Equilibrium Concepts for Social Interaction Models[†]

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Abstract

This paper describes the relationship between two different binary choice social interaction models. The Brock and Durlauf (2001) model is essentially a static Nash equilibrium model with random utility preferences. In the Blume (forthcoming) model is a population game model similar to Blume (1993), Kandori, Mailath, and Rob (1993) and Young (1993). We show that the equilibria of the Brock-Durlauf model are steady states of a differential equation which is a deterministic approximation of the sample-path behavior of Blume's model. Moreover, the limit distribution of this model clusters around a subset of the steady states when the population is large.

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1 Introduction

This paper describes the relationship between two different binary choice models with interdependent preferences in order to elucidate some general properties of models of this type. One model is due to Brock and Durlauf (2001) and is in essence a static Nash equilibrium model in which a random utility framework is extended to include an effects of the expected choices of others on individual payoffs. The second model is the strategy adjustment model of Blume (forthcoming), in which binary choice evolve in response to the past behavior of others via a stochastic population process similar to Blume (1993), Kandori, Mailath, and Rob (1993) and Young (1993).

At one level, of course, these social interaction models are nothing more than coordination games in which particular restrictions have been made on the state space of choices and the ways in which payoffs between agents are interdependent. What has made these types of models of general interest is their both their ability to elucidate theoretical questions such as the existence and local stability of multiple equilibria as well as their potential to be applied in a range of econometric contexts (cf. Glaeser, Sacerdote, and Scheinkman (1996) and Brock and Durlauf (2000)).

In our analysis, we link these two social interactions models by using the dynamic ideas developed in Blume (1994, 1997) to calculate differential equations whose steady states are precisely the equilibria studied by Brock and Durlauf (2001). We show that for large player populations, the solution path of the differential equation from given initial conditions closely approximates the sample path of the population process studied by Blume (forthcoming) from the same starting point. This result is well-known in the population biology literature and has also been demonstrated in some population game models.

More interesting is that the differential equation also carries information about the asymptotic behavior of the population process. As the population size becomes large, any (weak convergence) accumulation point of the sequence of invariant distributions has support contained in the set of stable steady states of the differential equation. We characterize (weak) accumulation points of the sequence of suitably scaled invariant distributions for the population process. In general, the limit distributions distribute their mass among the mean field equilibria. For two particular cases, the constant tremble probability model of Kandori, Mailath and Rob and Young and the logit choice model of Blume and Brock and Durlauf, we demonstrate that the sequence of invariant distributions converges and we compute the limit.

The typical population game analysis fixes a population size and investigates the limit behavior of the sequence of invariant distributions as the stochastic component of choice disappears. These so-called "stochastic stability results" have been used to justify a particular selection from the set of Nash equilibria of the static game which drives the population process. The noisy choice is just a means to an equilibrium selection technique. We take seriously both the dynamic models and noisy choice. Consequently for us the invariant distributions are interesting in their own right rather than as a means to an end, and we want to understand the behavior of these models when there is a significant random component to choice.

Density-dependent population processes arise frequently in economic analysis, and most often they are studied by examining a differential equation which describes the evolution of mean behavior. The rationale for this approach is an appeal to a law of large numbers. For a particular class of game-theoretic models we make the large numbers argument precise, and clarify what can be learned from it. We expect that our results can be extended to some of the literature on search and sorting which proceeds in this manner, and we believe this to be an important area for future research.

2 The Structure of Interactions-Based Models

The object of interactions-based models is is to understand the behavior of a population of economic actors rather than that of a single actor. The focus of the analysis is the externalities across actors. These externalities, the source of the social interactions, are taken to be direct. The decision problem of any one actor takes the decisions of other actors to be parametric. Hence the interactions approach treats aggregate social behavior as a statistical regularity of the individual interactions. A second feature of these models is that individual behavior is not as tightly modeled as it is in traditional economic equilibrium models. Individual choice is guided by payoffs, but has a random component. This randomness can be attributed to to some form of bounded rationality. In static equilibrium models it may also be interpreted as an unobserved agent characteristic.

In this paper we focus on a simple class of interaction models with strategic complementarities. Formally, consider a population of I individuals. Suppose that each individual chooses one of two actions, labeled -1 and $+1$. Suppose that each individual's utility is the sum of utilities from pairwise interactions with every other player. Actor i's expected utility is

$$V_i(\omega_i) = h_i\omega_i - E\big\{\sum_j J_{i,j}(\omega_i - \omega_j)^2\big\} + \epsilon(\omega_i)$$

This specification can be decomposed into a private component, $h_i\omega_i + \epsilon(\omega_i)$, and the interaction effect, $E\big\{\sum_j J_{i,j}(\omega_i - \omega_j)^2\big\}$. The private component can be further decomposed (without loss of generality) into its mean, $h_i\omega_i$, and a mean-0 stochastic deviation $\epsilon(\omega_i)$. The terms $J_{i,j}$ is a a measure of the disutility of non-conformance. When the $J_{i,j}$ are all positive there is an incentive to conform. The presence of positive conformity effects gives rise to multiple equilibria and interesting dynamics. Our methods also encompass the case of negative conformity effects, but the results are less interesting both economically and technically.

For binary choice, this specification of preferences is quite general. Any model in which the utility of action ω_i to individual i is the sum of the utilities from pairwise interactions with other players can be modeled this way. This specification does not include some interesting models of strategic complementarities, such as the stag hunt game. Multiplying out the quadratic and renormalizing,[1]

$$V_i(\omega_i) = h_i\omega_i + 2E\big\{\sum_j J_{i,j}\omega_i\omega_j\big\} + \epsilon(\omega_i) \tag{1}$$

The random terms are independent, and so the random variable $\epsilon_i(-1) - \epsilon_i(1)$ has mean 0. Let $F(z)$ be its cdf. Then

$$\begin{aligned} \text{Prob}(\omega_i = 1) &= \text{Prob}\big(V_i(1) > V_i(-1)\big) \\ &= F(2h_i + 4E\sum_j J_{i,j}\omega_j) \end{aligned}$$

Different specifications of the h_i and J_{ij} coefficients give rise to models with very different kinds of behavior. In this paper we will study *uniform global interaction*. That is, $J_{ij} \equiv J/2(I-1)$ and $h_i \equiv h$. Interactions with all other players are weighted equally, and so mean utility is the sum of a private effect, the $h_i\omega_i$ term, and a social effect which places a weight of J on the covariance of i's play with mean play of all other players. Also, the private terms are identical across players.[2] Under these assumptions, the individual choice probabilities are

$$\text{Prob}(\omega_i = 1) \;=\; F\biggl(2h + 2\frac{J}{I-1}\sum_{j\neq i} E\omega_j\biggr) \tag{2}$$

An important special case arises when the random terms are assumed to be distributed according to the extreme value distribution with parameter β_i. That is,

$$\text{Prob}\bigl(\epsilon(-1) - \epsilon(1) < z\bigr) = \frac{1}{1+\exp(-\beta_i z)}, \qquad \beta_i > 0$$

This model reduces to an instance of the standard logit binary choice framework when there are no interaction effects; that is, when $J_{i,j} \equiv 0$.

From the extreme value distribution the individual choice probabilities can be computed.

$$\text{Prob}(\omega_i = 1) \;=\; \frac{1}{1+\exp -2\beta\biggl(h + \frac{J}{I-1}E\sum_j \omega_j)\biggr)} \tag{3}$$

When β is very large, individual i will choose an action to maximize mean utility

$$EV_i(\omega_i) = \biggl(h + \frac{J}{I-1}\sum_{j\neq i} E\omega_j\biggr)\omega_i$$

with probability near 1. When β is 0 the player will choose by flipping a coin.

Another important special case is that where the random terms are assumed to be distributed such that they take on the values A, 0 and $-A$

with probability δ, $1-2\delta$ and δ respectively, where $A > |EV(1) - EV(-1)|$. Then[3]

$$\text{Prob}(\omega_i = 1) \;=\; \begin{cases} \delta(2-3\delta) & \text{if } EV_i(1) < EV_i(-1), \\ 1-\delta(2-3\delta) & \text{if } EV_i(1) > EV_i(-1), \\ 1/2 & \text{if } EV_i(1) = EV_i(-1). \end{cases} \tag{4}$$

This model is the "tremble" or "mistakes" model of Kandori, Mailath, and Rob (1993) and Young (1993). As ϵ becomes small, the probability of best responding approaches 1.

In the general model, equation (2) describes the probabilities with which the actions available to player i will be taken. This choice model is not closed, however, because we have not specified how the expectation in equation (2) is to be taken. In the special cases just examined, we have yet to specify how $EV_i(\omega_i)$ is computed. In fact it is a conditional probability, and different choices for on what it is conditioned give rise to the different models which we consider in this paper.

3 Static Equilibrium: The Mean Field Model

One approach to closing the model of equations (1) and (2) is that suggested by Nash equilibrium. That is, each individual i has beliefs about all the ω_j, and these beliefs are correct. This specification gives the Brock and Durlauf (2001) model. Formally, suppose that each player believes that the expectation of the action of each of his opponents is m. Equation (2) becomes

$$\text{Prob}(\omega_i = 1) \;=\; F(2h + 2Jm) \tag{5}$$

If this guess is to be correct, it must be that

$$m = 1 \cdot \text{Prob}(\omega_i = 1) + (-1) \cdot \text{Prob}(\omega_i = -1) \tag{6}$$

which is the equilibrium condition that closes the model. It will be convenient to rewrite this condition in terms of the log-odds function $g(z) = \log F(z) - \log\bigl(1 - F(z)\bigr)$. In all that follows we need to make sure that the support of $F(\,\cdot\,)$ is large enough that $g(z)$ is everywhere defined and that the population externality is present.

Axiom 1. *$F(z) > 0$ for all z in the interval $[2h - 2J, 2h + 2J]$ and $F(2h - 2J) < F(2h + 2J)$.*

Substituting into equation (6),

$$\begin{aligned} m &= \frac{\exp g(2h+2Jm)}{1+\exp g(2h+2Jm)} - \frac{1}{1+\exp g(2h+2Jm)} \\ &= \frac{\exp \frac{1}{2}g(2h+2Jm) - \exp\bigl(-\frac{1}{2}g(2h+2Jm)\bigr)}{\exp \frac{1}{2}g(2h+2Jm) + \exp\bigl(-\frac{1}{2}g(2h+2Jm)\bigr)} \\ &= \tanh\left(\frac{1}{2}g(2h+2Jm)\right) \end{aligned} \tag{7}$$

For the logit model, equation (5) becomes

$$\text{Prob}(\omega_i = 1) \;=\; \frac{1}{1+\exp\bigl(-2\beta(h+Jm)\bigr)} \tag{5'}$$

The log-odds function is $g(z) = \beta z$, and so the equilibrium condition (7) is

$$m = \tanh \beta(h + Jm) \tag{7'}$$

Equation ($7'$) is well-known in the world of statistical physics, where it has an important physical interpretation, and is known as the Curie-Weiss model of magnetization. The following theorem characterizes the solutions to ($7'$). See Brock and Durlauf (2001).

Theorem 1 (Static Equilibrium — Logit Model).

1. *If $\beta J \leq 1$ and $h = 0$, then $m = 0$ is the unique solution to ($7'$).*

2. *If $\beta J > 1$ and $h = 0$, then there are three solutions: $m = 0$ and $m = \pm\hat{m}(\beta J)$. Furthermore, $\lim_{\beta J \to \infty} \hat{m}(\beta J) = 1$.*

3. *If $h \neq 0$ and $J > 0$, then there is a threshold $C(h) > 0$ (which equals $+\infty$ if $h \geq J$) such that (a) for $\beta h < C(h)$, there is a unique solution, which agrees with h in sign; and (b) for $\beta h > C(h)$ there are three solutions, only one of which agrees with h in sign. Furthermore, as β becomes large the extreme solutions converge to ± 1.*

4. *If $J < 0$, then there is a unique solution which agrees with h in sign.*

This theorem illustrates both the nonlinearities and the multiple steady states which are the hallmarks of interacting systems. The model is nonlinear with respect to a change in h, the private component of preference, on the mean behavior m of the population. Indeed, the effect of a change in h may be to increase the number of equilibria, which will exceed one when the strength of interactions is great enough.

The underlying strategic situation for $J > |h|$ corresponds to a coordination game played by a population of opponents, wherein player i's preferences are the mean preferences $h + J\sum_{j\neq i}\omega_j/(I-1)$. The strategy choice $+1$ (-1) is risk-dominant if $h \geq 0$ $(h \leq 0)$. As β becomes large, the two extreme solutions converge to the pure strategy Nash equilibria. When $h \neq 0$ the middle equilibrium will not converge to the mixed Nash equilibrium because the choice probabilities $(5')$ impose a particular randomization when $V_i(1) = V_i(-1)$ which will be incompatible with that required to implement the mixed equilibrium.

For the mistakes model, equation (5) becomes

$$\text{Prob}(\omega_i = 1) \;=\; \begin{cases} \delta(2-3\delta) & \text{if } 2h + 2Jm < 0, \\ 1 - \delta(2-3\delta) & \text{if } 2h + 2Jm > 0, \\ 1/2 & \text{if } 2h + 2Jm = 0. \end{cases} \tag{5"}$$

Let $m_\delta = 1 - 2\delta(2-3\delta)$. The equilibrium condition is that m is any solution to the following equations:

$$m = \begin{cases} m_\delta & \text{if } h + Jm_\delta > 0, \\ -m_\delta & \text{if } h - Jm_\delta < 0, \\ 0 & \text{if } h = 0. \end{cases} \tag{7"}$$

Again multiple solutions are possible. If $J > h$ and $\delta \approx 0$ both $m = m_\delta$ and $m = -m_\delta$ are equilibria. Due to the discontinuities in the choice probabilities $(5'')$, there will typically either be one or two solutions, but never three solutions unless $h = 0$.

The parameter m is of interest to the modeler as well as to the actors. Because this model preserves the factorization of the joint distribution of choices into the product of the distribution of individual choices, a strong law of large numbers guarantees that m is approximately the (sample) average choice when I is large.

4 Dynamics

Since Blume (1993), Kandori, Mailath, and Rob (1993) and Young (1993), interest has developed in stochastic processes wherein individuals in a population of players adapt their strategic choice to the play of the population. At randomly chosen moments players observe the play of their opponents and respond by by choosing a new strategy according to a random utility model. The stochastic processes of individual response have implications for the emergent dynamics of population behavior.

4.1 The Population Process

We formalize this model by giving each individual a Poisson alarm clock. When it rings, she revises her choice. Formally, each actor i is endowed with a collection of random variables $\{\tau_n^i\}_{n=1}^{\infty}$ such that each $\tau_n^i - \tau_{n-1}^i$ is exponentially distributed with mean 1, and all such differences are independent of all others, hers and the other actors'. At each time τ_n^i individual i chooses a new action by applying the random utility model of equation (2). Here she takes the expectation given certain knowledge of the ω_j at time t. That is, she chooses according to the transition probability

$$\text{Prob}(\omega_{i\,t+} = 1|\omega_t) \;=\; F\left(2h + 2\frac{J}{I-1}\sum_{j\neq i}\omega_{jt}\right) \tag{8}$$

Implicit in this equation is the fact that players are myopic in (stochastically) best-responding to the current play of the population rather than some forecast of future paths of play. This assumption is much discussed in the literature and will not be defended here.

The process of individual strategy revision is a continuous time Markov process that changes state in discrete jumps. We are interested in tracking only the aggregate $S_t = \sum_{i=1}^{I}\omega_{it}$ rather than the behavior of each individual. The process $\{S_t\}_{t=0}^{\infty}$ is also a Markov jump process, whose states are $S^I = \{-I, -I+2, \ldots, I-2, I\}$. This process changes state whenever an actor changes her choice. If an actor changes from -1 to $+1$ S_t increases by 2, and it decreases by 2 whenever an actor changes in the opposite direction. These are the only possible transitions, and so the process $\{S_t\}_{t\geq 0}$ is a birth-death process. The transition rates can be computed from the conditional probability distribution (8). Suppose the system is in state S. It transits to state $S+2$ only when a revision opportunity comes to one of the $(S-I)/2$ actors currently choosing -1, and that actor chooses $+1$.[4] The probability of a -1 actor making this choice is[5]

$$F\left(2h + 2\frac{J}{I-1}(S_t+1)\right)$$

It will be convenient to make use of the log-odds function $g(z) = \log F(z) - \log\bigl(1 - F(z)\bigr)$. In terms of $g(z)$, $F(z) = \exp g(z)/\bigl(1 + \exp g(z)\bigr)$.

Putting this together, the transition rate from S to $S+2$ in a population of size I is

$$\lambda_S^I = \frac{I-S}{2}\frac{\exp g\left(2h + \dfrac{2J}{I-1}(S+1)\right)}{1 + \exp g\left(2h + \dfrac{2J}{I-1}(S+1)\right)}.$$

A similar computation gives the transition rate in the other direction. To transit from $S+2$ back to S requires that one of the $(S+2+I)/2$ actors choosing $+1$ switches to -1. The transition rate is

$$\mu_{S+2}^I = \frac{I+S+2}{2}\frac{1}{1 + \exp g\left(2h + \dfrac{2J}{I-1}(S+1)\right)}.$$

Since we will study the behavior of processes with different population sizes, we scale them so they all sit in the same state space, $[-1,1]$, by defining $m_t = S_t/I$. The process with population size takes values in $\{-1, -1 + 2/I, \ldots, 1 - 2/I, 1\} = M_I \subset [-1,1]$. The process has birth rates and death rates

$$\begin{aligned}\lambda_m^I &= \frac{I}{2}(1-m)\frac{\exp g\bigl(\Delta(m)\bigr)}{1 + \exp g\bigl(\Delta(m)\bigr)}\\ \mu_{m+2/I}^I &= \frac{I}{2}(1 + m + \frac{2}{I})\,\frac{1}{1 + \exp g\bigl(\Delta(m)\bigr)}\end{aligned} \tag{9}$$

respectively, where

$$\begin{aligned}\Delta(m) &= EV\{(1)\} - E\{V(-1)\} \\ &= 2\Big(h + \frac{J}{I-1}(S+1)\Big) \\ &= 2\Big(h + J\frac{I}{I-1}m + \frac{J}{I-1}\Big) \\ &\approx 2(h + Jm)\end{aligned}$$

for large I.

4.2 Short Run Dynamics

The birth and death rates are the time derivatives of the transition probabilities. Thus they can be used to characterize the rates of change of expected values of functions of the state. For any differentiable function $f : \mathbf{R} \to \mathbf{R}$,

$$\begin{aligned}\frac{d}{d\tau}E\{f(m_{t+\tau})|m_t = m\}\Big|_{\tau=0} &= \lambda_m^I\Big(f\big(m + \frac{2}{I}\big) - f(m)\Big) + \\ &\qquad \mu_m^I\Big(f\big(m - \frac{2}{I}\big) - f(m)\Big) \\ &= \frac{I}{2}\frac{(1-m)\exp g\big(\Delta(m)\big)\big(f(m+\frac{2}{I}) - f(m)\big)}{1+\exp g\big(\Delta(m)\big)} \\ &\qquad + \frac{I}{2}\frac{(1+m)\big(f(m-\frac{2}{I}) - f(m)\big)}{1+\exp g\big(\Delta(m)\big)} \\ &= \frac{(1-m)\exp g\big(\Delta(m)\big)f'(m) - (1+m)f'(m)}{1+\exp g\big(\Delta(m)\big)} \\ &\qquad + O(I^{-2}) \\ &= \Big(\tanh\Big(\frac{1}{2}g\big(\Delta(m)\big)\Big) - m\Big)f'(m) + O(I^{-2})\end{aligned}$$

When $f(m) = m$, this differential equation gives

$$\frac{d}{d\tau}E\{m_{t+\tau}|m_t = m\}\big|_{\tau=0} = \tanh\Big(\frac{1}{2}g\big(\Delta(m)\big)\Big) - m + O(I^{-2})$$

Taking the $I \to \infty$ limit suggests the following differential equation, called the *mean field equation*:

$$\dot{m} = \tanh\Big(\frac{1}{2}g(2h + 2Jm)\Big) - m \tag{10}$$

For large I solutions to the diffenrential equation (10) approximate the sample path behavior of the process $\{m^I_\tau\}$ on finite time intervals. The following theorem is an application of a standard strong law of large numbers for density dependent population processes. (An elementary proof is too long to be given here. A quick high-tech proof can be found in Chapter 11.2 of Ethier and Kurtz (1986).)

Theorem 2 (Sample-Path Behavior). *Let $\{m^I_t\}_{t\geq 0}$ refer to the average process with population size I. Suppose $m^I_0 = m_0$ and let $m(\tau)$ be the solution to the mean field equation* (10) *with initial condition $m(0) = m_0$. Then for every $t \geq 0$,*

$$\lim_{I\to\infty} \sup_{\tau\leq t} |m^I_\tau - m_\tau| = 0 \qquad a.s.$$

The content of the Theorem is that when I is large, the stochastic perturbations from individuals' random choices more or less averages out, and so the mean field path is nearly followed for some time. But notice that the theorem is about finite time intervals. For any time horizon t, the mean field solution $m(\tau)$ gives a good approximation to the sample path over a time interval of length t. In the long run, however, large deviations will occur, and ultimately the sample path will diverge from its mean field approximation.

4.3 Asymptotic Behavior

It is apparent that the steady states of the mean field equations (10) are precisely those states which satisfy the equilibrium condition (7) of the Brock-Durlauf model. Furthermore the sample path theorem suggests that there is motion towards at least the stable steady states of (10). This suggests that the long run behavior of the process should tend to be concentrated around the stable steady states of (10), a subset of the mean field equilibria.

The birth-death process with transition rates given by (9) is irreducible, and so for each population size I the population process has a unique invariant distribution ρ_I, which describes the long-run behavior of the process. The next Theorem shows that the intuition of the previous paragraph is correct. For large I the invariant distribution tends to pile up mass near one or more of the stable steady states of (10).

Since the state space $[-1, 1]$ is compact, the sequence of invariant measures $\{\rho_I\}$ is relatively compact, and so has weakly convergent subsequences. The next Theorem demonstrates properties about the subsequential limits. In all the applications we have examined the sequence $\{\rho_I\}$ converges, and the proof of the Theorem suggests a sufficient condition for convergence. Define the function

$$r(m) = \left(\left(\frac{1+m}{2}\right)^{\frac{1+m}{2}}\left(\frac{1-m}{2}\right)^{\frac{1-m}{2}}\right)^{-1} \exp\frac{1}{2}\int_{-1}^{m} g\big(\Delta(x)\big)\,dx \tag{11}$$

Theorem 3 (Asymptotic Behavior). *Let ρ be a weak subsequential limit of the sequence $\{\rho_I\}_{I=2}^{\infty}$ of invariant distributions for population processes with population size I. Then* $\operatorname{supp}\rho$ *is the set of global maxima of $r(m)$, and is contained in the open interval* $(-1, 1)$*. If the set of stationary states of the mean field equations* (10) *is the finite union of points and intervals, then* $\operatorname{supp}\rho$ *is the finite union of points and intervals, all of which are locally stable.*

This Theorem implies that if the population I is large, mean behavior is most often near the stable states of the mean-field equation.

Proof. For a population of size I the invariant measure ρ^I on M_I satisfies the relationship

$$\rho^I(m)\lambda^I_m = \rho^I\left(m + \frac{2}{I}\right)\mu^I_{m+\frac{2}{I}}$$

Consequently

$$\begin{aligned}\rho^I(m) &= z_I \binom{I}{I\frac{1+m}{2}} \exp\{g\big(\Delta(-1)\big) + \cdots + g\big(\Delta(m)\big)\} \\ &\approx \tilde{z}_I I^{-\frac{1}{2}} \sqrt{\frac{2}{\pi(1-m^2)}} r(m)^I \\ &\equiv \tilde{\rho}^I(m)\end{aligned}$$

where z_I and $\tilde{z}_I$ are normalizing factors.

The approximation comes from Stirling's formula and the Riemann sum approximation to the integral. The approximation is such that $\rho^I - \tilde{\rho}^I$ converges uniformly to 0 on compact subsets of the interior of M. Let m^* denote a global maximum of $r(m)$. The function $r(m)$ is strictly increasing in a neighborhood of $m = -1$, strictly decreasing in a neighborhood of $m = 1$, and $r(-1) = 1$. Consequently all its critical points are interior, and $\max_m r(m)$ exceeds 1. Let O be an open neighborhood of $\operatorname{argmax} r(m)$ and let C be a compact set disjoint from the closure of O. Then $\tilde{\rho}^I(O)/\tilde{\rho}^I(C) \to +\infty$, so $\lim_{I\to\infty} \tilde{\rho}^I(O) = 1$. Consequently $\lim_{I\to\infty} \rho^I(O) = 1$ and so $\operatorname{supp} \rho \subset \operatorname{argmax} r(m)$. This proves the first part of the Theorem.

It remains only to show that $\operatorname{argmax} r(x)$ is contained in the set of stable equilibria of (10). The derivative of $\log r(m)$ is

$$\begin{aligned}\frac{d}{dm} \log r(m) &= -\frac{1}{2} \log \frac{1+m}{1-m} + \frac{1}{2} g\big(\Delta(m)\big) \\ &= -\operatorname{arctanh}(m) + \frac{1}{2} g\big(\Delta(m)\big)\,,\end{aligned}$$

and so the critical points are those m which satisfy the equation

$$m = \tanh\left(\frac{g\big(\Delta(m)\big)}{2}\right).$$

By hypothesis, the solution set is the union of a finite collection of points $p_1, \ldots, p_K$ and intervals, $[a_1, b_1], \ldots, [a_L, b_L]$. This union is the set of all critical points of $\log r(m)$, and so the set of global maxima of $r(m)$ is the union of a sub-collection of these elements. Consider a point p_k or a left endpoint a_l in $\operatorname{supp}\rho$. For the point p in question there is an $\epsilon > 0$ such that on the interval $(p-\epsilon, p)$, $d/dm \log r(m) < 0$. Suppose at some point m in this interval, $\dot{m} \leq 0$. Then $m \geq \tanh\Bigl(g\bigl(\Delta(m)\bigr)\Bigr)/2$. Since arctanh is an increasing function, applying it to both sides of the inequality gives $d/dm \log r(m) \geq 0$, which is a contradiction. A similar argument works for the right side of all singletons and right endpoints to show that on some neighborhood to their right, $\dot{m} < 0$. Consequently they are all locally stable. $\square$

The next result follows from the proof of the Asymptotic Behavior Theorem.

Corollary 1 (Convergence). *If the function $r(m)$ defined in equation (11) has a unique global maximum m^*, then the sequence $\{\rho^I\}$ converges weakly to δ_{m^*}, point mass at m^*.*

4.4 Examples

In the following examples we assume $J > 0$. There is no loss of generality in assuming $h > 0$ because our examples treat the different strategies symmetrically. For more on this, see Blume (forthcoming).

For logit choice, $g(x) = \beta x$. The mean field equation (10) is

$$\dot{m} = \tanh \beta(h + Jm) - m$$

For generic values of the parameters there are either one or three equilibria, and if there are three, the center equilibrium is unstable. When β is small there is a unique stable equilibrium. When β is large and $h = 0$ the distributions ρ^I are symmetric, and so ρ^I converges to the distribution which places mass 1/2 on each stable steady state. If $h > 0$ then for all $m > 0$, $r(m) > r(-m)$. $r(m)$ has three critical points, two of which are negative and one which is positive. The positive critical point is the unique global maximizer of $r(m)$, and so ρ^I converges to point mass on the positive equilibrium. In the following picture, the bottom plot shows the mean field equilibria and the top plot shows invariant distribution probability functions for $I = 100$ (flatter) and $I = 400$ (more peaked), for the logit choice model with $\beta = 1.5$,

$h = 0.05$ and $J = 1$. The top figure is drawn so that the area under the each of the two curves is identical. The probability of states away from the upper equilibrium is negligible. The two insets show the invariant distribution conditioned on regions around the other equilibria: The interval $(-1.0, -0.6)$ on the left and the interval $(-0.24, -0.04)$ on the right. In each case the flatter curve corresponds to the smaller population. Notice that the unstable steady state is a local minumum of the probability functions, and that the stable steady state is a local maxima. This is a general feature of the models discussed here, and can be proved using similar techniques.

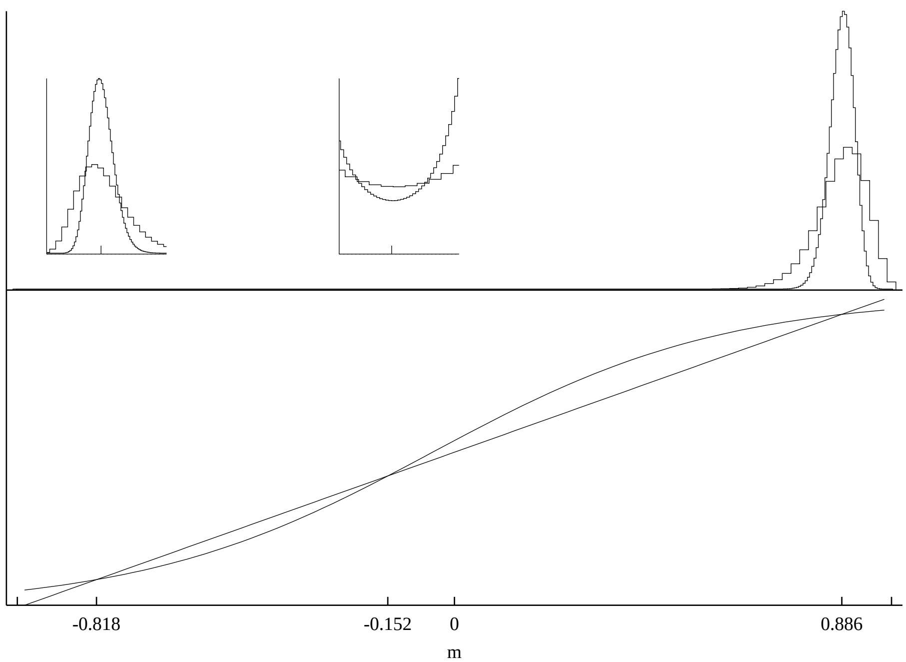

Figure 1: The Logit Model

For the tremble model, the mean field equation is

$$\dot{m} = \begin{cases} -m_\delta - m & \text{if } h + Jm < 0, \\ -m & \text{if } h + Jm = 0, \\ m_\delta - m & \text{if } h + Jm > 0. \end{cases}$$

Generically there are either one or two steady steady states, depending upon the size of h and J. Candidate steady states are at $\pm m_\delta$. Assuming $h > 0$

and that δ is small enough that $m_\delta > 0$, m_δ will be a stable steady state of the mean field equation. The other candidate, $-m_\delta$, will be a steady state iff $h - Jm_\delta < 0$. If it is a steady state, it is stable. The state $m = -h/J$ corresponds to the mixed equilibrium of the unperturbed game. In the logit model (and in any continuous model) it is a steady state, but not here unless $h = 0$. At $m = -h/J$, any individual is equally likely to choose $+1$ or -1 at a choice opportunity. But because if $h > 0$ (for instance), individuals with a choice are more likely to be -1-players, and so $\dot{m} > 0$.

In the trembles model the invariant distribution is easily calculated. Let $m' = \max\{m \in M_I : h + Jm \leq 0\}$. To avoid tedious explication of the boundary case, assume the generic condition that $h + Jm' < 0$.

$$\rho^I(m) = \binom{I}{\frac{mI+1}{2}} \cdot \begin{cases} \left(\frac{2\delta(2-3\delta)}{1-2\delta(2-3\delta)}\right)^{I\frac{1+m}{2}} & \text{if } m \leq m^*, \\ \left(\frac{2\delta(2-3\delta)}{1-2\delta(2-3\delta)}\right)^{I\frac{1+m'}{2}} \left(\frac{1-2\delta(2-3\delta)}{2\delta(2-3\delta)}\right)^{I\frac{1+m-m'}{2}} & \text{otherwise.} \end{cases}$$

If $m' < 0$, then for all $r(m) > 0$, $r(m) > r(-m)$, and so the invariant distribution will converge to point mass at the steady state $m = m_\delta$. Figure 2 displays the relationship between the short- and long-run dynamics for the mistakes model. Here $h = 0.1$, $J = 1.0$ and $m_\delta = 0.8$. The leftmost inset plots the conditional distributions on $(-0.9, -0.7)$ and the rightmost inset plots the log of the conditional distributions on $(-0.16, -0.06)$. The inset expanding the minimum of the stationary distributions is plotted in logs because the "valley" is so steep. The sample sizes are 100 and 400 and again, the steeper plots correspond to the larger sample size.

4.5 Risk-Dominance and Large Populations

The population games literature (Blume (1993), Kandori, Mailath, and Rob (1993), Young (1993), etc.) comes at these problems from a different point of view. We are given a symmetric 2×2 coordination game played by a population of players who are randomly matched against each other. Label the two strategies $+1$ and -1. The function $\Delta(m)$ measures the payoff difference between $+1$ and -1 to a random match when fraction $(mI + 1)/2I$ of the population plays $+1$. Players randomly receive opportunities to

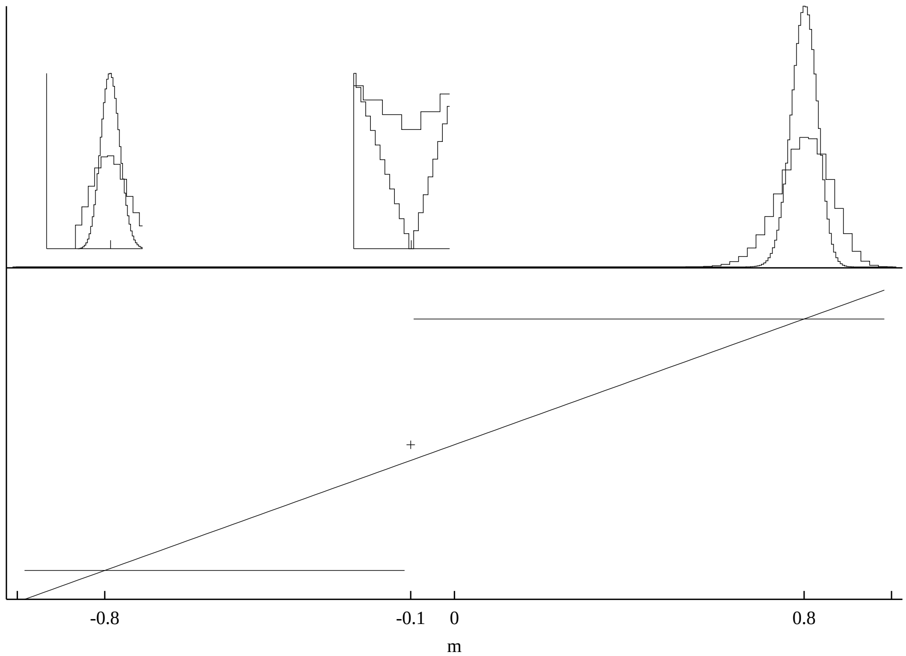

Figure 2: The Trembles Model

revise their choice. When they choose, they best-respond, but their choice is then stochastically perturbed. In the mistakes model players get their best response with probability $1-\epsilon$, and the other choice with probability ϵ. The logit model can be similarly interpreted.

The central question of the population games literature has been the behavior of population processes when the stochastic perturbation is small. For example, the parameter β in equation ($5'$) and ϵ in equation ($5''$) parametrize the size of the perturbation. As $\beta \to \infty$ and $\epsilon \to 0$, the probability of an individual choosing something other than a best response converges to 0. With strategic complementarities, $\Delta(m)$ is increasing. If $\Delta(-1) < 0$ and $\Delta(1) > 0$, then the game has three Nash equilibria: Pure equilibria at $m_1 = 1$, $m_2 = -1$ and a mixed equilibrium at $m_m = \Delta^{-1}(0)$. If $m_m < 0$, then m_1 is the *risk dominant* equilibrium.

The main result of this literature goes as follows: Suppose that $\Delta(m)$ is increasing, and that $m_m < -1/I$. Then as $\beta \to \infty$ (or $\epsilon \to 0$), the invariant distribution for the population process converges weakly to point mass at m_1. That is, as choice becomes less noisy, population play converges to the risk-dominant equilibrium. In other words, the risk-dominant equilibrium is said to be *stochastically stable.*

This result does not depend upon the details of the noise process. Following Blume (forthcoming), write $g(\Delta) = \beta\tilde{g}(\Delta)$. If the odds of choosing any one strategy over the other depend only upon the payoff difference Δ, then it follows that $h(\Delta)$ must be skew-symmetric; that is, $h(-\Delta) = -h(\Delta)$. Blume (forthcoming) shows that if $h(\Delta)$ is skew symmetric, the risk-dominant equilibrium is stochastically stable. The stability result applies to the logit and mistakes models. If $m_m < 0$, the support of the limit distribution is the strictly positive stable steady state. This property of the invariant distribution is general.

In this paper we have fixed the parameters of the noise distribution, and vary only the population size. Nonetheless here too we see a similar kind of stochastic stability result.

Theorem 4 (Risk dominance). *Suppose that $m_m < 0$ and $g(\Delta)$ is skew-symmetric. Then $\lim_I \rho^I\big((0,1]\big) = 1$.*

Proof. The function $r(m)$ is the product of two terms. The first term is symmetric around $m = 0$, where it takes a maximum. The second term is symmetric around $m = m_m$, where it takes a minimum. If $m_m < 0$, then for all $m > 0$, $r(m) > r(-m)$, because the first term takes the same values at $\pm m$ while the second term is always greater at m than at $-m$. Consequently the global maxima of $r(m)$ must all be positive. $\square$

In particular, g is skew-symmetric for both the logit and the mistakes models. If $h > 0$, then $m^* < 0$ and so the Risk Dominance Theorem implies that for these models, if $h > 0$ then the invariant distribution converges to point mass on the unique positive steady state of the mean field approximation.

5 Conclusion

We have shown that the equilibria of the Brock-Durlauf model are steady states of the mean-field differential equation of Blume's model, which is a deterministic approximation of sample-path behavior. Moreover, the limit distribution of this model clusters around a subset of the steady states when the population is large. This large-numbers result is an analog of the stochastic stability results of Blume (1993), Kandori, Mailath, and Rob (1993) and Young (1993). The relationship between deterministic and stochastic evolutionary dynamics uncovered here is similar to results in Binmore, Samuelson, and Vaughn (1995).

We conjecture that these results have a straightforward extension to n-ary choice when the underlying game described by the expected utilities has a potential. In this case, computing the invariant distribution to what is now a multitype birth-death process comes down to finding global maxima of a function like $r(m)$. This fact is true because if the game has a potential, the multitype birth-death process is reversible and the invariant distribution is characterized by the so-called detailed balance conditions, that the probability of state x times the flow rate from x to y equals the probability of state y times the flow rate from y to x. If the game has no potential, the population process is not reversible and the invariant distribution has no simple characterization. The role of the potential function is discussed more fully in Blume (forthcoming). It is important to note that the existence of the deterministic approximation, the mean field equation, is independent of the existence of a potential. The state of the art on deterministic approximations to stochastic processes for general n-ary social interaction models is Benaim and Weibull (2000). Among other things, they show that the asymptotic behavior of the stochastic process clusters around stable steady states of the deterministic approximation when the population size is large.[6] In our model these are the local maxima of $r(m)$. The existence of $r(m)$, however, allows us to get a sharper characterization of limit behavior.

An important substantive implication of our analysis is that the inefficient equilibria in the static Brock-Durlauf model will only be visited infrequently in dynamic versions of the model, at least from the perspective of invariant measures. Of course, this says nothing about lengths of time for passages across equilibria and so these models can still have much to say empirically. However, to the extent that models such as this are used to make claims about cross-section differences in group rates of high school graduation, nonmarital fertility, etc., our finding implies that these differences presumably are due to differences in initial conditions. This is of course plausible in cases such as differences between ethnic groups. However, our finding also implies that models of this type may not be empirically relevant for understanding how similar groups diverge in the short run, as may be observed in the paths of residential neighborhoods in large cities. For such cases, one will either need an alternative framework or an appeal to changes in fundamentals such as employment opportunities. Hence analyses of the type we have provided should have some value in guiding applied work on social interactions.

Notes

[1]We have subtracted off the constant term $-2\sum_j J_{ij}$.

[2]The other leading example of social interaction is *uniform local interaction*, studied by (Blume 1993) and (Ellison 1993). Here $h_i \equiv h$ and $J_{i,j} = J$ or 0 depending upon whether or not i and j are neighbors. (Ellison uses a different model of the stochastic component.)

[3]When $EV_i(1) = EV_i(-1)$, individual i draws until she gets a non-identical realizations of $\epsilon(-1) - \epsilon(1)$.

[4]There are other imaginable transitions, such as where two -1 actors switch to $+1$ and one $+1$ actor switches to -1, but these transitions all involve the simultaneous arrival of revision opportunities to more than one actor, and is thus a 0-probability transition.

[5]$S_t + 1$ rather than S_t because we are interested only in $\sum_{j \neq i} \omega_{jt}$, which equals $S_t + 1$ if $\omega_{it} = -1$.

[6]More generally, the Birkhoff center of the deterministic flow.

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