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Temporal Aggregation in a Multi-Sector Economy With Endogenous Growth

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ABSTRACT

We provide a theoretical treatment of temporal aggregation in models that exhibit long-term endogenously-generated steady growth; hence generalizing our previous analysis (*Econometrica* 62, 1994, pp. 635–56). We introduce the *property of steady-growth invariance*—that the long-term growth of the continuous-time economy not be affected by the discretization—which imposes consistency restrictions on the joint formulation of preferences and stock accumulation of the discrete-time approximation. We establish, under mild conditions, these restrictions in the form of necessary and sufficient conditions on the discretization.

Keywords: Temporal aggregation, endogenous growth, general equilibrium, intertemporal optimization

JEL Classification: C61, C63, C68, D58, O41

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1. Introduction

There appears to be little compelling theoretical reason for the widespread use of periodic analysis in economics. The main justification, besides analytical convenience, is presumably to be found in the necessary confrontation of model predictions with data, for which the sampling is typically periodic. This justification may not be entirely appropriate, however. Indeed, there is nothing to suggest that economic time series are collected at frequencies that enable the researcher to fully capture the movements of the economy. Neither is there any reason to believe that economic agents make decisions at fixed exogenously-specified intervals of time. If it is assumed that they do so, why would this interval of time coincide with the data sampling interval? Also, for many variables that are expensive to measure, data may be available at irregular dates only. This problem, known as temporal aggregation, is well recognized in particular in the time-series literature [e.g., Sims (1971), Geweke (1978), Marcet (1991)], and its potential perverse consequences on parameter estimates and hypothesis testing are amply documented in macroeconomics [e.g., Hansen and Sargent (1983), Christiano (1985), Christiano and Eichenbaum (1987), Christiano, Eichenbaum and Marshall (1991), Burdett, Coles and Van Ours (1994)].

To deal with this problem, researchers usually have resorted to the continuous-time framework as a limiting case which provides a natural benchmark. In macroeconomic time-series analysis, for instance, one uses the available discrete sampled data to infer the parameters of the underlying continuous-time generation process [e.g., Hansen and Sargent, (1980, 1981)]. Similar in essence is the problem of the formulation of discrete-time finite-horizon approximations of continuous-time infinite-horizon optimization programs. In a recent paper (Mercenier and Michel 1994a), we showed that this apparently simple exercise hides unexpected caveats overlooked in the applied literature and with potentially serious implications. Restricting our attention to optimal growth programs which have a steady state, we studied the conditions to be imposed on the discretization process for the property of steady-state invariance to hold; that is, the temporally aggregated problem is required to have the same steady state as the underlying continuous-time process. We established such conditions and showed that failure to take these restrictions into account introduces a dependency of the solution steady state (and, a fortiori, of the transitional dynamics) on the arbitrarily chosen sequence of time intervals.

The theoretical treatment of temporal aggregation in models that exhibit long-term endogenously-generated steady growth obviously requires even more care. In this paper, we extend our previous analysis (Mercenier and Michel 1994a) to this class of infinite-horizon optimization problems. More specifically, we introduce the *property of steady-growth invariance*--that the long-term growth of the continuous-time economy not be affected by the discretization--which imposes consistency restrictions on the joint formulation of preferences and stock accumulation of the discrete-time approximation. We establish, under mild conditions, these restrictions in the form of necessary and sufficient conditions on the discretization.

The results are potentially useful because of their generality. Firstly, they apply to multidimensional problems and are fit, therefore, for the analysis of an important class of general equilibrium models initiated by Uzawa (1965) and Lucas (1988) in which it is the interaction among technologies, that allow for the accumulation of physical and human capital, and consumers' preferences that determines endogenously the economy's rate of growth [see Caballé and Santos (1993) and Bond, Wang and Yip (forthcoming) for analyses of this class of models]. Secondly, our results apply to models in which there may be inequality restrictions on the decision set. Models of optimal growth with occasionally binding constraints have attracted considerable attention in recent years [see Christiano and Fisher (1994) and McGrattan (forthcoming)]. Thirdly, our results apply to models with converging externalities [such as those of Shell (1967), Romer (1986), Lucas (1988)]. If the interest is in equilibrium rather than in optimal allocations, then our results can be used after simple transformations of the problem [see Kehoe (1991) or Kehoe, Levine and Romer (1992)]. Finally, our analytical discussion fits into the nonlinear programming framework, for which robust and efficient algorithms are available [see Jones, Manuelli and Rossi (1993) for a recent advocation and use of such numerical techniques for solving endogenous-growth models].

In the next section, we present the continuous-time model and characterize its optimal solution and balanced growth. Section 3 deals with time-aggregation: we state and prove our theorem on the conditions for steady-growth invariance. In section 4 we discuss the case with externalities.

2. The Model

We consider the following continuous-time infinite-horizon multi-sectoral optimal-growth model with human capital:

$$\text{(P1)} \quad \begin{aligned} &\text{Max} \int_0^\infty e^{-\rho t}\, h(t)^a\, g(x(t),u(t))\, dt \\ &\text{s.t.} \quad \dot{x}(t) = f(x(t),u(t)), \ x(0) = x_0 \text{ given,} \\ &\qquad \frac{\dot{h}(t)}{h(t)} = \varphi(x(t),u(t)), \ h(0) = h_0 \text{ given,} \end{aligned}$$

where the state vector $x(t)$ is of dimension n, the vector of controls $u(t)$ is of dimension m, and the human capital $h(t)$ is one dimensional. This simple form of the class of endogenous-growth models considered by Lucas (1988) is obtained by trivial transformations of the original problem (see the appendix). Here, a balanced-growth path is characterized by constant values $\hat{x}$ and $\hat{u}$ of the state and control vectors $x(t), u(t)$; the endogenous rate of growth of human capital along this path is $\varphi(x,u)$. The assumption that functions $g(.)$, $f(.)$, and $\varphi(.)$ are continuously differentiable ensures that Pontryagin's maximum principle may be applied.

We now characterize the optimal balanced-growth path of this economy. Let $z = \log h$ so that $h^a = e^{az}$ and $\dot{h}/h = \dot{z}$. The discounted Hamiltonian associated with (P1) is

$$\widetilde{H}(x,u,z,t) = e^{-\rho t}\, e^{az}\, g(x,u) + \widetilde{p} f(x,u) + \widetilde{q}\,\varphi(x,u).$$

For paths $x^*(t)$, $u^*(t)$, and $z^*(t)$ to be optimal, it is necessary that

(i) the maximum of $\widetilde{H}(x^*(t), u, z^*(t), t)$ is reached at $u^*(t)$,

(ii) $\dot{\widetilde{p}}(t) = -\nabla_x \widetilde{H}(x^*(t),u^*(t),z^*(t),t)$,

(iii) $\dot{\widetilde{q}}(t) = -\nabla_z \widetilde{H}(x^*(t),u^*(t),z^*(t),t)$,

(iv) $\lim_{t\to\infty} \left(\widetilde{p}(t)x^*(t) + \widetilde{q}(t)z^*(t)\right) = 0$,

(v) and the dynamic constraints of (P1) are satisfied, i.e.,
$\dot{x}^*(t) = f(x^*(t),u^*(t))$, $x^*(0)$ given, and $\dot{z}^*(t) = \varphi(x^*(t),u^*(t))$, $z^*(0)$ given.

We now transform the problem so that the normalized shadow prices are constant along the optimal balanced-growth path. Define

$$p(t) = e^{\rho t} e^{-az^*(t)}\widetilde{p}(t), \qquad q(t) = e^{\rho t} e^{-az^*(t)}\widetilde{q}(t)$$

and the associated Hamiltonian

$$H(x,u) = g(x,u) + p f(x,u) + q\,\varphi(x,u).$$

The necessary conditions for optimality become

(i') the maximum of $H(x^*(t),u)$ is attained at $u^*(t)$,

(ii') $\dot{p}(t) = \left[\rho - a\,\varphi(x^*(t),u^*(t))\right] p(t) - \nabla_x H(x^*(t),u(t))$,

(iii') $\dot{q}(t) = \left[\rho - a\,\varphi(x^*(t),u^*(t))\right] q(t) - a\,g(x^*(t),u^*(t))$,

(iv') $\lim_{t\to\infty} e^{-\rho t} e^{az^*(t)}\big(p(t)x^*(t) + q(t)z^*(t)\big) = 0$,

(v') and stocks $x^*(t)$ and $z^*(t)$ move according to the differential constraints in (P1).

For constant values x^*, u^*, p^*, q^* of $x^*(t)$, $u^*(t)$, $p^*(t)$, and $q^*(t)$, the optimality conditions can be particularized further:

(i'') the maximum of $H^*(x^*,u) = g(x^*,u) + p^* f(x^*,u) + q^*\varphi(x^*,u)$ is at u^*,

(ii'') $\left[\rho - a\,\varphi(x^*,u^*)\right] p^* = \nabla_x H^*(x^*,u^*)$,

(iii'') $\left[\rho - a\,\varphi(x^*,u^*)\right] q^* = a\,g(x^*,u^*)$.

The equations governing the motion of stocks imply that $f(x^*,u^*) = 0$ and that the long-term growth rate of human capital is $\gamma^* = \varphi(x^*,u^*)$. The transversality condition imposes that $a\gamma^* < \rho$.

3. Temporal Aggregation

Our aim is to uncover the relationship that exists between the balanced-growth rates of (P1) and of its discrete-time version. More specifically, we look for conditions to be imposed on the discretization so as to preserve the asymptotic growth rate of the continuous-time formulation. We refer to this desirable property of the discretization as the *property of steady-growth invariance*.

Consider any strictly increasing infinite sequence of dates $t_0=0, t_1, \cdots, t_n, \cdots$ with limit equal to $+\infty$. We approximate the motion of $x(t)$ between two successive dates t_n and t_{n+1} by a difference equation of the form

$$x(t_{n+1}) - x(t_n) = \Delta_n f(x(t_n),u(t_n))$$

where Δ_n is a scalar factor that converts the continuous flow into a stock increment; we place no restriction on the choice of Δ_n.

<u>Theorem</u>

(a) The discrete-time infinite-horizon optimization problem (P2),

$$\text{(P2)}\qquad \begin{aligned} &\operatorname{Max} \sum_{n=0}^{\infty} \frac{\Delta_n}{y(t_{n+1})}\, g(x(t_n),u(t_n)) \\ &\text{s.t.}\quad x(t_{n+1}) - x(t_n) \;=\; \Delta_n f(x(t_n),u(t_n)), \qquad\qquad x(t_0) = x_0 \textit{ given}, \\ &\qquad\;\; y(t_{n+1}) \;=\; y(t_n)\left\{1+\Delta_n(\rho - a\varphi(x(t_n),u(t_n))\right\}, \quad y(t_0) = y_0 \textit{ given}, \end{aligned}$$

has the same stationary values (x^*, u^*) *and the same stationary shadow prices* (p^*, q^*) *as the continuous-time problem* (P1).

(b) Consequently, (P1) *and* (P2) *generate the same steady growth. In the discrete-time formulation,*

$$\log h^*(t_{n+1}) - \log h^*(t_n) \;=\; z^*(t_{n+1}) - z^*(t_n) \;=\; (t_{n+1} - t_n)\gamma^*$$

where $\gamma^* = \varphi(x^*, u^*)$.

Observe that in (P2), the optimal values of $x(t_n)$ and $u(t_n)$ do not depend on the initial condition y_0: we set $y_0{=}1$.

Proof. For convenience, we note x_n, u_n, and y_n as the values of x, u, and y at t_n. Write the Lagrangian of (P2) as

$$L \;=\; \sum_{n=0}^{\infty} \frac{1}{y_{n+1}} L_n$$

where

$$L_n \;=\; \Delta_n g(x_n,u_n) + p_n\,(\Delta_n f(x_n,u_n) + x_n - x_{n+1}) + q_n\,(\frac{y_{n+1}}{a y_n} - \frac{1}{a} - \frac{\Delta_n}{a}\rho + \Delta_n\,\varphi(x_n,u_n))\,.$$

The necessary conditions for a path (x_n^*, u_n^*, y_n^*) to be optimal are

(a) The maximum value of $L_n(x_n^*, u_n)$ with respect to u_n is attained at u_n^*; this value coincides with the max_{u_n} of $H_n(x_n^*, u_n)$ where

$$H_n(x_n,u_n) \;=\; g(x_n,u_n) + p_n f(x_n,u_n) + q_n\,\varphi(x_n,u_n)\,.$$

(b) The gradient of L with respect to x_n is null so that

$$0 = \frac{1}{y^*_{n+1}} \nabla_{x_n} L_n + \frac{1}{y^*_n} \nabla_{x_n} L_{n-1}.$$

Using $y^*_{n+1}/y^*_n = 1 + \Delta_n\big(\rho - a\,\varphi(x^*_n, u^*_n)\big)$, it follows that

$$p_{n-1}\left[1 + \Delta_n\big(\rho - a\,\varphi(x^*_n, u^*_n)\big)\right] = p_n + \Delta_n \nabla_{x_n} H_n(x^*_n, u^*_n).$$

(c) The derivative of L with respect to y_{n+1} equals zero

$$0 = \frac{1}{y^*_{n+2}} \frac{\partial L_{n+1}}{\partial y_{n+1}} - \frac{1}{y^{*\,2}_{n+1}} L_n + \frac{1}{y^*_{n+1}} \frac{\partial L_n}{\partial y_{n+1}}$$

which, after multiplication by $y^{*\,2}_{n+1}$, yields

$$\frac{q_{n+1}}{a} + L_n = \frac{y^*_{n+1}}{a y^*_n} q_n = \frac{1}{a} q_n \left[1 + \Delta_n\big(\rho - a\varphi(x^*_n, u^*_n)\big)\right].$$

With $L_n(x^*_n, u^*_n) = \Delta_n\, g(x^*_n, u^*_n)$, we get

$$q_{n+1} + a\,\Delta_n\, g(x^*_n, u^*_n) = q_n\left[1 + \Delta_n\big(\rho - a\,\varphi(x^*_n, u^*_n)\big)\right].$$

Specialized for the constant values $(x^*_n, u^*_n, p_n, q_n) = (x^*, u^*, p^*, q^*)$, these conditions are identical to *(i")*, *(ii")*, and *(iii"')* of the continuous-time problem (P1). Since the continuous-time and discrete-time economies both asymptotically grow at the same constant rate, the transversality condition $a\,\varphi(x^*, u^*) < \rho$ is satisfied.

If the optimal growth problem is concave, the necessary conditions for optimality, including the transversality condition, are also sufficient (see Remark 1 below).

Q.E.D.

Remark 1. As is well known, the necessary optimality conditions are also sufficient if the problem is *concave*. (P1) is obviously not concave, as e^z is a nonconcave function of z. However, under standard assumptions (see the appendix), the original multi-sectoral endogenous-growth model before its transformation into (P1) is a concave optimization problem for which the first order conditions are also sufficient. It is the case that the optimality conditions obtained after a change of variables are equivalent to those of the original problem.

Therefore, if the latter is concave, the necessary conditions used in the demonstration of the theorem are also sufficient conditions for optimality.

Remark 2. In a problem with additional constraints of the form $l_j(x(t),u(t)) = 0$ (resp. ≥ 0), the associated multipliers appear in the necessary conditions for optimality. Write these constraints as $\Delta_n l_j(x_n,u_n) = 0$ (resp. ≥ 0) in the discrete-time version so that the same stationary values are obtained for the multipliers as in the continuous-time case, and the theorem therefore applies.

Remark 3. The theorem provides an intuitive generalization to the first proposition of Mercenier and Michel (1994a). To see this, define $\alpha_n = 1/y(t_{n+1})$ and substitute in the equation governing the motion of y. Rearranging, we get

$$\alpha_{n+1} = \frac{\alpha_n}{1 + \Delta_{n+1}\,(\rho - a\varphi(x(t_{n+1}),u(t_{n+1})))}.$$

With $\varphi \equiv 0$--the exogenous-growth case--this is the recurrence relation that defines the discount factors in their first proposition.

4. Endogenous Growth With Externalities

Introducing *converging externalities* $X(t) = (x^*(t),u^*(t))$ into (P1) does not affect the conditions for steady-growth invariance. The modified continuous-time problem is

$$\text{(P1')}\quad \begin{aligned} &\text{Max} \int_0^\infty e^{-\rho t}\, h(t)^a\, g(x(t),u(t),X(t))\, dt \\ &\text{s.t.}\quad \dot{x}(t) = f(x(t),u(t),X(t)),\ x(0) = x_0 \text{ given,} \\ &\qquad\ \ \frac{\dot{h}(t)}{h(t)} = \varphi(x(t),u(t),X(t)),\ h(0) = h_0 \text{ given,} \end{aligned}$$

and its discrete-time form is

$$\text{(P2')}\quad \begin{aligned} &\text{Max} \sum_{n=0}^{\infty} \frac{\Delta_n}{y(t_{n+1})}\, g(x(t_n),u(t_n),X(t_n)) \\ &\text{s.t.}\quad x(t_{n+1}) - x(t_n) = \Delta_n f(x(t_n),u(t_n),X(t_n)), \qquad x(t_0) = x_0 \textit{ given,} \\ &\qquad\ \ y(t_{n+1}) = y(t_n)\left\{1+\Delta_n(\rho - a\varphi(x(t_n),u(t_n),X(t_n))\right\}, \qquad y(t_0) = y_0 \textit{ given.} \end{aligned}$$

$X(t)$ is held fixed in the optimization so the first order conditions are unchanged, and the equilibrium steady-growth path is determined after substitution of (x^*, u^*, x^*, u^*) for $(x^*(t), u^*(t), X(t))$ in these conditions.

The case of *non-converging externalities* requires some preliminary work before the theorem can be applied. In absence of a canonic form for this type of externality, we illustrate the point using Lucas' (1988) well known model:

$$\text{Max} \int_0^\infty e^{-\rho t} N(t)\, c(t)^a\, dt$$

$$\begin{aligned} \text{s.t.} \quad \dot{K}(t) &= F(K(t), N(t)u(t)H(t)\overline{H}(t)^{\gamma}) - N(t)C(t) - \mu K(t), \quad && K(0) = K_0 \text{ given}, \\ \dot{H}(t) &= \delta(1-u(t))H(t), && H(0) = H_0 \text{ given}, \end{aligned}$$

where N is population (assumed to grow at constant rate n), c is individual consumption, K and H are physical and human capital, u is the share of non-leisure time devoted to the production of goods and $1-u$ the share devoted to human capital accumulation, an overbar denotes an externality, $F(.)$ has constant returns, and γ, δ, and μ are positive scalars. Using the following transformations

$$\begin{aligned} h(t) &= N(t)\, H(t)\, \overline{H}(t)^{\gamma}, \\ x(t) &= K(t)/h(t), \\ v(t) &= c(t)/h(t), \end{aligned}$$

and rearranging, we get

$$\text{Max} \int_0^\infty e^{-(\rho-n)t}\, h(t)^a\, v(t)^a\, dt$$

$$\begin{aligned} \text{s.t.} \quad \dot{x}(t) &= F(x(t), u(t)) - (\mu+n)x(t) - \delta(1-u(t))x(t) - \gamma\delta(1-\overline{u}(t))x(t) - v(t), \\ \frac{\dot{h}(t)}{h(t)} &= n + \delta(1-u(t)) + \gamma\delta(1-\overline{u}(t)), \\ x(0) &= x_0 \text{ given}, \;\; h(0) = h_0 \text{ given}, \end{aligned}$$

which is now of the form (P1) so that the theorem applies. A parameterized version of this model is used to illustrate the effectiveness of the *property of steady-growth invariance* (see

Figure 1).[1] There, we report percentage aggregation errors on computed optimal values of $u(t_n)$ when using two alternative draws of unequally dense samples of irregularly spaced points on the time axis.[2] We see that the *property of steady-growth invariance* is clearly very effective in "protecting" the economy's long-term endogenous-growth rate from temporal-aggregation errors.

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1 A quarterly model serves as a benchmark. The horizon is truncated at 200 quarters, and post-terminal constant growth is assumed ever after. The economy is calibrated in steady-growth state and shocked by setting $x_0 = 2/3\, x^*$. The functional forms and parameter values used are

$$F(x^*,u^*) = x^{*\beta}\, u^{*(1-\beta)} \text{ with } \beta = .25,\ \beta \frac{F(x^*,u^*)}{x^*} = .25,\ u^* = .75,$$

$$a = .1,\ n = .01,\ \gamma = .01,\ \delta = .05/4,\ \rho = .04/4,\ \mu = .04/4.$$

Numerical optimization is performed using mathematical programming techniques.

2 The two grids have respectively 30 and 50 draws. For the sampling, we use the following formula:

$$t_n = \frac{\log\left(1 - \frac{n}{N+1}\right)}{\frac{1}{N}\log\left(\frac{1}{N+1}\right)},$$

which builds on a criterion we developed for exogenous growth models (Mercenier and Michel, 1994b).

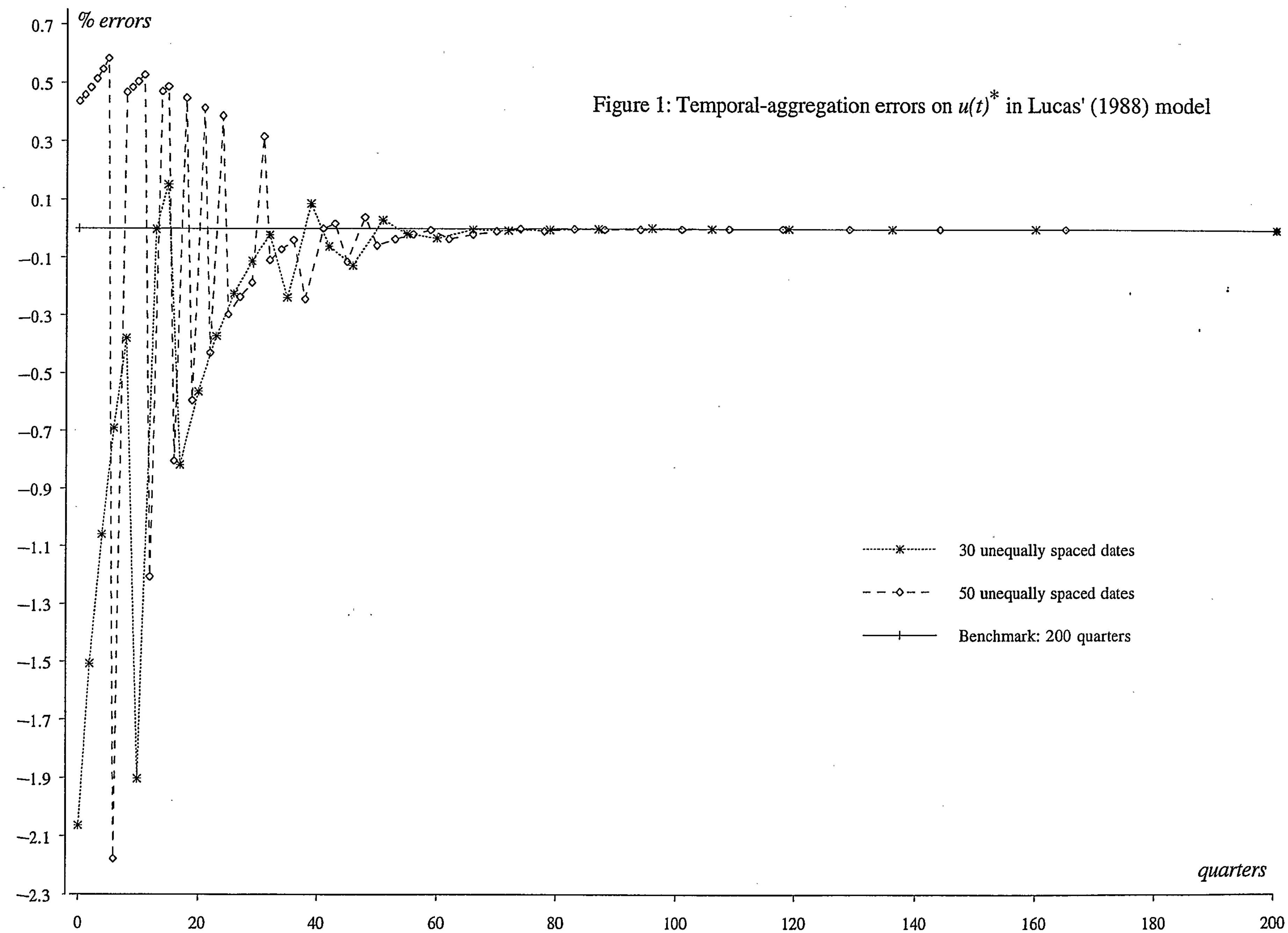
Figure 1: Temporal-aggregation errors on $u(t)^*$ in Lucas' (1988) model
% errors
quarters
30 unequally spaced dates
50 unequally spaced dates
Benchmark: 200 quarters
0.7
0.5
0.3
0.1
−0.1
−0.3
−0.5
−0.7
−0.9
−1.1
−1.3
−1.5
−1.7
−1.9
−2.1
−2.3
0
20
40
60
80
100
120
140
160
180
200

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Appendix: A Multi-Sectoral Optimal Endogenous-Growth Model

Consider the following multi-sectoral endogenous-growth model:

$$\text{Max} \int_0^\infty e^{-\rho t}\, U(C_1(t),\dots,C_m(t))\, dt$$

$$\begin{aligned}
\text{s.t.} \quad \dot{K}_i(t) &= V_i(t) - \mu_i K_i(t), && K_i(0) = K_{i0} \text{ given, } i=0,\dots,m,\\
\dot{H}(t) &= F_0(K_0(t),H_0(t)) - vH(t), && H(0) = H_0 \text{ given,}\\
F_j(K_j(t),H_j(t)) &= C_j(t) + \sum_{i=1}^{m} \eta_{ji}\, V_i(t), && j=1,\dots,m,\\
H(t) &= \sum_{i=0}^{m} H_i(t).
\end{aligned}$$

Physical capital K_i is sector specific, whereas human capital H is assumed mobile. Depreciation is exponential at rates μ_i, v. The technologies $F_i(.)$ for producing goods and human capital have constant returns. Output serves for consumption and investment. Instantaneous utility $U(.)$ is homogeneous of degree a>0.

We express all variables per unit of human capital and use lowercase letters. In these new notations, the model becomes

$$\text{Max} \int_0^\infty e^{-\rho t}\, H(t)^a\, U(c_1(t),\dots,c_m(t))\, dt$$

$$\begin{aligned}
\text{s.t.} \quad \dot{k}_i(t) &= v_i(t) - (\mu_i + F_0(k_0(t),h_0(t)) - v)\, k_i(t), && k_i(0) = k_{i0} \text{ given, } i=0,\dots,m,\\
\frac{\dot{H}(t)}{H(t)} &= F_0(k_0(t),h_0(t)) - v, && H(0) = H_0 \text{ given,}\\
F_j(k_j(t),h_j(t)) &= c_j(t) + \sum_{i=1}^{m} \eta_{ji}\, v_i(t), && j=1,\dots,m,\\
1 &= \sum_{i=0}^{m} h_i(t).
\end{aligned}$$

The variables $c_j(t)$ and $h_0(t)$ may be substituted out by making use of the market equilibrium conditions. This yields a model of the form (P1) in the text, with

$$\begin{aligned}
x &= (k_0,k_1,\cdots,k_m),\\
u &= (h_1,\cdots,h_m,\ v_1,\cdots,v_m).
\end{aligned}$$