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Technology Diffusion and Growth*

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ABSTRACT

Suppose firms are subject to decreasing returns and permanent idiosyncratic productivity shocks. Suppose also firms can only stay in business by continuously paying a fixed cost. New firms can enter. Firms with a history of relatively good productivity shocks tend to survive and others are forced to exit. This paper identifies assumptions about entry that guarantee a stationary firm size distribution and lead to balanced growth. The range of technology diffusion mechanisms that can be considered is greatly expanded relative to previous work. High entry costs slow down the selection process and imply slow aggregate growth. They also push the firm size distribution in the direction of Zipf's law.

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1. Introduction

This paper describes a competitive economy in which aggregate productivity growth is endogenous and driven by firm-level experimentation, selection, and imitation by new entrants. The structure of the economy combines elements of Luttmer [2007, 2010], who in turn builds on an extensive literature on size distributions.[1] The assumption of perfectly competitive markets is a simplifying assumption that shuts down idiosyncratic firm product demand as a source of heterogeneity. The focus is instead on highlighting the types of assumptions about entry that can ensure the existence of a stationary firm size distribution and a balanced growth path when the process of individual firm productivity growth is highly non-stationary.

There are two ingredients. First, there is a mechanism by which new entrants can benefit from the successes of incumbent firms.[2] The assumption here is that entering firms can improve on the technology used by firms at the very low end of the incumbent productivity distribution. Firms can become really productive only through stochastic post-entry improvements in productivity. Second, as in Luttmer [2010], the supply of new firms created by entrepreneurs is not perfectly elastic with respect to the market value of new firms. This ensures that the growth rate of the economy and the number of firms are jointly determined by an entry condition and a labor market clearing condition. This forces average firm demand for labor to be finite at the equilibrium growth rate. This is typically not guaranteed if a zero-profit entry condition by itself is sufficient to compute the equilibrium growth rate of the economy. In such an economy, a balanced growth path may fail to exist even if a stationary firm size distribution can be constructed.

The employment size distribution of US firms is quite stable over time. Parametric approximations of this distribution (in most studies, a Pareto or Fréchet distribution, or something very similar) are very close to implying an infinite mean. With the technology diffusion and entry assumptions just described, this happens naturally whenever the initial technology available to entrants is sufficiently unproductive or entry is sufficiently costly. Without the equilibrating forces implied by these assumptions, otherwise arbitrary restrictions on firm and aggregate growth would be needed to account for this phenomenon.

[1] Classic models of the firm size distribution are Simon and Bonini [1958], Steindl [1965], Lucas [1978]. Gabaix [1999] gives an interpretation of Zipf's Law for cities.

[2] The mechanism will be an externality in this paper, but it does not have to be. See Boldrin and Levine [2002, 2009].

This paper emphasizes the combination of trial and error and selection as an important driving force of aggregate growth. The focus is on the role of selection at the level of populations of firms or organizations. Alchian [1950] argued for the importance of trial and error, imitation, and selection in understanding the behavior of producers. Nelson and Winter [1982] describe models of growth based on selection. The classic paper on selection at the industry level is Jovanovic [1982]. His firms learn about a fixed productivity parameter, and those who learn they are productive remain in the industry, while those who learn they are not will exit. Selection is a transitory phenomenon. Here firms are subject to new shocks all the time, resulting in perpetual selection and growth.

Consumers and incumbent firms are introduced in Sections 2 and 3. The key technical material about stationary size distributions is contained in Section 3.1. It is more explicit about entry than previous treatments. Section 4 shows that an economy in which entry productivity is exogenous may not have a balanced growth path. Section 5 does the same for an economy with endogenous technological progress and entry decisions that are perfectly elastic with respect to firm value. Section 6 provides a simple remedy. Section 7 presents a calibration and Section 8 concludes.

2. Consumers

The population of consumers is $H_t = He^{\eta t}$ and everyone has one unit of labor. Population growth is non-negative. There is one consumption good at all times. Dynastic preferences over per-capita consumption flows c_t are

$$\int_0^\infty e^{-\rho t} \ln(c_t)\mathrm{d}t.$$

The subjective discount rate ρ may depend on η. The use of logarithmic utility here is mostly for simplicity. More general homothetic preferences can be considered.

Everyone is a price taker and subject to a standard dynastic present-value budget constraint. Throughout, the focus will be on balanced growth. Along a balanced growth path, per-capita consumption and wages are

$$[c_t, w_t] = [c, w]e^{\kappa t},$$

and the resulting interest rate is $r = \rho + \kappa$. It is assumed that $\rho > \eta$ so that the present value of aggregate consumption is finite.

3. Incumbent Firms

A firm is a technology for producing consumption goods using labor that is subject to decreasing returns to scale. Firms are the same except for a productivity index. This productivity index changes continuously, as a result of firm-specific random shocks.

3.1 Type-z Firms

A type-z firm at time t can produce zl^β units of output with l units of labor, where $\beta \in (0,1)$. Type-z firms at time t solve

$$v_t[z] = \max_l \left\{ zl^\beta - w_t l \right\}.$$

This behaves like a profit function. Since z multiplies a production function exhibiting decreasing returns, $v_t[z]$ is a convex function of z. Let $l_t[z]$ and $y_t[z]$ be the optimal levels of employment and output. Then

$$\begin{bmatrix} l_t[z] \\ v_t[z]/w_t \\ y_t[z]/w_t \end{bmatrix} = \frac{1}{\beta} \begin{bmatrix} \beta \\ 1-\beta \\ 1 \end{bmatrix} \left(\frac{\beta z}{w_t} \right)^{1/(1-\beta)}.$$

Output and labor inputs are also convex functions of z. Fixing the number of firms and the aggregate labor supply, a mean-preserving spread of productivity will raise aggregate output and wages in this economy.

To survive, a firm must incur a flow cost of λ_{F} units of labor. Interruption of this cost causes its productivity to permanently collapse to zero. Technology is like the volatile memory of a computer. It will be convenient to define $s_t[z]$ via

$$e^{s_t[z]} = \frac{v_t[z]}{\lambda_{\mathrm{F}} w_t} = \frac{1}{\lambda_{\mathrm{F}}} \frac{1-\beta}{\beta} \left(\frac{\beta z}{w_t} \right)^{1/(1-\beta)}.$$

With this definition, firm employment and profits can be written as

$$\begin{bmatrix} w_t l_t[z] + w_t \lambda_{\mathrm{F}} \\ v_t[z] - w_t \lambda_{\mathrm{F}} \end{bmatrix} = w_t \lambda_{\mathrm{F}} \begin{bmatrix} \left(\frac{\beta}{1-\beta} \right) e^{s_t[z]} + 1 \\ e^{s_t[z]} - 1 \end{bmatrix}. \tag{1}$$

Thus $s_t[z] = 0$ corresponds to zero profits. Clearly, employment and profitability are perfectly correlated for this technology, and thus $s_t[z]$ can be viewed as a measure of both. The fact that employment, profits, and output are convex functions of z makes them convex functions of $s_t[z]$ as well.

3.2 Productivity Dynamics

New firms can enter with an initial productivity given by $Z_t = Ze^{\kappa t}$. The cost of entry and the determination of Z and κ are described in later sections. Following entry, the productivity of a particular time-t entrant evolves with age according to

$$Z_{t,a} = Z_t \exp\left(\theta a + \sigma_{\mathrm{Z}} W_a\right),$$

where W_a is a standard Brownian motion that is independent across firms. As noted, $Z_{t,a}$ drops permanently to zero if the firm stops paying the fixed cost. The parameters θ and $\sigma_{\mathrm{Z}} > 0$ are taken as exogenous throughout.

Since productivity and wages grow at the same rate κ, $s_t[Z_t]$ is constant over time. Define $S = s_t[Z_t]$. Then

$$e^S = \frac{1}{\lambda_{\mathrm{F}}} \frac{1-\beta}{\beta} \left(\frac{\beta Z}{w}\right)^{1/(1-\beta)}. \tag{2}$$

Entrant employment and profitability are inversely related to the level of wages in the economy. Since $w_t l_t[z] = \beta y_t[z]$, a stationary firm employment distribution combined with a number of firms that grows at the same rate η as aggregate employment results in per-capita output that grows at the same rate κ as wages.

The fact that wages trend with entry productivity also implies that $s_t[Z_{t-a,a}]$ only depends on a and not on t. Thus the state of a firm of age a is $s_a = s_t[Z_{t-a,a}]$, and this evolves with firm age according to $s_a = S + [(\theta - \kappa)a + \sigma_{\mathrm{Z}} W_a]/(1-\beta)$. Along a balanced growth path, there is no aggregate state to keep track of. It will be convenient to write

$$\begin{bmatrix} \mu \\ \sigma \end{bmatrix} = \frac{1}{1-\beta} \begin{bmatrix} \theta - \kappa \\ \sigma_{\mathrm{Z}} \end{bmatrix}$$

so that

$$s_a = S + \mu a + \sigma W_a,$$

as long as the firm does not stop paying the fixed cost.

3.3 Exit

Apart from static production decisions, the only choice the firm faces is whether or not to continue. Because of (1), the value of a firm of size s can be written as $w_t \lambda_{\mathrm{F}} V(s)$. The value function $V(s)$ is given by

$$V(s) = \sup_{\tau} \mathrm{E}_0 \left[\int_0^{\tau} e^{-\rho a}(e^{s_a} - 1)\mathrm{d}a \right],$$

where $s_0 = s$, and τ is a stopping time that depends on the observed history of s_a. The value of the firm is finite if and only if $\rho > \mu + \sigma^2/2$, which says that the present value of $\{\mathrm{E}_0[e^{s_a}]\}_{a \geq 0}$ discounted at the rate ρ is finite. The solution to the stopping problem is to exit when s reaches an exit barrier B, defined by

$$e^B = \frac{\xi}{1+\xi}\left(1 - \frac{\mu + \sigma^2/2}{\rho}\right), \quad \xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\rho}{\sigma^2/2}}. \tag{3}$$

The resulting value function is

$$V(s) = \frac{1}{\rho}\frac{\xi}{1+\xi}\left(e^{s-B} - 1 - \frac{1 - e^{-\xi(s-B)}}{\xi}\right) \tag{4}$$

for all $s \geq B$ and $V(s) = 0$ otherwise (Dixit and Pindyck [1994], Luttmer [2007]). Observe that this value function, measured in units of labor, only depends on ρ/σ^2 and μ/σ^2.

Remark *The exponent ξ and the value function $V(\cdot)$ are increasing in μ. The exit barrier B is decreasing in μ.*

Rapid firm growth lowers the exit barrier B and raises the value of a firm. The level of productivity X_t at which exit at time t takes place can be written as

$$X_t = Z_t e^{-(1-\beta)(S-B)}. \tag{5}$$

The exit level of productivity trends up with entry productivity and wages. Firms that cannot keep up are driven out of business.

3.4 The Size Distribution

There is a continuum of firms measured by N_t at time t. Assuming that $S > B$, firms can enter at $s_0 = S$, evolve according to $\mathrm{d}s_a = \mu \mathrm{d}a + \sigma \mathrm{d}W_a$, and then exit if s_a reaches B. Without entry, there is a continuous flow of firms that exit at B, and the number of firms declines. Sufficiently large entry rates result in a rising number of firms. The following gives conditions under which a stationary size distribution exists and describes the relation between the entry rate ε and the resulting growth rate ω of the number of firms. Along the balanced growth paths to be constructed later, the number of firms has to grow at the same rate η as aggregate employment.[3]

[3] An example in Luttmer [2010] has balanced growth with no entry and a non-stationary size distribution that spreads out forever. But in that example, there are no fixed costs and all firms are equally productive.

3.4.1 Constant Entry Rates

Suppose there is a non-negative flow εN_t of new firms entering at time t. Conjecture that there is a stationary size distribution with density f so that the number of firms grows at some constant rate ω. It can be shown that $\frac{1}{2}\sigma^2 \mathrm{D}f(B)$ measures the flow of firms crossing the exit boundary B. The entry rate must therefore satisfy

$$\varepsilon = \omega + \frac{1}{2}\sigma^2 \mathrm{D}f(B). \tag{6}$$

The density f and ω must also satisfy the Kolmogorov forward equation

$$\omega f(s) = -\mu \mathrm{D}f(s) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(s), \quad s \in (B, S) \cup (S, \infty), \tag{7}$$

and

$$\lim_{s \uparrow S} f(s) = \lim_{s \downarrow S} f(s), \tag{8}$$

and

$$f(B) = f(\infty) = 0, \tag{9}$$

and f must integrate to 1. Integrating (7) using (8) and (9) gives

$$\omega = \frac{1}{2}\sigma^2 \left[\mathrm{D}_- f(S) - \mathrm{D}f(B) - \mathrm{D}_+ f(S) \right].$$

The entry condition (6) is therefore equivalent to

$$\varepsilon = \frac{1}{2}\sigma^2 \left[\mathrm{D}_- f(S) - \mathrm{D}_+ f(S) \right]. \tag{10}$$

As a result of entry, f will have a kink at S.

The differential equation (7) is linear and homogeneous with constant coefficients. On (B, S) and (S, ∞), it has solutions that are linear combinations of $e^{-\alpha s}$ and $e^{\alpha_* s}$, where

$$\alpha = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\omega}{\sigma^2/2}}, \quad \alpha_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\omega}{\sigma^2/2}}. \tag{11}$$

If α and α_* are complex, then any real-valued linear combination of $e^{-\alpha s}$ and $e^{\alpha_* s}$ will change signs indefinitely on (S, ∞). If α is real and non-positive then α_* is real and non-negative. In that case, no linear combination of $e^{-\alpha s}$ and $e^{\alpha_* s}$ converges to zero as s becomes large. Stationary densities can be constructed only if α is positive, or

$$\omega \geq -\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \text{ if } \mu < 0, \quad \omega > 0 \text{ if } \mu \geq 0. \tag{12}$$

This defines a lower bound on how fast a stationary population of firms can decline. There can be no stationary distribution with a declining population of firms if μ is non-negative.

A Zero Entry Rate Without entry, the population of firms can only decline, and thus μ must be negative for a stationary distribution to exist. At $\varepsilon = 0$, (10) implies that f is differentiable at S. Solving (7) with the boundary conditions $f(B) = 0$ and $f(\infty) = 0$ gives

$$f(s) = \alpha(-\alpha_*)e^{-\alpha(s-B)} \left(\frac{e^{(\alpha+\alpha_*)(s-B)} - 1}{\alpha + \alpha_*} \right). \tag{13}$$

This is a well defined density if and only if $\alpha > -\alpha_* > 0$. Given that μ is negative, this will be the case for any ω in $[-(\mu/\sigma)^2/2, 0)$. Thus there is a multiplicity of ω and associated stationary densities f. At the boundary $\omega = -(\mu/\sigma)^2/2$ the density (13) becomes $f(s) = \alpha^2(s - B)e^{-\alpha(s-B)}$, with $\alpha = -\mu/\sigma^2$. An argument given in Luttmer [2007] indicates that this is the limiting distribution when the initial size distribution has a compact support.[4]

A Positive Entry Rate With positive entry, (10) implies that f is not differentiable at S. Thus (7) defines two second-order differential equations, one on (B, S) and one on (S, ∞). Conditions for the endpoints are given in (8) and (9), except that the level of $f(S)$ is unrestricted. But f integrates to 1. Ignoring (6) and (10), this implies

$$f(s) = \frac{\alpha\alpha_* e^{-\alpha(s-B)}}{e^{\alpha_*(S-B)} - 1} \min\left\{ \frac{e^{(\alpha+\alpha_*)(s-B)} - 1}{\alpha + \alpha_*}, \frac{e^{(\alpha+\alpha_*)(S-B)} - 1}{\alpha + \alpha_*} \right\} \tag{14}$$

for all $s \geq B$. This is a well defined density as long as $\alpha > 0$, which is equivalent to (12). Conditional on $s \geq S$, the resulting distribution for e^s is Pareto, and α is its tail index.

The ω and f that correspond to a particular entry rate $\varepsilon > 0$ can be determined by imposing (6) or (10). From (14),

$$\frac{1}{2}\sigma^2 \mathrm{D}f(B) = \frac{\frac{1}{2}\sigma^2\alpha\alpha_*}{e^{\alpha_*(S-B)} - 1}, \tag{15}$$

which is positive because α is positive. Imposing (6) and simplifying gives

$$\varepsilon = \frac{\omega}{1 - e^{-(S-B)\left(\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\omega}{\sigma^2/2}}\right)}}. \tag{16}$$

Only real solutions for ω can be interpreted as growth rates. Figure I shows (16) for various μ. For $\mu < 0$ the $\varepsilon = 0$ entry rate that corresponds to (13) is also indicated.

[4] If firms replicate themselves at a rate ε and the population of firms grows at the rate η as a result, then the size density will satisfy (7) with $\omega = \eta - \varepsilon$. This is an alternative interpretation for (13), and the stability argument gives $\eta = \varepsilon - (\mu/\sigma)^2/2$.

The right-hand side of (16) is strictly increasing in ω and approaches ω from above for large ω. Exit at B becomes negligible if ω is large. If μ is non-negative, then the right-hand side of (16) starts at 0 when $\omega = 0$. Thus a positive entry rate implies positive growth rate ω, and it is unique. If μ is negative, then the right-hand side of (16) attains a positive minimum when $\omega < 0$ reaches the lower bound (12). The entry rate has to be large enough. If it is, there is a unique ω, and this growth rate turns positive when ε rises above $(S - B)/(-\mu)$.

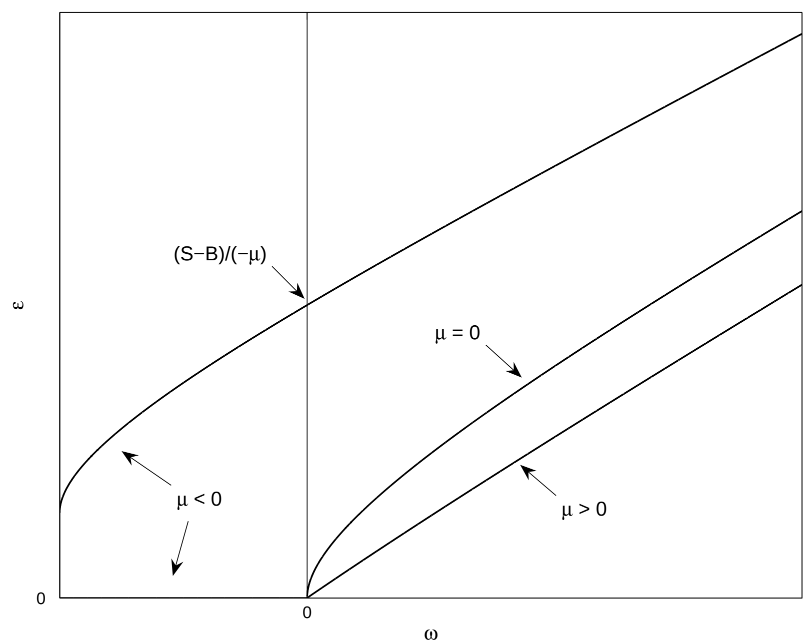

FIGURE I Entry and Growth of the Number of Firms

Conversely, given a growth rate ω and an associated stationary density f, one can simply compute ε from (6). The conditions for existence of a stationary density f given ω can be summarized as follows.

PROPOSITION 1 *Assume $S > B$. If ω satisfies (12) then there is a stationary density f for which the number of firms grows at the rate ω. The associated entry rate ε is given by (6). If $\omega < 0$ then there are two stationary densities, one of which implies $\varepsilon = 0$. If $\omega \geq 0$ then the stationary density is unique.*

It is possible to take a limit in (14) as $S \downarrow B$, holding ω fixed. This yields $f(s) = \alpha e^{-\alpha(s-B)}$ for any $s > B$. This means that the limiting distribution of e^s on $[B, \infty)$ is Pareto. The expression for the exit rate (15) implies that $\frac{1}{2}\sigma^2 \mathrm{D}f(B) \to \infty$ as $S \downarrow B$, and hence the required amount of entry ε is infinite as well. The above limiting density is also the density of a regulated Brownian motion on $[B, \infty)$. The infinite entry rate matches the fact that the regulator process needed to keep this Brownian motion above B is not a differentiable function of time (see Harrison [1985]).

3.5 The Mean of e^s

Variable firm employment is proportional to e^s. Given a positive measure of firms, e^s has to have a finite mean, or else aggregate employment would be infinite. Since $f(s)$ behaves like $e^{-\alpha s}$ for large s, e^s has a finite mean if and only if $\alpha > 1$. If there is positive entry at S,

$$\int_B^\infty e^{s-B} f(s) \mathrm{d}s = \frac{\alpha \alpha_*}{(\alpha - 1)(\alpha_* + 1)} \frac{e^{(S-B)(\alpha_*+1)} - 1}{e^{\alpha_*(S-B)} - 1}, \tag{17}$$

and this diverges as $\alpha \downarrow 1$. If $\omega = 0$ this simplifies to $(\alpha/(\alpha - 1)) \times (e^{S-B} - 1)/(S - B)$, which is decreasing in α, and therefore increasing in μ. Calculations reported in Appendix A show that this holds for $\omega \neq 0$ as well.

LEMMA 1 *Holding fixed B and S, the mean of e^s is increasing in μ.*

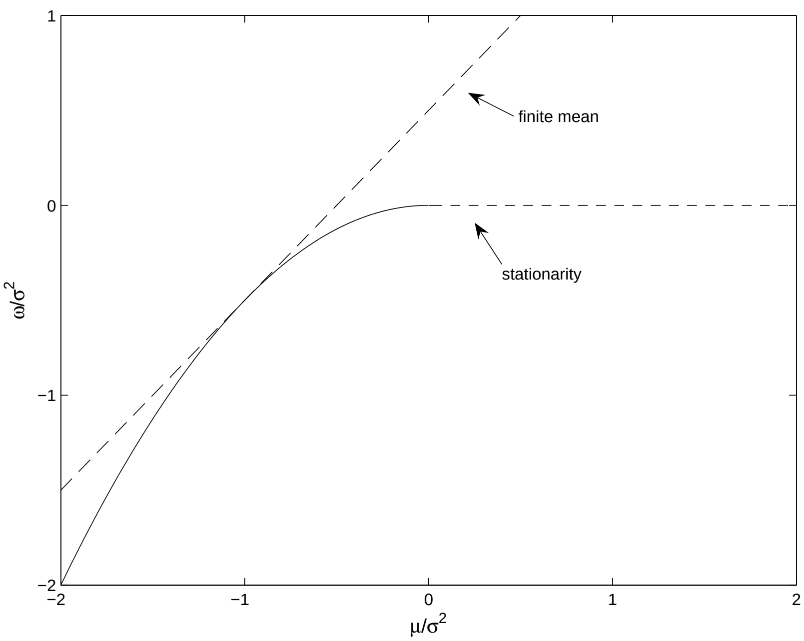

FIGURE II Conditions for Stationarity and a Finite Mean

The restriction $\alpha > 1$ is equivalent to

$$\omega > \mu + \frac{1}{2}\sigma^2. \tag{18}$$

The condition (12) for stationarity and the condition for a finite mean given in (18) are shown in Figure II. The right-hand side of (18) corresponds to the right-hand side of (12) at $\mu/\sigma^2 = -1$, but one boundary is linear in μ and the other a quadratic in μ. Furthermore, (12) can hold with equality while (18) has to hold strictly. Thus (18) is a stronger condition (12) for any μ. Note that

$$\lim_{\Delta \downarrow 0} \mathrm{E}_a \left[e^{s_{a+\Delta} - s_a} \right] = \mu + \frac{1}{2}\sigma^2$$

for any $s_a > B$. Thus (18) says that the number of firms must grow faster than variable employment at incumbent firms that are not about to exit. Clearly, for given μ and σ^2, it is a condition that can be met by having the number of firms grow fast enough.

4. Entry and Technology Adoption

Suppose now that entry productivity $Z_t = Ze^{\kappa t}$ is completely exogenous. Since θ is also exogenous, this implies that μ is an exogenous parameter.

Although anyone can access Z_t, to set up new firms does take labor and time. Applying λ_{E} units of labor continuously generates a new firm following an exponentially distributed waiting time with mean 1. Suppose the project of setting up a firm is initiated at time t and leads to success at time $t + \tau$. The cost of labor is $w_{t+a}\lambda_{\mathrm{E}}$ per unit of time, $a \in [0, \tau]$, and the value of the firm will be $w_{t+\tau}\lambda_{\mathrm{F}}V(S)$ when it arrives at time $t + \tau$. The interest rate is $r = \rho + \kappa$ and wages grow at the rate κ. The present-value of the project is thus $(\lambda_{\mathrm{F}}V(S) - \lambda_{\mathrm{E}})/(1 + \rho)$. Since anyone can start a project, these profits cannot be positive.

Without entry, the number of firms has to decline. If the distribution of firms is stationary, then aggregate firm employment declines as well. But we are assuming that the population grows at a non-negative rate η and everyone has one unit of labor. This means there can be no balanced growth path without entry. Thus entry is positive, and the profits from a project to start a firm must be zero. That is,

$$\lambda_{\mathrm{E}} = \lambda_{\mathrm{F}}V(S). \tag{19}$$

Since $V(s)$ is increasing and convex, with an asymptote that behaves like e^s, a unique solution for S is guaranteed. This solution varies with entry and fixed costs only to the extent that $\lambda_{\mathrm{E}}/\lambda_{\mathrm{F}}$ is affected.

If the number of firms N_t grows at some rate ω that satisfies (18), then so does the amount of labor $\lambda_{\mathrm{E}}\varepsilon N_t$ required to set up new firms, as well as the amount of labor employed by all firms. The only way this can be part of a balanced growth path is if $\omega = \eta$. The resulting entry rate ε is then determined by (6). Recall from (1) that a firm with productivity z employs $\lambda_{\mathrm{F}} + l_t[z]$ units of labor. Labor market clearing therefore requires

$$\frac{H}{N} = \lambda_{\mathrm{E}} \left(\eta + \frac{1}{2}\sigma^2 \mathrm{D} f(B) \right) + \lambda_{\mathrm{F}} \left(1 + \frac{\beta}{1-\beta} \int_B^\infty e^s f(s) \mathrm{d}s \right). \tag{20}$$

A balanced growth path can now be constructed as follows. The density f is defined by (11) and (14), evaluated at $\omega = \eta$, and given B and S. The exit boundary B is determined by (3) and the entry size S is determined by the zero-profit condition (19). The labor market clearing condition (20) then determines the number of firms. The level of wages follows from S and its definition (2).

This construction works only as long as (18) holds for $\omega = \eta$. Without this condition, the right-hand side of (20) would not be finite, and this is inconsistent with a positive measure of firms.

5. Endogenizing Growth

If (18) is violated at $\omega = \eta$, there can be no balanced growth path. Condition (18) depends on parameters that have so far all been taken as exogenous. In this section, the growth rate κ of the entry productivity of new firms is endogenous. This makes $\mu = (\theta - \kappa)/(1-\beta)$ endogenous and sets the stage for finding equilibrium mechanisms that ensure the existence of a balanced growth path.[5]

5.1 A Spillover at the Bottom

Continue to assume that the drift of incumbent productivity is given exogenously by θ. As before, suppose that entrepreneurs can expend a flow of λ_{E} units of labor to create new firms at a unit rate. Rather than assuming entrant productivity grows exogenously, suppose that an entrepreneur who creates a new firm can copy the technology of firms that are about to exit, and improve the productivity of this technology by a factor $\Gamma^{1-\beta}$, for some $\Gamma > 1$. The resulting entry productivity is

$$Z_t = \Gamma^{1-\beta} X_t.$$

As in Arrow [1962], this introduces a knowledge spillover. In contrast to the quality-ladder model of Grossman and Helpman [1991], new firms here do not enter at the top

[5]See Atkeson and Burstein [2007] for a related economy in which θ is endogenous.

of a productivity ladder, ahead of everyone else. Instead, assuming Γ is relatively close to 1, entrants only get to skip the bottom few rungs, occupied by firms that are about to exit. From there they have to climb the productivity ladder, by trial and error.[6]

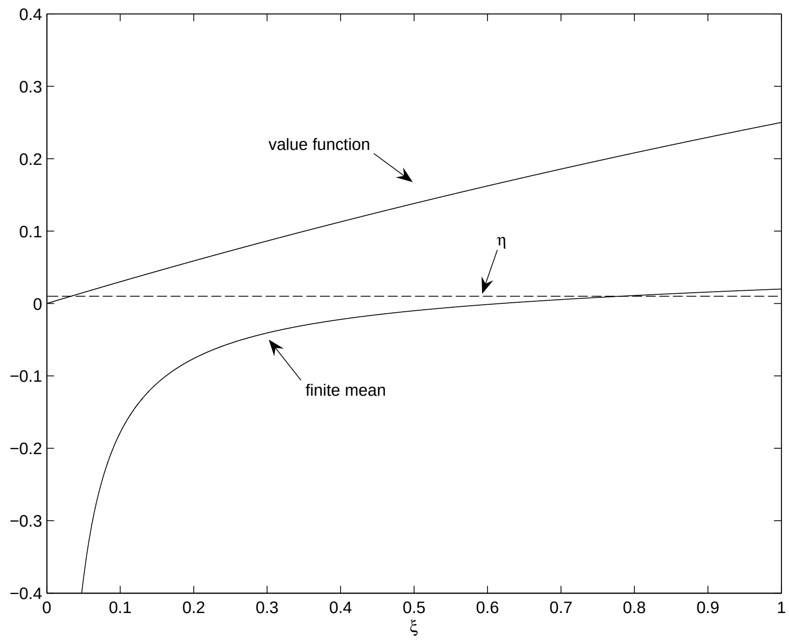

FIGURE III Existence of a Balanced Growth Path

Recall from (5) that $X_t/Z_t = e^{-(1-\beta)(S-B)}$, and hence

$$S = B + \ln(\Gamma). \tag{21}$$

The zero-profit condition for entrepreneurs is still $\lambda_{\mathrm{E}} = \lambda_{\mathrm{F}} V(S)$. Combining this with the solution (4) for $V(\cdot)$ and (21) yields

$$\lambda_{\mathrm{E}} = \frac{\lambda_{\mathrm{F}}}{\rho} \frac{\xi}{1+\xi} \left(\Gamma - 1 - \frac{1 - \Gamma^{-\xi}}{\xi} \right). \tag{22}$$

This is a function of ξ, which in turn is a function only of κ, via $\mu = (\theta - \kappa)/(1-\beta)$. Thus the requirement that entrepreneurs make zero profits implies an equilibrium condition that depends only on the balanced growth rate κ.

The right-hand side of (22) approaches 0 from above as $\xi \downarrow 0$. It is increasing in ξ, with a large-ξ asymptote equal to $(\Gamma - 1)\lambda_{\mathrm{F}}/\rho$. The size of the productivity improvement

[6]For alternative models of spillovers with productivity distributions, see Kortum [1997], Eaton and Kortum [1999], Luttmer [2007], Lucas [2008], and Alvarez, Buera and Lucas [2008].

entrants can make puts an upper bound on the value of a new firm. Thus (22) will have a solution for ξ as long as

$$\frac{\lambda_{\mathrm{E}}}{\lambda_{\mathrm{F}}} < \frac{\Gamma - 1}{\rho},$$

and this solution is unique. Entry costs cannot be too high, or the improvements available to entrants over the firms that are just exiting cannot be too small. In particular, taking a limit and letting $\Gamma \downarrow 1$ so that $S \downarrow B$ is guaranteed to rule out a solution to (22) unless entry costs also go to zero.

Although μ is now an equilibrium variable, nothing about the equilibrium condition (22) guarantees that (18) holds at $\omega = \eta$. Solving the definition (3) of ξ for μ and combining the solution with the restriction (18) gives

$$\frac{1}{\xi}\left(\frac{1}{2}\sigma^2\xi(1+\xi) - \rho\right) < \eta. \tag{23}$$

Figure III shows the left-hand side of (23) as a function of ξ, together with $\rho/\lambda_{\mathrm{F}}$ times the right-hand side of the zero-profit condition (22), for $\Gamma = 2$, $\sigma = .2$ and $\rho = .02$. If $\eta = .01$ then ξ cannot be much above .78, and hence $\rho\lambda_{\mathrm{E}}/\lambda_{\mathrm{F}}$ cannot be much above .2. Slightly higher values of $\rho\lambda_{\mathrm{E}}/\lambda_{\mathrm{F}}$ lead to non-existence of a balanced growth path, even though $\lambda_{\mathrm{E}}/\lambda_{\mathrm{F}}$ is still much smaller than $(\Gamma - 1)/\rho$ and (22) has a unique solution.

5.2 Random Imitation

One possible mechanism for ensuring that (18) holds in equilibrium is given in Luttmer [2007]. There, entrants at time t do not start with a common $Z_t = \Gamma^{1-\beta}X_t$, but with idiosyncratic $\delta^{1-\beta}z$, where the z are random draws from the population of incumbent producers. The fact that entrepreneurs draw at random from the time-t population means that their incentives to enter are driven by the population average of $V(s_t[z])$. Since $V(s_t[z])$ behaves like $e^{s_t[z]}$, this population average will explode precisely when $\mu + \sigma^2/2$ approaches η from below. Imposing the zero-profit condition therefore forces the equilibrium growth rate to be such that the mean of e^s is finite, which corresponds to condition (18). Given this, the right-hand side of (20) is finite, and one can use this condition to solve for the number of firms N.[7]

This captures the intuition that the presence of large and profitable firms should induce entry and reduce the growth rate of incumbent firms. Note also that even though

[7] If $\delta = 0$, the size distributions consistent with balanced growth are the ones defined in (13). There is entry, not at a point S, but throughout (B, ∞), with an intensity that is proportional to the incumbent size density.

potential entrants are drawing incumbents randomly, actual entry is selective if $\delta < 1$: all random draws at time t satisfy $z > X_t$, but only those potential entrants for whom $\delta^{1-\beta}z > X_t$ actually enter. A difficulty with this mechanism is that $\delta^{1-\beta}z$ will still be very large for some fraction of the entrants, even if the parameter δ is set far below 1 so that the ability of entrants to imitate is very poor. In the data, almost without exception, new firms are very small.

6. Less-Than-Perfectly Elastic Entry

In the economies described so far, the zero-profit condition for entrepreneurs is $\lambda_{\mathrm{E}} = \lambda_{\mathrm{F}}V(S)$ and $V(\cdot)$ depends on μ. With exogenous technological progress, μ is exogenous and the zero-profit condition determines S. In the spillover economy, $S = B + \ln(\Gamma)$ and the zero-profit condition pins down μ, and hence the growth rate $\kappa = \theta - (1-\beta)\mu$ of the economy. But it does so without reference to the labor-market clearing condition (20). A balanced growth path may then not exist because there is no guarantee that $\mu + \sigma^2/2 < \eta$.

In a setting of exogenous aggregate growth but endogenous firm growth, Luttmer [2010] avoids this problem by using a Roy model of occupational choice to make the entry of new firms less than perfectly elastic with respect to the market value of new firms. As a result, the number of firms and their growth rate are jointly determined by an entry condition and a labor market clearing condition. This forces average firm labor demand to be finite. This device can also be used here, with endogenous aggregate growth and exogenous growth of incumbent firms.

6.1 Workers and Entrepreneurs

Suppose that agents in the economy vary in their skills in supplying labor and creating new firms. Specifically, suppose a type-(x, y) agent can supply x units of labor or create new firms at a Poisson rate y. The distribution of talent in the population is described by some distribution function $T(x, y)$. Comparative advantage determines occupational choice. Wages are w_t per unit of labor and, along a balanced growth path, the value of a new firm is $w_t\lambda_{\mathrm{F}}V(S)$. In units of labor, this equals

$$q = \lambda_{\mathrm{F}}V(S).$$

A type-(x, y) agent will choose to be a worker if $qy < x$ and an entrepreneur if $qy > x$. The per-capita supplies of labor and new firms are therefore

$$\begin{pmatrix} L(q) \\ E(q) \end{pmatrix} = \int \begin{pmatrix} x\iota[qy < x] \\ y\iota[qy \geq x] \end{pmatrix} \mathrm{d}T(x,y).$$

Suppose that T is continuous with a finite mean and full support in $\mathbb{R}^2_{++}$. This implies that the supplies of labor and new firms are positive and vary continuously with q. Clearly, $L(q)$ is a decreasing function and $E(q)$ an increasing function. As q goes to zero, $E(q)$ goes to zero while $L(q)$ converges to the mean endowment of labor. On the other hand, if q becomes large, then the per-capita labor supply goes to zero while the supply of new firms grows to the mean of ability to create new firms. Hence $L(q)/E(q) \to \infty$ as $q \to 0$ and $L(q)/E(q) \to 0$ as $q \to \infty$. Thus the relative supplies of labor and new firms vary inversely with q and range throughout $(0, \infty)$. The ratio $L(q)/E(q)$ will be proportional to a negative power of q if labor and entrepreneurial skills are given by two independent Fréchet distributions.

6.2 Balanced Growth

Along a balanced growth path, the de-trended supply of new firms is $E(q)H = \varepsilon N$, and this supply must sustain a number of firms that grows at the rate η. Accounting for exit using (6), this means that

$$\frac{N}{H} = \frac{E(q)}{\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(B)}. \tag{24}$$

Labor is now used only by incumbent firms, and hence the labor market clearing condition simplifies from (20) to

$$\frac{N}{H} = \frac{L(q)}{\lambda_{\mathrm{F}} \left(1 + \frac{\beta}{1-\beta} \int_B^\infty e^s f(s) \mathrm{d}s\right)}. \tag{25}$$

New firms enter with a productivity that implies $S = B + \ln(\Gamma)$, as in (21). Combining this with $q = \lambda_{\mathrm{F}} V(S)$ yields

$$q = \frac{\lambda_{\mathrm{F}}}{\rho} \frac{\xi}{1+\xi} \left(\Gamma - 1 - \frac{1 - \Gamma^{-\xi}}{\xi}\right). \tag{26}$$

The parameter ξ and the exit barrier B are a functions only of μ, defined in (3). Hence q is a function only of μ. Because of $S = B + \ln(\Gamma)$, the stationary density f, defined in (14), also depends only on μ. Using (3), (14) and $S = B + \ln(\Gamma)$, the equilibrium conditions (24)-(26) therefore jointly determine the value of a new firm q, the number of firms N, and the employment growth rate $\mu + \sigma^2/2$ of incumbent firms. From there the rest of the balanced growth path follows. If there is a solution to (24)-(26), then (25) forces μ to be such that $\mu + \sigma^2/2 < \eta$.

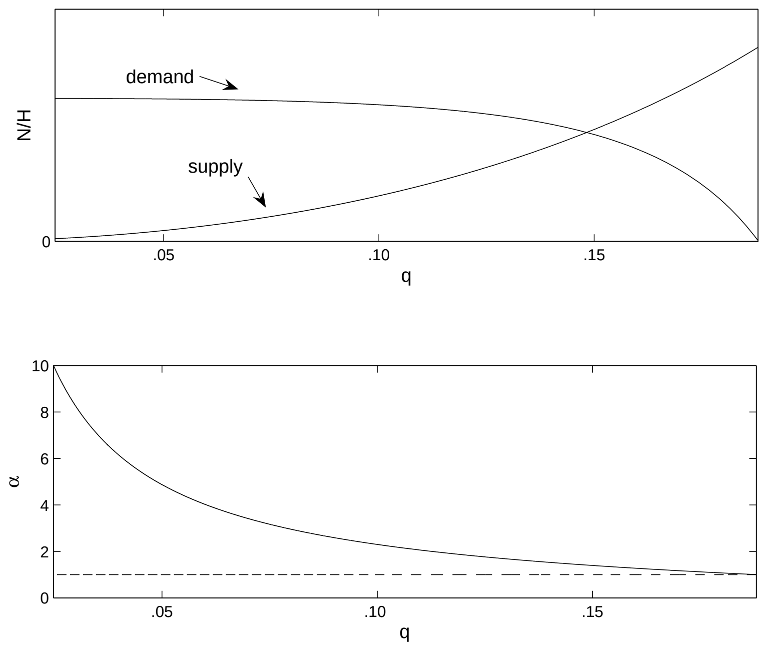

FIGURE IV The Supply and Demand for Firms

One can view (24) and (25) as, respectively, the balanced growth supply and demand of firms. The supply depends on the entrepreneurial supply of new firms and the rate at which new firms need to be created to ensure the number of firms grows at the rate η. The demand for firms depends on the supply of labor and the size of the average firm. Both supply and demand depend directly on the price of new firms, through the occupational choices made by agents, and also indirectly: via μ, the price of a new firm depends the growth rate of the economy, and this growth rate determines the exit rate and average firm demand for labor.

6.2.1 Existence and Uniqueness

A balanced growth path exists and is unique if the demand and supply of firms intersect uniquely. The following lemma collects some key derivatives that can be used to show that this is indeed the case.

LEMMA 2 *Given the definitions (3), (14), and (26), of B, f and q,*

$$\frac{\partial}{\partial\mu}e^B < 0, \quad \frac{\partial q}{\partial\mu} > 0, \quad \frac{\partial}{\partial\mu}\mathrm{D}f(B) < 0, \quad \frac{\partial}{\partial\mu}\int_B^\infty e^s f(s)\mathrm{d}s > 0,$$

when $\omega = \eta$ and the entry condition $S = B + \ln(\Gamma)$ holds.

More rapid firm growth implies a lower exit barrier B and a higher value q for new firms that enter with $S = B + \ln(\Gamma)$. The effect on B follows immediately from (3), and the effect on q from (3) and (26). Evaluating the entry rate (6) at $\omega = \eta$ and using (15) and $S = B + \ln(\Gamma)$ gives $\varepsilon = \eta + \frac{1}{2}\sigma^2 \mathrm{D}f(B) = \eta/(1 - \Gamma^{-\alpha_*})$, and this is easily seen to be decreasing in μ when $\Gamma > 1$. Rapidly growing firms are less likely to exit and so it takes less entry to maintain a growing population of firms. As shown in Lemma 1, holding fixed B and S, the mean of e^s is increasing in μ. But here B declines and $S - B = \ln(\Gamma)$, which tends to lower the mean of e^s. It is shown in Appendix B that the effect described in Lemma 1 dominates.

The supply of firms (24) is increasing in μ, since $E(q)$ is an increasing function, q is increasing in μ, and the entry rate $\varepsilon = \eta + \frac{1}{2}\sigma^2 \mathrm{D}f(B)$ is decreasing in μ. Furthermore, letting $\mu \to -\infty$ will cause q to approach 0 and $\varepsilon = \eta/(1 - \Gamma^{-\alpha_*})$ to approach ∞. Thus the supply of firms can be made arbitrarily close to 0 by taking μ sufficiently small.

The demand for firms (25) is decreasing in μ, since $L(q)$ is a decreasing function, q is increasing in μ, and the mean firm size is increasing in μ. In addition, letting $\mu + \frac{1}{2}\sigma^2$ approach the upper bound $\eta < \rho$ will cause the mean firm size to diverge, while q remains well defined (the value of a firm only diverges when $\mu + \frac{1}{2}\sigma^2$ reaches ρ) and $L(q)$ remains bounded. Thus the demand for firms will shrink to zero for $\mu + \frac{1}{2}\sigma^2$ close to η.

PROPOSITION 2 *There is a unique equilibrium growth rate $\kappa = \theta - (1 - \beta)\mu$.*

The demand and supply of firms are shown in Figure IV as a function of q, which is a monotone function of μ implied by (3) and (26). The second panel of Figure IV shows the relation between the tail index α and the price q of new firms.

6.2.2 Incumbent and Entrant Innovation

The incumbent productivity growth rate θ only appears in the expression $\kappa = \theta - (1 - \beta)\mu$. Changes in the growth rate of incumbent productivity therefore translate one-for-one into changes in the growth rate of aggregate productivity. They leave no mark on the underlying distribution of productivity, the resulting size distribution of firms, entry and exit rates, or the market value of new and existing firms.

Changes in the distribution of skills in this economy do give rise to shifts in the demand and supply curves in Figure IV. This typically has growth effects, not just level effects. An increase in the supply of entrepreneurial effort lowers q, which increases α

and lowers μ, leading to an increase the growth rate κ of aggregate productivity. More entrepreneurial effort at a given q implies a size distribution with a thinner tail and a higher growth rate of aggregate productivity. The tail index of the size distribution will be close to the $\alpha = 1$ asymptote if entrepreneurial talent sufficiently scarce.

To study the effects of an increase in the step Γ by which entrepreneurs can improve the technology of exiting producers, eliminate q from (24) and (25) using (26), to obtain a supply of firms that is increasing in μ and a demand for firms that is decreasing in μ, by Lemma 2. How these supply and demand curves shift is determined by

$$\frac{\partial q}{\partial \Gamma} > 0, \quad \frac{\partial}{\partial \Gamma} \mathrm{D} f(B) < 0, \quad \frac{\partial}{\partial \Gamma} \int_B^\infty e^s f(s) \mathrm{d}s > 0,$$

given the entry condition $S = B + \ln(\Gamma)$. The fact that the value of new firms rises with Γ is immediate from (26). The entry rate $\varepsilon = \eta/(1 - \Gamma^{-\alpha_*})$ is clearly decreasing in Γ. A higher Γ means that new firms enter farther away from the exit barrier and thus survive longer. It then requires less entry to maintain a growing population of firms. The fact that the mean size is increasing in Γ follows from (17) evaluated at $S - B = \ln(\Gamma)$. With this, one can use (25)-(26) to argue that an increase in Γ causes the supply of firms to rise and the demand for firms to fall, for given μ. It follows that μ must fall. An increase in Γ makes new firms more valuable, causing agents to substitute from supplying labor (demanding firms) to supplying entrepreneurial effort (supplying firms), and the effects on the supply and demand for firms are magnified by the fact that less entry is required and firms will be larger. Equilibrium can only be restored with a decline in μ, and hence a rise in the growth rate κ of aggregate productivity. In this economy, making it easier for entrants to make productivity improvements relative to exiting firms causes growth to accelerate.

In the data, $\alpha > 1$ is close to 1. This can be interpreted to mean that entrants can make only small productivity improvements over exiting firms.

Corollary *The tail index $\alpha > 1$ converges to 1, Zipf's Law, as $\Gamma > 1$ converges to 1.*

To see this, first note that the supply of firms at any given μ converges to zero as $\Gamma \downarrow 1$, for two reasons. By (26), q goes to zero as $\Gamma \downarrow 1$ and holding fixed μ. This means that the supply of entrepreneurial effort goes to zero. In addition, the entry rate $\eta/(1-\Gamma^{-\alpha_*})$ explodes as Γ approaches 1 from above. To examine the limiting demand for firms, note from (17) that the mean of e^{s-B} converges to $\alpha/(\alpha - 1)$ as $\Gamma \downarrow 1$, for any given $\mu + \sigma^2/2 < \eta$. Since q goes to zero, the supply of labor converges to its maximum. In the limit, the demand for firms (25) becomes a function of μ only via the effect of μ on the average firm size. Since the mean of e^{s-B} diverges as $\alpha \downarrow 1$, this limiting demand curve still has the property that $N/H \downarrow 0$ as $\mu + \sigma^2/2 \uparrow \eta$. It follows that the intersection of the supply and demand curves must occur at a $\mu + \sigma^2/2 \uparrow \eta$ as the supply converges to zero because of $\Gamma \downarrow 1$.

6.2.3 Comparison with Perfectly Elastic Entry

In terms of μ, the structure of the equilibrium conditions (24)-(26) is

$$E(q) = G_1(\mu)N/H, \quad L(q) = G_2(\mu)N/H, \quad q = G_3(\mu),$$

where $G_1(\mu)$ is the required entry rate, $G_2(\mu)$ is the average firm demand for labor, and $G_3(\mu)$ is the value of a new firm. By condition (18), $G_2(\mu)$ is finite if and only if $\mu + \sigma^2/2 < \eta$. With less-than-perfectly-elastic entry, $E(q)$ and $L(q)$ vary continuously and Proposition 2 shows that a unique solution is guaranteed. With perfectly elastic entry, the pair $[E(q), L(q)]$ equals $[0,1]$ if $\lambda_{\mathrm{E}} > q$ and $[1/\lambda_{\mathrm{E}}, 0]$ if $\lambda_{\mathrm{E}} < q$. This means that $\lambda_{\mathrm{E}} = q = G_3(\mu)$ is the only candidate equilibrium. But $G_2(\mu)$ may not be finite at the implied μ, and in that case there will be no balanced growth path. The random imitation process in Luttmer [2007] implies that $G_2(\mu)$ if finite whenever $G_3(\mu)$ is, ensuring the existence of a balanced growth path.

7. A Calibration

US Census data compiled by the Small Business Administration show that the employment size distribution of employer firms is stable over time and close to Pareto for firms with more than about 20 employees. A point estimate of the tail index α is about 1.06. There is some uncertainty about this estimate. A value of 1.1 could work too, but larger values of α are hard to reconcile with the observed number of large firms. The total number of firms grows at roughly the US population growth rate of around 1%, although there are some fluctuations. Entry of new firms is at a stable rate of about 11% per annum. Employment among firms with 500 or more employees grows at an annual rate of around .36% over the period 1988-2006, although this number is not precisely estimated (serially uncorrelated measurement or approximation error implies a standard error of about .38%.) Consistent with the class of models considered here, not many of these firms exit over the course of a year.[8]

[8] The number is about 2.5%, higher than one might expect if firms only exit at B. No doubt some of these exits are due to takeovers and mergers.

Solving the formula (11) for the tail index α for μ gives

$$\mu + \frac{1}{2}\sigma^2 = \frac{\eta}{\alpha} - \frac{1}{2}\sigma^2(\alpha - 1). \tag{27}$$

Over small periods of time, the left-hand side is the growth rate of employment at incumbent firms. A tail index close to 1 implies that this growth rate will be close to η, but the discrepancy will be larger the larger the variance of productivity shocks. If the employment growth rate of .36% per annum at firms with more than 500 employees is used to infer $\mu + \sigma^2/2$, then (27) yields $\sigma = .44$. Davis et al. [2006] report estimates of the annual standard deviation of firm employment growth that range widely, roughly from 15% to 65%, depending on whether a firm is publicly traded or not, and on the sample period. Their most recent estimate for the whole economy is about 37%.

Recall that the entry rate is $\varepsilon = \eta/(1 - \Gamma^{-\alpha_*})$ and note that (11) implies $\alpha\alpha^* = \eta/(\sigma^2/2)$. Using this to solve for Γ gives

$$\ln(\Gamma) = \frac{1}{2}\sigma^2 \times \frac{\alpha}{\eta} \ln\left(\frac{\varepsilon}{\varepsilon - \eta}\right). \tag{28}$$

Thus $\ln(\Gamma)$ is proportional to $\sigma^2/2$, with a coefficient that can be inferred from the tail index $\alpha \approx 1.06$, the population growth rate $\eta \approx .01$, and the entry rate $\varepsilon \approx .11$. The resulting coefficient is 10.1, and then $\sigma \approx .44$ implies $\ln(\Gamma) \approx .98$. Hence, variable employment at entering firms is about 2.7 times variable employment at exiting firms. The implicit difference in productivity will be much smaller if β is close to 1 (there is no physical capital in this economy.) At $\beta = .9$, entrants have a productivity advantage over exiting firms of about 10%, and the standard deviation σ_Z of incumbent productivity growth is about 4.4% per annum.

The first and second panel of Figure V show the incumbent employment growth rate (27) and the spillover parameter (28) as functions of σ. Included in the first panel is a further decomposition of the growth rate η of aggregate employment,

$$\eta \quad = \quad \underbrace{\frac{\varepsilon e^S - (\varepsilon - \eta)e^B}{\int_B^\infty e^s f(s)\mathrm{d}s}}_{\text{entry} - \text{exit}} \quad + \quad \underbrace{\mu + \frac{1}{2}\sigma^2}_{\text{incumbents}}.$$

This can be verified mechanically using (11) and (17). The first term is the entry rate times the ratio of entry employment over average employment, which accounts for variable employment created by entry. The second term represents variable employment lost as a result of exiting firms, and the third term is the growth rate of employment at incumbent firms. In SBA data, the gross employment flows from entry and exit hover around 3% of aggregate employment, which significantly exceeds the gross flows shown in Figure V. In the data, the ratio of entry and exit employment over aggregate employment is higher than implied by σ in the range reported here. One likely reason for this is that $\int_B^\infty e^{s-S} f(s)\mathrm{d}s$ and $\int_B^\infty e^{s-B} f(s)\mathrm{d}s$ are very sensitive to small changes in the tail index α near its asymptote $\alpha = 1$. Another is heterogeneity in the exit and entry points B and S.

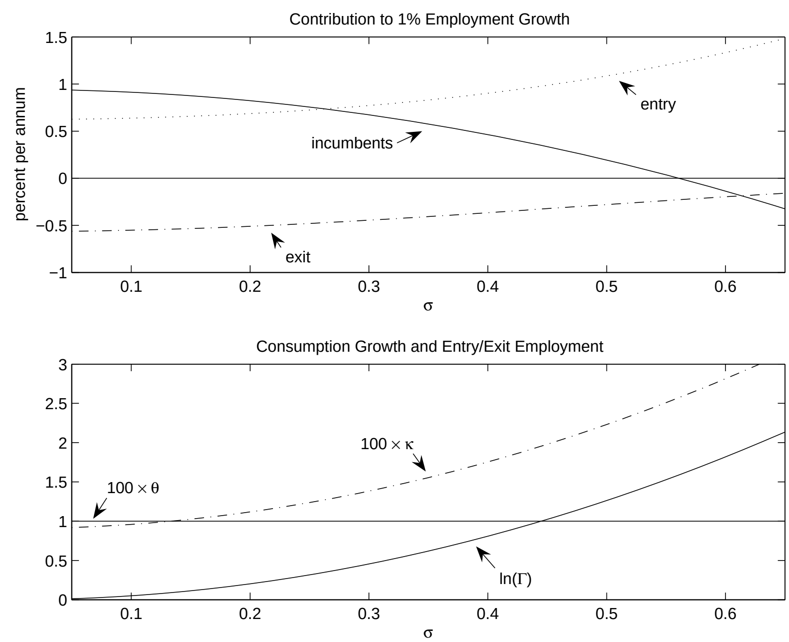

FIGURE V Growth Decomposition and Implied Spillover

The second panel of Figure V also shows the implied growth rate of per-capita consumption $\kappa = \theta + (1-\beta)(\frac{1}{2}\sigma^2\alpha - \eta/\alpha)$ at $\beta = .9$. The Cobb-Douglas technology used by all firms implies that a firm with productivity z produces $y_t[z] = w_t l_t[z]/\beta$ units of consumption. The contributions of incumbent firms to aggregate consumption growth can therefore be calculated by simply adding the κ shown in the second panel to the $\mu + \sigma^2/2$ shown in the first panel. The result is $\kappa + \mu + \sigma^2/2 = \theta + \beta\eta/\alpha + (1-\alpha\beta)\,\sigma^2/2$. Thus the implied contribution of incumbent firms to aggregate consumption growth is linear and increasing in σ^2, with a small slope when β is close to 1.

At $\sigma = .44$, an incumbent employment growth rate of .36% implies $\mu \approx -.0932$. Thus, according to these estimates, the positive incumbent employment growth rate is the combination of a strong negative drift in incumbent employment, and a lot of reallocation of employment driven by productivity shocks. This means that $\kappa = \theta + (1 -$

$\beta) \times .0932$, and hence randomness and re-allocation add almost 1% to the logarithmic drift θ of incumbent productivity growth if $\beta = .9$, or about half the 2% growth rate of US per-capita consumption. This is consistent with the important role for re-allocation found in Restuccia and Rogerson [2008] and Hsieh and Klenow [2009].

8. Concluding Remarks

Clearly, the assumption that entering firms are small only because their technology is relatively close to that of the small firms that are about to exit is difficult to maintain. So is the assumption, Gribrat's Law, that firm growth is independent of size. Google and Wal-Mart did not grow at comparable rates over the last decade: from 8 to 20,000 and from 1.1 million to 2.1 million employees, at respective annual rates of 78% and 6.5%. As emphasized in Luttmer [2010] many large firms have gone through a sustained period of extremely rapid growth following entry. The correct interpretation is likely to be that these firms early on actually had a very productive technology (or at least the building blocks for such a technology), but creating the organization to implement this technology on a large scale took significant resources and time. Incorporating these elements into tractable and quantitatively plausible models of technology diffusion and endogenous growth remains an important task for further research.

A Proof of Lemma 1

Write (17) as

$$\int_B^\infty e^s f(s)\mathrm{d}s = \frac{\alpha e^B}{\alpha - 1}\frac{\frac{e^{(1+\alpha_*)(S-B)}-1}{(1+\alpha_*)(S-B)}}{\frac{e^{\alpha_*(S-B)}-1}{\alpha_*(S-B)}} = \frac{\alpha e^B}{\alpha - 1} \times \frac{\frac{e^{S-B+z}-1}{S-B+z}}{\frac{e^z-1}{z}} \tag{29}$$

where $z = \alpha_*(S-B)$. The definitions of α and α_* imply that $\partial\alpha/\partial\mu < 0$ and $\partial\alpha_*/\partial\mu > 0$. Thus, for fixed B, the first factor on the right-hand side is increasing in μ. We need to show that the second factor is increasing in z. Define $g(y) = (e^y - 1)/y$ for any $y > 0$. We need to show that $g(S - B + z)/g(z)$ is increasing in z when $S > B$. That is, we need $\partial \ln(g(S - B + z))/\partial z > \partial \ln(g(z))/\partial z$ when $S > B$. This is true if $\mathrm{D}g(z)/g(z)$ is increasing in z. Now,

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{D}g(z)}{g(z)} = \frac{1}{(e^z-1)^2}\left(\left(\frac{e^z-1}{z}\right)^2 - e^z\right).$$

To see that this is positive, note that $e^{z/2} - e^{-z/2} - z$ is an increasing function for $z > 0$, and equal to 0 at $z = 0$. Hence $e^{z/2} > e^{-z/2} + z$ for $z > 0$. It follows that $(e^z - 1)/z > e^{z/2}$ for all $z > 0$, and this proves the desired result.

B Proof of Lemma 2

Only the derivative of the mean of e^s with respect to μ remains to be shown. By Lemma 1, the second factor on the right-hand side of (29) is increasing in μ when $S - B = \ln(\Gamma)$ is fixed. Write $a = \eta/(\sigma^2/2)$, $b = \rho/(\sigma^2/2)$ and recall that $b > a > 0$. Also write $x = \mu/\sigma^2$ and note that $\alpha > 1$ corresponds to $a > 1 + 2x$. Using the definitions of α and B one can verify that

$$\frac{\alpha e^B}{\alpha - 1} = \frac{\frac{-x+\sqrt{x^2+a}}{-(1+x)+\sqrt{x^2+a}}}{\frac{-x+\sqrt{x^2+b}}{-(1+x)+\sqrt{x^2+b}}}.$$

Differentiating with respect to x gives

$$\frac{\partial}{\partial x}\frac{\alpha e^B}{\alpha - 1} = \frac{-x + \sqrt{x^2+a}}{-x + \sqrt{x^2+b}} \frac{b - (1+x)\sqrt{x^2+b} - \left(a - (1+x)\sqrt{x^2+a}\right)}{\sqrt{x^2+a}\sqrt{x^2+b}\left(-(1+x) + \sqrt{x^2+a}\right)^2}.$$

Since $b > a$, the question is thus if $y - (1+x)\sqrt{x^2+y}$ is increasing in y when $y > 1 + 2x$. The derivative of this function is

$$\frac{\partial}{\partial y}\left(y - (1+x)\sqrt{x^2+y}\right) = 1 - \frac{1+x}{2\sqrt{x^2+y}}.$$

This is clearly positive if $1 + x \leq 0$. The derivative is also positive if $1 + x > 0$ and $3(x^2 + y) + y > 1 + 2x$. This is true when $y > 1 + 2x$ and $y > 0$, and hence for $y \in \{a, b\}$.

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