Optimal Control and Spatial Heterogeneity: Pattern Formation in Economic-Ecological Models

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Abstract

This paper extends Turing analysis to standard recursive optimal control frameworks in economics and applies it to dynamic bioeconomic problems where the interaction of coupled economic and ecological dynamics under optimal control over space creates (or destroys) spatial heterogeneity. We show how our approach reduces the analysis to a tractable extension of linearization methods applied to the spatial analog of the well known costate/state dynamics. We explicitly show the existence of a non-empty Turing space of diffusive instability by developing a linear-quadratic approximation of the original non-linear problem. We apply our method to a bioeconomic problem, but the method has more general economic applications where spatial considerations and pattern formation are important. We believe that the extension of Turing analysis and the theory associated with the dispersion relationship to recursive infinite horizon optimal control settings is new.

JEL Classification Q2, C6

Keywords: Spatial analysis, Pattern formation, Turing mechanism, Turing space, Pontryagin's principle, Bioeconomics.

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1 Introduction

In economics the importance of space has long been recognized in the context of location theory,[1] although as noted by Krugman (1998) there has been some neglect in the systematic analysis of spatial economics, associated mainly with difficulties in developing tractable models of imperfect competition which are essential in the analysis of location patterns. After the early 1990s there was a renewed interest in spatial economics, mainly in the context of *new economic geography,*[2] which concentrates on issues such as the determinants of regional growth and regional interactions, or the location and size of cities (e.g. Krugman, 1993).

In environmental and resource management problems the majority of the analysis has been concentrated on taking into account the temporal variation of the phenomena, and has been focused on issues such as the transition dynamics towards a steady state, or the steady-state stability characteristics. However, it is clear that when renewable and especially biological resources are analyzed, the spatial variation of the phenomenon is also important. Biological resources tend to disperse in space under forces promoting "spreading", or "concentrating" (Okubo, 2001); these processes along with intra and inter species interactions induce the formation of spatial patterns for species. In the management of economic-ecological problems, the importance of introducing the spatial dimension can be associated with a few attempts to incorporate spatial issues, such as resource management in patchy environments (Sanchirico and Wilen, 1999, 2001; Sanchirico, 2004; Brock and Xepapadeas, 2002), the study of control models for interacting species (Lenhart and Bhat, 1992; Lenhart et al., 1999), the control of surface contamination in water bodies (Bhat et al. 1999), or the creation of marine reserves (Neubert, 2003).

[1] See for example Alfred Weber (1909), Harold Hotelling (1929), Walter Christaller (1933), and August Löcsh (1940) for early analysis.

[2] Krugman (1998) attributes this new research to: the ability to model monopolistic competition using the well known model of Dixit and Stiglitz (1977); the proper modeling of transaction costs; the use of evolutionary game theory; and the use of computers for numerical examples.

In the economic-ecological context, a central issue that this paper attempts to explore is under what conditions interacting processes characterizing movements of biological resources, and economic variables which reflect human actions on the resource (e.g. harvesting effort), could generate steady-state spatial patterns for the resource and the associated economic variables. That is, a steady-state concentration of the resource and the economic variable which is different at different points in a given spatial domain. We will call this formation of spatial patterns *spatial heterogeneity,* in contrast to *spatial homogeneity* which implies that the steady state concentration of the resource and the economic variable is the same at all points in a given spatial domain.[3]

As stated by Levin (2002) pattern formation and the emergence of robust patterns as asymptotic outcomes of dynamical systems is the first aspect of the two main processes characterizing complex adaptive systems,[4] the other being evolution. A common framework for studying pattern formation is the use of the concept of diffusion as central concept in modelling the movements in space-time of populations of species, chemicals or other state variables, which are interacting locally and redistribute via random movements.. Diffusion is thus defined as a process whereby the microscopic irregular movement of particles such as cells, bacteria, chemicals, or animals results in some macroscopic regular motion of the group (Okubo and Levin, 2001; Murray, 1993, 2003). Biological diffusion is based on random walk models, which when coupled with population growth equations, lead to general reaction-diffusion systems.[5] As stated by Okubo, et al. (2001, p. 348),

> "In general a diffusion process in an ecosystem tends to give rise to a uniform density of population in space, [that is spatial homogeneity]. As a consequence it may be expected that

[3]Trivially all dynamic models where spatial characteristics and dispersal are ignored lead to spatial homogeneity.

[4]Following Levin (1999) complex adaptive systems can be defined by three properties: (i) diversity and individuality of components; (ii) localized interactions among those components; and (ii) an autonomous process that uses the outcomes of those interactions to select a subset of those components for replication or enhancement.

[5]When only one species is examined the coupling of classical diffusion with a logistic growth function leads to the so-called Fisher-Kolmogorov equation.

diffusion, when it occurs, plays the general role of increasing stability in a system of mixed populations and resources. ...However there is an important exception known as *diffusion induced instability*, or *diffusive instability.* This exception might not be a rare event especially in aquatic systems."

It was Alan Turing (1952) who suggested that under certain conditions reaction-diffusion systems, which have an asymptotically stable equilibrium in the absense of diffusion can generate spatially heterogeneous patterns under diffusion.This is the so-called *Turing mechanism* or *Turing effect* for generating diffusion instability.[6] The Turing effect implies that an initially spatially homogeneous state can be transformed into a stable patterned state under purturbations induced by diffusion.

Levin (2002) presents other mechanisms that can act as pattern generators, althought the Turing mechanism has a central part in his discussion. However Levin, and as far as we know other researchers in the field, do not treat optimal management of a Turing dynamical mechanism as we do this in the current paper. We use the classical problem of optimal harvesting of a renewable resource as a leading example, but we believe our paper will help in formulating an analytically tractible approach to the optimal management of general complex adaptive systems as discussed by Levin.

In this context the purpose of this paper is to explore the impact of the Turing mechanism on the emergence of diffusive instability in optimal control problems in space-time using as a leading example a unified economic/ecological model of optimal resource management. This is a different approach to the one most commonly used to address spatial issues, which is the use of metapopulation models in discrete patchy environments with dispersal among patches (e.g. Sanchirico and Willen, 1999; Sanchirico, 2004). The use of the Turing mechanism allows us to analyze in detail conditions under which diffusion could produce spatial

[6] It should also be noted that the emergence of spatial homogeneity or not depends on boundary conditions associated with the spatial domain. If there is no flux on the boundary of the spatial domain (zero flux conditions), then spatial homogeneity might be expected, although as it will be shown, the Turing mechanism under appropriate conditions can generate spatial heterogeneity with zero flux conditions.

heterogeneity and generation of spatial patterns, or spatial homogeneity. Thus the Turing mechanism can be used to reveal conditions which generate spatial heterogeneity in models where ecological variables interact with economic variables. When spatial heterogeneity emerges the concentration of variables of interest (e.g. resource stock or harvesting effort) in a steady state, are different in different locations of a given spatial domain.

The importance of the Turing mechanism in spatial economics has been recognized by Fujita et al. (1999, chapter 6) in the analysis of core-periphery models. Our analysis extends this approach mainly by explicit introduction of diffusion processes governing interacting economic and ecological variables in continuous time space in optimal management models, and by developing the ideas for the emergence of spatial heterogeneity in an optimizing context by an appropriate modification of Pontryagin's maximum principle.

In particular we consider the emergence of spatial heterogeneity in the context of an optimizing model, where the objective of a social planner is to maximize a welfare criterion subject to resource dynamics that include a diffusion process. We present a suggestion for extending Pontryagin's maximum principle to the optimal control of diffusion. Although conditions for the optimal control of partial differential equations have been derived either in abstract settings (e.g. Lions 1971) or for specific problems,[7] our derivation not only makes the paper self contained, but it is also close to the optimal control formalism used by economists, so it can be used for analyzing other types of economic problems, where state variables are governed by diffusion processes. Furthermore, the Pontryagin principle developed in this paper allows for an extension of the Turing mechanism for generation of spatial patterns, to the optimal control of systems under diffusion.

A new - to our knowledge - characteristic of our continuous space-time approach is that we are able to embed Turing analysis in an optimal control recursive infinite horizon approach in a way that allows us to

[7]See for example Lenhart and Bhat (1992); Lenhart et al. (1999); Bhat et al. (1999); Raymond and Zidani (1998, 1999).

locate sufficient conditions on parameters of the system (for example, the discount rate on the future, and interaction terms in the dynamics) for diffusive instability to emerge even in systems that are being optimally controlled. This mathematically challenging problem becomes tractable by exploiting the recursive structure of the utility and the dynamics in our continuous space/time framework in contrast to the more traditional approach of discrete patch optimizing models. This is so because the symmetries in the spatial structure coupled with the recursivity in the temporal structure of our framework reduce the potentially very large number of state and costate variables to a pair of "sufficient" variables that describe the dynamics of the whole system. We believe that our framework will be quite easily adaptable to other applications, including an extension of the classical Ramsey-Solow growth model to include spatial externalities. Colin Clark's classic volume (1990), as well as the work of Sanchirico and Wilen (1999, 2001), is very suggestive, but they do not contain the unification of Turing analysis with infinite horizon temporally recursive optimal control problems that we present here.[8]

Here, we use our methodology to study an optimal fishery management problem under biomass diffusion. For the fishery problem, our results suggest that diffusion could alter the usual saddle point characteristics of the spatially homogeneous steady state as defined by the modified Hamiltonian dynamic system. In an analog to the Turing mechanism for an optimizing system, spatial heterogeneity in a steady state could be the result of optimal management. In particular we locate conditions for the Turing set of parameters inducing diffusive instability to be non-empty in the case where we have, under a positive discount rate, a saddle point steady state when diffusion is zero. On the other hand diffusion could stabilize, in the saddle point sense, an unstable steady state of an optimal control problem.

[8] We would note again that the Turing mechanism is not the only source of spatial heterogeneity in resource management models. As shown by Neubert (2003), a spatially heterogeneous steady state emerges in the temporal equilibrium of a bioeconomic model of optimal harvesting and marine reserve design, where the associated Hamiltonian function is linear in harvesting effort.

2 On the Optimal Control of Diffusion: An Extension of Pontryagin's Principle

In this section we explicitly introduce optimization and we analyze the effects of the optimal control of diffusion processes in the emergence of spatial heterogeneity through diffusion driven instability.

We start by considering an optimal control problem defined in the spatial domain $z \in Z = [z_0, z_1]$ and the time domain $t \in [t_0, t_1]$. Let $x(t,z), u(t,z)$ be the scalar state and control variables respectively at time t and spatial point z, taking values in compact sets $\mathcal{X}$ and $\mathcal{U}$. Let $f(x(t,z), u(t,z))$ be a net benefit function satisfying standard concavity assumptions and consider the following optimal control problem:

$$\max_{\{u(t,z)\}} \int_{z_0}^{z_1} \int_{t_0}^{t_1} f(x(t,z), u(t,z))\, dtdz \tag{1}$$

$$\text{s.t. } \frac{\partial x(t,z)}{\partial t} = g(x(t,z), u(t,z)) + D\frac{\partial^2 x(t,z)}{\partial z^2} \tag{2}$$

$$x(t_0, z) \text{ given, } \left.\frac{\partial x(t,z)}{\partial z}\right|_{z=z_0} = \left.\frac{\partial x(t,z)}{\partial z}\right|_{z=z_1} = 0: \text{ zero flux} \tag{3}$$

$$x(t, z_0) = x(t, z_1) = 0: \text{ hostile boundary, } x(t_0, z) \;\; z \in (z_0, z_1) \text{ given} \tag{4}$$

In the above problem the transition equation (2) states that the rate of change of the state variable, e.g the concentration of a biological resource or some other stock, at a given spatial point is determined by a general growth function $g(x(t,z), u(t,z))$ which reflects the kinetics of the state variable, and by dispersion reflected by $D\frac{\partial^2 x(t,z)}{\partial z^2}$. In (2) $D > 0$ is diffusivity or the diffusion coefficient and the basic assumptions regarding diffusion are those of the classical approach (or Ficksian diffusion), stating that the flux of the resource is proportional to the gradient of the resource concentration and that the movement is from high to low concentration. The first part of (3) provides initial conditions, while the second part is a zero flux condition. By zero flux condition it is assumed that there is no external biomass or effort input on the boundary of the spatial domain.[9] Conditions (4) are an alternative set of boundary conditions indicating that the exterior of the spatial domain (z_0, z_1) is completely hostile to the resource (e.g. Murray, 2003, Vol II, p. 120; Neubert, 2003). So if x denotes a species, (4) imply that individuals that cross the boundary die.

Problem (1) is an optimal control problem in fixed and finite time and spatial domains. The zero flux terminal condition (3) corresponds to a "free endpoint problem" for the state variable, since the terminal value of the state variable is not *a priori* specified at terminal time or terminal space. The hostile boundary condition (4) can be associated with a type of a "fixed endpoint problem" for the state variable, since the terminal value of the state variable is zero at terminal space for all t. These terminal conditions will be used to specify the appropriate transversality conditions for the problem.

Problem (1) to (4) has been analyzed in more general forms (e.g. Lions, 1971). We however choose to present here an extension of Pontryagin's principle for this problem, because it is in the spirit of optimal control formalism used by economists, and thus can be used for other applications, but also because it makes the whole analysis in the paper self contained.[10] Furthermore, as noted in the introduction, the use of Pontryagin's principle in continuous time space allows for a drastic reduction in the dimensionality of the dynamic system describing the phenomenon and makes the problem tractable. Our results are presented below, with proofs in the Appendix.

Maximum Principle under diffusion: Necessary Conditions - Finite time horizon (MPD-FT). *Let $u^* = u^*(t,z)$ be a choice of instrument that solves problem (1) to (4) and let $x^* = x^*(t,z)$ be the associate path for the state variable. Then there exists a function $\lambda(t,z)$ such that for each t and z.*[11]

[9] The zero flux boundary conditions is imposed so that the organizing pattern is self-organizing and not driven by boundary conditions (Murray 2003, Vol II, p.82).

[10] Similar conditions have been derived for other cases. such as the control of parabolic equations (Raymond and Zidani,1998, 1999), boundary control (Lenhart et al., 1999), or distributed parameter control (Dean Carlson et al., 1991; Lenhart and Bhat, 1992).

[11] In some cases in order to simplify notation, and when no confusion arises, sub-

1. $u^* = u^*(t,z)$*maximizes the generalized Hamiltonian function*

$$\mathbf{H}\left(x\left(t,z\right),u\left(t,z\right),\lambda\left(t,z\right)\right) =$$
$$f\left(x\left(t,z\right),u\left(t,z\right)\right)+\lambda\left(t,z\right)\left[g\left(x\left(t,z\right),u\left(t,z\right)\right)+D\frac{\partial^2 x\left(t,z\right)}{\partial z^2}\right]$$

or under appropriate concavity assumptions:

$$f_u + \lambda\left(t,z\right)g_u = 0 \tag{5}$$

2.

$$\frac{\partial \lambda\left(t,z\right)}{\partial t} = -\frac{\partial \mathbf{H}}{\partial x} - D\frac{\partial^2 \lambda\left(t,z\right)}{\partial z^2} = -\left(f_x + \lambda\left(t,z\right)g_x + D\frac{\partial^2 \lambda\left(t,z\right)}{\partial z^2}\right) \tag{6}$$

$$\frac{\partial x\left(t,z\right)}{\partial t} = g\left(x\left(t,z\right),u^*\left(t,z\right)\right)+D\frac{\partial^2 x\left(t,z\right)}{\partial z^2} \tag{7}$$

evaluated at $u^* = u^*\left(x\left(t,z\right),\lambda\left(t,z\right)\right).$

3. *The following transversality conditions hold*

$$\int_{z_0}^{z_1}\lambda\left(t_1,z\right)x\left(t_1,z\right)dz = 0, \;\; \Rightarrow \;\; \lambda\left(t_1,z\right)=0, \;\; z\in\left[z_0,z_1\right] \tag{8}$$

For zero flux boundary conditions (3) it also holds that

$$\frac{\partial \lambda\left(t,z_1\right)}{\partial z} = \frac{\partial \lambda\left(t,z_0\right)}{\partial z} = 0 \tag{9}$$

The result can also be extended to infinite time horizon problems

scripts associated with functions denote partial derivatives.

with discounting. In this case the problem is:

$$\int_{z_0}^{z_1}\int_{t_0}^{\infty} e^{-\rho t} f\left(x\left(t,z\right),u\left(t,z\right)\right)dtdz \quad ,\rho>0 \tag{10}$$

$$\text{s.t } \frac{\partial x}{\partial t}=g\left(x\left(t,z\right),u\left(t,z\right)\right)+D\frac{\partial^2 x\left(t,z\right)}{\partial z^2} \tag{11}$$

$$x\left(t_0,z_0\right) \text{ given, } \left.\frac{\partial x\left(t,z\right)}{\partial z}\right|_{z=z_0}=\left.\frac{\partial x\left(t,z\right)}{\partial z}\right|_{z=z_1}=0: \text{ zero flux} \tag{12}$$

$$x\left(t,z_0\right)=x\left(t,z_1\right)=0: \text{hostile boundary,} x\left(t_0,z\right),z\in\left(z_0,z_1\right) \text{ given} \tag{13}$$

Maximum Principle under diffusion: Necessary Conditions - Infinite time horizon (MPD-IT). *Let $u^*=u^*\left(t,z\right)$ be a choice of instrument that solves problem (10) to (13) and let $x^*=x^*\left(t,z\right)$ be the associate path for the state variable. Then there exists a function $\lambda\left(t,z\right)$ such that for each t and z*

1. $u^*=u^*\left(t,z\right)$*maximizes the generalized current value Hamiltonian function*

$$\begin{aligned}&\mathbf{H}\left(x\left(t,z\right),u,\lambda\left(t,z\right)\right)=\\&f\left(x,u\right)+\lambda\left(t,z\right)\left[g\left(x\left(t,z\right),u\left(t,z\right)\right)+D\frac{\partial^2 x\left(t,z\right)}{\partial z^2}\right],\end{aligned}$$

or under appropriate concavity assumptions:

$$f_u+\lambda\left(t,z\right)g_u=0 \tag{14}$$

2.

$$\begin{aligned}\frac{\partial\lambda\left(t,z\right)}{\partial t}=&\rho\lambda\left(t,z\right)-\frac{\partial\mathbf{H}}{\partial x}-D\frac{\partial^2\lambda\left(t,z\right)}{\partial z^2}= \\&\rho\lambda\left(t,z\right)-\left(f_x+\lambda\left(t,z\right)g_x+D\frac{\partial^2\lambda\left(t,z\right)}{\partial z^2}\right)\end{aligned} \tag{15}$$

$$\frac{\partial x\left(t,z\right)}{\partial t}=g\left(x\left(t,z\right),u^*\left(t,z\right)\right)+D\frac{\partial^2 x\left(t,z\right)}{\partial z^2} \tag{16}$$

evaluated at $u^*=u^*\left(x\left(t,z\right),\lambda\left(t,z\right)\right)$

3. *Transversality conditions at infinity are part of the sufficient conditions given below.*

It is clear that conditions (5)-(9) or (14)-(16) can characterize the whole dynamic system in continuous time space. It is interesting to note that (15) - (16) is a modified dynamic Hamiltonian system defined in continuous space time. In this system the diffusion coefficient for the costate variable is negative, and it is the opposite of the state variable's diffusion coefficient. Since the costate variable can be interpreted as the shadow value of the resource stock, negative diffusion implies that the movement in space is from low shadow values to higher shadow values. Furthermore, the opposite signs of the diffusion coefficient for the state and the costate variable imply that time 'runs backward' in the state variable and 'runs forward' in the costate variable which is a *forward capitalization* type variable in capital theoretic terms.

The conditions derived above are essentially necessary conditions. Sufficiency conditions can also be derived by extending sufficiency theorems of optimal control. Proofs are provided in the Appendix.

Maximum Principle under diffusion: Sufficient conditions - Finite time horizon

Assume that functions $f(x,u)$ *and* $g(x,u)$ *are concave differentiable functions for problem (1) to (4) and suppose that functions* $x^*(t,z), u^*(t,z)$ *and* $\lambda(t,z)$ *satisfy necessary conditions (5)-(9) for all* $t \in [t_0, t_1], z \in [z_0, z_1]$ *and that* $x(t,z)$ *and* $\lambda(t,z)$ *are continuous with*

$$\lambda(t,z) \geq 0 \text{ for all } t \text{ and } z. \tag{17}$$

Then the functions $x^*(t,z), u^*(t,z)$ *solve the problem (1) to (4). That is, the necessary conditions (5) - (9) are also sufficient.*

The result can also be extended along the lines of Arrow's sufficiency theorem. We state here the infinite horizon case.

Maximum Principle under diffusion: Sufficient conditions - Infinite time horizon

Let $\mathbf{H}^0$ *denote the maximized Hamiltonian, or* $\mathbf{H}^0(x,\lambda) = \max_u \mathbf{H}(x,u,\lambda)$. *If the maximized Hamiltonian is a concave function of* x *for given* λ, *then* *functions* $x^*(t,z)$, $u^*(t,z)$ *and* $\lambda(t,z)$ *that satisfy conditions (14)-(16) for all* $z \in [z_0, z_1]$ *and the transversality conditions*

$$\lim_{t\to\infty} e^{-\rho t} \int_{z_0}^{z_1} \lambda(t,z)\,dz \geq 0,\ \lim_{t\to\infty} e^{-\rho t} \int_{z_0}^{z_1} \lambda(t,z)\,x(t,z)\,dz = 0 \quad (18)$$

or

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t,z)\,x(t,z) = 0 \text{ when } (\lambda(t,z), x(t,z)) \geq 0\ \forall\ t,z \quad (19)$$

solve the problem (10) to (13).

3 Optimal Harvesting under Biomass Diffusion

Having established the optimality conditions, we are interested in the implications of diffusion on optimally controlled systems regarding mainly the possibility of emergence of spatial heterogeneity under optimal control, but also the possibility of diffusion acting as a stabilizing force for unstable steady states under optimal control. To illustrate our approach we use a classical case from ecological economics, namely the optimal harvesting of a renewable biological resource (e.g. fishery). Let $x(t,z)$ denote the concentration of the biomass of a renewable resource (e.g. fish) at spatial point $z \in \mathcal{Z}$, at time t, with x taking non-negative values in a compact set $\mathcal{X}$, and $\mathcal{Z}$ a one-dimensional spatial domain such that $0 \leq z \leq a$. Boundary conditions could be either zero flux at $z=0$ and $z=a$, that is, $\left.\frac{\partial x(t,z)}{\partial z}\right|_{z=0} = \left.\frac{\partial x(t,z)}{\partial z}\right|_{z=a} = 0$, or of the hostile type that is, $x(t,0) = x(t,a) = 0$, implying that fish do not survive outside the spatial domain. Biomass grows according to a standard concave growth function $F(x)$ and disperses in space with a constant diffusion coefficient D, or

$$\frac{\partial x(t,z)}{\partial t} = F(x(t,z)) - H(t,z) + D\frac{\partial^2 x(t,z)}{\partial z^2}$$

Harvesting $H(t,z)$ of the resource is determined as $H(t,z) = qx(t,z)\,E(t,z)$, where $E(t,z)$ denotes harvesting effort (e.g. boats) at spatial point z and time t, taking non-negative values in a compact set $\mathcal{E}$, and $q > 0$ is the catchability coefficient. The total cost of applying effort $E(t,z)$ at location z is given by an increasing and convex function $c(E(t,z))$ in effort. Let benefits from harvesting at each point in space be given by an increasing and concave function $S(H(t,z))$. The optimal harvesting problem in space-time is then defined as:

$$\max_{E(t,z)} \int_0^\infty \int_{\mathcal{Z}} e^{-\rho t} \left[S(H(t,z)) - c(E(t,z)) \right] dz dt \tag{20}$$

$$\text{s.t. } \frac{\partial x(t,z)}{\partial t} = F(x(t,z)) - qx(t,z)E(t,z) + D\frac{\partial^2 x(t,z)}{\partial z^2} \tag{21}$$

$$x(0,z) \quad \text{given, and zero flux on } 0, a, \text{ or} \tag{22}$$

$$x(t,0) = x(t,a) = 0, \; x(0,z), \; z \in (0,a) \text{ given} \tag{23}$$

Following the results of the previous section, **MPD-IT** implies that the optimal control maximizes the generalized current value Hamiltonian for each location z,

$$\begin{aligned} \mathcal{H} =& S(H(t,z)) - c(E(t,z),z) + \\ & \mu(t,z)\left[x(t,z)(s - rx(t,z)) - qx(t,z)E + D\frac{\partial^2 x(t,z)}{\partial z^2} \right] \end{aligned} \tag{24}$$

Setting $S^{'}(H(t,z)) = p(z) > 0$, necessary conditions for the **MPD-IT**, omitting t to simplify notation, imply

$$\frac{\partial \mathcal{H}}{\partial E(z)} = 0 \text{ or } (p(z) - \mu(z))qx(z) = c^{'}(E(z)) \tag{25}$$

$$E^0(z) = E(x(z), \mu(z)), \; E^0(z) \geq 0, \text{ if } p(z) - \mu(z) \geq 0, \tag{26}$$

$$\frac{\partial E}{\partial x} = \frac{(p-\mu)q}{c''} > 0, \; \frac{\partial E}{\partial \mu} = -\frac{qx}{c''} < 0 \text{ for all } z \tag{27}$$

Then, the Hamiltonian system in space time becomes:

$$\frac{\partial x}{\partial t} = F(x) - qxE(x,\mu) + D\frac{\partial^2 x}{\partial z^2} = G_1(x,\mu) + D\frac{\partial^2 x}{\partial z^2} \tag{28}$$

$$\frac{\partial \mu}{\partial t} = \left[\rho - F^{'}(x) + qE(x,\mu) \right] \mu - pqE(x,\mu) - D\frac{\partial^2 \mu}{\partial z^2} = G_2(x,\mu) - D\frac{\partial^2 \mu}{\partial z^2} \tag{29}$$

The Hamiltonian system (28) - (29) indicates that in the optimally controlled system the resource's biomass moves from high concentration to low concentration, while the biomass shadow value moves in space from points of low value to points of high value. The purpose of our analysis is to examine conditions under which the *optimally controlled* diffusion system (28) - (29) could either produce a spatially heterogeneous pattern that will persist in the steady state, in the sense that the biomass concentration and the biomass shadow value will be different in different points of the spatial domain, or that the system will settle to a spatially homogeneous, or 'flat', state where the biomass concentration and the biomass shadow value are the same in every point of the spatial domain. We will explore the possibility of the Turing mechanism acting as a driver for inducing spatial heterogeneity.

3.1 The Turing mechanism in optimally controlled systems

The Turing mechanism for generating diffusion instability in reaction diffusion systems relies on the analysis of the stability of a spatially homogeneous (or 'flat') steady state of the associated dynamical system under perturbations induced by diffusion. In the optimally controlled system this implies that the Turing effect should be examined in association with the stability of the spatially homogeneous steady state of the Hamiltonian system (28) - (29). A "flat" steady state (x^*, μ^*) for this system is determined as the solution of $\frac{\partial x}{\partial t} = \frac{\partial \mu}{\partial t} = 0$ for $D = 0$. Given the nonlinear nature of (28) - (29), although it is possible to derive general conditions for the emergence of Turing instability, it not possible to derive closed form solutions and verify whether the conditions for the emergence of Turing instability are satisfied in a non-empty parameter set.

Since we feel it is important at this stage to verify the emergence of Turing instability in an optimally controled system under diffusion, a task which to our knowledge has not been performed, we will replace the non-linear control problem with its linear quadratic approximation and verify the emergence of Turing instability for the linear quadratic model. In this way we can derive precise conditions under which Turing instability can emerge in linear quadratic models or models that can be formulated in terms of their linear quadratic approximations.

We start by replacing problem (20) - (23) with its linear quadratic approximation. In doing so we extend the method developed by Fleming (1971), and Magill (1977)[12] - by which a non-linear optimal stochastic control problem is replaced by a simpler linear quadratic optimal stochastic control problem - to the case in which a deterministic control problem (such as a resource management problem), where the transition of the system is described by a partial differential equation with a diffusion term, and not by an ordinary differential equation, is replaced by a linear quadratic approximation.

Proposition 1 *Let (x^*, μ^*) be a flat steady state of the Hamiltonian system (28) - (29) satisfying the optimality conditions (14)-(16). Let E^* be the corresponding steady-state effort, and $H^* = qE^*x^*$. Then under certain conditions problem (20) - (23) can be replaced by the linear quadratic (LQ) problem:*

$$\max_{u(t,z)} \int_0^\infty \int_{\mathcal{Z}} e^{-\rho t} \left[-\frac{Q}{2} y^2 - \frac{R}{2} u^2 \right] dz dt \quad Q, R, \rho > 0 \tag{30}$$

$$s.t. \ \frac{\partial y(t,z)}{\partial t} = Sy(t,z) - Gu(t,z) + D\frac{\partial^2 y(t,z)}{\partial z^2}, S, G > 0 \tag{31}$$

$$y(0,z) \ \ \textit{given, and zero flux on } 0 \textit{ and } a, \textit{ or} \tag{32}$$

$$\textit{hostile boundary } y(t,0) = y(t,a) = 0, \ y(0,z), \ z \in (0,a) \ \textit{given} \tag{33}$$

where

$$(y(t,z), \gamma(t,z), p(t,z)) = \tag{34}$$
$$(x(t,z) - x^*, E(t,z) - E^*, \mu(t,z) - \mu^*)$$

$$\textit{and } u(t,z) = \gamma(t,z) + \frac{N}{B} y(t,z), N \gtreqless 0, B < 0 \tag{35}$$

and the initial state $x_0 = x(0,z)$ is close to x^* for all $z \in \mathcal{Z}$.

For the derivation and the definitions of the parameters of the LQ problem see Appendix.

Following the results of the previous section, **MPD-IT** implies that optimal control maximizes the generalized current value Hamiltonian for

[12] See also Judd (1996) for a similar approach.

the LQ problem for each location z,

$$\mathcal{H} = -\frac{Q}{2}y^2 - \frac{R}{2}u^2 + p(t,z)\left[Sy - Gu + D\frac{\partial^2 y}{\partial z^2}\right] \tag{36}$$

The necessary conditions for the **MPD-IT**, omitting t to simplify notation, imply

$$\frac{\partial \mathcal{H}}{\partial E(z)} = -Ru - pG = 0 \Rightarrow u^0 = -\frac{G}{R}p \tag{37}$$

$$\text{with } E = u^0 + E^* - N/B \geq 0 \quad \left(= 0 \text{ if } u^0 + E^* < N/B\right) \tag{38}$$

Then, the Hamiltonian system in space time becomes:

$$\frac{\partial y(t,z)}{\partial t} = Sy(t,z) - \frac{G}{R}p(t,z) + D\frac{\partial^2 y(t,z)}{\partial z^2} \tag{39}$$

$$\frac{\partial p(t,z)}{\partial t} = [\rho - S]\,p(t,z) + Qx(t,z) - D\frac{\partial^2 p(t,z)}{\partial z^2} \tag{40}$$

3.1.1 Existence of the Turing mechanism in optimally controlled LQ system

The flat steady state for (y^*, p^*) for the LQ problem is determined as the solution of $\frac{\partial y}{\partial t} = \frac{\partial p}{\partial t} = 0$ of (39) - (40) for $D = 0$. It is clear by the homogeneity of the flat system (39) - (40) that the origin is the steady state, or $(y^*, p^*) = (0,0)$. The stability of this steady state depends on the Jacobian matrix

$$J = \begin{bmatrix} S & G/R \\ Q & \rho - S \end{bmatrix}$$

Therefore for the flat steady state we have $\operatorname{tr}(J) = \rho > 0$ and $\det J = (\rho - S)\,S - G/R$. Hence, if $\det J > 0$ the steady state is unstable, while if $\det J < 0$ the steady state has the local saddle point property. In the saddle point case there is a one-dimensional manifold such that for any initial value of y there is an initial value for p, such that the system converges to the origin along the manifold.

The idea behind the Turing mechanism for diffusion driven instability and pattern formation is that an asymptotically stable, in the absence of diffusion, spatially homogeneous steady state, can be destabilized locally by perturbations induced by diffusion. The result of this instability could be the emergence of a regular stable patterned distribution of biomass and its shadow value across the spatial domain.

To analyze the impact of diffusion consider the Jacobian of the full Hamiltonian system (39) - (40), to obtain:

$$\mathbf{w}_t = J\mathbf{w} + \tilde{D}\mathbf{w}_{zz} \ , \tag{41}$$

$$\mathbf{w} = \begin{pmatrix} y(t,z) \\ p(t,z) \end{pmatrix}, \ \mathbf{w}_t = \begin{pmatrix} \partial y/\partial t \\ \partial p/\partial t \end{pmatrix} \ \mathbf{w}_{zz} = \begin{pmatrix} \partial^2 y/\partial z^2 \\ \partial^2 p/\partial z^2 \end{pmatrix}, \ \tilde{D} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \tag{42}$$

Following Murray (2003) we consider the time-independent solution of the spatial eigenvalue problem

$$\mathbf{W}_{zz} + k^2\mathbf{W} = 0, \ \mathbf{W}_z = \mathbf{0}, \ \text{for } z = 0, a \tag{43}$$

where k is the eigenvalue. For the one-dimensional domain $(0, a)$ we have solutions for (43) which are of the form

$$\mathbf{W}_k(z) = \mathbf{A}_n \cos\left(\frac{n\pi z}{a}\right) \ , \ n = \pm 1, \ \pm 2, ..., \tag{44}$$

where $\mathbf{A}_n$ are arbitrary constants. Solution (44) satisfies the zero flux condition at $z = 0$ and $z = a$.[13] The eigenvalue is $k = n\pi/a$, and $1/k = a/n\pi$ is a measure of the wave-like pattern. The eigenvalue k is called the *wavenumber* and $1/k$ is proportional to the wavelength $\omega : \omega = 2\pi/k = 2\alpha/n$. Let $\mathbf{W}_k(z)$ be the eigenfunction corresponding to the wavenumber k. We then look for solutions of (41) of the form

$$\mathbf{w}(t,z) = \sum_k c_k e^{\lambda t}\mathbf{W}_k(z) \tag{45}$$

Substituting (45) into (41), using (43) and canceling $e^{\lambda t}$ we obtain for

[13] If we are to use the hostile boundary conditions (4) then the solution would be of the form $\mathbf{W}_k(z) = \mathbf{A}_n \sin\left(\frac{n\pi z}{a}\right)$, $n = \pm 1, \pm 2, ...$, so that boundary conditions are satified at 0 and a.

each k or equivalently each n, that $\lambda \mathbf{W}_k = J\mathbf{W}_k - Dk^2\mathbf{W}_k$. Since we require non-trivial solutions for $\mathbf{W}_k$, λ must solve

$$\left|\lambda I - J + \tilde{D}k^2\right| = 0$$

Then the eigenvalue $\lambda(k)$ as a function of the wavenumber is obtained as the roots of

$$\lambda^2 - \rho\lambda + h\left(k^2\right) = 0 \tag{46}$$

$$h\left(k^2\right) = -D^2k^4 + D\left(2S - \rho\right)k^2 + \det J \tag{47}$$

where the roots are given by:

$$\lambda_{1,2}\left(k^2\right) = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4h\left(k^2\right)}\right)$$

It should be noted that the flat (no diffusion) case corresponds to $k^2 = 0$, so that $h\left(k^2 = 0\right) = \det J$, and $\lambda_{1,2} = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4\det J}\right)$. We examine the implication of diffusion in the case where the spatially homogeneous steady state is a saddle point, that is $\lambda_2 < 0 < \lambda_1$ for $k^2 = 0$, and diffusion generates spatial heterogeneity through the Turing mechanism.

In this case $\det J < 0$. Since tr $J > 0$ the spatially homogeneous system converges to the flat steady state $(x^*, p^*) = (0, 0)$ along the stable manifold. On this manifold and in the neighborhood of the steady state, for any initial value of y there is an initial value of p such that the spatially homogeneous system converges to the flat steady state. For the optimally-controlled system the optimal solution in the neighborhood of the steady state is such that

$$\begin{pmatrix} y^0\left(t, z\right) \\ p\left(t, z\right) \end{pmatrix} = C_2\mathbf{v}_2e^{\lambda_2 t}\text{, for all } z \tag{48}$$

where C_2 is a constant determined by initial conditions on y and transversality conditions, and $\mathbf{v}_2$ is the eigenvector corresponding to λ_2. In particular for the linearized system the transversality condition at infinity, $\lim_{t\to\infty} e^{-\rho t}\int_0^\infty p\left(t, z\right) y^0\left(t, z\right) = 0$ for all z, forces the constant C_1 associated with positive root λ_1 to be zero. Thus by choosing C_2 such that initial conditions on y and transversality conditions at infinity are satisfied, the initial conditions for p are selected such that the linearized system ends on the stable manifold. The corresponding path for the optimal control u is given by $u^0 = (-G/R)\, p(t,z)$ for all z. Solution (48) can be used to define the stable manifold as a function $p = \phi(y)$, and the associated optimal policy function $u^0 = \psi(y)$. By choosing appropriate values for y in the neighborhood of the steady state, such that $y_L < y^* < y_U$, the stable manifold can be represented by the set

$$M_S = \{(y,p) : p = \phi(y), y \in (y_L, y_U)\} \tag{49}$$

For any point along the manifold the state-costate system converges to the spatially homogeneous steady state.

We consider now the impact of a perturbation induced by diffusion. Under diffusion the smallest root λ_2 is given by

$$\lambda_2\left(k^2\right) = \frac{1}{2}\left(\rho - \sqrt{\rho^2 - 4h\left(k^2\right)}\right), k^2 = \frac{n^2\pi^2}{a^2} \tag{50}$$

Then,

- If $0 < h\left(k^2\right) < \rho^2/4$ for some k, then λ_2 becomes real and positive.
- If $h\left(k^2\right) > \rho^2/4$ for some k, then both roots corresponding to λ_2 are complex with positive real parts.

In both cases above, the linearly stable steady state $(y^*, p^*) \in M_S$ becomes unstable to *spatial* disturbances. Therefore if $h\left(k^2\right) > 0$ for some k, then $\lambda_2\left(k^2\right) > 0$ and the optimally controlled Hamiltonian system becomes unstable to spatial perturbations, in the neighborhood of the flat steady state and along the stable manifold. From (47) the quadratic function $h\left(k^2\right)$ is concave, and therefore has a maximum. Furthermore, $h(0) = \det J < 0$ and $h'(0) = (2S - \rho)$. Then $h\left(k^2\right)$ has a maximum for

$$k^2_{\max} : h'\left(k^2_{\max}\right) = 0, \text{ or } k^2_{\max} = \frac{(2S-\rho)}{2D} > 0, \text{ for } (2S-\rho) > 0 \tag{51}$$

If $h\left(k_{\max}^2\right) > 0$ or $-D^2 k_{\max}^4 + D\left(2S-\rho\right)k_{\max}^2 + \det J > 0$, and $2S-\rho > 0$, then there exist two positive roots $k_1^2 < k_2^2$ such that $h\left(k^2\right) > 0$ and $\lambda_2\left(k^2\right) > 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$. Using (51) the existence of two positive roots $k_1^2 < k_2^2$ requires

$$\frac{\left(2S-\rho\right)^2}{4} + \det J > 0 \text{ , or} \tag{52}$$

$$\frac{\left(2S-\rho\right)^2}{4} + \left(\rho - S\right)S - \frac{QG}{R} = \frac{\rho^2}{4} - \frac{QG}{R} > 0 \tag{53}$$

The interval (k_1, k_2) determines the range of the unstable modes associated with the spatial heterogeneous solution, while $h\left(k^2\right)$ is the dispersion relationship associated with the optimal control problem.[14] Diffusion driven instability in the optimally controled system emerges if the maximum of the dispersion relationship is in the positive quadrant along with the negative condition on the Jacobian of the flat system. These conditions are summarized below.

$$\left(\rho - S\right)S - \frac{QG}{R} < 0 \tag{54}$$

$$2S - \rho > 0 \tag{55}$$

$$\frac{\rho^2}{4} - \frac{QG}{R} > 0 \tag{56}$$

with

$$k_{1,2}^2 = \frac{\left(2S-\rho\right) \pm \sqrt{\left(\rho^2 - 4QG/R\right)}}{2} > 0 \tag{57}$$

The set of parameters for which (54)-(56) is satisfied is the *Turing space.* It is clear that for $\rho = 0$ the Turing space is empty and diffusion driven instability does not emerge. However for higher discount rates and for appropriate values of Q, G, S and R, the Turing space need not be empty. This is shown in figure 1 where the Turing space is defined in the (ρ, R) space for given values of Q, G, S.

[Figure 1]

The inequality (54) is satisfied above line BB, the inequality (55)

[14] For a detailed analysis of the dispersion relationship in problems without optimization, see Murray (2003).

is satisfied below the line $2SCD$, while the inequality (56) is satisfied above the line AA. Thus the Turing space is the area DCB.

Assume that for a parameter constellation (ρ, S, R, Q, G) the Turing set is not empty. Then the optimal spatially heterogeneous solution, under zero flux boundary conditions emerging from (44) and (45), is the sum of unstable modes or

$$\mathbf{w}^0(t,z) \sim \sum_{n_1}^{n_2} \mathbf{B}_n \exp\left[\lambda_2\left(\frac{n^2\pi^2}{a^2}\right)t\right]\cos\frac{n\pi z}{a}, k^2 = \left(\frac{n\pi}{a}\right)^2$$

where $\lambda_2(k^2) > 0$ for $k^2 \in (k_1^2, k_2^2)$, n_1 is the smallest integer greater or equal to ak_1/π and n_2 is the largest integer less than or equal to ak_2/π, and the wavenumbers k_1 and k_2 are such that $h(k^2) > 0$. Since $\lambda_2(k^2) > 0$ for $k^2 \in (k_1^2, k_2^2)$ only these modes grow with time; all the remaining modes for which $\lambda_2(k^2) < 0$ tend to zero exponentially. Assume that the spatial domain is such that there is only one unstable wave number, or $n = 1$. Then the only unstable mode is $\cos(\pi z/a)$, and the growing instability is determined by

$$\mathbf{w}^0(t,z) \sim \mathbf{B}_1 \exp\left[\lambda_2\left(\frac{\pi^2}{a^2}\right)t\right]\cos\frac{\pi z}{a} \tag{58}$$

where the vector of constants $\mathbf{B}_1$ is determined by initial conditions. Since the instability occurs on the stable manifold of the linearized system (49) it would be natural to choose initial conditions for y and p on this manifold. Take $\mathbf{B}_1 = (\epsilon_x, \epsilon_p)$, then using the definition of $\mathbf{w}$ from (42) we have that the optimal spatially heterogeneous solution evolves approximately as:

$$y^0(t,z) \sim \epsilon_x \exp\left[\lambda_2\left(\frac{\pi^2}{a^2}\right)t\right]\cos\frac{\pi z}{a}, \ \frac{\pi^2}{a^2} = k^2 \tag{59}$$

$$p(t,z) \sim \epsilon_p \exp\left[\lambda_2\left(\frac{\pi^2}{a^2}\right)t\right]\cos\frac{\pi z}{a} \tag{60}$$

Solutions (59) - (60) indicate that diffusion causes the spatially homogeneous steady state to be transformed into a wave-like pattern as t increases. This of course is spatial heterogeneity since the biomass and its shadow value will, at any given point in time, have different values in different spatial points. Then the path for optimal effort in the neighborhood of the flat steady state will be determined as $u^0(t,z) = (-G/R)\, p(t,z)$, while the spatially heterogeneous optimal effort is determined, using (34) and (35) as:

$$E^0(t,z) = E^* + u^0(t,z) - \frac{N}{B} = E^* - \frac{G}{R} p(t,z) - \frac{N}{B},\ E^0(t,z) \geq 0 \quad (61)$$

Furthermore a conjecture can be stated. For the optimal paths $(y^0(t,z), u^0(t,z))$ of the solution to the LQ problem, an analog in time-space of a Michel-type transversality condition (Michel, 1982) is verified. This transversality conditions implies that the maximum of the Hamiltonian of the LQ problem for every spatial point is zero when t goes to infinity. Following Michel (1982) the maximum of the Hamiltonian should verify for every (t,z) that:

$$\begin{aligned}\mathcal{H}^{\max}(t,z) = e^{-\rho t} \Bigg[& -\frac{Q}{2}\left(y^0(t,z)\right)^2 - \frac{R}{2}\left(u^0(t,z)\right)^2 + \\ & p(t,z)\left(Sy^0(t,z) - Gu^0(t,z) + D\frac{\partial^2 y(t,z)}{\partial z^2}\right)\Bigg]\end{aligned} \quad (62)$$

Substituting (59) - (60) into (62), taking the limit as $t \to \infty$ and noting that, by the definition of a steady state, for all z the term

$$p(t,z)\left[Sy^0(t,z) - Gu^0(t,z) + D\frac{\partial^2 y(t,z)}{\partial z^2}\right] \quad (63)$$

is zero as $t \to \infty$, we obtain

$$\begin{aligned}& \lim_{t\to\infty} \mathcal{H}^{\max}(t,z) = \\ \lim_{t\to\infty} & \left[-\frac{1}{2} e^{(2\lambda_2 - \rho)t}\left[Q\epsilon_x^2 + \frac{G^2}{R}\epsilon_p^2\right]\cos^2\left(\frac{\pi z}{a}\right)\right] = 0\end{aligned} \quad (64)$$

since, as can be seen from (50), $2\lambda_2 < \rho$.

The value function of the LQ problem

$$V(y(0,z), 0, 0) = \sup \int_0^\infty \int_{\mathcal{Z}} e^{-\rho t}\left[-\frac{Q}{2}y^2 - \frac{R}{2}u^2\right] dz dt \quad (65)$$

should verify that:

$$V\left(y^0(0,z),0,0\right) = \tag{66}$$
$$\int_0^\infty \int_{\mathcal{Z}} e^{-\rho t}\left[-\frac{Q}{2}\left(y^0(t,z)\right)^2 - \frac{R}{2}\left(u^0(t,z)\right)^2\right] dzdt =$$
$$-\frac{1}{2}\int_0^\infty e^{(2\lambda_2-\rho)t}\int_{\mathcal{Z}} e^{(2\lambda_2-\rho)t}\left[Q\epsilon_x^2 + \frac{G^2}{R}\epsilon_p^2\right]\cos^2\left(\frac{\pi z}{a}\right) dtdz \tag{67}$$

which is finite since $2\lambda_2 < \rho$, indicating that the LQ problem is well posed. These results can be summarized in the following proposition.

Proposition 2 *For an optimal harvesting system of an LQ form or for a non-linear system that can be adequately approximated by an LQ system, which exhibits the saddle point property at a steady state in the absence of diffusion, it is optimal, under biomass diffusion and for a certain set of parameter values, to have emergence of diffusive instability, induced by the Turing mechanism. Diffusive instability leads to a spatially heterogeneous optimal path where the biomass and its shadow value will, at any given point in time, have different values in different spatial points.*

The significance of this proposition, which extends the concept of the Turing mechanism to the optimal control of diffusion, is that spatial heterogeneity and pattern formation, resulting from diffusive instability, might be an optimal outcome under certain circumstances. For regulation purposes and for the harvesting problem examined above, it is clear that the spatially heterogeneous steady-state shadow value of the resource stock, and the corresponding harvesting effort, can be used to define *optimal regional* fees or quotas. Although the full characterization of the spatially heterogeneous steady state is outside the purpose of this paper, since our target is to show the existence of the Turing mechanism in optimally controlled systems, there are some inferences that can be heuristically made from the results obtained by the LQ problem.

If the LQ approximation is an adequate one for the non-linear system, it is expected that a saddle point steady state of the non-linear system (x^*, μ^*) will also be destabilized by perturbations caused by diffusion through the Turing effect.[15] With a non-empty Turing space, spatially heterogenous solutions similar to (59) - (60) grow exponentially. This however cannot be valid for all t, since then exponential growth would imply that $(x, \mu) \to \infty$ at $t \to \infty$. However, the kinetics of the Hamiltonian system (28) - (29) and the transversality conditions at infinity (18) should bound the solution in the positive quandrant.[16] This implies that for a subset of the spatial domain the resource stock and its shadow value are above the flat steady-state levels and for another subset they are below the flat steady-state levels, in a wave-like pattern. In this case an ultimate steady-state spatially heterogeneous solution for the optimally controlled system will emerge.[17] This steady state can be characterized by taking the steady state of (28) - (29) and defining the dynamic system in the spatial domain $[0, a]$.

$$0 = F(x) - qxE(x, \mu) + D\frac{\partial^2 x}{\partial z^2}, \text{or } -G_1(x, \mu) = D\frac{\partial^2 x}{\partial z^2} \tag{68}$$

$$0 = \left[\rho - F'(x) + qE(x, \mu)\right]\mu - pqE(x, \mu) - D\frac{\partial^2 \mu}{\partial z^2}, \tag{69}$$

$$\text{or } G_2(x, \mu) = D\frac{\partial^2 \mu}{\partial z^2} \tag{70}$$

Setting $v = \frac{\partial x}{\partial z}$, $u = \frac{\partial \mu}{\partial z}$, we obtain the first-order system

$$-G_1(x, \mu) = D\frac{\partial v}{\partial z} \tag{71}$$

$$v = \frac{\partial x}{\partial z} \tag{72}$$

$$G_2(x, \mu) = D\frac{\partial u}{\partial z} \tag{73}$$

$$u = \frac{\partial \mu}{\partial z} \tag{74}$$

Under zero flux boundary conditions the boundary conditions for this system are $v(0) = v(a) = 0$, and $u(0) = u(a) = 0$ from zero flux,

[15] It should be noticed that it was around this steady state that the LQ approximation was carried out.

[16] See Murray (2003, Vol II, pp. 93-94) for this type of argument.

[17] In this context it may be shown (Segel and Levin, 1976) that the destabilized spatially homogeneous pattern is replaced asymptotically by a stable spatially heterogeneous solution.

while under hostile boundary conditions we have $x(0) = x(a) = 0$ and $\mu(0) = \mu(a) = 0$.

3.1.2 Diffusion as a stabilizer

We examine now the case where the spatially homogeneous steady state is unstable, that is $\operatorname{Re}\lambda_{1,2} > 0$ for $k^2 = 0$, and diffusion acts as a stabilizing form. Since $\operatorname{tr} J > 0$, this implies that $\det J > 0$. Let $\Delta_D = \rho^2 - 4[\det J] > 0$ so that we have two positive real roots at the flat steady state. Diffusion can stabilize the system in the sense of producing a negative root. For the smallest root to turn negative or $\lambda_2 < 0$, it is sufficient that $h(k^2) < 0$. The quadratic function (47) is concave, and therefore has a maximum. Furthermore $h(0) = \det J > 0$ and if $h'(0) = (2F - \rho) > 0$ there is a root $k_2^2 > 0$, as shown in figure 2, such that for $k^2 > k_2^2$, we have $\lambda_2 < 0$. The solutions for $y(t,z)$ and $p(t,z)$ will be determined by the sum of exponentials of λ_1 and λ_2. Since we want to stabilize the system we set the constant associated with the positive root λ_1 equal to zero. Then the solution will depend on the sum of unstable and stable modes associated with λ_2.

[Figure 2]

Following the previous procedure the solutions for y and p will be of the form:

$$\begin{pmatrix} y(t,z) \\ p(t,z) \end{pmatrix} \sim \sum_{0}^{n_2} \mathbf{C}_{\hat{n}} \exp\left[\lambda_2\left(\frac{n^2\pi^2}{a^2}\right)t\right]\cos\frac{n\pi z}{a} + \sum_{n_2}^{N} \mathbf{C}_n \exp\left[\lambda_2\left(\frac{n^2\pi^2}{a^2}\right)t\right]\cos\frac{n\pi z}{a}, \tag{75}$$

where n_2 is the smallest integer greater or equal to ak_2^2/π and $N > n_2$. Since $\lambda_2\left(\frac{n^2\pi^2}{a^2}\right) < 0$ for $n > n_2$, all the modes of the second term of (75) decay exponentially. So to converge to the steady state we need to set $\mathbf{C}_{\hat{n}} = 0$, then the spatial patterns corresponding to the second term of (75) will die out with the passage of time and the system will converge to the spatially homogeneous steady state $(y^*, p^*) = (0,0)$.

This result can be summarized in the following proposition.

Proposition 3 *For an optimal harvesting system of an LQ form or for a nonlinear system that can be adequately approximated by an LQ system, with an unstable steady state in the absence of diffusion, it is optimal, under biomass diffusion and for a certain set of parameter values, to stabilize the steady state. Stabilization is in the form of saddle point stability where spatial patterns decay and the system converges along one direction to the previously unstable spatially homogeneous steady state.*

The significance of this proposition is that it shows that under diffusion it is optimal to stabilize a steady state which was unstable under spatial homogeneity.

4 Concluding Remarks

The present paper seeks to provide a conceptual framework for studying pattern formation in optimally controlled systems associated with economic applications. Considering the Turing mechanism as the pattern generator we develop the optimal control of a dynamical system under diffusion by appropriately extending Pontryagin's maximum principle. Using as our leading example the classical problem of harvesting of a renewable resource (fishery) we show that, when we have a saddle point equilibrium with zero diffusion for a positive discount rate, then there exists a non-empty parameter set such that the Turing mechanism acting on the associated Hamiltonian Dynamic System implies that the optimal choice of control (harvesting effort) in time-space leads to the emergence of a spatial pattern for both the resource stock (state variable) and its corresponding shadow value (costate variable). In the same context we show that, when we have an unstable steady state with zero diffusion, then the presence of diffusion in the optimal harvesting problem can, in certain cases, stabilize an unstable spatially homogeneous steady state.

The methodological approach developed in this paper can be linked to further research in the optimal management and the design of optimal policies for general complex adaptive systems arising in economics, where self organizing aspects reflected in notions such as 'the invisible hand' or Pareto optimality are complemented by policy interactions aiming at directing the system to a desired outcome (Levin, 2002). The spatial and pattern formation aspect of these complex adaptive systems, with the Turing mechanism acting as a pattern generator, when coupled with policy interventions produce the type of optimal control problem in space-time studied in this paper.

In more general terms the Turing mechanism is one pattern generator that can be used in the study of socio-economic systems in the context of developing statistical mechanics approaches aiming at exploring how individual microscopic interactions give rise to macroscopic phenomena (Durlauf, 1997). It should be noted that the application of pattern generators to complex socio-economic systems has yet to overcome tractability issues, although there are some exceptions such as the Large Type Limit concept (Brock et al., 2005) and its generalization (Diks and Vanderweide, 2003) that provide an analytically tractable pattern generator for stock market applications. The use of the Turing mechanism as pattern generator in recursive infinite horizon optimal control developed in this paper, apart from its usefulness in studying other economic applications and pattern formation in time-space, can also be useful as a basis for extending the analysis to general pattern generating systems where patterns emerge from individual agent heterogeneity into macroscopic dynamics and macroscopic patterns.

Appendix

Extension of Pontryagin's Principle: Necessary conditions

We develop a variational argument along the lines of Kamien and Schwartz (1981, pp. 115-116). Problem (1) to (4) can be written as:

$$J = \int_{z_0}^{z_1} \int_{t_0}^{t_1} f\left(x\left(t,z\right), u\left(t,z\right)\right) dtdz = \int_{z_0}^{z_1} \int_{t_0}^{t_1} \left\{ f\left(x\left(t,z\right), u\left(t,z\right)\right) \right. \\ \left. \lambda\left(t,z\right) \left[g\left(x\left(t,z\right), u\left(t,z\right)\right) + D\frac{\partial^2 x}{\partial z^2} - \frac{\partial x}{\partial t} \right] \right\} dtdz \qquad (76)$$

We integrate by parts the last two terms of (76). The $\lambda\left(t,z\right)\frac{\partial x}{\partial t}$ term becomes

$$(-1) \int_{z_0}^{z_1} \int_{t_0}^{t_1} \lambda\left(t,z\right) \frac{\partial x}{\partial t} dt = \\ \int_{z_0}^{z_1} \left[-\lambda\left(t_1\right) x\left(t_1\right) + \lambda\left(t_0\right) x\left(t_0\right) + \int_{t_0}^{t_1} x\left(t,z\right) \frac{\partial \lambda}{\partial t} dt \right] dz \qquad (77)$$

The term $\lambda\left(t,z\right) D\frac{\partial^2 x}{\partial z^2}$ becomes

$$D \int_{z_0}^{z_1} \int_{t_0}^{t_1} \lambda\left(t,z\right) \frac{\partial^2 x\left(t,z\right)}{\partial z^2} = \\ D \int_{t_0}^{t_1} \left[\lambda\left(t,z_1\right) \left.\frac{\partial x\left(t,z\right)}{\partial z}\right|_{z=z_1} - \lambda\left(t,z_0\right) \left.\frac{\partial x\left(t,z_0\right)}{\partial z}\right|_{z=z_0} - \int_{z_0}^{z_1} \frac{\partial x\left(t,z\right)}{\partial z} \frac{\partial \lambda\left(t,z\right)}{\partial t} dz \right] dt = \qquad (78) \\ -D \int_{t_0}^{t_1} \left[\int_{z_0}^{z_1} \frac{\partial x\left(t,z\right)}{\partial z} \frac{\partial \lambda\left(t,z\right)}{\partial t} dz \right] dt$$

by the zero flux conditions (3) on the state variable, or by setting $\lambda\left(z_1\right) = \lambda\left(z_0\right) = 0$ if we use the hostile boundary conditions $x\left(z_1\right) = x\left(z_0\right) = 0$.

Integrating by parts once more we have

$$
\begin{aligned}
&(-1)\,D\int_{t_0}^{t_1}\left[\int_{z_0}^{z_1}\frac{\partial x\,(t,z)}{\partial z}\frac{\partial\lambda\,(t,z)}{\partial t}dz\right]dt = \\
&D\int_{t_0}^{t_1}\left[-\frac{\partial\lambda\,(t,z_1)}{\partial z}x\,(t,z_1)+\frac{\partial\lambda\,(t,z_0)}{\partial z}x\,(t,z_0)\right. \\
&\left.+\int_{z_0}^{z_1}x\,(t,z)\,\frac{\partial^2\lambda\,(t,z)}{\partial z^2}\right]dzdt
\end{aligned}
\tag{79}
$$

Thus (76) becomes

$$
\begin{aligned}
&\int_{z_0}^{z_1}\int_{t_0}^{t_1}f\,(x\,(t,z)\,,u\,(t,z))\,dtdz = \\
&\int_{z_0}^{z_1}\int_{t_0}^{t_1}\left[f\,(x\,(t,z)\,,u\,(t,z))+\lambda\,(t,z)\,g\,(x\,(t,z)\,,u\,(t,z))\right. \\
&\left.+x\,(t,z)\,\frac{\partial\lambda\,(t,z)}{\partial t}+x\,(t,z)\,D\frac{\partial^2\lambda\,(t,z)}{\partial z^2}\right]dtdz \\
&+\int_{z_0}^{z_1}\left[-\lambda\,(t_1,z)\,x\,(t_1,z)+\lambda\,(t_0,z)\,x\,(t_0,z)\right]dz+ \\
&D\int_{t_0}^{t_1}\left[-\frac{\partial\lambda\,(t,z_1)}{\partial z}x\,(t,z_1)+\frac{\partial\lambda\,(t,z_0)}{\partial z}x\,(t,z_0)\right]dt
\end{aligned}
\tag{80}
$$

We consider a one parameter family of comparison controls $u^*\,(t,z)+\epsilon\eta\,(t,z)\,,$ where $u^*\,(t,z)$ is the optimal control, $\eta\,(t,z)$ is a fixed function and ϵ is a small parameter. Let $y\,(t,z,\epsilon)\,,\ t\in[t_0,t_1]\,,\ z\in[z_0,z_1]$ be the state variable generated by (2) and (3) or (4) with control $u^*\,(t,z)+\epsilon\eta\,(t,z)\,,\ t\in[t_0,t_1]\,,\ z\in[z_0,z_1]\,.$ We assume that $y\,(t,z,\epsilon)$ is a smooth function of all its arguments and that ϵ enters parametrically. For $\epsilon=0$ we have the optimal path $x^*\,(t,z)\,;$ furthermore all comparison paths must satisfy initial and zero flux or hostile boundary conditions. Thus,

$$
y\,(t,z,0)=x^*\,(t,z)\,,\ \ y\,(t_0,z,\epsilon)=x\,(t_0,z)\ \text{ fixed} \tag{81}
$$

$$
\left.\frac{\partial y\,(t,z)}{\partial z}\right|_{z=z_0}=\left.\frac{\partial y\,(t,z)}{\partial z}\right|_{z=z_1}=0\ ,\ \text{ zero flux} \tag{82}
$$

$$
y\,(t,z_1,\epsilon)=y\,(t,z_0,\epsilon)=0\ \text{hostile boundary} \tag{83}
$$

When the functions $u^*, x^*$and η are held fixed, the value of (1) evaluated along the control function $u^*(t,z)+\epsilon\eta(t,z)$ and the corresponding state function $y(t,z,\epsilon)$ depend only on the single parameter ϵ. Therefore,

$$J(\epsilon)=\int_{z_0}^{z_1}\int_{t_0}^{t_1}\left[f\left(y(t,z,\epsilon),u^*(t,z)+\epsilon\eta(t,z)\right)\right]dtdz$$

or using (80)

$$\begin{aligned}J(\epsilon)=&\int_{z_0}^{z_1}\int_{t_0}^{t_1}\Big[f\left(y(t,z,\epsilon),u^*(t,z)+\epsilon\eta(t,z)\right)\\&+\lambda(t,z)\,g\left(y(t,z,\epsilon),u^*(t,z)+\epsilon\eta(t,z)\right)\\&+y(t,z,\epsilon)\frac{\partial\lambda(t,z)}{\partial t}+Dy(t,z,\epsilon)\frac{\partial^2\lambda(t,z)}{\partial z^2}\Big]dtdz\\&+\int_{z_0}^{z_1}\left[-\lambda(t_1,z)\,y(t_1,z,\epsilon)+\lambda(t_0,z)\,y(t_0,z,\epsilon)\right]dz\\&+D\int_{t_0}^{t_1}\left[-\frac{\partial\lambda(z_1)}{\partial z}y(t,z_1,\epsilon)+\frac{\partial\lambda(z_0)}{\partial z}y(t,z_0,\epsilon)\right]dt\end{aligned}\tag{84}$$

Since u^* is a maximizing control the function $J(\epsilon)$ assumes the maximum when $\epsilon=0$. Thus $\left.\frac{dJ(\epsilon)}{d\epsilon}\right|_{\epsilon=0}$ or

$$\begin{aligned}&\left.\frac{dJ(\epsilon)}{d\epsilon}\right|_{\epsilon=0}=\\&\int_{z_0}^{z_1}\int_{t_0}^{t_1}\left[\left(f_x+\lambda g_x+\frac{\partial\lambda(t,z)}{\partial t}+D\frac{\partial^2\lambda(t,z)}{\partial z^2}\right)y_\epsilon+(f_u+\lambda g_u)\,\eta(t,z)\right]dtdz+\\&\int_{z_0}^{z_1}\left[-\lambda(t_1,z)\,y_\epsilon(t_1,z,\epsilon)+\lambda(t_0,z)\,y_\epsilon(t_0,z,\epsilon)\right]dz+\\&D\int_{t_0}^{t_1}\left[-\frac{\partial\lambda(z_1)}{\partial z}y_\epsilon(t,z_1,\epsilon)+\frac{\partial\lambda(z_0)}{\partial z}y_\epsilon(t,z_0,\epsilon)\right]dt=0\end{aligned}\tag{85}$$

- In (85) $y_\epsilon(t_0,z,\epsilon)=0$, since $y(t_0,z,\epsilon)=x(t_0,z)$ fixed by initial conditions. Next we impose the condition

$$\int_{z_0}^{z_1}\lambda(t_1,z)\,\beta(t_1,z)=0\tag{86}$$

  for all $\beta(t_1,z)$ piecewise continuous functions in $[z_0,z_1]$. It follows, using Athans and Falb's (1996, p260) fundamental lemma that

$$\lambda(t_1, z) = 0 \ , z \in [z_0, z_1] \tag{87}$$

Furthermore if we impose zero flux conditions on λ, then,

$$\frac{\partial \lambda(t, z_1)}{\partial z} = \frac{\partial \lambda(t, z_0)}{\partial z} = 0 \tag{88}$$

Conditions (86) or (87) and (88) can be used as transversality conditions. Then we obtain from ((85))

$$\frac{\partial \lambda}{\partial t} = -\left(f_x + \lambda g_x + D\frac{\partial^2 \lambda}{\partial z^2}\right) \tag{89}$$

$$f_u + \lambda g_u = 0 \tag{90}$$

- If we use hostile boundary conditions then from (83), $y(t, z_1, \epsilon) = y(t, z_0, \epsilon) = 0$ fixed, and $y_\epsilon(t_0, z, \epsilon) = y_\epsilon(t, z_1, \epsilon) = 0$ in (85). Then (89) and (90) are obtained by imposing transversality conditions (86) or (87).

So if we define a generalized Hamiltonian function

$$\mathbf{H} = f(x, u) + \lambda\left[g(x, u) + D\frac{\partial^2 x}{\partial z^2}\right]$$

then by (89) and (90) optimality conditions become conditions (5) - (15), along with the appropriate transversality conditions.

The infinite horizon case with discounting is obtained by following the same approach and using Arrow and Kurz (1970, Chapter II.6).□

Extension of Pontryagin's Principle: Sufficiency

Suppose that $x^*(t, z), u^*(t, z), \lambda(t, z)$ satisfy conditions (5) and (15) and let $x(t, z), u(t, z)$ functions satisfy (2). Let f^*, g^* denote functions evaluated along $(x^*(t, z), u^*(t, z))$ and let f, g denote functions evaluated along the feasible path $(x(t, z), u(t, z))$. To prove sufficiency we need to show that

$$W \equiv \int_{z_0}^{z_1} \int_{t_0}^{t_1} (f^* - f)\, dt dz \geq 0$$

From the concavity of f it follows that

$$f^* - f \geq (x^* (t,z) - x(t,z))\, f_x^* + (u^* (t,z) - u(t,z))\, f_u^* \tag{91}$$

Then

$$W \geq \int_{z_0}^{z_1} \int_{t_0}^{t_1} \left[(x^* (t,z) - x(t,z))\, f_x^* + (u^* (t,z) - u(t,z))\, f_u^* \right] dt dz \tag{92}$$

$$= \int_{z_0}^{z_1} \int_{t_0}^{t_1} \left[(x^* (t,z) - x(t,z)) \left(-\frac{\partial \lambda (t,z)}{\partial t} - \lambda (t,z)\, g_x^* - D \frac{\partial^2 \lambda (t,z)}{\partial z^2} \right) \right. \tag{93}$$

$$\left. + \, (u^* (t,z) - u(t,z)) \left(-\lambda (t,z)\, g_u^* \right) \right] dt dz$$

$$= \int_{z_0}^{z_1} \int_{t_0}^{t_1} \lambda \left[g^* - g - (x^* (t,z) - x(t,z))\, g_x^* - (u^* (t,z) - u(t,z)) \right] g_u^* dt dz \geq 0 \tag{94}$$

Condition (93) follows from (92) by using conditions (5) and (15) to substitute for f_x^* and f_u^*. Condition (94) is derived by integrating first by parts the terms involving $\frac{\partial \lambda}{\partial t}$, substituting for $\frac{\partial x}{\partial t}$ from (2), and using the transversality conditions, as has been done above, then by integrating twice the terms involving $\frac{\partial^2 \lambda}{\partial z^2}$ and using again the zero flux or the hostile boundary conditions. The non-negativity of the integral in (94) follows from (17) and the concavity of g.

The result can be easily extended along the lines of Arrow's sufficiency theorem (Arrow and Kurz, 1970, Chapter II.6) with a transversality condition at infinity.

$$\lim_{t \to \infty} e^{-\rho t} \int_{z_0}^{z_1} \lambda (t,z)\, dz \geq 0, \; \lim_{t \to \infty} e^{-\rho t} \int_{z_0}^{z_1} \lambda (t,z)\, x(t,z)\, dz = 0, \text{or} \tag{95}$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda (t,z)\, x(t,z) = 0 \text{ when } (\lambda (t,z), x(t,z)) \geq 0 \text{ for all } t, z \tag{96}$$

□

Linear - Quadratic Approximation of the Optimal Control Problem under Diffusion

First we derive the LQ approximation for the general problem and then we apply it to the problem of optimal harvesting.

Consider the general optimal control problem under diffusion.

$$\max_{\{u(t,z)\}} \int_{z_0}^{z_1} \int_{t_0}^{t_1} f\left(x\left(t,z\right), u\left(t,z\right)\right) dt dz \tag{97}$$

$$\text{s.t.} \quad \frac{\partial x\left(t,z\right)}{\partial t} = g\left(x\left(t,z\right), u\left(t,z\right)\right) + D\frac{\partial^2 x\left(t,z\right)}{\partial z^2} \tag{98}$$

$$x\left(t_0,z\right) \text{ given, } \left.\frac{\partial x\left(t,z\right)}{\partial z}\right|_{z=z_0} = \left.\frac{\partial x\left(t,z\right)}{\partial z}\right|_{z=z_1} = 0 \text{ , zero flux} \tag{99}$$

$$x\left(t,z_0\right) = x\left(t,z_1\right) = 0, \text{ hostile boundary } x\left(t_0,z\right), \ z \in \left(z_0,z_1\right) \text{ given} \tag{100}$$

with the Hamiltonian function

$$\mathbf{H}\left(x\left(t,z\right), u\left(t,z\right), \lambda\left(t,z\right)\right) = f\left(x,u\right) + \lambda\left[g\left(x,u\right) + D\frac{\partial^2 x}{\partial z^2}\right] \tag{101}$$

For problem (97) - (100) let (x^*, u^*, λ^*) be a flat optimal steady state associated with the Hamiltonian system (5)-(6) for $D = 0$. This optimal steady state satisfies the optimality conditions (5)-(9). Our approach is to extend the method developed by Fleming (1971) and Magill (1977), by which a non-linear optimal stochastic control problem is replaced by a simpler linear quadratic optimal stochastic control problem, to the case of a deterministic control problem, such as (97) - (100) where the transition of the system is described by a PDE with a diffusion term and not by a stochastic ODE. Assume that the diffusion process (98) starts close to the steady state or that $x_0 = x(0,z)$ starts close to x^* for all $z \in Z$, and let $(y(t,z), \gamma(t,z), p(t,z)) = (x(t,z) - x^*, u(t,z) - u^*, \lambda(t,z) - \lambda^*)$. Perturb the control u by letting

$$u\left(t,z\right) = u^* + \varepsilon\left(u\left(t,z\right) - u^*\right) = u^* + \varepsilon\gamma\left(t,z\right) \tag{102}$$

For a control of the form (102) we adapt Athans and Falb (1966 page 261) to focus on perturbations of the form below,

$$x(t,z) = x^* + \varepsilon y(t,z) + \varepsilon^2 \xi(t,z) + o\left(\varepsilon^2, t, z\right) \tag{103}$$

where y and ξ are first and second order state perturbations respectively and $o(\varepsilon^2, t, z) \to 0$ as $\varepsilon^2 \to 0$ uniformly in (t, z).

Athans and Falb (1966, pp. 254-265) show that control perturbations of the form (102) lead to state perturbations of the form (103) under appropriate regularity conditions for the case where $\mathcal{Z}$ is one point. We proceed heuristically here. Substituting (103) and (102) into (98), the $g(x,u)$ function describing the kinetic of the state variable, we obtain

$$g\left(x^* + \varepsilon y(t,z) + \varepsilon^2 \xi(t,z) + o\left(\varepsilon^2, t, z\right), u^* + \varepsilon \gamma(t,z)\right) \tag{104}$$

Substituting also for $x(t,z)$ in the derivative $\frac{\partial x(t,z)}{\partial t}$ and $\frac{\partial^2 x(t,z)}{\partial z^2}$, using (103) and expanding as a Taylor series around (x^*, u^*), we obtain[18]

$$\begin{aligned} &\varepsilon \frac{\partial y(t,z)}{\partial t} + \varepsilon^2 \frac{\partial \xi(t,z)}{\partial t} = g(x^*, u^*) + g_x\left(\varepsilon y + \varepsilon^2 \xi\right) + g_u(\varepsilon \gamma) + w' W w + \\ &+\varepsilon D \frac{\partial^2 y(t,z)}{\partial z^2} + \varepsilon^2 D \frac{\partial \xi^2(t,z)}{\partial z^2} + \text{ higher order terms} \\ &w = \left(\varepsilon y + \varepsilon^2 \xi, \varepsilon \gamma\right)', W = \begin{pmatrix} g_{xx}\, g_{xu} \\ g_{ux}\, g_{uu} \end{pmatrix} \end{aligned} \tag{105}$$

where all derivatives are evaluated at the flat steady state. Divide (105) by ε and then take the limit as $\varepsilon \to 0$, and note that $g(x^*, u^*) = 0$ because (x^*, u^*) is a steady state, to obtain the linear approximation of (98) around the flat steady state as

$$\frac{\partial y(t,z)}{\partial t} = +g_x y(t,z) + g_u \gamma(t,z) + D \frac{\partial^2 y(t,z)}{\partial z^2} \tag{106}$$

$$\text{with } y(t_0, z) = 0 \text{ for all } z. \tag{107}$$

If, using the equality of the ε-terms in (105) we cancel these terms, divide by ε^2 and then take the limit $\varepsilon^2 \to 0$, we obtain a differential equation

[18] Subscripts denote derivatives.

in the second-order state perturbation

$$\frac{\partial \xi (t,z)}{\partial t} = g_x \xi (t,z) +$$

$$g_{xx}\xi (t,z) + g_{uu}\gamma (t,z) + 2g_{xu}\xi (t,z)\gamma (t,z) + D\frac{\partial^2 \xi (t,z)}{\partial z^2} \quad (108)$$

$$\text{with } \xi (t_0, z) = 0 \text{ for all } z. \quad (109)$$

Write the performance functional (97) using the Hamiltonian function (101) with $x(t,z)$ and $u(t,z)$ given by the perturbations (103) and (102) and with $\lambda (t,z)$ evaluated along the optimal path $\lambda^* (t,z)$, as

$$J(u) = \int_{z_0}^{z_1} \int_{t_0}^{t_1} \left[\mathbf{H}(x(t,z), u(t,z), \lambda^* (t,z)) - \lambda^* (t,z) \frac{\partial x(t,z)}{\partial t} \right] dtdz \quad (110)$$

Write the performance functional along an optimal path as

$$J(u^*) = \int_{z_0}^{z_1} \int_{t_0}^{t_1} \left[\mathbf{H}(x^* (t,z), u^* (t,z), \lambda^* (t,z)) - \lambda^* (t,z) \frac{\partial x^* (t,z)}{\partial t} \right] dtdz \quad (111)$$

then

$$J(u) - J(u^*) =$$

$$\int_{z_0}^{z_1} \int_{t_0}^{t_1} [\mathbf{H}(x(t,z), u(t,z), \lambda^* (t,z)) - \mathbf{H}(x^* (t,z), u^* (t,z), \lambda^* (t,z))$$

$$\left. -\lambda^* (t,z) \frac{\partial (x(t,z) - x^* (t,z))}{\partial t} \right] dtdz \quad (112)$$

By expanding around the optimal steady state (x^*, u^*, λ^*) we obtain, with derivatives evaluated at the optimal steady state,[19]

$$J(u) - J(u^*) = \int_{z_0}^{z_1} \int_{t_0}^{t_1} \Bigg[\mathbf{H}_x \varepsilon y(t,z) + \mathbf{H}_u \varepsilon \gamma(t,z) + \frac{1}{2}\mathbf{v}'\mathbf{Q}\mathbf{v}$$

$$+o\left(\varepsilon^2, t, z\right) + \lambda^*(t,z) \frac{\partial \left(x(t,z) - x^*(t,z)\right)}{\partial t} \Bigg] dt dz \tag{113}$$

$$\mathbf{v} = \left(\varepsilon y(t,z), \varepsilon \gamma(t,z)\right)' \tag{114}$$

$$\mathbf{Q}(y,\gamma) = \begin{pmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xu} \\ f_{ux} & f_{uu} \end{pmatrix} + \lambda^* \begin{pmatrix} g_{xx} & g_{xu} \\ g_{ux} & g_{uu} \end{pmatrix} \tag{115}$$

In (113) integrate by parts the term $\lambda^*(t,z) \frac{\partial(x(t,z)-x^*(t,z))}{\partial t}$ as in (77) to obtain

$$\int_{z_0}^{z_1} \Bigg[-\lambda(t_1,z)\left[x(t_1,z) - x^*(t_1,z)\right] + \lambda(t_0,z)\left[x(t_0,z) - x^*(t_0,z)\right]$$

$$+ \int_{t_0}^{t_1} \left[x(t,z) - x^*(t,z)\right] \frac{\partial \lambda^*(t,z)}{\partial t} dt \Bigg] dz \tag{116}$$

In (116) the first term in the bracket is zero by transversality conditions, the second term is zero by initial conditions on the state perturbation, while the third term under the integral can be written, using (103) and the optimality conditions, as:

$$-H_x \frac{\partial \left(x^* + \varepsilon y(t,z) + \varepsilon^2 \xi(t,z) + o(\varepsilon^2, t, z) - x^*(t,z)\right)}{\partial t} \tag{117}$$

Furthermore $H_u = 0$ by the optimality conditions. Substituting into (113) dividing by ε and taking limits we obtain

$$J(u) - J(u^*) = \int_{z_0}^{z_1} \int_{t_0}^{t_1} \frac{1}{2}\mathbf{v}'\mathbf{Q}\mathbf{v} \, dt dz \tag{118}$$

Therefore a "good approximation" of problem (97) - (100) can be obtained if we replace in problem (97) - (100) the function $f(x(t,z), u(t,z))$ with $\frac{1}{2}\mathbf{v}'\mathbf{Q}\mathbf{v}$ and the transition equation (98) with the linearized diffusion equation (106).

[19] See Athans and Falb (1966) for such an expansion in the context of deriving necessary conditions for standard control problems without diffusion.

The same substitution can be made in an infinite horizon problem by replacing the transversality condition used to simplify (116) with the requirement that we require controls that produce solutions for the state variable that grow by less than discounting. This approximation is similar to the one produced by Magill (1977) for the optimal stochastic control problem. It should be noted that since the diffusion coefficient D is independent of the state and the control, this term drops out from the approximation of the objective, but enters the problem through the linearized diffusion equation. It is clear that extra terms including the diffusion coefficient should be added into the approximating matrix $\mathbf{Q}$ in the general case where $D = D(x, u)$.

Application to the Optimal Harvesting Problem

We apply this result to the optimal harvesting problem (20) - (23). Let (x^*, E^*, μ^*) be the flat steady state for this problem, and define $U(x, E) = [S(H(t, z)) - c(E(t, z))]$. Following our results above we obtain

$$\frac{1}{2}\mathbf{v}'\mathbf{Q}\mathbf{v} = L^0(y, \gamma) = \frac{1}{2}\begin{bmatrix} y \\ \gamma \end{bmatrix}^T \begin{bmatrix} A & N \\ N & B \end{bmatrix} \begin{bmatrix} y \\ \gamma \end{bmatrix} = \tag{119}$$

$$\frac{1}{2}\left[Ay^2 + 2Ny\gamma + B\gamma^2\right] \tag{120}$$

$$A = U_{xx}(x^*, E^*) = S''(H^*)(H^*)^2 + \mu^* F''(x^*) \tag{121}$$

$$N = U_{xE}(x^*, E^*) = q\left[S''(H^*)H^* + S'(H^*)\right] \tag{122}$$

$$B = U_{EE}(x^*, E^*) = S''(H^*)(qx^*)^2 - c''(E^*) \tag{123}$$

$$H^* = qE^*x^* \tag{124}$$

where, $A < 0$ by the concavity of benefit and growth functions and the fact that the shadow value of the resource is non-negative at the steady state, $N \gtrless 0$, and $B < 0$ by the concavity of the benefit function and the convexity of the cost function.

Following Brock and Malliaris (1989) we make a change in units, so that

$$u = \gamma + \frac{N}{B}y \tag{125}$$

Then we have

$$L^0(y,u) = \frac{1}{2}\left[\left(A - \frac{N^2}{B}\right)y^2 + Bu^2\right] \tag{126}$$

Furthermore, the linearized transition equation becomes

$$\frac{\partial y(z,t)}{\partial t} = \left[F'(x^*) - qE^*\right]y(z,t) - qx^*\gamma(z,t) + D\frac{\partial y^2(z,t)}{\partial z^2} = \tag{127}$$

$$Fy(z,t) - G\gamma(z,t) + D\frac{\partial y^2(z,t)}{\partial z^2}, \text{ or} \tag{128}$$

$$\frac{\partial y(t,z)}{\partial t} = \left(F - \frac{GN}{B}\right)y(t,z) - Gu(t,z) + D\frac{\partial^2 y(t,z)}{\partial z^2} \tag{129}$$

where $G = qx^* > 0$ for a positive steady state for the resource. In order to have a well posed LQ problem with a concave net benefit function and a transition equation with positive growth for the resource, we assume:

$$\left(A - \frac{N^2}{B}\right) = -Q < 0 \tag{130}$$

$$B = -R < 0 \tag{131}$$

$$\left(F - \frac{GN}{B}\right) = S > 0 \tag{132}$$

The LQ approximation around the flat steady state of the original spatial problem is

$$\max_{u(t,z)} \int_0^\infty \int_{\mathcal{Z}} e^{-\rho t}\left[-\frac{Q}{2}x^2 - \frac{R}{2}u^2\right] dtdz \quad Q,R,\rho > 0 \tag{133}$$

$$\text{s.t. } \frac{\partial y(t,z)}{\partial t} = Sy(t,z) - Gu(t,z) + D\frac{\partial^2 x(t,z)}{\partial z^2} \quad , F,G > 0 \tag{134}$$

$$y(0,z) \quad \text{given, and zero flux on } 0,a, \text{ or} \tag{135}$$

$$y(t,0) = y(t,a) = 0,\; y(0,z),\; z \in (0,a) \text{ given} \tag{136}$$

□

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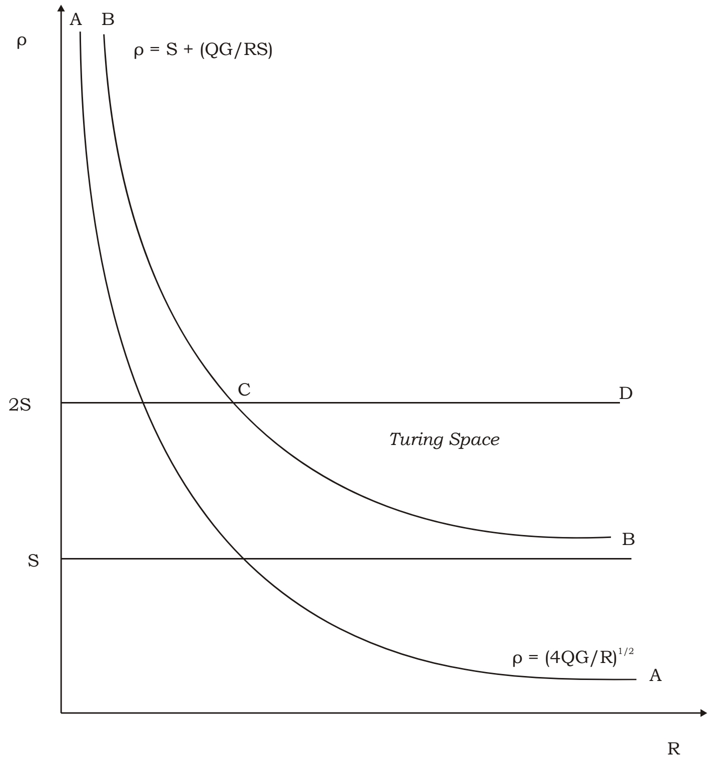

Figure 1: Turing Space

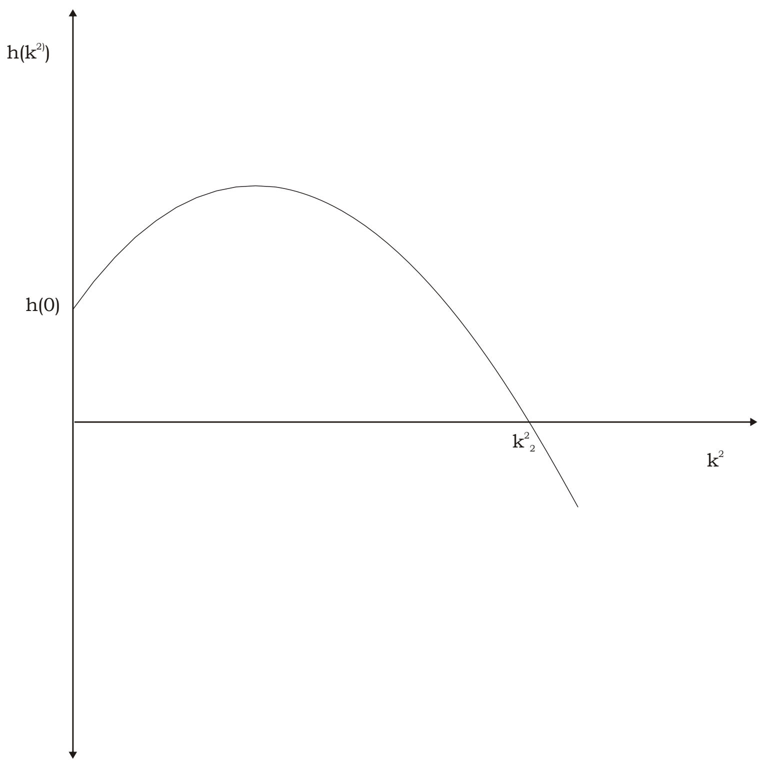

Figure 2: Diffusion as Stabilizer