

The publicly observable events are the aggregate investments and the messages. The public history $h_t = (x_0, x_1, \ldots, x_{t-1}, m_0, m_1, \ldots, m_t)$ records the aggregate investment in each period up through the end of period $t - 1$ and the messages in each period up through and including the message sent at the beginning of period t. The histories, strategies and beliefs of newly informed investors, previously informed investors and uninformed investors are defined analogously to those in Section 1.

We impose the following assumptions.

$$(45)\quad 1 < e^{R(T-T_1)}P(0)$$

$$(46)\quad 1 > e^{\bar{R}T}P(-1)$$

$$(47)\quad e^{\bar{R}T}P(0) < \nu_H(P(0))e^{R(T-1)}P(1) + \nu_L(P(0))$$

Assumptions (45)-(47) play the same role as before.

Under these assumptions truth-telling for all histories is clearly not an equilibrium. To see this consider a newly informed investor at T_1 who inherits a prior of $P(-1)$ from period $T_1 - 1$. If truth-telling were an equilibrium then if the informed investors sends a message H, the priors of all other investors rise to $P(0)$ and they invest, while if the informed investors sends a message L then the priors of all other investors fall to $P(-2)$ and they do not invest. Clearly, if the informed investor gets a high signal and lies by sending a message L he gains the early-mover advantage. Thus, truth-telling cannot be an equilibrium for all histories.

It is useful to define $H_t = (h_{t-1}, x_{t-1})$ to be the public history at the beginning of period t, before the period t message is sent, and to define the inherited prior at t to be the prior based on such a history. The beginning of period t history for any investor is $H_{it} = (H_t, s_r, r)$.

We define a symmetric stationary equilibrium of the communication game to be one with strategies for investment of the following form. There are functions x_U, x_I, and m such that for $t \leq T_1 - 1$ the investment strategies of the uninformed and previously informed investors are of the form

$$(48)\quad x_t(H_{it}, m_t) = x_U(p_t(H_{it}), m_t),$$

the investment strategy for newly informed investors is of the form

$$(49) \quad x_t(H_{it}, s_t) = x_I(p_t(H_{it}, s_t))$$

while the message strategy is of the form

$$(50) \quad m_t(H_{it}, s_t) = m(p_t(H_{it}, s_t), s_t).$$

(Notice that we allow the strategies at T_1 to differ from those for $t \leq T_1 - 1$.) We will say that a message is uninformative if it is the same for both signals.

We can define a symmetric stationary equilibrium of the game without communication analogously. It is easy to show that such an equilibrium is unique and has the following form. There are functions x_U and $x_I$such that for $t \leq T_1 - 1$ the investment strategies of the uninformed and previously informed investors are of the form

$$(51) \quad x_t(h_{it}) = x_U(p_t(h_{it})) = \left\{ \begin{array}{c} 0 \text{ if } p_t(h_{it}) \leq P(1) \\ 1 \text{ otherwise} \end{array} \right\},$$

the investment strategy for newly informed investors are of the form

$$(52) \quad x_t(h_{it}) = x_I(p_t(h_{it})) = \left\{ \begin{array}{c} 0 \text{ if } p_t(h_{it}) \leq P(0) \\ 1 \text{ otherwise} \end{array} \right\}.$$

and beliefs are defined in the obvious fashion. We then have

Proposition 7. *(Herds with Information Sharing)* Under (45)-(47), the symmetric stationary equilibrium of the communication game is essentially unique, in that there is a unique investment outcome. This outcome equals that of the symmetric stationary equilibrium of the no-communication game.

Proof. We will show that $x_U = 1$ if and only if $p_t(H_{it}) \geq P(1)$ for all m_t, $x_I = 1$ if and only if $p_t(H_{it}, s_t) \geq P(0)$, and the message strategy is uninformative when the inherited prior is $P(0), P(-1)$, or $P(-2)$. (Notice that if these strategies are followed, the only inherited priors at which there will be active investors at the end of the period are $P(0), P(-1)$ and $P(-2)$. Off the equilibrium path, the message strategies and beliefs can be filled in an analogous manner to that in the basic model.)

Consider period T_1. By (45) and (46) it follows that all investors invest if and only if their beliefs are greater than or equal to $P(0)$. Furthermore, the uninformed and the previously informed investors' decisions are independent of the newly informed investor's message. To see this suppose that the inherited prior is $P(0)$ or $P(-1)$. If the uninformed investors' decisions depend on the newly informed investor's message then the newly informed investor has an incentive to mislead.

Consider next period $T_1 - 1$. Consider first the newly informed investor at $T_1 - 1$. Since this investor's decision at T_1, if he waits, is independent of the messages at T_1, he can learn nothing by waiting and hence invests if and only if his beliefs are greater than or equal to $P(0)$. Thus, $x_I = 1$ if and only if $p_t(H_{it}, s_t) \geq P(0)$. If the inherited prior is either $P(0)$ or $P(-1)$ the uninformed and the previously informed investors' decisions must be independent of the newly informed investor's message, otherwise the newly informed investor has an incentive to mislead. Hence, in any symmetric stationary equilibrium at any date the message must be uninformative at an inherited prior of $P(0)$ or $P(-1)$. It remains to be shown that at an inherited prior of $P(-2)$ the message is uninformative. We consider this case below. If the inherited prior is $P(1)$ or higher, everyone invests and thus the message is irrelevant.

Consider next an uninformed investor at $T_1 - 1$. Suppose this investor has an inherited prior $P(0)$ or $P(-1)$. If this investor waits, he can observe the newly informed investor's investment decision at $T_1 - 1$ and hence can infer the signal and take the optimal decision in period T_1. At inherited prior $P(0)$, (47) implies that it is optimal to wait. At inherited prior $P(-1)$, (46) implies that is optimal to wait. By (45), since under the conjectured strategy all other investors are investing, it is optimal to invest if the inherited prior is $P(1)$ or higher. This shows that $x_U = 1$ if and only if the inherited prior is greater than or equal to $P(1)$.

We now show that at an inherited prior of $P(-2)$ the newly informed investor sends an uninformative message. We will show that if this is not true at t then it gives the newly informed investor at $t-1$ an incentive to deviate from the conjectured strategy. To see this suppose by way of contradiction that at inherited prior $P(-2)$, truth-telling is part of the equilibrium at t. To show that this supposition is false, consider period $t-1$ and suppose that the newly informed investor has inherited prior $P(-1)$ and receives a high signal. This investor's prior rises to $P(0)$ and he is supposed to invest (and send an uninformative message). Suppose that he deviates and waits. The uninformed investors' priors fall to $P(-2)$ at the beginning of period t. Under our posited strategy the newly informed investor at t sends a truthful message, the deviating investor's prior either rise to $P(1)$ or fall to $P(-1)$. By (47) this deviation is profitable. It follows at inherited prior $P(-2)$ the message must be uninformative. *Q.E.D.*

The communication game has nonstationary equilibria, in much the same way as the no communication game. More interesting to us are equilibria in which the investment decisions are stationary but the message decisions are nonstationary in that they depend on the whole history. We can construct an example of such an equilibrium which supports more investment than does the corresponding stationary equilibrium of the no communication game.

The construction is as follows. Suppose that if investment is zero in all periods before $T_1 - 2$, the newly informed investor in period $T_1 - 2$ sends a truthful message, otherwise he sends an uninformative message. Let the rest of the strategies and messages be the same as in the stationary equilibrium. Consider then the equilibrium behavior in which the initial signals at date 0 and 1 are low. In the stationary equilibrium these signals set off a herd of no investment. In this nonstationary equilibrium, no one invests up through $T_1 - 2$. If the newly informed investor at $T_1 - 2$ gets a high signal he tells the truth and all investors' priors rise to $P(-1)$. If the newly informed investor in period $T_1 - 1$ gets a high signal he invests and sends an uninformative message. At the beginning of period T_1, all investors invest. Thus, we have an example in which communication partially overcomes the private information problems.

This construction depends on the discounting to be relatively high relative to the value of information. If discounting is small then in the above example the investor at date 1 would wait for the information at date $T_1 - 2$ regardless of his signal. Hence no such equilibrium could exist. When discounting is small relative to value of information one can show that if investment decisions are stationary but message decisions are not, then the unique equilibrium outcomes coincide with the stationary ones in the game without messages.