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The Use of Debt and Equity in Optimal Financial Contracts

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ABSTRACT

We consider an environment in which risk-neutral firms must obtain external finance. They have access to two kinds of linear, stochastic investment opportunities. For one, return realizations are costlessly observed by all agents. For the other, return realizations are costlessly observed only by the investing firm; however, they can be (privately) observed by outsiders who bear a fixed verification cost. Thus, the second investment opportunity is subject to a standard costly state verification (CSV) problem of the type considered by Townsend (1979), Gale and Hellwig (1985), or Williamson (1986, 1987).

We examine the optimal allocation of investment between the two kinds of projects, as well as the optimal contract used to finance it. We show that the optimal contractual outcome can be supported by having firms issue appropriate (and determinate) quantities of debt and equity securities to outside investors.

The optimal debt-equity ratio necessarily depends (in part) on the firm's asset structure. Investments in projects subject to CSV problems are associated (in a sense to be made precise) with the use of debt—as might be expected from the existing CSV literature. Investments in projects with publicly observable returns are associated with the use of external equity.

We examine in detail the relationship between the optimal asset and liability structure of the firm. We also describe conditions under which an increase in the cost of state verification shifts the composition of investment towards projects with observable returns, and reduces the optimal debt-equity ratio. Interestingly, the optimal debt-equity ratio is also shown to depend on factors that are irrelevant to asset allocations.

Finally, a large part of the interest in CSV environments has been due to the fact that they may result in equilibrium credit rationing. Our analysis has strong implications for the possibility of equilibrium credit rationing in more general CSV models.

Introduction

One of the great successes of modern contract theory is the costly state verification (CSV) environment, developed originally by Townsend (1979), and subsequently extended by Diamond (1984), Gale and Hellwig (1985), Williamson (1986, 1987), and many others. Under the assumptions of risk neutral agents and fixed verification costs, this environment delivers standard debt contracts as an optimal contractual form. In addition, it delivers a well-defined optimal capital structure for a particular sort of firm. More specifically, internal funds, provided by an informed entrepreneur, are exchanged for a residual claim, which resembles "equity." External funds, provided by lenders who must monitor (at cost) to become informed, are exchanged for contracts that promise payments which are a piece-wise linear function of firm performance. These claims resemble "debt." In this environment—and in some more complicated ones such as those studied by Boyd and Prescott (1986), Bernanke and Gertler (1989), or Boyd and Smith (1993)—the firm is optimally financed with this combination of "debt" and "equity."

This kind of model, where "equity" is owned entirely by insiders (say management), and where all external funding takes the form of debt, is a reasonable description of a very limited class of firms, however—for example sole proprietorships or other small owner-managed firms. Moreover, as discussed by Bernanke and Gertler (1989), the equity investment decision is rather trivial, since it is always optimal for firm management to invest its entire wealth endowment in the firm.

This sort of capital structure does not look much like that employed by large corporations in the United States or most other developed nations. In particular, major corporations typically have a large class of outside equity investors who are not necessarily any better informed than are debt holders. In most cases, there are also inside owner-managers and directors who hold equity and who, presumably, are better informed than outsiders.[1]

In this paper, we attempt to provide a formal analysis of this kind of capital structure using the same contract theoretic approach that has proved so successful in understanding the use of debt contracts. It turns out that it is possible to do so with only a fairly slight modification of the standard CSV environment. In particular, instead of presenting entrepreneurs with only a single class of investment opportunity, we allow entrepreneurs to make use of convex combinations of two classes of investments. The first is exactly like that available in the standard CSV setup. Return realizations are costlessly observed only by an entrepreneur, and all other agents must pay a fixed monitoring cost to become informed about return realizations. The second (and new) type of stochastic investment opportunity is one in which return realizations are costlessly observed by *all* agents. For obvious reasons, we assume that the expected rate of return on the former exceeds that on the latter.

This minor modification of the underlying environment results in a considerable difference in the optimal financial structure of the firm. In order to characterize it, we begin by solving an optimization problem that determines each firm's investment in each type of project (that is, we determine its asset structure), and that determines the aggregate payments made by the firm as a function of the returns it receives. We then show that an optimal contractual arrangement can be supported by having the firm issue an appropriately chosen amount of debt (of the standard CSV variety) and an appropriately chosen quantity of equity which is sold externally. External equity-holders are true residual claimants, receiving some (state contingent) fraction of the returns on a firm's assets—less payments to debt-holders. In addition, firm managers (entrepreneurs), whose endowment is their access to investment opportunities, are also residual claimants, being subordinate (in a sense to be made precise) both to debt-holders and outside equity-holders.

What is of greatest interest about this environment, of course, is that it delivers an optimal mix of debt and equity to be used in obtaining external finance. This optimal mix of claims depends on several factors, some of which the firm chooses, and some of which are exogenous to the firm (or the economy). For given parameters, we show that there is a functional relationship between the firm's asset structure and its liability structure. In particular, as the insight provided by the CSV literature would suggest, the use of investment technologies with unobservable returns is associated with debt, while the use of investment technologies with observable returns enables equity finance to be viable. The relationship between the structure of assets and liabilities validates the ancient belief that certain kinds of investments are optimally associated with certain kinds of liability issues.

In terms of factors which are exogenous to the firm, we provide conditions under which the optimal amount of debt finance decreases with monitoring costs. When these costs are sufficiently high, it may be optimal for firms to be financed entirely with equity, and indeed it may be infeasible for them to be financed entirely with debt. Indeed, we state conditions under which 100 percent debt finance is infeasible for a firm; under these conditions a firm which was precluded from issuing liabilities with a sufficient degree of state contingency would be unable to operate. This provides an additional sense in which the ability to issue equity—even in potentially small amounts—can be highly beneficial to a firm.

We also show that factors that are irrelevant to resource allocations are *not* irrelevant to the liability structure of the firm. For example, with risk neutral agents all resource allocations depend on the distribution of returns on the observable return investment technology *only* through its mean. However, the debt-equity ratio depends on the entire distribution of returns on this technology. Thus, for example, mean preserving spreads in this distribution are irrelevant to asset allocations, but are far from irrelevant to the composition of firm liabilities. We investigate the relationship between risk and liability structure in some detail.

As should be apparent from this discussion, the environment we describe violates the Modigliani-Miller theorem, and delivers an optimal debt-equity ratio for each firm (or for all firms). Why is it useful to firms to issue one particular mix of debt and equity as opposed to any other? The answer has to do with the nature of investors' responses to the presence of the CSV problem. By assumption, the observable return investment technology has a lower expected return, gross of monitoring costs, than the unobservable return technology does. However, its use permits firms to reduce verification costs. In particular, payments to equity-holders can be conditioned on the returns to investments in the observable return technology. In order to enable the firm to reduce its expected monitoring costs, equity-holders give up resources when the return on the observable return technology is low. In exchange, they receive compensating payments when the return on this technology is high. Moreover, the amount that equity-holders give up (or receive) varies with the outcome of investments in the observable return technology; on average a lot is given up when this return is low, and by way of compensation a lot is received when this return is high. These "transfers" of resources across different return states permit the firm to optimally "smooth" monitoring costs across states. In effect, outside equity-holders allow the firm to use the funds they provide as "collateral" or "internal finance;" the value of both devices in mitigating CSV problems is well known (Bernanke-Gertler 1989 and Boyd-Smith 1994b,c). In this way it is useful to a firm to have a class of external equity claimants, while it also will wish to issue some debt for reasons that are familiar from the standard CSV literature.

The analysis we have just described also provides several ingredients for a sequel (Boyd-Smith 1995), which will examine the evolution of debt and equity market activity as an economy moves along a growth path. In particular, the economy of this paper can be embedded in the neoclassical growth model of Diamond (1965). Under some assumptions about the form taken by monitoring costs, as an economy accumulates capital firms will perceive relative costs of monitoring that change over time. These shifts in perceived (relative) monitoring costs will alter the equilibrium composition of investment, and the equilibrium debt/equity ratio as an economy develops. Boyd and Smith (1995) suggests that the typical expected pattern of development will be that equity markets become increasingly active as an economy grows, and that the composition of investment will become increasingly heavily weighted in favor of the observable return technology. Certainly the former prediction is well-supported by casual observation.

The rest of the study proceeds as follows. Section 1 sets out the model environment. Section 2 describes the financing of investment and defines funding contracts. This section also describes the optimal composition of investment. Section 3 reviews optimal contracts in the standard CSV environment, and contrasts them with optimal contracts in our modified environment. It also shows how an optimal contractual arrangement can be supported by having a firm issue appropriately selected amounts of debt and equity. Section 4 provides some numerical examples indicating how the amount of debt finance (and the debt/equity ratio) depend on the cost of monitoring, and on the risk associated with the various investment technologies. Finally, Section 5 summarizes, offers some conclusions, and relates our results to those obtained in some of the other literature on the optimal composition of firm financial structure.

1. The Model

A. Environment

We consider an essentially static, two period economy. Agents in this economy are divided into two types, which we term borrowers and lenders. All borrowers (lenders) are identical, ex ante, and borrowers (lenders) constitute a fraction $\alpha \in (0,1)$ $(1-\alpha)$ of the population. Lenders are endowed with one unit of a single good in the first period. Borrowers are endowed with none of this good, but they do own a firm which is endowed with access to individual-specific, high return investment projects, which are described below. All agents, both borrowers and lenders, are assumed to be risk neutral, and to care only about second period consumption.[2]

Investment can occur in the first period using one or more of the following three technologies. First, there is a commonly available, nonstochastic linear technology whereby one unit invested in the first period yields $r > 0$ units of output in period two. In addition, there are two stochastic linear technologies that convert current investment into future output. Technology o (for observable return) produces y units of output in period two per unit invested in period one, where y is an iid random variable which is realized at the beginning of the second period. We assume that $y \in \{y_1, y_2, \ldots, y_N\}$, and we let $p_n \equiv \text{prob}(y = y_n)$. Obviously $0 \leq p_n \leq 1$ for all n, and $\Sigma p_n = 1$. Finally, we assume that for any investor the return yielded by investments in technology o is publicly observable (at zero cost).

Technology u (for unobservable return) is assumed to produce w units in period 2 per unit invested at the first date. w is assumed to be a continuous, iid random variable with cdf G and pdf g. g is assumed to be continuously differentiable, and to have support $[0,\bar{w}]$. In addition, the return on investments in technology u can be observed (by any agent other than the initiating investor) only by bearing a fixed cost of $\gamma > 0$ units of the current consumption good. Thus a standard costly state verification (CSV) problem arises for investments in technology u.

Only borrowers/firms are endowed with access to the investment technologies o and u, and ownership of these investment opportunities cannot be traded. In particular, only the agent who is endowed with a specific project can operate it. Moreover, as is typical in CSV models (with fixed verification costs) we will impose an upper bound on the scale at which any firm can operate the investment technologies. Thus we let i^o (i^u) be investment in technology o(u) by a representative firm, and we let $i \equiv i^o + i^u$. Then each firm faces the maximum scale of operation constraint $i \leq q$, where q is an exogenously given parameter. Finally, we also assume that $i^o \geq 0$ and $i^u \geq 0$ must hold. For future reference, we define $\theta \equiv i^o/i$ to be the fraction of total investment done in technology o by a representative firm. Clearly $\theta \in [0,1]$ must hold.

In the analysis that follows, we assume that, at the beginning of period 1, each borrower is endowed with ownership of a single firm, as well as of a single investment project. We also assume that each borrower can create additional firms (but *not* additional projects) if so desired. However, the creation of each additional firm results in a fixed cost $F > 0$. For our purposes F can be taken to be arbitrarily small.[3]

With respect to the investment technologies, define $\hat{y} \equiv \Sigma_n p_n y_n$ to be the expected gross return on investments in technology o, and $\hat{w} \equiv \int_0^{\bar{w}} wg(w)dw$ to be the expected gross return, not inclusive of verification costs, on investments in technology u. We assume that

(a.1) $\hat{w} > \hat{y} > r.$

Thus the commonly available investment technology is relatively unproductive. Since borrowers have no wealth endowment, they must obviously rely on external finance in order to fund their investments. Lenders can either provide this external finance to borrowers, or invest in the inferior, commonly available production technology, or both. We now describe how this trade occurs.

2. Financing of Investment

At this point we assume that borrowers obtain external funds by announcing general contracts specifying how repayments will occur in various contingencies. These contracts should be regarded as abstract objects; in Section 3 we describe how equilibrium contracts can be supported through the use of debt and equity markets. We also assume throughout that borrowers confront no credit rationing,[4] or in other words, that the supply of funds by lenders is always at least as great as the demand for funds by borrowers. The supply of funds by lenders is $(1-\alpha)$, while the maximum demand for funds by borrowers is αq. Hence we assume that

(a.2) $(1-\alpha) \geq \alpha q.$

When (a.2) holds, any marginal savings must be invested in the commonly available technology, yielding r units of consumption per unit invested. Thus the opportunity cost of funds is r, which is a parameter of the economy.

A. Funding Contracts

A funding contract specifies a quantity of resources that will be transferred to a particular borrower (i), as well as how these resources will be allocated among technologies o and u.[5] In addition, the contract specifies a set of state contingent repayments. Since the return y is publicly observable, repayments can always be made contingent on it. However, since the return w can only be observed externally if monitoring (or verification) occurs, repayments can only meaningfully be made contingent on w if state verification occurs. Hence a contract must specify a set of states A(y) ($[0,\bar{w}] - A(y) \equiv B(y)$) in which monitoring will (will not) occur; this set can obviously be conditioned on y. In monitoring states, repayment can be made contingent on w. We let R(w,y) be the promised repayment, per unit borrowed, if $w \in A(y)$. In addition, we let x(y) be the promised repayment, per unit borrowed, if $w \in B(y)$. Notice that we have restricted attention to nonstochastic monitoring.[6] Furthermore, we assume that the funding contract specifies a set of agents who are responsible for monitoring (if necessary), so that there is never a necessity for a duplication of monitoring effort.[7]

Funding contracts are assumed to be announced by borrowers, and such contracts are then either accepted or rejected by lenders. In order to avoid rejection, such contracts must satisfy a set of constraints, which we now describe.

First, contracts must be feasible or, in other words, specify nonnegative consumption levels for borrowers. Thus

(1.a) $R(w,y) \leq [\theta y + (1-\theta)w]; \quad w \in A(y)$

$$(1.b) \quad x(y) \leq \inf_{w \in B(y)} [\theta y + (1-\theta)w]$$

must hold. Second, contracts must be incentive compatible, so that borrowers have an incentive to announce truthfully when a monitoring state has occurred. They will do so iff repayments are lower in monitoring than in nonmonitoring states. Therefore the incentive constraint takes the form

$$(2) \quad R(w,y) \leq x(y), \quad \forall\, w \in A(y).$$

Finally, since lenders can always invest in the commonly available technology, the expected repayment—per unit borrowed—must at least equal the opportunity cost of funds r plus monitoring costs (which without loss of generality we assume are born by lenders). Thus contractual repayments must satisfy the expected return constraint

$$(3) \quad \left\{ \sum_n p_n \int_{w \in A(y_n)} R(w,y_n)g(w)dw + \sum_n p_n \int_{w \in B(y_n)} x(y_n)g(w)dw \right\} i$$

$$- \gamma \sum_n p_n \int_{w \in A(y_n)} g(w)dw \geq ri.$$

Borrowers announce funding contract terms to maximize their own expected utility, subject to the constraints (1)-(3). The expected utility of a borrower, as a function of contract terms, is simply the expected return on the borrower's investments, less the expected repayment implied by the contract. Thus borrowers choose contract terms to maximize

$$i \left\{ [\theta \hat{y} + (1-\theta)\hat{w}] - \sum_n p_n \int_{w \in A(y_n)} R(w,y_n)g(w)dw - \sum_n p_n \int_{w \in B(y_n)} x(y_n)g(w)dw \right\}$$

subject to (1)-(3).

Observing that (3) must hold with equality at an optimum, this problem can be transformed as follows. Substituting (3)—at equality—into the objective function yields the alternative maximand

$$i[\theta\hat{y} + (1-\theta)\hat{w} - r] - \gamma \sum_n p_n \int_{A(y_n)} g(w)dw$$

which the borrower seeks to optimize subject to (1)–(3). We now make several observations about the solution to the borrower's problem, which are summarized in the following lemma.

Lemma 1. The solution to the borrower's problem has

(a) $i = q$

(b) $A(y_n) = \left[0, \dfrac{x(y_n) - \theta y_n}{1 - \theta}\right]$

and

(c) $R(w,y_n) = \theta y_n + (1-\theta)w; \quad w \in A(y_n)$.

The Proof of Lemma 1 appears in the Appendix.[8] Lemma 1 asserts that borrowers always invest up to their maximum scale of operation. In addition, they repay $x(y_n)$ if it is feasible to do so, in which case no monitoring occurs. If it is infeasible to repay $x(y_n)$ (which occurs iff $w \in A(y_n)$, as defined in (b)), then monitoring takes place. Part (c) of the lemma says that if monitoring occurs, providers of external funding receive the entire net of monitoring cost proceeds of the borrower's investments. All of these aspects of an optimal contract mirror results in more standard CSV environments.

In order to further characterize optimal funding contracts, it will be useful to make the following transformation. Define

(4) $z_n \equiv \{x(y_n) - \theta y_n\}/(1-\theta); \quad n = 1, \ldots, N.$

Then Lemma 1 implies that $A(y_n) = [0,z_n)$ for all n, and in addition it implies that the borrowers' problem can be reformulated as maximizing

$$[\theta\hat{y} + (1-\theta)\hat{w} - r]q - \gamma \sum_n p_n G(z_n)$$

subject to

$$(5) \qquad \sum_n p_n \int_0^{z_n} [\theta y_n + (1-\theta)w]g(w)dw + \sum_n p_n x(y_n)[1 - G(z_n)] - \psi \sum_n p_n G(z_n) = r$$

where $\psi \equiv \gamma/q$ is the cost of monitoring, relative to the size of the investment project. Now rewrite equation (5) as

$$(5') \qquad \sum_n p_n \theta y_n G(z_n) + (1-\theta) \sum_n p_n \int_0^{z_n} wg(w)dw + \sum_n p_n x(y_n)[1 - G(z_n)]$$
$$- \psi \sum_n p_n G(z_n) = r.$$

Adding and subtracting $\Sigma\, p_n \theta y_n[1 - G(z_n)]$ from the left-hand side of (5′), we can reformulate the borrower's problem as choosing $\theta \in [0,1]$ and $z_n \in [0,\bar{w}]$, $n = 1, \ldots, N$, to solve the problem

$$(P) \qquad \max\ \theta\hat{y} + (1-\theta)\hat{w} - r - \psi \sum_n p_n G(z_n)$$

subject to

$$(6) \qquad \theta\hat{y} + (1-\theta) \sum_n p_n \int_0^{z_n} wg(w)dw + (1-\theta) \sum_n p_n z_n[1 - G(z_n)] - \psi \sum_n p_n G(z_n) = r.$$

It will prove useful to have the following result concerning the solution to the problem (P).

Lemma 2. Define the function $\bar{z}(\theta;\psi)$ by

$$\bar{z}(\theta;\psi) \equiv \underset{z}{\operatorname{argmax}} \left\{ (1-\theta) \int_0^{z} wg(w)dw + (1-\theta)z[1 - G(z)] - \psi G(z) \right\}.$$

Then:

(a) the vector $(z_1, z_2, \ldots, z_N)$ solving (P) satisfies $z_n \leq \bar{z}(\theta;\psi)$, for all n.

(b) $\bar{z}(\theta;\psi) \leq \bar{z}(0;\psi)$ holds, for all θ. Thus $z_n \leq \bar{z}(0;\psi)$ also holds, for all n.

The Proof of Lemma 2 appears in the Appendix.

Our strategy will now be to fix a value $\theta \in [0,1)$ arbitrarily, and for that value of θ to find the optimal values $(z_1,z_2,\ldots,z_N)$.[9] Having obtained these values as a function of θ, we will then turn to the problem of finding the value of θ that maximizes the borrower's expected utility. In doing so we will typically focus on interior optima, although we will periodically comment on how the analysis would be affected by relaxing this focus.

In order to pursue this strategy, it will be useful to make assumptions implying that the borrower's objective function is concave in, and that the constraint set is convex with respect to $(z_1,z_2,\ldots,z_N)$, at least over "relevant" values of this vector. The following assumption implies the satisfaction of both conditions[10]

(a.3) $\quad g'(w) \geq 0, \quad \forall\, w \in [0,\bar{z}(0;\psi)]$.

We now solve the problem

$$(\mathrm{P}') \qquad \max\ \theta\hat{y} + (1-\theta)\hat{w} - r - \psi \sum_n p_n G(z_n)$$

subject to (6) and $z_n \in [0,\bar{z}(0;\psi)]$, taking $\theta \in [0,1)$ as given. Let λ denote the Lagrange multiplier associated with the constraint (6). Then, under (a.3), the following conditions are necessary and sufficient for an interior optimum:[11]

$$(7) \qquad g(z_n)/\{(1-\theta)[1 - G(z_n)] - \psi g(z_n)\} = \lambda; \quad n = 1, \ldots, N.$$

Since (a.3) implies that the left-hand side of (7) is strictly increasing in z_n, the N conditions in (7) imply that z_n is independent of n or, in other words, that

$$(8) \qquad z_1 = z_2 = \ldots = z_N = z(\theta;\psi).$$

The function $z(\theta;\psi)$ in (8) is the common value of z_n, given θ and ψ, that satisfies the expected return constraint at equality. Thus, in particular, $z(\theta;\psi)$ is implicitly defined by

$$(9) \qquad \theta\hat{y} + (1-\theta) \int_0^{z(\theta;\psi)} wg(w)dw + (1-\theta)z(\theta;\psi)\{1 - G[z(\theta;\psi)]\} - \psi G[z(\theta;\psi)] \equiv r.$$

If we define the function $\pi(z,\theta;\psi)$ by

$$(10) \qquad \pi(z,\theta;\psi) \equiv (1-\theta) \int_0^{z} wg(w)dw + (1-\theta)z[1 - G(z)] - \psi G(z)$$

then (9) reduces to

$$(11) \qquad \pi[z(\theta;\psi),\theta;\psi] = r - \theta\hat{y}.$$

The function π, and the determination of $z(\theta;\psi)$, are depicted in Figure 1. Assumption (a.3) implies that π is concave in z (for "relevant" values of z, and for all θ), and

$$(a.4) \qquad \pi_1(0,\theta;\psi) > 0$$

implies that π is increasing at $z = 0$. Moreover, if equation (11) has more than one solution, then the criterion of minimizing expected monitoring costs dictates that the smallest solution is the relevant one. Thus (since credit is not rationed),

$$(12) \qquad \pi_1[z(\theta;\psi),\theta;\psi] > 0$$

holds. Evidently, $z(\theta;\psi) < \bar{z}(\theta;\psi)$ is also satisfied at an interior optimum.

It is now possible to describe the borrower's optimal choice of θ. Define the function $H(\theta;\psi)$ by

$$H(\theta;\psi) \equiv \theta\hat{y} + (1-\theta)\hat{w} - r - \psi G[z(\theta;\psi)].$$

Then, substituting (8) into the maximand in (P′), it is apparent that a borrower will choose θ to solve the problem

(P.1) $\max_{\theta \in [0,1]} H(\theta;\psi)$.

The solution to this problem gives the optimal composition of investment for a representative borrower. This composition is characterized in the next subsection.

B. The Optimal Composition of Investment

Prior to describing the borrower's optimal choice of θ, it will be useful to make some observations about the conditions which that choice must satisfy. These conditions are stated in the following lemmas. (Proofs appear in the Appendix.)

Lemma 3. $\theta \leq r/\hat{y} < 1$ holds.

As shown in Appendix B, $z(r/\hat{y};\psi) = 0$ holds, so when $\theta = r/\hat{y}$, no monitoring costs are incurred. It can therefore never be optimal to set $\theta > r/\hat{y}$, since this decreases the expected return on investments $[\theta\hat{y} + (1-\theta)\hat{w}]$ without reducing the expected costs of state verification.

Lemma 3 implies that $\theta \in [0,r/\hat{y}]$. It will now be useful to state conditions implying the existence of an interior optimum for θ.

Lemma 4. $\theta \in (0,r/\hat{y})$ holds if

(a) $\hat{y} - \hat{w} - \psi g(0)z_1(r/\hat{y};\psi) < 0$

and

(b) $\hat{y} - \hat{w} - \psi g(\tilde{z})z_1(0;\psi) > 0$

both hold, where $\tilde{z}$ is defined by[12]

(13) $\pi(\tilde{z},0;\psi) \equiv r$.

Condition (a) is equivalent to

(14) $(\hat{w}-\hat{y})[(\hat{y}-r)/\hat{y} - \psi g(0)] > \psi\hat{y}g(0)$.

Condition (b) is equivalent to

(15) $[\hat{y} - r - \psi G(\bar{z})]\psi g(\bar{z}) > (\hat{w}-\hat{y})[1 - G(\bar{z}) - \psi g(\bar{z})]$.

Condition (a) asserts that $H(\theta;\psi)$ is decreasing in θ at $\theta = r/\hat{y}$, while (b) implies that the same function is increasing in θ at $\theta = 0$. Thus the maximand in (P.1) has something like the configuration depicted in Figure 2, and the optimal choice of θ is in the interior of $[0,r/\hat{y}]$.

An alternative strategy for guaranteeing the existence of an interior optimum is to assume that $H(\theta;\psi)$ is decreasing in θ at $\theta = r/\hat{y}$, and to assume that

(16) $H(0;\psi) \leq H(r/\hat{y};\psi)$.

Equation (16) implies that it cannot be optimal to set $\theta = 0$, while (14) again implies that it cannot be optimal to set $\theta = r/\hat{y}$. (16) is equivalent to the more primitive assumption

(16′) $r(\hat{w}-\hat{y})/\hat{y} \leq \psi G(\bar{z})$.

(16) (or (16′)) implies that the function H has the configuration depicted in Figure 2a. We henceforth assume that (14) and either (15) or (16) hold.[13]

Under assumptions implying the existence of an interior optimum, the choice of θ that maximizes the expected utility of a representative borrower—denoted here by θ^*—satisfies the first order condition

(17) $-\psi g[z(\theta^*;\psi)]z_1(\theta^*;\psi) = \hat{w} - \hat{y}$

where

(18) $z_1(\theta;\psi) \equiv -\{\hat{y} + \pi_2[z(\theta;\psi),\theta;\psi]\}/\pi_1[z(\theta;\psi),\theta;\psi]$.

In addition, θ^* must satisfy the second-order condition

(19) $\quad -\psi g[z(\theta^*;\psi)]z_{11}(\theta^*;\psi) - \psi g'[z(\theta^*;\psi)][z_1(\theta^*;\psi)]^2 < 0.$

The equilibrium choice of θ^* is depicted in Figures 2 and 2a.

1. The Dependence of θ^* on Costs of Monitoring

It will be useful for future reference to know how θ^* depends on the relative monitoring cost variable $\psi \equiv \gamma/q$. Equations (17) and (19) imply the existence of a function T such that

(20) $\quad \theta^* = T(\psi).$

$T'(\psi)$ is obtained by implicitly differentiating equation (17); doing so yields

(21) $$\{\psi g[z(\theta^*;\psi)]z_{11}(\theta^*;\psi) + \psi g'[z(\theta^*;\psi)][z_1(\theta^*;\psi)]^2\}T'(\psi)$$
$$= -g[z(\theta^*;\psi)]z_1(\theta^*;\psi) - \psi g[z(\theta^*;\psi)]z_{12}(\theta^*;\psi) - \psi g'[z(\theta^*;\psi)]z_1(\theta^*;\psi)z_2(\theta^*;\psi).$$

We now state

Lemma 5.

(a) Suppose that

(22) $\quad z_1(\theta^*;\psi) + \psi z_{12}(\theta^*;\psi) + \psi\{g'[z(\theta^*;\psi)]/g[z(\theta^*;\psi)]\}z_1(\theta^*;\psi)z_2(\theta^*;\psi) < 0.$

Then $T'(\psi) > 0$.

(b) Equation (22) holds if $z_{12}(\theta^*;\psi) \leq 0$. A sufficient condition for this is

(23) $\quad (\hat{w}-\hat{y})\{[(\hat{y}-r)/\hat{y}] - \psi g[z(\theta^*;\psi)]\} \geq \psi/4.$

Lemma 5 describes conditions under which $T'(\psi) > 0$ or, in other words, under which an increase in the relative cost of state verification leads to heavier use of the investment technology with an observable return. Certainly it is intuitively plausible that as state verification becomes more expensive, investors should optimally make greater use of the investment technology that does not require state verification. The Proof of Lemma 5 appears in the Appendix.[14]

3. External Financing Through Debt and Equity Issues

In this section we show how a borrower can implement the optimal funding contract described in Section 2 through the issuance of debt and equity. We also show how the debt-equity ratio of an individual borrower is related to the asset side of the borrower's balance sheet. In particular, the debt-equity ratio of any borrower will be related to the composition of that borrower's investments in technologies o and u in a manner we now describe.

Before doing so, we begin with a discussion of standard debt contracts in CSV models in order to motivate our interpretation of borrowers' external financing.

A. Standard Debt Contracts

The environment we consider differs from the more typical CSV models of Gale-Hellwig (1985) and Williamson (1986, 1987) in only one respect. In those models, borrowers have access to only a single investment technology (technology u). In our context, this could be interpreted as constraining θ to be identically zero at each date. Then $z = z(0;\psi) = \tilde{z}$ would hold, and from equation (4), so would $x = \tilde{z}$. Hence contracts between borrowers and lenders will specify a fixed repayment (x) if this is feasible. If it is infeasible to make this fixed repayment, lenders verify the project return, and retain the entire net of monitoring cost proceeds of the project. x is often interpreted as a gross interest rate, and a failure to repay principal plus interest is often interpreted as bankruptcy. Under this interpretation, the optimal funding contract is associated with a "standard debt contract," and if this contract requires investments by more than one lender, it is typical to think of lending as being intermediated in order to avoid duplication of monitoring effort (Diamond 1984 and Williamson 1986).

When investors have access to two technologies, we think that the interpretation just given extends naturally to interpreting an optimal funding contract as involving the issue of debt and equity.

In particular, applying conventional terminology from the CSV literature, define debt to be a liability with the following properties. Debt either pays a return which is not contingent on firm performance or, if a fixed obligation is not met, debt-holders monitor the firm and retain the entire (net of monitoring cost) proceeds of the firm's investments. Of course firm debt may be held by a bank—as in Diamond (1984) and Williamson (1986)—to avoid duplication of monitoring activity by debt-holders.

This (standard) definition of debt, along with the optimal repayment schedules derived above, implies that each firm has a class of external residual claimants. The claim they hold resembles conventional equity. In particular, if firm profits are too low to fully honor the firm's debt obligations, then equity-holders receive nothing. On the other hand, when all debt claims are honored, external equity-holders receive a payment based on firm performance. The connection between firm performance and payments to this class of claimant validates the interpretation of these liabilities as equity.

Under these definitions of debt and equity, the debt-equity ratio of an individual firm (borrower) is uniquely determined. We now describe how the composition of firm liabilities is determined.

B. Debt and Equity Issues

Recall that an optimal funding contract calls for a (per unit borrowed) repayment of x(y) if the return on investments in technology o is y, and if state verification does *not* occur. Then from equation (4),

$$(24)\qquad x(y) = (1-\theta^*)z(\theta^*;\psi) + \theta^* y$$

in nonmonitoring states. Notice that this promised repayment is the sum of two terms; one $[(1-\theta^*)z(\theta^*;\psi)]$ which is *not* contingent in any way on firm performance, and one $(\theta^* y)$ which depends on that component of firm performance (y) which is observable in a nonmonitoring state. The definition of a standard debt contract directs us to term the first component "payments to debt-holders" (per unit borrowed) in nonmonitoring states, while the second component is payments to external equity-holders in such states. When each of these payments is made by the firm, all additional firm output is retained by the insider owner/managers.

Now suppose that the return on the firm's investment in technology u is $w \in A(y)$. We must then describe how monitoring occurs. Following the definition of a standard debt contract, debt-holders are "paid first." If it is infeasible to make total payments to debt-holders of $q(1-\theta^*)z(\theta^*;\psi)$, then the owners of this debt (the "owners" may be a bank, as described by Williamson 1986) monitor and become residual claimants on investment returns. Monitoring effort is not duplicated.

If the returns on investments are large enough to repay $q(1-\theta^*)z(\theta^*;\psi)$, but are *not* large enough to repay $qx(y)$, then debt-holders are fully repaid while equity-holders receive less than $q\theta^*y$. Since an optimal funding contract stipulates that monitoring must occur, and since debt-holders are full repaid, this monitoring must be performed by the firm's equity-holders. Equity-holders therefore verify the return on investments in technology u, and they retain all firm income (net of monitoring costs), less payments to debt-holders. We assume that equity-holders can coordinate their monitoring activity, again avoiding duplication of monitoring effort.[15]

This situation is depicted in Figure 3 for a particular fixed value of y. Given the repayment schedule specified in part (c) of lemma 1, it is infeasible to pay debt-holders $q(1-\theta^*)z(\theta^*;\psi)$ iff

(25) $$w \le z(\theta^*;\psi) - [\theta^*/(1-\theta^*)]y.$$

Define the function $J(\theta,y;\psi)$ by

(26) $$J(\theta,y;\psi) \equiv \max\{0, z(\theta;\psi) - [\theta/(1-\theta)]y\}.$$

Then debt-holders are responsible for monitoring if $w \in [0,J(\theta^*,y;\psi))$; in this region equity-holders receive no repayments. If $w \in [J(\theta^*,y;\psi),z(\theta^*;\psi))$, then equity-holders are responsible for monitoring, and the aggregate payment they receive is simply proportional to the vertical distance between the solid total repayment schedule, and the horizontal line $(1-\theta^*)z(\theta^*;\psi)$ in the figure. Notice that, for a fixed value of y, equity resembles a type of subordinated debt. However, when y varies, so does the contingent payment to equity-holders in nonmonitoring states.

This interpretation of monitoring, and of which agents perform it, is inconsistent with the terminology employed in the existing literature in one significant way. Specifically, equity-holders cannot literally be construed as forcing firms into bankruptcy proceedings, and therefore we can no longer refer to all states in which monitoring occurs as "bankruptcy states." Nonetheless, the interpretation that some monitoring is done by outside equity-holders does reflect what is observed in practice when firms experience low profits, but are still able to cover their payments to debt-holders. Under these circumstances there is likely to be a conflict between the outside equity-holders and the inside owner-managers. Outside equity-holders cannot force bankruptcy, but they can undertake a variety of costly actions against management which, among other things, have the effect of uncovering information about the firm. Such actions include the hiring of outside auditors, various attempts to force changes in firm policies, or even attempts to replace the incumbent management. These kinds of actions may be channeled through the board of directors, or even through formal class action suits, and hence are coordinated among external equity-holders. In the model, of course, all of these activities are represented by costly monitoring.

1. The Volume of Debt and Equity Issued

Let d denote the value of the debt issued by a representative firm, and let e be the value of the external equity that the firm sells. In this section we demonstrate that both d and e are continuous functions of θ and ψ, and hence that for a fixed value of ψ, there is a continuous relationship between the composition of a firm's assets and of its liabilities. We begin by discussing how d and e are determined.

d and e must satisfy two conditions. First, the firm must raise enough external financing to fund its investments, so that

(27) $\quad d + e = q$

must hold. Second, given the way that the responsibilities for monitoring are allocated among debt and equity-holders, and given the allocation of state-contingent payments to each group, debt and equity-holders must each receive an expected return of r. Thus, for example, expected payments to equity-holders, net of monitoring costs, must equal er.

We now describe how much equity (and, by implication, how much debt) a firm will issue for each possible value of $\theta \in [0,r/\hat{y}]$. Under our interpretation of payments to debt and equity in nonmonitoring states, debt-holders are promised a payment of $(1-\theta)z(\theta;\psi)$ in such states, while equity-holders are promised θy.

Suppose first that $w \in [0,J(\theta,y;\psi))$. Then debt-holders are responsible for monitoring, and equity-holders receive no payments. However, when $w \in [J(\theta,y;\psi),z(\theta;\psi))$, equity-holders are responsible for monitoring and they receive the entire net-of-monitoring-cost proceeds of the firm's investments, less payments to debt-holders. Thus, in these states, the total payment to equity-holders is $q\{\theta y + (1-\theta)w - (1-\theta)z(\theta;\psi) - \psi\}$. The probability of this contingency occurring, conditional on $y = y_n$, is of course simply $G[z(\theta;\psi)] - G[J(\theta,y_n;\psi)]$. Finally, if $w \geq z(\theta;\psi)$, which occurs with probability $1 - G[z(\theta;\psi)]$, no monitoring occurs, and total payments to equity-holders in state y are $q\theta y$. Thus equity-holders earn an expected return of r iff e satisfies[16]

$$(28) \quad re/q = \theta\hat{y}\{1 - G[z(\theta;\psi)]\} + \sum_n p_n \int_{J(\theta,y_n;\psi)}^{z(\theta;\psi)} [\theta y_n + (1-\theta)w - (1-\theta)z(\theta;\psi)]g(w)dw$$

$$- \sum_n p_n\psi\{G[z(\theta;\psi)] - G[J(\theta,y_n;\psi)]\}.$$

Equations (27) and (28) determine the amount of debt and equity issued by any firm as a function of its choice of θ. It is straightforward to show that, for all $\theta \in [0,r/\hat{y}]$, $e/q \in [0,1]$.

We now establish how the quantity of external equity issued by a representative firm is related to its choice of θ.

PROPOSITION 1.

(a) e satisfies

$$(29) \quad e = (q/r)\left\{r - \sum_n p_n(1-\theta)[z(\theta;\psi) - J(\theta,y_n;\psi)] - \sum_n p_n\pi[J(\theta,y_n;\psi),\theta;\psi]\right\} \equiv E(\theta;\psi).$$

(b) The function E is differentiable at (θ,ψ) iff $z(\theta;\psi)[(1-\theta)/\theta] \neq y_n$ for any n.

(c) E is differentiable at (θ,ψ) if θ is sufficiently near zero or sufficiently near $r/\hat{y}$. For θ near zero or $r/\hat{y}$, $E_1(\theta;\psi) > 0$.

(d) $E(0;\psi) = 0$. $E(r/\hat{y};\psi) = q$.

(e) Suppose $z_1(\theta;\psi) \leq 0$ holds. Then if E is differentiable at (θ,ψ), $E_1(\theta;\psi) > 0$ holds if

$$(30) \quad \pi_1[z(\theta;\psi),\theta;\psi] \geq \theta(1-\theta)\{1 - G[J(\theta,y_n;\psi)]\}$$

is satisfied for all n such that $J(\theta,y_n;\psi) > 0$. A sufficient condition for (30) is

$$(31) \quad \pi_1[z(\theta;\psi),\theta;\psi] \geq 0.25.$$

(f) Suppose $E(\theta;\psi)$ is differentiable at (θ,ψ). Then $E_2(\theta;\psi) \leq 0$.

The Proof of Proposition 1 is given in the Appendix.

Proposition 1 demonstrates that, from the standpoint of an individual borrower, the choice of θ is equivalent to the choice of how much equity to issue. Equity issues vary continuously with θ, and Proposition 1 describes conditions under which they vary monotonically with θ as well. The conditions stated in proposition 1 are also merely sufficiently conditions for e and θ to be monotonically related, and they are generally much stronger than what would be needed for such a relationship. Thus we conjecture that a monotonic relationship between e and θ will typically be observed, and this is certainly the case in the examples we examine in Section 4.

Intuitively, the use of technology u is associated with debt financing, as in the standard CSV literature, while the use of technology o allows equity finance to be viable, since it does not require state verification. Thus the more heavily firms invest in technology o, the more equity they will issue.

Parenthetically, the fact that equity-holders (in the aggregate) receive payments of $q\theta^*y$ in nonmonitoring states may suggest that equity-holders receive a claim against only a subset of the firm's assets. This is, in fact, not the case, since in monitoring states payments to equity holders equal $q\{\theta y + (1-\theta)w - (1-\theta)z(\theta;\psi) - \psi\}$ whenever $w > J(\theta,y;\psi)$. Clearly, then, equity-holders have a claim against the complete set of returns on firm investments. Payments to equity-holders, however, can (at most) be conditioned on y in nonmonitoring states, since that is the only aspect of firm performance that is observable to them in such states.

2. Debt and Equity Issues in Equilibrium

In equilibrium, the amount of equity issued by a firm is given by

$$E(\theta^*;\psi) \equiv E[T(\psi);\psi] \equiv \tilde{E}(\psi). \tag{32}$$

If E is differentiable at (θ^*,ψ), then $\tilde{E}$ is differentiable at ψ, and

$$(33) \quad \tilde{E}'(\psi) = E_1(\theta^*;\psi)T'(\psi) + E_2(\theta^*;\psi).$$

If $E_1(\theta^*;\psi) > 0$ and $T'(\psi) > 0$ both hold, then how the quantity of equity issued varies with monitoring costs will generally be ambiguous. In Section 4 we explore via example how $\tilde{E}(\psi)$ varies with ψ. Our examples indicate that higher monitoring costs will typically lead to the increased use of equity finance, so that $\tilde{E}$ is an increasing function.

4. Some Examples

In this section we produce a class of numerical examples illustrating the results discussed in Sections 2 and 3. The examples are all based on the following triangular distribution for w:

$$(34) \quad g(w) = \begin{cases} \zeta w; & 0 \leq w \leq \bar{w}/2 \\ \zeta(\bar{w}-w); & \bar{w}/2 \leq w \leq \bar{w}. \end{cases}$$

In addition, we set $\zeta \equiv 4/\bar{w}^2$, so that $G(\bar{w}) = 1$. For this distribution, $\hat{w} = \bar{w}/2$.

The density function described in equation (34) is depicted in Figure 4. We have opted to use a triangular distribution for w because it represents a simple (and tractable), piece-wise linear, symmetric distribution for w which places relatively little weight in the tails of the distribution. It seems apparent to us that this is an empirically more interesting distribution of returns to use in generating examples than, say, the even simpler uniform distribution.

The specification in (34), of course, violates one of our maintained assumptions: that g is everywhere differentiable. In addition, it is necessary to describe conditions that must be satisfied by parameter values in order to guarantee that assumption (a.3) is satisfied. Clearly, satisfaction of (a.3) requires that

$$(35) \quad \bar{z}(0;\psi) < \bar{w}/2.$$

If (35) is satisfied, then g' exists for all "relevant" values of w, so that this class of examples satisfies our assumptions "where they matter."

We now establish when (35) holds.

PROPOSITION 2. Equation (35) (and hence assumption (a.3)) is satisfied iff $\psi > \bar{w}/4$.

Proposition 2 is proved in the Appendix. It asserts that (35) is satisfied iff monitoring costs are sufficiently large.

For this class of examples, and for $w \leq \bar{w}/2$, $G(w) = (\zeta/2)w^2$ holds. In addition, if (35) holds,

$$(36)\qquad \pi(z,\theta;\psi) = (1-\theta)z - (1-\theta)(\zeta/6)z^3 - \psi(\zeta/2)z^2.$$

Equations (11) and (36) implicitly define the function $z(\theta;\psi)$, which is of course the smallest nonnegative solution to (11) when (36) is used. Moreover, from (36) and the first order condition (17), it is possible to obtain $z(\theta^*;\psi)$ and θ^* for each combination of parameter values $(\hat{w},\hat{y},r,\psi)$, while $E(\theta^*;\psi)$ is then given by proposition 1 once a distribution for y is specified. Notice that $z(\theta^*;\psi)$ and θ^* depend on the distribution of y only through its mean ($\hat{y}$), but as is apparent from equation (29), $E(\theta^*;\psi)$ depends on the entire distribution of y. Thus changes in the distribution of y, such as mean-preserving spreads, that leave $\hat{y}$ unaltered may affect the composition of firm liabilities without affecting the composition of the firm's investment activities. This can occur because such changes redistribute the allocation of monitoring responsibilities between debt and equity-holders, and the amount of debt and equity issued must respond accordingly.

Example 1. For our first set of examples we selected the following parameter values: $\hat{w} = 1.6$ $(=\bar{w}/2)$, $r = 0.6$, $y_1 = 0.01$, $y_2 = 2.79$, and $p_1 = p_2 = 0.5$ (so that $\hat{y} = 1.4$). In addition, ψ was allowed to vary from 0.81 to 1.03. These parameter values were chosen to satisfy equations (14) and (15), and $\psi > \bar{w}/4$. In addition, for these parameters, it is feasible to set θ anywhere in the unit interval. Finally, these parameter values were selected because they deliver the result that anywhere from 54 to 73 percent of external funding is raised by issuing equity (depending on the value of ψ). This is similar in magnitude to the percentage of external funds raised through equity issues by a wide variety of large U.S. corporations.

Table 1 reports the equilibrium values of θ^*, $z^* \equiv z(\theta^*;\psi)$, and $E(\theta^*;\psi)$ corresponding to each value of ψ. Also, for comparison, we report the value $\tilde{z} = z(0;\psi)$, which is the value of z (in effect, the gross rate of interest) that would have to be chosen if θ was constrained to be zero (or, if there was only one investment opportunity available, as in the standard CSV literature).

As is apparent from Table 1, for these parameter values, $\tilde{z}$ is substantially larger than z^*, indicating that the use of technology o allows a significant reduction in monitoring costs. This reduction in monitoring costs results in a very large welfare gain for borrowers; indeed it is easy to compute that $H(0;0.81) = 0.845$ while $H(\theta^*;0.81) = 1.529$. Thus for $\psi = 0.81$, optimal use of the observable return investment technology allows for a utility increase of almost 81 percent over what could be obtained if only technology u were available. (For $\psi = 1.03$, the comparable increase in borrower expected utility is 84 percent.)

As is also clear from Table 1, for this example increases in ψ lead to significant increases in θ^*. Thus, as is intuitively plausible, increases in monitoring costs lead to increased use of the observable return investment technology. Moreover, here a monotonic association between e/q and θ is apparent. In particular, increases in ψ with θ held fixed would lead to a decline in e/q (as suggested by Proposition 1). However, increases in θ^* raise the amount of equity-finance here, and they do so sufficiently strongly that $e/q = \tilde{E}(\psi)$ is increasing in ψ.

It is also interesting to note that $\tilde{z}$ is increasing in ψ. This result is to be expected, since $z_2(\theta;\psi) > 0$ (and hence $z_2(0;\psi) > 0$) holds. In particular, when θ is held fixed, an increase in ψ

requires that z be increased to satisfy the expected return constraint. Thus, as in the standard CSV literature, with θ held constant an increase in monitoring costs requires that the probability of monitoring— $G[z(\theta;\psi)]$—increase as well.

As Table 1 indicates, this result can be overturned when firms are allowed to alter their asset structures. More specifically, increases in ψ uniformly result in reductions in z^* for this example. Thus as monitoring costs increase here, the probability of monitoring—$G[z(\theta^*;\psi)]$—actually declines. This illustrates the significance of allowing firms to adjust their investments, and demonstrates (for this example) just how powerful an instrument θ can be for the firm.

Finally, as we have noted, increases in ψ in Table 1 raise (quite substantially) the fraction of external funding obtained through issuing equity. This reflects the fact that equity issues provide the same kinds of advantages as collateral or internal finance. It is certainly intuitive that these advantages increase—and hence are relied upon more heavily—as monitoring costs rise, and the CSV problem becomes correspondingly more severe.

Example 2. For this example we set $\hat{w} = 1.6$, $r = 0.6$, and $\hat{y} = 1.4$, as in Example 1. In addition, we set $\psi = 0.85$, so that our parameter values again satisfy (14), (15), and the condition of Proposition 2. Here we explore the consequences of allowing y_1 and y_2 to vary in a mean preserving spread, with p_1 and p_2 fixed, and we also examine the consequences of changing p_1 and p_2 along with y_1 and y_2.

Table 2 reports the equilibrium value of e/q, for various values of y_1 and y_2, when $p_1 = 0.7$ and $p_2 = 0.3$. (θ^* and z^* continue to be as reported in Table 1.) When $y_1 = y_2 = \hat{y}$, 57.55 percent of external finance is raised by issuing equity. As y_1 is reduced (and y_2 is increased), the percentage of external finance raised by issuing equity rises (weakly) monotonically, and when $y_1 = 0.07$ (with $y_2 = 4.5033$), $e/q = 0.5972$.

Table 3 repeats this experiment, but with $p_1 = 0.95$ and $p_2 = 0.05$. Evidently, when $y_1 = y_2$ this makes no difference to the amount of equity issued. As before, when y_1 is reduced (y_2 is increased), the percentage of equity issued rises monotonically, reaching a value of 60.5 percent when $y_1 = 0.07$.

Why does the percentage of external finance raised through equity issues behave in the manner displayed in Tables 2 and 3? It is easy to show that, for $y_1 \geq 0.9718$, equity-holders perform *all* of the monitoring that occurs for these parameter values. Thus, as y_1 varies between 0.98 and 1.4, e/q does not vary, as both the expected monitoring costs born and the expected payments received by equity-holders are independent of y_1 and y_2.

Once $y_1 < 0.9718$ holds, debt-holders become responsible for some monitoring when $y = y_1$, and they become responsible for monitoring with higher probability as y_1 falls. Thus equity-holders shed monitoring responsibility as y_1 is reduced, but they also lose payments for the states in which they do not monitor. In both Tables 2 and 3 the former effect is dominated by the latter. Therefore, if e/q were not increased (equity-holders did not receive a greater claim on project returns), equity-holders would receive an expected return of less than r.

Relative to Table 2, this effect is more powerful in Table 3 because the probability that $y = y_1$ is larger. Hence, for each $y_1 < 0.9718$, e/q is larger in Table 3 than in Table 2.

Clearly spreading y_1 and y_2—with p_1 and p_2 fixed—increases the variance of investment returns. If y_1 and y_2 are not too far apart (here if $y_1 \geq 0.9718$), equity-holders will appear to be risk neutral, as they in fact are. Increasing the variance of investment returns does not require that any actions be taken to compensate equity-holders for the increased risk they face. But, once y_1 and y_2 are far enough apart, increasing the variance of investment returns does require equity-holders to be compensated, so they will appear to behave in a risk averse manner. Thus, to an external observer who does not take account of the distribution of monitoring costs, the risk aversion of equity-holders will appear to depend on the amount of risk they face.

In comparing Tables 2 and 3, in each row of Table 3 we increase p_1, reduce p_2, and raise y_2 to keep $\hat{y}$ fixed. It is easy to show that this increases the variance of project returns. As before, with $y_1 < 0.9718$, this increase in the variance of project returns requires equity-holders to be compensated with a greater claim on overall project returns.

Our next two examples are designed to explore numerically an issue which is relatively difficult to explore theoretically: what happens if it is not feasible for the firm to set θ "too low."[17] In particular, we now consider the possibility that

$$(37) \qquad \pi[\bar{z}(0;\psi),0;\psi] < r$$

holds, at least for some values of ψ.[18] When (37) does hold, it is not feasible to set $\theta = 0$ and, indeed, there will be a minimum feasible value of θ, which we denote by $\underline{\theta}$. It is easy to show that $\underline{\theta}$ satisfies[19]

$$(38) \qquad \pi[\bar{z}(\underline{\theta};\psi),\underline{\theta};\psi] + \underline{\theta}\hat{y} = r.$$

When (37) holds, borrowers will seek to maximize $H(\theta;\psi)$ subject to $\theta \geq \underline{\theta}$.

Parenthetically, our analysis bears on the issue of the use of CSV models to study "credit rationing." In standard CSV models—with a single type of investment opportunity—credit is said to be rationed when $1 - \alpha < \alpha q$ (the potential demand for credit exceeds its supply), and when $\bar{z} = \bar{z}(0;\psi)$, so that it is impossible for a borrower who is denied credit to alter funding contract terms in a way that raises the expected return to a lender. (See Gale and Hellwig 1985, Williamson 1986, 1987, or Boyd and Smith 1994a,b for a discussion of this kind of credit rationing.) Here, however, it is always feasible for a borrower to offer an expected return of at least $\hat{y}$, so credit can never be rationed *unless* $1 - \alpha < \alpha q$ and the expected return to a lender is bid above $\hat{y}$. In this situation, it is apparent that $\theta^* = 0$, so credit can only be rationed when borrowers are driven to a corner with respect to their investment choices. Moreover, in order for it to be feasible for borrowers to obtain credit in this situation, it is necessary that $\pi[\bar{z}(0;\psi), 0;\psi] > \hat{y}$ hold. Thus the conditions under which credit rationing can be observed will be far more stringent in CSV models with multiple investment opportunities than in CSV models with only single investment opportunities.

The analysis of this section also raises the possibility that it is not feasible for a firm to be 100 percent debt financed. Suppose, for example, that firms are prohibited from issuing liabilities that offer state contingent repayments in nonmonitoring states.[20] Then firms are constrained to issue only debt. It is straightforward but tedious to show that 100 debt finance is infeasible under *any* (not just an optimal) debt contract if (37) and

(39) $\quad r + p_n\psi G(r) > \hat{y}$

hold for some n with $y_n < r$. We will consider an example where these conditions are satisfied.

Our next example examines what happens when (37) *fails* for relatively "low" values of ψ, but holds at higher values of ψ. We then turn to an example where (37) holds for all $\psi > \bar{w}/4$.

Example 3. For this example we set $\hat{w} = 1.5$, $\hat{y} = 1.4$, $r = 0.75$, and we vary ψ from 0.76 to 1.08. The results are reported in Table 4. For $\psi \geq 1.04$, it becomes infeasible to set $\theta = 0$ (there is no nonnegative solution for $\bar{z} \equiv z(0;\psi)$). Thus, in a single investment opportunity world, there would be a severe discontinuity at $\psi = 1.04$. However, with two investment opportunities, θ^* and $z^* \equiv z(\theta^*;\psi)$ vary continuously with ψ, even as ψ rises above 1.04.

For these parameter values, the upper bound for θ is $r/\hat{y} = 0.5357$. Thus the unobservable return investment technology will always be used, and indeed, equation (14) implies that $\theta^* < r/\hat{y}$ will always obtain—no matter how large ψ is—when $g(0) = 0$ holds, as it does here. Thus, even for monitoring costs which would make the use of technology u prohibitively costly if technology u were the only one available, technology u can still be used—quite extensively, in fact—when technology o is utilized optimally. This again serves to illustrate the potential gains associated with the use of technology o to reduce costly monitoring.

Example 4. This example is identical to example 3, except that $r = 1.2315$. For this value of r, (37) holds for all $\psi > \bar{w}/4$, so that it is always infeasible to set $\theta = 0$. Table 5 reports the minimum feasible value of θ corresponding to each value of ψ; obviously $\underline{\theta}$ increases with ψ. Table 5 also reports the optimal value θ^* corresponding to each ψ. Evidently, even though the set of feasible values of θ is fairly small, we continue to have an interior optimum.

If $p_1 \geq 2/3$ and $y_1 < 1.2315$ hold, then (39) holds along with (37) for this example, and for all values $\psi > \bar{w}/4$. In this case it would not be feasible for the firm to be 100 percent debt financed using *any* kinds of debt contracts. Thus, under this parameter configuration, it is essential that the firm be permitted to issue liabilities with state contingent payments in nonmonitoring states; if it was unable to do so, it could not operate at all. And, obviously, an optimal contract involves the firm using equity finance very heavily.

5. Conclusion

We have examined an environment in which risk-neutral firms must obtain external financing for capital investments. There are two investment technologies available; one with a high expected return that is subject to a CSV problem, and one with a lower expected return, but for which project returns are fully observable. We have characterized the firm's optimal utilization of each investment technology, and described conditions under which the observable return technology will be employed more heavily as the costs of state verification rise.

The optimal contract between a firm and the agents who finance it specifies a repayment schedule that is a piece-wise linear function of all the project returns that can be observed by outsiders. This contract can be supported by having the firm issue an appropriately selected—and determinate—amount of external debt and equity. The amount of equity issued depends on the composition of the firm's investments, on the costs of state verification, and on factors—like the distribution of returns on technology o—that are irrelevant to the allocation of resources.

The debt issued by a firm in this world is identical to the debt issued (by wholly debt-financed firms) in standard CSV models. The equity issued makes equity-holders true residual claimants; equity-holders receive some fraction of project returns less payments to debt-holders. In nonmonitoring states payments to equity-holders are a function of the return on technology o alone, which is of course the only return they observe in such states. In monitoring states equity-holders receive potentially complicated repayments, which are a function of the returns on both technologies o and u.

Our examples indicate that the optimal debt-equity ratio for a firm will typically fall as the costs of state verification increase, and as technology o becomes more risky (for a give value of $\hat{y}$). A further examination of the generality of these properties is an important topic for future investigation. Another such topic would involve extending the model to allow inside equity-holders (project owners) to contribute some internal finance for their own projects. The existence of internal finance will typically alter any firm's optimal ratio of debt to equity. The relationship between internal finance and other dimensions of a firm's capital structure is also deserving of further consideration.

A particularly important extension of this study, and one which is undertaken in Boyd and Smith (1995), is to embed the analysis of this paper in a dynamic model which allows for the possibility of economic growth. By modifying the model in this way, it is possible to investigate two issues of central importance in economic development: how does the level of activity in debt and equity markets evolve as an economy develops, and how does the composition of investment change as capital is accumulated? The results obtained by Boyd-Smith (1995) suggest that economic growth will typically be accompanied by an increasingly heavy reliance on equity markets, and by an increased use of the observable return investment technology. The former result is certainly strongly supported by casual observation.

Several of these results are deserving of some additional discussion. For example, the determinacy of the debt-equity ratio in this environment depends on the assumption that there is a positive cost (F) to the establishment of additional corporations. In particular, if $F = 0$ were to hold, there would not be a determinate equity ratio. To see this, note that in the absence of a cost to creating (fictitious) "new firms," a borrower could create one firm utilizing only the observable return technology, and could sell a set of equity claims against it (raising $\theta\hat{y}/r$ in the process). These proceeds could then be invested in a second firm, which operated the unobservable return technology. The second firm would be entirely debt-financed (as a conventional CSV firm), and would issue debt with a value of $q - (\theta\hat{y}/r)$. The result is an equity ratio, for the two firms together, of $\theta\hat{y}/rq$. At the same time, the equity ratio described in equation (28) would continue to constitute an equilibrium for the firm. Thus the equity ratio would not be pinned down by the model.

The firm's indifference between these two strategies (or between any convex combinations of them) depends crucially on the absence of any costs to the creation of an additional entity against which equity claims can be sold. With any positive costs of doing so, a borrower will strictly prefer to operate a single firm. Then, since debt-holders must be prior claimants, any given borrower will have a determinate ratio of debt to total liabilities. Since we think it implausible that the spin-off of new corporate entities is completely costless, we also believe that the factors we have described are adequate to deliver a determinate ratio of debt to equity.

Of course while equity-holders in this environment are residual claimants, they do not perform another function of real world equity owners. In particular, they exercise no control rights over firms.[21] However, a generalized version of our analysis—modified to allow for the presence of a managerial incentive problem—is likely to suggest that the appropriate sale of debt and equity to outside investors provides inside owner/managers with a strong set of performance incentives. Indeed, our owner/managers are compensated—after external debt and equity-holders are paid—in a manner that is strongly related to firm performance. Thus a combination of debt and equity that is sold to address a CSV problem may only require minor modifications when managerial incentive problems are added to the model as well.

We now conclude by discussing the relationship between our analysis of firm financial structure, and two previous contributions to the literature.

Chang (1987) considers the optimal liability structure of a firm which has a *fixed* investment composition that yields a partially (freely) observable and a partially unobservable (observable only at a cost) investment return.[22] Chang then shows how an optimal contract can be supported through the issue of something like debt and equity.

Relative to our formulation, Chang's yields some counter-intuitive results. For example, with θ fixed, an increase in monitoring costs will typically lead to an increased use of debt (proposition 1.f). When incorporated into a growth model (see Boyd-Smith 1995), Chang's formulation would then imply that debt is more heavily—and equity less heavily—utilized as an economy develops. Such a result is clearly contrary to observation. Moreover, Chang's interpretation of debt issues typically requires that payments to debt-holders be contingent on firm performance, even in nonmonitoring states. These results suggest the importance of treating θ as endogenous.

Seward (1990) also considers an environment with two investment technologies, with one being subject to and one not being subject to a CSV problem. Furthermore, Seward allows the firm to choose the composition of its investments, as we do. Seward then shows that the *first best allocation* of investment can be supported by employing a contract that resembles the use of debt and equity finance. However, Seward does not consider the possibility that the firm might (and generally should) alter the composition of its investment relative to that in a frictionless environment in order to reduce the level of its expected monitoring costs.[23] Obviously this is the essence of our analysis.

Appendix

Proof of Lemma 1.

(a) Fix θ, $R(w,y_n)$, $x(y_n)$, and $A(y_n)$; $n = 1, \ldots, N$. Assumption (a.1) implies that increases in i always raise the value of the objective function. Moreover, since constraints (1) and (2) are in "per unit borrowed terms", these are unaffected by increasing i. Finally, it is easy to see that increases in i, ceteris paribus, relax constraint (3). Thus, at an optimum, i must be as large as possible.

(b) Fix $i = q$ and θ. For $y \neq y_n$, fix $R(w,y)$, $x(y)$, and $A(y)$. It is evident that feasibility requires that

$$\text{(A1)} \quad A(y_n) \subset \left[0, \frac{x(y_n) - \theta y_n}{1 - \theta}\right].$$

Now suppose that

$$\text{(A2)} \quad \left[0, \frac{x(y_n) - \theta y_n}{1 - \theta}\right] \subsetneq A(y_n).$$

Then it is possible to change $R(w,y_n)$, $x(y_n)$, and $A(y_n)$ in a way that satisfies (1)–(3), while reducing the set of monitoring states when $y = y_n$. (See Gale-Hellwig 1985 or Williamson 1986, 1987 for a constructive proof of this assertion.) Since a reduction in the set of monitoring states always raises the value of the borrower's objective, (A.2) is inconsistent with an optimal contract. Moreover, this is true for all $n = 1, \ldots, N$.

(c) Fix $i = q$, θ, and $A(y_n)$ as in part (b) of the lemma. In addition, for $y \neq y_n$ fix $R(w,y)$ and $x(y)$. If $R(w,y_n) < \theta y_n + (1-\theta)w$ holds for some subset of $A(y_n)$ which occurs with positive probability, then it is possible to raise $R(w,y_n)$ on this subset, and to reduce $x(y_n)$ in a way that satisfies (1)–(3). (Again, see Gale-Hellwig 1985 and Williamson 1986, 1987.) Part (b) of the lemma implies that reductions in $x(y_n)$ reduce expected monitoring costs, and hence raise the value of the borrower's objective function. Thus $R(w,y_n) < \theta y_n + (1-\theta)w$ is inconsistent with optimality, establishing the result. □

Proof of Lemma 2.

(a) Suppose to the contrary that $z_n > \bar{z}(\theta;\psi)$, for some n. Then, holding θ fixed, replace z_n with $\bar{z}(\theta;\psi)$ in the problem (P). This increases the value of the objective function and it relaxes constraint (6). But this contradicts the assumption that $z_n > \bar{z}(\theta;\psi)$ is optimal.

(b) $\bar{z}(\theta;\psi)$ satisfies the first-order condition

(A3) $$(1-\theta)\{1 - G[\bar{z}(\theta;\psi)]\} - \psi g[\bar{z}(\theta;\psi)] = 0$$

and the second-order condition

(A4) $$(1-\theta)g[\bar{z}(\theta;\psi)] + \psi g'[\bar{z}(\theta;\psi)] > 0.$$

Then the implicit function theorem guarantees the differentiability of $\bar{z}(\theta;\psi)$; and differentiating (A.3) with respect to θ yields

(A5) $$\{(1-\theta)g[\bar{z}(\theta;\psi)] + \psi g'[\bar{z}(\theta;\psi)]\}\bar{z}_1(\theta;\psi) = -\{1 - G[\bar{z}(\theta;\psi)]\} < 0.$$

(A4) and (A5) imply that $\bar{z}_1(\theta;\psi) < 0$, so that $\bar{z}(\theta;\psi) \leq \bar{z}(0;\psi)$ holds, for all θ. This establishes (b). □

Proof of Lemma 3.

From equation (11), clearly

(A6) $$\pi[z(r/\hat{y};\psi),r/\hat{y};\psi] \equiv 0.$$

As is apparent from (10), $\pi(0,\theta;\psi) = 0$, for all (θ,ψ). Thus

(A7) $$z(r/\hat{y};\psi) = 0.$$

For all $\theta \geq r/\hat{y}$ no monitoring costs are born. Then, since $\hat{w} > \hat{y}$, for $\theta \geq r/\hat{y}$ the borrower's objective function in (P.1) is strictly decreasing in θ. Thus $\theta \leq r/\hat{y}$ at an optimum. □

Proof of Lemma 4.

The derivative of the objective function in (P.1) is $\hat{y} - \hat{w} - \psi g[z(\theta;\psi)]z_1(\theta;\psi)$. Since $z(r/\hat{y};\psi) = 0$, the left derivative of this function at $\theta = r/\hat{y}$ is given by the left-hand side of condition (a). If this is negative, clearly $\theta < r/\hat{y}$ holds. Similarly, since $z(0;\psi) = \tilde{z}$, the right derivative of this function at $\theta = 0$ is given by the left-hand side of condition (b). If this is positive, clearly $\theta > 0$ holds.

From equation (18),

$$(A8)\quad z_1(r/\hat{y};\psi) = -\{\hat{y} + \pi_2[z(r/\hat{y};\psi),r/\hat{y};\psi]\}/\pi_1[z(r/\hat{y};\psi),r/\hat{y};\psi]$$

$$= -\{\hat{y} + \pi_2(0,r/\hat{y};\psi)\}/\pi_1(0,r/\hat{y};\psi).$$

Now

$$(A9)\quad \pi_1(z,\theta;\psi) = (1-\theta)[1-G(z)] - \psi g(z)$$

and

$$(A10)\quad \pi_2(z,\theta;\psi) = -z[1-G(z)] - \int_0^z wg(w)dw.$$

Thus

$$(A11)\quad \pi_1(0,r/\hat{y};\psi) = (\hat{y}-r)/\hat{y} - \psi g(0)$$

and

$$(A12)\quad \pi_2(0,r/\hat{y};\psi) = 0.$$

Therefore

$$\text{(A13)}\quad z_1(r/\hat{y};\psi) = -\hat{y}/\{[(\hat{y}-r)/\hat{y}] - \psi g(0)\}.$$

Substituting (A13) into condition (a) and rearranging terms yields equation (14). Similarly

$$\text{(A14)}\quad \pi_1(\tilde{z},0;\psi) = 1 - G(\tilde{z}) - \psi g(\tilde{z})$$

and

$$\text{(A15)}\quad \pi_2(\tilde{z},0;\psi) = -\tilde{z}[1-G(\tilde{z})] - \int_0^{\tilde{z}} wg(w)dw.$$

Therefore

$$\text{(A16)}\quad z_1(0;\psi) = -\left\{\hat{y} - \tilde{z}[1 - G(\tilde{z})] - \int_0^{\tilde{z}} wg(w)dw\right\}/[1 - G(\tilde{z}) - \psi g(\tilde{z})].$$

Moreover, the expected return constraint (11) implies that

$$\text{(A17)}\quad \tilde{z}[1 - G(\tilde{z})] + \int_0^{\tilde{z}} wg(w)dw = r + \psi G(\tilde{z}).$$

Substituting (A17) into (A16), substituting the result into condition (b), and rearranging terms yields equation (15). □

Proof of Lemma 5.

(a) The second-order condition for the problem (P.1)—equation (19)— implies that the term multiplying $T'(\psi)$ on the left-hand side of equation (21) is positive. Therefore, since $-g'(-) < 0$, $T'(\psi) > 0$ holds iff (22) is satisfied.

(b) It is obvious from (17) that $z_1(\theta^*;\psi) < 0$. Moreover, equation (11) implies that

$$\text{(A18)}\quad z_2(\theta^*;\psi) = -\pi_3[z(\theta^*;\psi),\theta^*;\psi]/\pi_1[z(\theta^*;\psi),\theta^*;\psi].$$

The assumption that credit is not rationed implies that

(A19) $\pi_1[z(\theta^*;\psi),\theta^*;\psi] > 0$

(see Figure 1), while from (10)

(A20) $\pi_3[z(\theta^*;\psi),\theta^*;\psi] = -G[z(\theta^*;\psi)] < 0.$

Thus, $z_2(\theta^*;\psi) > 0$. Therefore, it follows from (a.3) that (22) is satisfied if $z_{12}(\theta^*;\psi) \leq 0$. Differentiating equation (18) with respect to ψ gives

(A21) $$z_{12}(\theta;\psi) = -[\pi_{23}(-)/\pi_1(-)] - [\pi_{21}(-)/\pi_1(-)]z_2(\theta;\psi) - z_1(\theta;\psi)\{\pi_{11}(-)z_2(\theta;\psi) + \pi_{13}(-)\}/\pi_1(-).$$

It is straightforward to verify that $\pi_{23}(z,\theta;\psi) \equiv 0$. Moreover, if we evaluate (A21) at $\theta = \theta^*$ and use (A18)-(A20), it is apparent that $z_{12}(\theta^*;\psi) \leq 0$ holds iff

(A22) $$-z_1(\theta^*;\psi)\{[\pi_{13}(-)/\pi_3(-)] - [\pi_{11}(-)/\pi_1(-)]\} \geq -\pi_{21}(-)/\pi_1(-).$$

From (A9), (A10), and (A20) it follows that

(A23) $$[\pi_{13}(-)/\pi_3(-)] - [\pi_{11}(-)/\pi_1(-)] = g(\hat{z})/G(\hat{z}) + \{(1-\theta^*)g(\hat{z}) + \psi g'(\hat{z})\}/\{(1-\theta^*)[1-G(\hat{z})] - \psi g(\hat{z})\}$$

and

(A24) $$-\pi_{21}(-)/\pi_1(-) = [1-G(\hat{z})]/\{(1-\theta^*)[1-G(\hat{z})] - \psi g(\hat{z})\}$$

where $\hat{z} \equiv z(\theta^*;\psi)$. Substituting (A23) and (A24) into (A22) then yields that $z_{12}(\theta^*;\psi) \leq 0$ holds iff

(A25) $$-z_1(\theta^*;\psi)\{(1-\theta^*)g(\hat{z}) - \psi g(\hat{z})^2 + G(\hat{z})\psi g'(\hat{z})\} \geq G(\hat{z})[1-G(\hat{z})].$$

Now substitute (17) into (A25). Upon rearranging terms we obtain that $z_{12}(\theta^*;\psi) \leq 0$ holds iff

(A26) $(\hat{w}-\hat{y})\{(1-\theta^*) - \psi g(\hat{z}) + \psi G(\hat{z})[g'(\hat{z})/g(\hat{z})]\} \geq \psi G(\hat{z})[1-G(\hat{z})]$.

Since $\theta^* \leq r/\hat{y}$ holds, obviously a sufficient condition for (A26) is

(A27) $(\hat{w}-\hat{y})\{[(\hat{y}-r)/\hat{y}] - \psi g[z(\theta^*;\psi)]\} \geq \psi G(\hat{z})[1-G(\hat{z})]$.

Moreover, $G(\hat{z})[1-G(\hat{z})] \leq 1/4$ holds, yielding equation (23). □

Proof of Proposition 1.

(a) Rewrite equation (28) as

$$\text{(A28)}\quad re/q = \theta\hat{y}\{1 - G[z(\theta;\psi)]\} + \sum_n p_n \int_0^{z(\theta;\psi)} [\theta y_n+(1-\theta)w-(1-\theta)z(\theta;\psi)]g(w)dw - \psi G[z(\theta;\psi)]$$

$$+ \sum_n p_n\psi G[J(\theta,y_n;\psi)] - \sum_n p_n \int_0^{J(\theta,y_n;\psi)} [\theta y_n + (1-\theta)w - (1-\theta)z(\theta;\psi)]g(w)dw$$

$$= \theta\hat{y} - (1-\theta)z(\theta;\psi)G[z(\theta;\psi)] + \int_0^{z(\theta;\psi)} (1-\theta)wg(w)dw - \psi G[z(\theta;\psi)]$$

$$+ \sum_n p_n\psi G[J(\theta,y_n;\psi)] - \sum_n p_n \int_0^{J(\theta,y_n;\psi)} [\theta y_n + (1-\theta)w - (1-\theta)z(\theta;\psi)]g(w)dw.$$

Equation (9) implies that

$$\text{(A29)}\quad \theta\hat{y} + \int_0^{z(\theta;\psi)} (1-\theta)wg(w)dw - (1-\theta)z(\theta;\psi)G[z(\theta;\psi)] - \psi G[z(\theta;\psi)] = r - (1-\theta)z(\theta;\psi).$$

Substituting (A29) into (A28) yields

$$\text{(A30)}\quad re/q = r - (1-\theta)z(\theta;\psi) + \sum_n p_n\psi G[J(\theta,y_n;\psi)] - \sum_n p_n \int_0^{J(\theta,y_n;\psi)} (1-\theta)wg(w)dw$$

$$+ \sum_n p_n(1-\theta)[z(\theta;\psi) - (\theta/(1-\theta))y_n]G[J(\theta,y_n;\psi)]$$

$$= r - (1-\theta)z(\theta;\psi) + \sum_n p_n\psi G[J(\theta,y_n;\psi)] - \sum_n p_n \int_0^{J(\theta,y_n;\psi)} (1-\theta)wg(w)dw$$

$$+ \sum_n p_n(1-\theta)J(\theta,y_n;\psi)G[J(\theta,y_n;\psi)]$$

where the last equality follows from the fact that $J(\theta,y_n;\psi) = z(\theta;\psi) - [\theta/(1-\theta)]y_n$ holds iff $J(\theta,y_n;\psi) \neq 0$.

Now observe that, by definition,

$$\text{(A31)}\quad \sum_n p_n\psi G[J(\theta,y_n;\psi)] - \sum_n p_n \int_0^{J(\theta,y_n;\psi)} (1-\theta)wg(w)dw + \sum_n p_n(1-\theta)J(\theta,y_n;\psi)G[J(\theta,y_n;\psi)]$$

$$= \sum_n p_n(1-\theta)J(\theta,y_n;\psi) - \sum_n p_n\pi[J(\theta,y_n;\psi),\theta;\psi].$$

Substituting (A31) into (A30) we obtain

$$\text{(A32)}\quad (r/q)e = r - \sum_n p_n(1-\theta)[z(\theta;\psi) - J(\theta,y_n;\psi)] - \sum_n p_n\pi[J(\theta,y_n;\psi),\theta;\psi].$$

But this is equation (29).

(b) Clearly E is differentiable iff $J(\theta,y_n;\psi)$ is differentiable for all n. $J(\theta,y_n;\psi)$ is differentiable for all n iff $[(1-\theta)/\theta]z(\theta;\psi) \neq y_n$ holds for all n.

(c) Define the correspondence $M(\theta;\psi)$ by

$$\text{(A33)}\quad M(\theta;\psi) \equiv \{n = 1,\ldots,N\colon J(\theta,y_n;\psi) = 0\}.$$

Then write that

$$\text{(A34)}\quad (r/q)E(\theta;\psi) \equiv r - \sum_{n\in M(\theta;\psi)} p_n(1-\theta)z(\theta;\psi) - \sum_{n\notin M(\theta;\psi)} p_n(1-\theta)[z(\theta;\psi) - J(\theta,y_n;\psi)]$$

$$- \sum_{n\notin M(\theta;\psi)} p_n\pi[J(\theta,y_n;\psi),\theta;\psi]$$

$$= r - \sum_{n\in M(\theta;\psi)} p_n(1-\theta)z(\theta;\psi) - \sum_{n\notin M(\theta;\psi)} \theta p_n y_n - \sum_{n\notin M(\theta;\psi)} p_n\pi[J(\theta,y_n;\psi),\theta;\psi].$$

Then, at points of differentiability,

$$\text{(A35)}\quad (r/q)E_1(\theta;\psi) = \sum_{n\in M(\theta;\psi)} p_n[z(\theta;\psi) - (1-\theta)z_1(\theta;\psi)] - \sum_{n\notin M(\theta;\psi)} p_n y_n$$

$$- \sum_{n\notin M(\theta;\psi)} p_n\pi_1[J(\theta,y_n;\psi),\theta;\psi]J_1(\theta,y_n;\psi) - \sum_{n\notin M(\theta;\psi)} p_n\pi_2[J(\theta,y_n;\psi),\theta;\psi]$$

$$= \sum_{n\in M(\theta;\psi)} p_n[z(\theta;\psi) - (1-\theta)z_1(\theta;\psi)] - \sum_{n\notin M(\theta;\psi)} p_n y_n$$

$$- \sum_{n\notin M(\theta;\psi)} p_n\pi_1[J(\theta,y_n;\psi),\theta;\psi)]\{z_1(\theta;\psi) - (1-\theta)^{-2}y_n\}$$

$$- \sum_{n\notin M(\theta;\psi)} p_n\pi_2[J(\theta,y_n;\psi),\theta;\psi].$$

Case 1. Suppose that θ is large enough that

$$\text{(A36)}\quad [(1-\theta)/\theta]z(\theta;\psi) < y_n$$

holds, for all n. Then E is differentiable, and $M(\theta;\psi) = \{1,2,\ldots,N\}$. Thus $E_1(\theta;\psi) > 0$ holds if $z_1(\theta;\psi) \leq 0$. But $z_1(r/\hat{y};\psi) = -\{\hat{y} + \pi_2(0,r/\hat{y};\psi)\}/\pi_1(-) = -\hat{y}/\pi_1(-) < 0$. Thus (A36) and $z_1(\theta;\psi) \leq 0$ hold if θ is sufficiently close to $r/\hat{y}$. It follows that $E_1(\theta;\psi) > 0$.

Case 2. Suppose $\theta = 0$. Then we have $J(0,y_n;\psi) = z(0;\psi) > 0$, that $M(0;\psi) = \phi$, and that E is differentiable. Moreover, evaluating (A35) at $\theta = 0$ gives

$$\text{(A37)}\quad (r/q)E_1(0;\psi) = -\hat{y} - \pi_1[z(0;\psi),0;\psi]z_1(0;\psi) + \pi_1[z(0;\psi),0;\psi]\hat{y} - \pi_2[z(0;\psi),0;\psi].$$

In addition, equation (18) implies that

$$\text{(A38)}\quad -z_1(0;\psi)\pi_1[z(0;\psi),0;\psi] = \hat{y} + \pi_2[z(0;\psi),0;\psi].$$

Substituting (A38) into (A37) yields

$$(r/q)E_1(0;\psi) = \pi_1[z(0;\psi),0;\psi]\hat{y} > 0.$$

Then $E_1(\theta;\psi) > 0$ for θ sufficiently near zero follows from continuity.

(d) $E(r/\hat{y};\psi) = q$ follows from the fact that $z(r/\hat{y};\psi) = J(r/\hat{y},y_n;\psi) = 0$. $E(0;\psi) = 0$ follows from the facts that $z(0;\psi) = J(0,y_n;\psi)$ and that $\pi[z(0;\psi),0;\psi] \equiv r$.

(e) If $z_1(\theta;\psi) \leq 0$ holds for some (θ,ψ), and if E is differentiable at that point, equation (A35) implies that $E_1(\theta;\psi) > 0$ holds if

$$\text{(A39)}\quad -\sum_{n\notin M(\theta;\psi)} p_n\pi_1[J(\theta,y_n;\psi),\theta;\psi]\{z_1(\theta;\psi) - (1-\theta)^{-2}y_n\} - \sum_{n\notin M(\theta;\psi)} p_n\pi_2[J(\theta,y_n;\psi),\theta;\psi]$$
$$- \sum_{n\notin M(\theta;\psi)} p_n y_n \geq 0.$$

Moreover, $z_1(\theta;\psi) \leq 0$, $z(\theta;\psi) \geq J(\theta,y_n;\psi)$ for all $n \notin M(\theta;\psi)$, and $\pi_{11}(-) < 0$ imply that (A39) holds if

$$\text{(A40)}\quad -\sum_{n\notin M(\theta;\psi)} p_n\pi_1[z(\theta;\psi),\theta;\psi]\{z_1(\theta;\psi) - (1-\theta)^{-2}y_n\} - \sum_{n\notin M(\theta;\psi)} p_n\pi_2[J(\theta,y_n;\psi),\theta;\psi]$$
$$- \sum_{n\notin M(\theta;\psi)} p_n y_n \geq 0.$$

Equation (18) implies that

$$-\pi_1[z(\theta;\psi),\theta;\psi]z_1(\theta;\psi) = \hat{y} + \pi_2[z(\theta;\psi),\theta;\psi].$$

Substituting this expression into (A40) yields the equivalent condition

$$\text{(A40')}\quad \hat{y}\sum_{n\notin M(\theta;\psi)} p_n + \sum_{n\notin M(\theta;\psi)} (1-\theta)^{-2}\, p_n\pi_1[z(\theta;\psi),\theta;\psi]y_n - \sum_{n\notin M(\theta;\psi)} p_n\{\pi_2[J(\theta,y_n;\psi),\theta;\psi]$$
$$- \pi_2[z(\theta;\psi),\theta;\psi]\} - \sum_{n\notin M(\theta;\psi)} p_n y_n \geq 0.$$

Now (A40′) holds if

$$(A41)\quad \hat{y} \geq \left(\sum_{n\notin M(\theta;\psi)} p_n y_n\right) \Big/ \sum_{n\notin M(\theta;\psi)} p_n$$

and

$$(A42)\quad \sum_{n\notin M(\theta;\psi)} (1-\theta)^{-2} p_n \pi_1[z(\theta;\psi),\theta;\psi] y_n - \sum_{n\notin M(\theta;\psi)} p_n\{\pi_2[J(\theta,y_n;\psi),\theta;\psi] - \pi_2[z(\theta;\psi),\theta;\psi]\} \geq 0$$

both hold. (A41) must be satisfied, since $n \notin M(\theta;\psi)$ holds only for "smaller" values of y_n. Thus $E_1(\theta;\psi) > 0$ holds if (A42) is satisfied.

From (A10)

$$(A43)\quad -\pi_2(s,\theta;\psi) = s[1 - G(s)] + \int_0^s wg(w)dw = s - \int_0^s G(w)dw$$

where the second equality follows via integration by parts. Using (A43) in (A42) yields

$$(A44)\quad \sum_{n\notin M(\theta;\psi)} (1-\theta)^{-2} p_n \pi_1[z(\theta;\psi),\theta;\psi] y_n + \sum_{n\notin M(\theta;\psi)} p_n[J(\theta,y_n;\psi) - z(\theta;\psi)]$$

$$+ \sum_{n\notin M(\theta;\psi)} p_n \int_{J(\theta,y_n;\psi)}^{z(\theta;\psi)} G(w)dw \geq 0$$

as a sufficient condition for $E_1(\theta;\psi) > 0$. (A44) is equivalent to

$$(A44')\quad \sum_{n\notin M(\theta;\psi)} p_n\{(1-\theta)^{-2}\pi_1[z(\theta;\psi),\theta;\psi] - [\theta/(1-\theta)]\} y_n + \sum_{n\notin M(\theta;\psi)} p_n \int_{J(\theta,y_n;\psi)}^{z(\theta;\psi)} G(w)dw \geq 0.$$

Since

$$\int_{J(\theta,y_n;\psi)}^{z(\theta;\psi)} G(w)dw \geq [z(\theta;\psi) - J(\theta;\psi)]G[J(\theta,y_n;\psi)] = [\theta/(1-\theta)]G[J(\theta,y_n;\psi)]y_n$$

holds, a sufficient condition for (A44′) is that

$$(A45)\quad (1-\theta)^{-2}\pi_1[z(\theta;\psi),\theta;\psi] \geq [\theta/(1-\theta)]\{1 - G[J(\theta,y_n;\psi)]\}$$

hold for all n $\notin M(\theta;\psi)$. (A45) is, of course, equivalent to (30). Since $\theta(1-\theta) \leq 0.25$, obviously (31) is sufficient for (30). □

(f) If $E(\theta;\psi)$ is differentiable at (θ,ψ), then differentiating equation (29) yields

$$\text{(A46)}\quad E_2(\theta;\psi) = -\sum_{n\in M(\theta;\psi)} p_n(1-\theta)z_2(\theta;\psi) - \sum_{n\notin M(\theta;\psi)} p_n\pi_1[J(\theta,y_n;\psi),\theta;\psi]J_3(\theta,y_n;\psi)$$

$$- \sum_{n\notin M(\theta;\psi)} p_n\pi_3[J(\theta,y_n;\psi),\theta;\psi].$$

From (A18)-(A20), $z_2(\theta;\psi) \geq 0$, while $J_3(\theta,y_n;\psi) = z_2(\theta;\psi)$ if $J(\theta,y_n;\psi) > 0$. Using these facts and (A18) in (A46), we obtain

$$\text{(A47)}\quad E_2(\theta;\psi) = -\sum_{n\in M(\theta;\psi)} p_n(1-\theta)z_2(\theta;\psi)$$

$$+ \sum_{n\notin M(\theta;\psi)} p_n\pi_1[J(\theta,y_n;\psi),\theta;\psi]\pi_3[z(\theta;\psi),\theta;\psi]/\pi_1[z(\theta;\psi),\theta;\psi]$$

$$- \sum_{n\notin M(\theta;\psi)} p_n\pi_3[J(\theta,y_n;\psi),\theta;\psi].$$

Now $\pi_1[J(-),\theta;\psi]/\pi_1[z(-),\theta;\psi] \geq 1$ holds (since $\pi_{11} < 0$), and $-\pi_3[z(-),\theta;\psi] \geq -\pi_3[J(-),\theta;\psi]$ holds, since $z(\theta;\psi) \geq J(\theta,y_n;\psi)$ for all n (see equation (A20)). Therefore each term on the right-hand side of (A47) is nonpositive, so $E_2(\theta;\psi) \leq 0$. □

Proof of Proposition 2.

$\bar{z}(0;\psi)$ solves the equation

$$\text{(A48)}\quad 1 = G(z) + \psi g(z)$$

and satisfies the second-order condition

$$\text{(A49)}\quad g(z) + \psi g'(z) \geq 0.$$

Since $G(0) + \psi g(0) = 0$ when (34) holds, $1 > G(0) + \psi g(0)$. Moreover, (A49) holds for $z \in [0,\bar{w}/2)$, so (A48) has a unique solution in the interval $[0,\bar{w}/2)$ iff

(A50) $G(\bar{w}/2) + \psi g(\bar{w}/2) > 1$.

For the triangular distribution, (A50) is equivalent to $\psi > \bar{w}/4$.

We now show that (A48) has exactly one solution in the interval $(\bar{w}/2,\bar{w}]$, and that this solution represents a local *minimum* of $\pi(z,0;\psi)$. Thus $\bar{z}(0;\psi) < \bar{w}/2$, as claimed.

To establish that there is at most one solution to (A48) in the interval $(\bar{w}/2,\bar{w}]$, note that $G(\bar{w}/2) + \psi g(\bar{w}/2) > G[\bar{z}(0;\psi)] + \psi g[\bar{z}(0;\psi)] = 1$ (see Figure 4). Moreover, for $z > \bar{w}/2$, $G(z) + \psi g(z)$ is nonincreasing in z iff $z \geq \bar{w} - \psi$. It is therefore the case that any solution to (A48) satisfying $z > \bar{w}/2$ must also satisfy $z > \bar{w} - \psi$, and that (A48) can have at most one solution in the interval $(\max[\bar{w}/2,\bar{w} - \psi],\bar{w}]$. Moreover there clearly is such a solution, since $z = \bar{w}$ satisfies (A48).

At $z = \bar{w}$, of course, (A49) is violated. Therefore, setting $z = \bar{w}$ attains a local minimum of $\pi(z,0;\psi)$. Thus $\bar{z}(0;\psi) < \bar{w}/2$, as was to be shown. □

Footnotes

[1]Large corporations are also likely to have multiple classes of debt securities, but that is not the focus of this study.

[2]There is no difficulty created by giving lenders more general utility functions and letting them make a nontrivial consumption-savings decision. However, this adds to notational requirements without yielding any additional insights. Thus we retain the simpler specification of the text.

[3]F represents any costs associated with the process of establishing a new firm. We will discuss below the significance of assuming that it is costly to establish additional corporations. Parenthetically, it does not affect our results if a borrower also faces a (small) fixed cost in establishing an initial firm.

[4]Credit rationing in this context is described by Gale and Hellwig (1985) and Williamson (1986, 1987).

[5]Since $\hat{y} > r$, a borrower will never invest in the commonly available technology. Also, note that we are assuming that i and θ are observable objects for external investors.

[6]While this is a real restriction, Boyd and Smith (1994a) have shown that—for realistic parameter values—the utility gains resulting from stochastic monitoring are negligible. We discuss in more detail below how a duplication of monitoring activity can be avoided.

[7]For a further discussion of the issue of duplication of monitoring, see Diamond (1984), Boyd and Prescott (1986), Williamson (1986), or Krasa and Villamil (1992).

[8]Parenthetically, it is straightforward but tedious to show that $x(y_n) > \theta y_n$ must hold for all n, so that $A(y_n)$ is well-defined.

[9]Note that we are implicitly assuming that a solution exists for all $\theta \in [0,1]$. A necessary (but not sufficient) condition for this to be the case is that $\pi[\bar{z}(0;\psi),0;\psi] \geq r$. If this condition is

violated, then (P) will have no solution if we set θ "too low." Our assumptions do imply that (P) always has a solution if θ is sufficiently large.

For the present we assume that it is feasible to set θ anywhere in the unit interval. We explore—in the context of some examples—what happens when this is not feasible in Section 4.

[10]According to standard usages in the CSV literature, a firm with $\theta = 0$ is "bankrupt" iff $w < \bar{z}(0;\psi)$. For a symmetric, unimodal return distribution, assumption (a.3) asserts merely that a bankruptcy cannot occur when the firm's return exceeds the median return. Thus we do not regard (a.3) as a strong assumption.

[11]If there is no interior optimum, then it is straightforward to show that $z_n = \bar{z}(\theta;\psi)$ holds, for all n.

[12]$\bar{z}$ is the critical value of w that triggers monitoring when $\theta = 0$. As such, it is simply the conventional gross interest rate associated with a standard debt contract in more typical CSV models (for example Gale-Hellwig 1985 and Williamson 1986, 1987).

[13]The consequences of relaxing (15) or (16) are discussed in Section 4.

[14]For noninterior optima, it is straightforward to show that $T'(\psi) \geq 0$ necessarily holds.

[15]For instance, it may be possible for equity-holders to file a class-action suit against the firm, so that monitoring costs are only incurred once. Alternatively, we could think of equity as being owned indirectly through a mutual fund, which conducts monitoring activities on behalf of its own share-holders. Or, the same end result could be obtained by assuming that monitoring outcomes are publicly observable, and that monitoring costs are shared among a large number of equity investors.

Parenthetically, it is necessary that equity ownership be arranged so that equity owners always have enough income to cover their obligations with respect to monitoring costs. This can be assured either by assuming that equity-holders are perfectly diversified, or by assuming that equity ownership is intermediated, say through a mutual fund. Both arrangements are consistent with the equilibrium we describe.

[16]If the expected return on equity is r, the expected return constraint (11) implies that the expected return on debt is also r, since (11) guarantees that all external funds earn that return in the aggregate.

[17]It is not feasible for the firm to choose a particular value for θ if, at that θ, it is not feasible to deliver an expected return of r to a lender.

[18]Obviously if (37) *fails* to hold for some values $\psi > \bar{w}/4$, it will fail to hold only for relatively small values of ψ.

[19]When (37) holds, it is easy to show that (38) has a unique solution $\underline{\theta} \in (0,r/\hat{y})$. Also, if there is no interior optimum, $\theta^* = \underline{\theta}$ and $z^* = \bar{z}(\underline{\theta};\psi)$ hold.

[20]Such restrictions against equity-like securities are not unknown historically. See, for instance, Bencivenga and Smith (1991).

[21]See Dewatripont and Tirole (1994) for an analysis of firm liability structure that is based on control rights. Boot and Thakor (1993) emphasize the informational aspects of debt and equity, as we do, although their analysis is predicated on the existence of informational asymmetries affecting the providers of external finance.

[22]That is, Chang exogenously fixes θ.

[23]Seward also assumes that the two investment technologies have the same expected returns. In our environment, that would obviously lead to $\theta^* = 1$, and to 100 percent equity finance.

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Table 1

Solutions for Example 1

ψ	$\tilde{z}$	z^*	θ^*	e/q
.81	.700	.383	.239	.536
.83	.703	.363	.252	.567
.85	.706	.345	.262	.593
.87	.708	.330	.271	.616
.89	.711	.316	.279	.636
.91	.715	.304	.286	.653
.93	.718	.292	.293	.669
.95	.721	.282	.299	.683
.97	.724	.273	.304	.696
.99	.727	.264	.309	.708
1.01	.731	.256	.313	.718
1.03	.734	.248	.317	.728

Other parameter values are: $\hat{w} = 1.6$, $r = 0.6$, $y_1 = 0.01$, $y_2 = 2.79$, and $p_1 = p_2 = 0.5$.

Table 2

Solutions for Example 2*

p_1	p_2	y_1	y_2	e/q
.7000	.3000	1.4000	1.4000	.5755
.7000	.3000	1.3300	1.5633	.5755
.7000	.3000	1.2600	1.7267	.5755
.7000	.3000	1.1900	1.8900	.5755
.7000	.3000	1.1200	2.0533	.5755
.7000	.3000	1.0500	2.2167	.5755
.7000	.3000	.9800	2.3800	.5755
.7000	.3000	.9100	2.5433	.5756
.7000	.3000	.8400	2.7067	.5759
.7000	.3000	.7700	2.8700	.5765
.7000	.3000	.7000	3.0333	.5774
.7000	.3000	.6300	3.1967	.5785
.7000	.3000	.5600	3.3600	.5798
.7000	.3000	.4900	3.5233	.5815
.7000	.3000	.4200	3.6867	.5834
.7000	.3000	.3500	3.8500	.5856
.7000	.3000	.2800	4.0133	.5880
.7000	.3000	.2100	4.1767	.5908
.7000	.3000	.1400	4.3400	.5939
.7000	.3000	.0700	4.5033	.5972

Table 3

Solutions for Example 2*

p_1	p_2	y_1	y_2	e/q
.9500	.0500	1.4000	1.4000	.5755
.9500	.0500	1.3300	2.7300	.5755
.9500	.0500	1.2600	4.0600	.5755
.9500	.0500	1.1900	5.3900	.5755
.9500	.0500	1.1200	6.7200	.5755
.9500	.0500	1.0500	8.0500	.5755
.9500	.0500	.9800	9.3800	.5755
.9500	.0500	.9100	10.7100	.5756
.9500	.0500	.8400	12.0400	.5761
.9500	.0500	.7700	13.3700	.5769
.9500	.0500	.7000	14.7000	.5780
.9500	.0500	.6300	16.0300	.5795
.9500	.0500	.5600	17.3600	.5814
.9500	.0500	.4900	18.6900	.5836
.9500	.0500	.4200	20.0200	.5862
.9500	.0500	.3500	21.3500	.5892
.9500	.0500	.2800	22.6800	.5925
.9500	.0500	.2100	24.0100	.5963
.9500	.0500	.1400	25.3400	.6004
.9500	.0500	.0700	26.6700	.6050

*In Tables 2 and 3 other parameter values are $\hat{w} = 1.6$, $r = 0.6$, $\hat{y} = 1.4$, and $\psi = 0.85$.

Table 4

Solutions for Example 3

ψ	z^*	θ^*	$\tilde{z}$
.76	.1061	.4991	.9842
.80	.0999	.5014	1.0044
.85	.0932	.5039	1.0336
.90	.0873	.5060	1.0691
.95	.0821	.5079	1.1156
1.00	.0775	.5095	1.1869
1.01	.0766	.5098	1.2088
1.02	.0758	.5101	1.2375
1.03	.0750	.5104	1.2877
1.04	.0741	.5107	—
1.05	.0734	.5110	—
1.06	.0726	.5113	—
1.07	.0718	.5115	—
1.08	.0711	.5118	—

$\hat{w} = 1.5$, $\hat{y} = 1.4$, and $r = 0.75$.

Table 5

Solutions for Example 4

ψ	$\underline{\theta}$	θ^*	e/q
.76	.8592	.8776	.9976
.85	.8620	.8778	.9978
.95	.8643	.8780	.9981
1.05	.8661	.8782	.9983

$\hat{w} = 1.5$, $\hat{y} = 1.4$, $r = 1.2315$

Figure 1

THE FUNCTION π

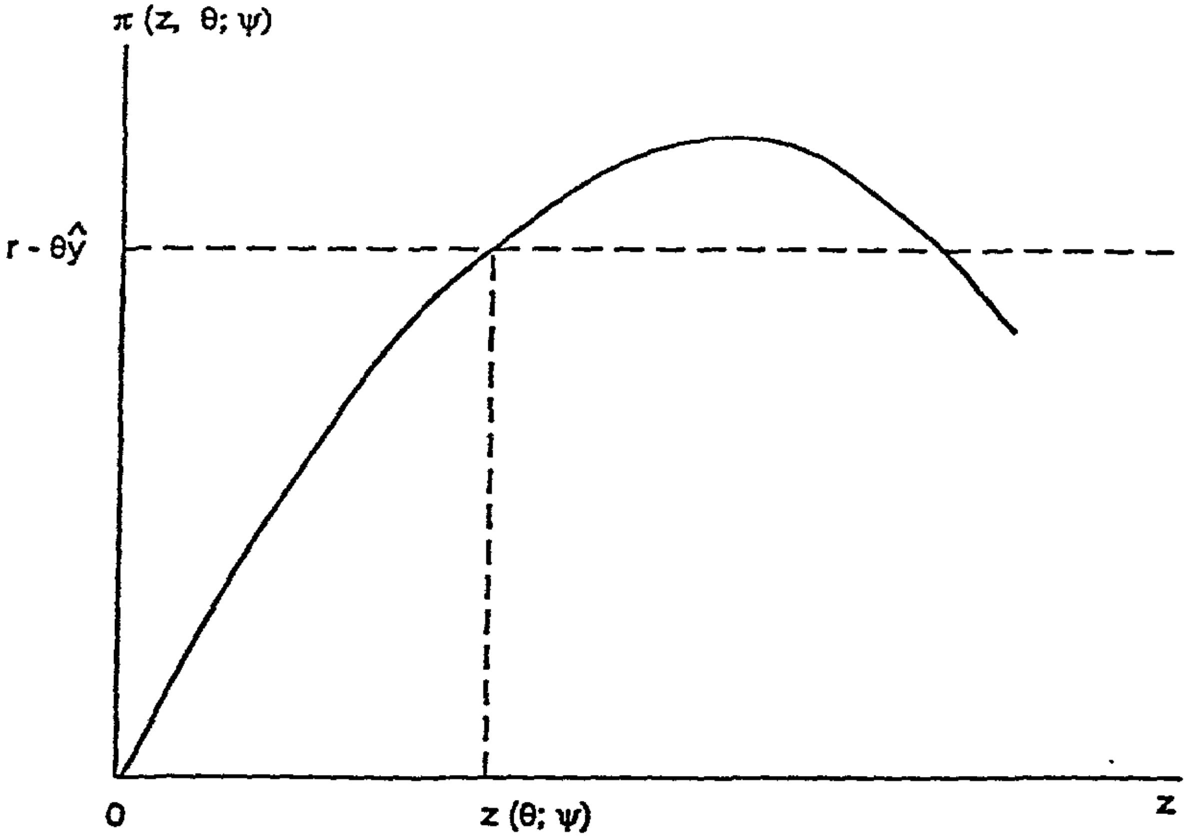

Figure 2

BORROWER'S OBJECTIVE FUNCTION

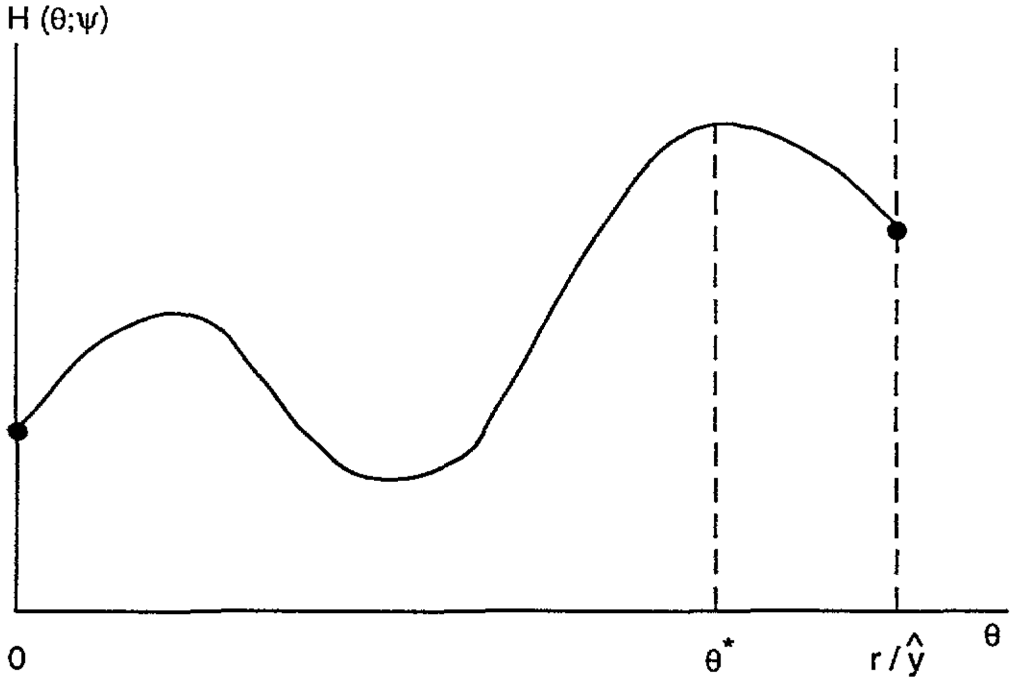

Figure 2.a

BORROWER'S OBJECTIVE FUNCTION

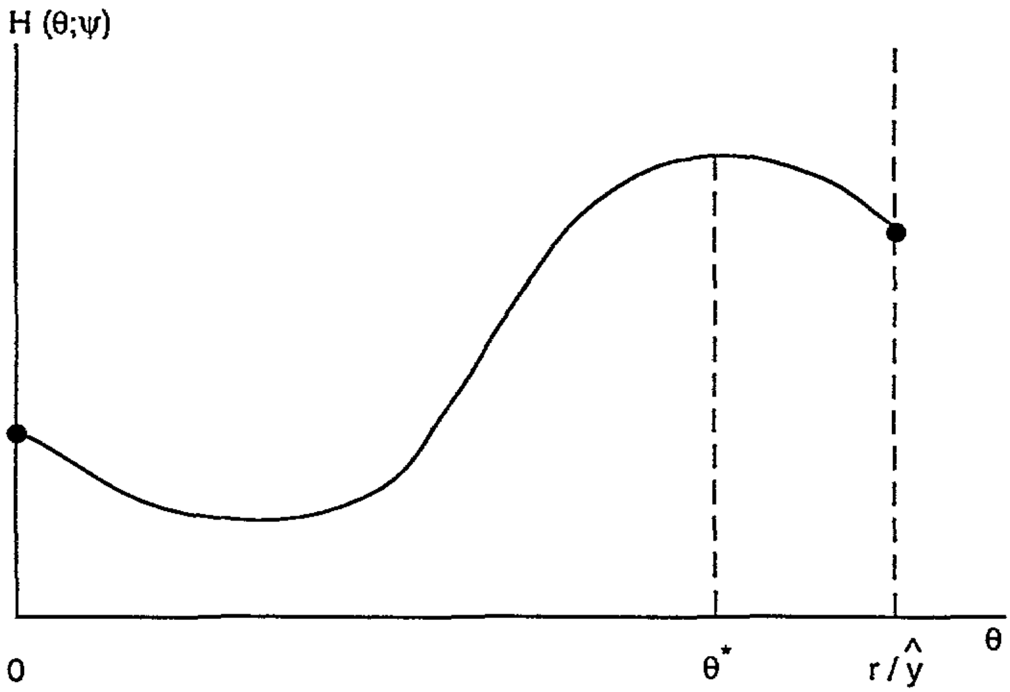

Figure 3

PAYMENTS TO DEBT AND EQUITY HOLDERS (y FIXED)

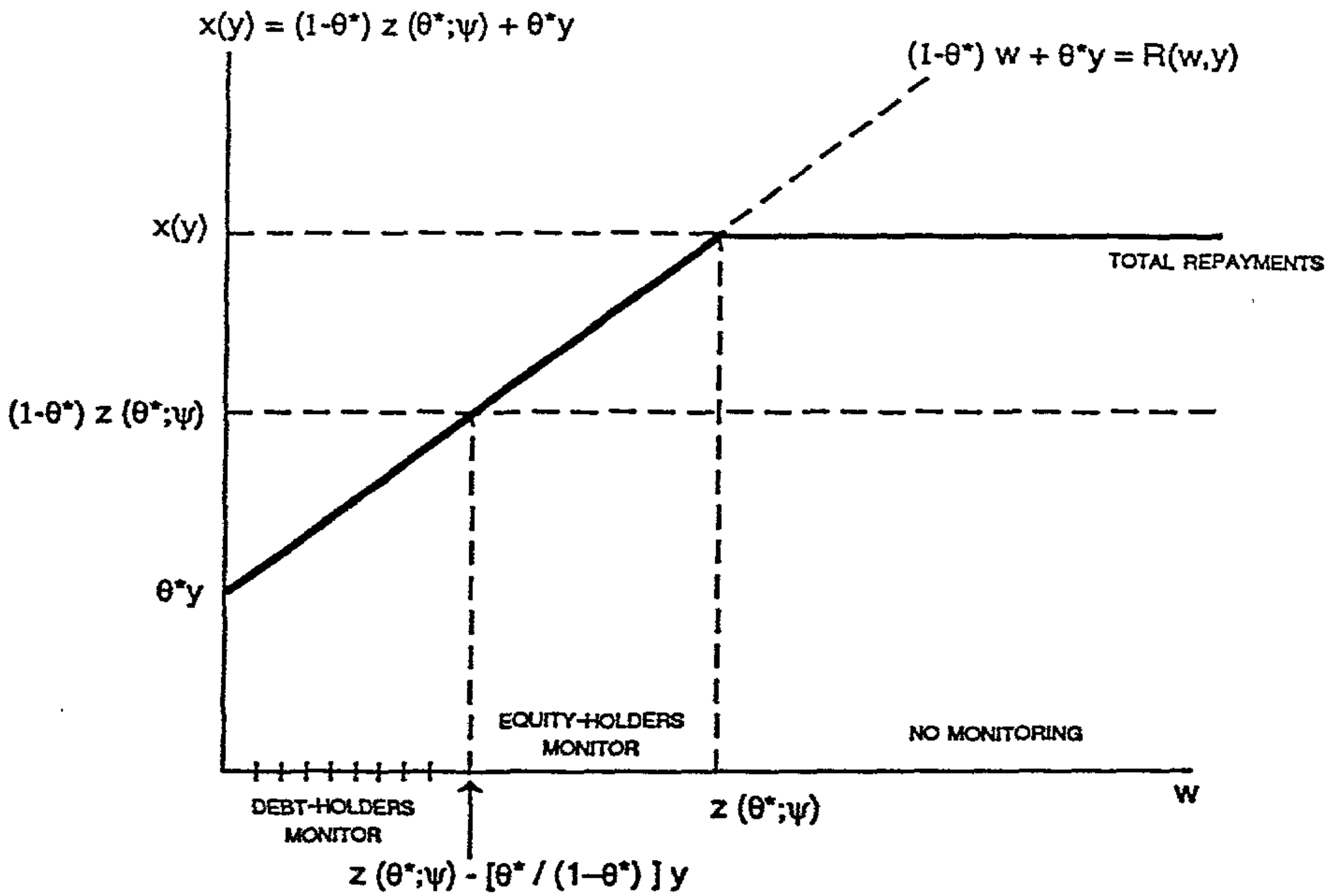

Figure 4

TRIANGULAR DISTRIBUTION