Evolutionary dynamics in financial markets with many trader types

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Abstract

This paper develops the notion of a Large Type Limit (LTL) describing the average behavior of adaptive evolutionary systems with many trader types. It is shown that generic and persistent features of adaptive evolutionary systems with many trader types are well described by the large type limit. Stability and bifurcation routes to instability and strange attractors are studied. An increase in the "intensity of adaption" or in the diversity of beliefs may lead to deviations from the RE fundamental benchmark and excess volatility. Simple examples of LTL are able to generate important stylized facts, such as volatility clustering and long memory, observed in real financial data.

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1 Introduction

Do expectations matter or do asset prices fully reflect economic fundamentals? Do heterogeneous beliefs average out due to aggregation, or can optimistic or pessimistic views cluster together and cause prices to deviate from underlying economic fundamentals? These questions have been a matter of heavy debate among economists as well as financial practitioners for many decades already. Keynes, for example, argued that stock prices are not governed by an objective view of 'fundamentals', but by 'what average opinion expects average opinion to be'. In Keynes words " *Investment based on genuine long-term expectation is so difficult as to be scarcely practicable. He who attempts it must surely lead much more laborious days and run greater risks than he who tries to guess better than the crowd how the crowd will behave; and, given equal intelligence, he may make more disastrous mistakes*" (Keynes, 1936, p.157).

In contrast, new classical economists have viewed "market psychology", "investors sentiment" and "trend following speculation" as being *irrational* and therefore inconsistent with the rational expectations hypothesis (REH) and the efficient market hypothesis (EMH). Friedman, for example, argued that irrational speculative traders would be driven out of the market by rational traders, who would trade against them by taking long opposite positions, thus driving prices back to fundamentals. In Friedman's words: *"People who argue that speculation is generally destabilizing seldom realize that this is largely equivalent to saying that speculators* lose *money, since speculation can be destabilizing in general only if speculators on the average sell when the currency is low in price and buy when it is high"* [emphasis added] (Friedman, 1953, p.175).

In empirical work much attention has been paid to the closely related question whether asset prices exhibit *excess volatility*, that is, whether volatility of asset prices is larger than volatility of underlying economic fundamentals. In particular the work by Shiller (1987, 2000) has emphasized the possibility of excess volatility and persistent deviations of asset prices from a fundamental RE benchmark. In behavioral finance, for example, Thaler (1994) has argued that *quasi-rationality* may be a key source of deviations from the RE fundamental benchmark and excess volatility. Quasi rationality means less than fully rational behavior, for example, due to investors' sentiment, overconfidence or overreaction. However work by Boldrin and Levine (2001) argues that patterns of returns behavior that look like evidence of "irrationality" or "bounded rationality" may be simply rational reaction of stock markets to earnings profiles generated by technological change. Our paper is focused more on higher frequency fluctuations than Boldrin and Levine (2001), but related arguments must always make one wary of concluding that patterns of booms and crashes in stock market values are evidence of any kind of irrational pricing. See the discussion of Kleidon's work below.

Experimental work has also addressed the question of possible deviations from the benchmark RE fundamental in speculative asset markets. For example, Smith, Suchanek and Williams (1988) showed that speculative bubbles and deviations from a benchmark REE

fundamental are frequently observed in experimental asset markets. These temporary bubbles tend to disappear however towards the end of the experimental asset market and also tend to become smaller as traders become more experienced. Kleidon (1994) has written a recent review paper about market "pathologies" such as crashes, blow offs, excess volatility, anomalies, etc.; see also Plott and Sunder (1988) and Camerer (1989). In particular he points out how the experimental literature in an asymmetric information setting has shown that REE is not achieved immediately but tends to be achieved by experienced traders. Kleidon states, "The laboratory results proved insight into when imperfect aggregation is more likely to occur, namely an absence of common information about preferences or beliefs of other traders and a lack of traders' experience in the market setting." Kleidon (1994, p. 4, 5) stresses the careful differentiation between modeling "new 'external' information about fundamentals, that is, information about expected future cash flows or discount rates that reaches any trader or investor for the first time" and modeling "rational changes in internal information about fundamentals."

The debate whether expectations affect asset prices and may lead to excess volatility should be viewed in the light of two important developments in the recent literature: *bounded rationality* and *heterogeneous agents* systems. Bounded rationality and heterogeneous agents may be viewed as building blocks in behavioral finance, where traders are viewed as boundedly rational agents using simple, habitual rule of thumb rules, see e.g. Shefrin (2000). Although rational expectations remains an important benchmark, work on bounded rationality in the past decade may be viewed as an attempt to explore deviations from this benchmark. General surveys on bounded rationality in expectations and learning are e.g. Sargent (1993), Grandmont (1998) and Evans and Honkapohja (2001). In particular, Sargent (1999) argues that many of the bounded rational expectations equilibria may be viewed as 'approximate' rational expectations equilibria.

In the last decade, a rapidly increasing interest in multi-agent systems can be observed. Markets viewed as evolutionary adaptive systems with boundedly rational interacting agents have e.g. been studied in Arthur et al. (1997), Brock and Hommes (1997,1998), Chiarella and He (1999, 2000) Farmer (1998), Gaunersdorfer and Hommes (2000), Kirman (1991), LeBaron et al. (1999), Lux (1995) and Lux and Marchesi (1999ab). These developments in multi-agent systems are closely related to recent work in finance on 'smart money' and noise traders e.g. by Frankel and Froot (1988), De Long, Shleifer, Summers and Waldmann (1989, 1990), and Wang (1994). A common feature of these contributions is that there are two different classes of investors that can also be observed in financial practice: fundamentalist and technical analysts. Fundamentalists base their forecasts of future prices and returns upon economic fundamentals, such as dividends, interest rates, price-earning ratio's, etc.. In contrast, technical analysts are looking for patterns in past prices and base their forecasts upon extrapolation of these patterns. An interesting outcome of these evolutionary heterogeneous agent systems is that the models mimic a number of stylized facts frequently observed in financial series, such as unpredictability of returns, fat tails, volatility clustering and long memory. Price deviations from the RE fundamental and excess volatility are triggered by uncertainty about economic fundamentals

but may be *amplified* by evolutionary interaction of competing, boundedly rational trading strategies. Most of this work is computationally oriented and based upon computer simulations of complex adaptive systems.

The present paper provides a theoretical framework for evolutionary markets with many different trader types. We introduce the notion of a so-called *Large Type Limit (LTL)*, which is a simple, low dimensional system describing the evolutionary dynamics in a market with many trader types. The concept of LTL was sketched in Brock (1997), Brock and Hommes (1999) and Brock and de Fontnouvelle (2000) but is actually rigorously developed in this paper[1]. The LTL is a type of ensemble limit rather like thermodynamic limits in statistical mechanics. It is motivated by observing that market equilibrium equations tend to have a form which is a function of expressions that look like sample moments. The LTL is then simply obtained by replacing sample moments by population moments. For particular distributions of characteristics, one may obtain closed form solutions for these population moments and thus obtain closed form expressions for objects that appeared intractable. The LTL can also be viewed as a device that effectively reduces a large number of parameters inherent in models with a large number of different belief types by introducing an underlying characteristics distribution which itself can be characterized by a small number of parameters. Thus empirical work can be assisted by this type of parameter reduction technique. We think this is a new method of analysis which is of independent interest. The LTL theory developed here can be used to form a bridge between analytical results and the literature on evolutionary simulation of asset trading, which has become very popular in the last few years (see e.g. Arthur *et al.* (1997), LeBaron (2000) and Hommes (2001) for reviews). In fact, the main result of the present paper is that the LTL well approximates the evolutionary dynamics in a market with many trader types in the sense that all *generic* and *persistent* features of an evolutionary adaptive system with many trader types are preserved in the LTL. Hence, the LTL can be used to study local stability of the benchmark RE fundamental steady state as well as possible deviations from the RE benchmark and bifurcations routes to instability and complicated periodic or even chaotic dynamics in adaptive evolutionary systems with many trader types. Furthermore, we will present a simple example of an LTL buffeted with dynamic noise, generating some important stylized facts observed in real financial data. In particular we present a simple example, where the autocorrelation patterns of noisy LTL generated time series of returns, absolute returns and squared returns match the corresponding autocorrelation patterns of 20 years of S&P 500 data.

Let us now discuss some closely related recent work. Brock and Hommes (1997) introduced a tractable form of evolutionary dynamics in the cobweb demand-supply model with a costly, sophisticated forecasting rule such as rational expectations competing against a

[1]The notion of LTL is related to the well known measure space approach in general equilibrium theory (See Kirman (1981) and Hildenbrand (1982) for reviews). The main difference lies in locating intertemporal *dynamical* equilibrium relationships that resemble sample moment conditions to suggest the form of the appropriate limit and establishing which dynamical phenomena are "persistent‘' under "finite" economy limits, so that the limit itself can be directly used to construct bifurcation diagrams and predict the dynamical bifurcation behavior for large enough "samples." Details will follow.

cheap habitual rule of thumb forecasting rule such as adaptive expectations. Brock and Hommes (1998), henceforth BH, called this evolutionary dynamics an *Adaptive Belief System (ABS)* and applied the framework to asset pricing theory in a model with one risky asset and one risk free asset. Tractability is achieved by the use of random utility models to model 'fitness' and 'natural selection.' Predictor choice across the set of predictors is evolutionarily rational in the sense that agents choose the predictors according to highest fitness, such as past profits. In the ABS studied by BH there were only *few* trader types, typically only two, three or four. The present paper generalizes the ABS to an evolutionary system with *many* different trader types and proposes to approximate the evolutionary dynamics with many trader types by a LTL. Gaunersdorfer and Hommes (2000) investigate time series properties of a simple ABS with two trader types, fundamentalists versus trend followers, buffeted with dynamic noise and match the autocorrelation patterns of returns, absolute returns and squared returns to 40 years of S&P 500 data. Simple versions of the ABS thus may explain observed stylized facts such as unpredictability of asset returns, fat tails, clustered volatility and long memory. Gaunersdorfer, Hommes and Wagener (2000) investigate bifurcation routes to complicated dynamics of this system, and show in particular that co-existence of a stable steady state and a stable cycle, may explain the observed stylized facts of the model buffeted with noise.

An important feature of the BH framework is that the RE benchmark is nested within the evolutionary model as a special case. The benchmark fundamental represents the traders' response to common knowledge exogenous news, such as interest rate movements, growth of the firms profits, earnings or dividends, a repeal of the capital gains tax, etc.. Traders' beliefs about future prices are formulated in terms of deviations from a benchmark RE fundamental. These deviations can be viewed as each RE trader's belief about how the deviations from RE by the rest of the trading community might show up in equilibrium prices. In this sense our theory is fully rational in the sense that truly rational traders must take into account the behavior of other traders in the trading community. The BH framework is motivated by the kinds of evidence reviewed above and starts the task of building a theory that "backs off" from RE in the manner of Sargent (1993), but nests RE in such a way that RE-econometric technology such as methods based upon orthogonality conditions can be readily adapted to test the "significance" of the extra "free parameters" that our theory adds to RE theory. Pro-RE economists loosely argue that the situations in which these extra free parameters are significant may not be frequent. Anti-RE types argue that such departures may be frequent for anti-RE types. We think a theory is needed that nests both views in a way such that econometric methods are suggested to test the null of RE against the alternative. We wish to contribute to the task of building a theory in which the data can speak to the controversy on RE and excess volatility. It is worthwhile noting that Baak (1999) and Chavas (2000) have run empirical tests for heterogeneity in expectations in agricultural data and indeed find evidence for the presence of boundedly rational traders in the hog and cattle markets. The theory developed in this paper also yields a decomposition of returns into a Martingale Difference Sequence, hereafter "MDS", and an "extra endogenous dynamics" part. The MDS component of returns corresponds

to conventional rational expectations theory (RE) and Efficient Markets Theory (EMH). The "endogenous dynamics part" is rather like "endogenous uncertainty" in the sense of Kurz's book (1997) which develops the theory of rational beliefs (RBE). Recent empirical work by Sharma (2001) on financial markets which is patterned after that of Chavas (2000) adduces some preliminary evidence for the presence of some boundedly rational traders in an econometric setting where REE/EMH is "nested" within a general evolutionary model rather like the one we are building here.

The plan of the paper is as follows. In section two, we recall the notion of *Adaptive Belief System* (ABS) in an asset pricing framework. Section 3 defines the notion of Large Type Limit (LTL). Section 4 proves a theorem about convergence of an ABS to a LTL, when the number of trader types goes to infinity. Section 5 discusses the consequences of the convergence theorem, and shows in particular that all generic and persistent features of an ABS with many trader types occur with high probability in the LTL. Section 6 presents a simple example of an LTL obtained when beliefs are randomly drawn from linear forecasting rules. Local stability as well as global bifurcation routes to instability and strange attractors are investigated for this case. Section 7 investigates time series properties of a LTL buffeted with noise, and shows that an LTL can match the autocorrelation patterns of returns, squared returns and absolute returns of 20 years of S&P 500 data. Finally, section 8 concludes.

2 Adaptive Belief Systems

In order to be self contained, this section recalls the notion of an *Adaptive Belief System (ABS)*, as introduced in Brock and Hommes (1998), applying the evolutionary framework developed in Brock and Hommes (1997); see also Brock (1997) and Hommes (2001) for extensive discussions. An ABS is in fact a standard discounted value asset pricing model derived from mean-variance maximization, extended to the case of *heterogeneous beliefs*. Agents can either invest in a risk free asset or in a risky asset. The risk free asset is perfectly elastically supplied and pays a fixed rate of return r; the risky asset, for example a large stock or a market index, pays an uncertain dividend. Let p_t be the price per share (ex-dividend) of the risky asset at time t, and let y_t be the stochastic dividend process of the risky asset. Wealth dynamics is given by

$$\tilde{W}_{t+1} = RW_t + (\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t)z_t, \tag{1}$$

where $R = 1 + r$ is the gross rate of risk free return, variables with a tilde such as $\tilde{y}_{t+1}$ denote random variables at date $t+1$ and z_t denotes the number of shares of the risky asset purchased at date t. Let E_t and V_t denote the conditional expectation and conditional variance based on a publically available information set such as past prices and past dividends. Let E_{ht} and V_{ht} denote the 'beliefs' or forecasts of trader type h about

conditional expectation and conditional variance. Agents are assumed to be myopic mean-variance maximizers so that the demand z_{ht} of type h for the risky asset solves

$$\text{Max}_{z_t}\{E_{ht}[\tilde{W}_{t+1}] - \frac{a}{2}V_{ht}[\tilde{W}_{t+1}]\}, \tag{2}$$

where a is the risk aversion parameter. The demand z_{ht} for risky assets by trader type h is then

$$z_{ht} = \frac{E_{ht}[\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t]}{aV_{ht}[\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t]} = \frac{E_{ht}[\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t]}{a\sigma^2}, \tag{3}$$

where the conditional variance $V_{ht} = \sigma^2$ is assumed to be equal for all types and constant.[2] Let z^s denote the supply of outside risky shares per investor, assumed to be constant, and let n_{ht} denote the fraction of type h at date t. Equilibrium of demand and supply yields

$$\sum_{h=1}^{H} n_{ht}\frac{E_{ht}[\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t]}{a\sigma^2} = z^s, \tag{4}$$

where H is the number of different trader types. The forecasts $E_{ht}[\tilde{p}_{t+1}+\tilde{y}_{t+1}]$ of tomorrows prices and dividends are made *before* the equilibrium price p_t has been revealed by the market and therefore will depend upon a publically available information set $I_{t-1} = \{p_{t-1}, p_{t-2}, ...; y_{t-1}, y_{t-2}\}$ of past prices and dividends. Market equilibrium (4) then implies that the realized market price p_t will be the unique price for which demand equals supply. The market equilibrium equation can be rewritten as

$$Rp_t = \sum_{h=1}^{H} n_{ht}E_{ht}[\tilde{p}_{t+1} + \tilde{y}_{t+1}] - a\sigma^2 z^s. \tag{5}$$

The quantity $a\sigma^2 z^s$ may be interpreted as a risk premium for traders to hold all risky assets.

2.1 The EMH benchmark with rational agents

Let us first discuss the EMH-benchmark with rational expectations. In a world where all traders are identical and expectations are *homogeneous* the arbitrage market equilibrium equation (5) reduces to

$$Rp_t = E_t[\tilde{p}_{t+1} + \tilde{y}_{t+1}] - a\sigma^2 z^s, \tag{6}$$

[2]Gaunersdorfer (2000) investigates the case with time varying beliefs about variances and shows that the results are quite similar to those for constant variance. Under assumptions B2 and B3 below, it can be shown that constant conditional variance beliefs is not contradictory with heterogeneous conditional mean beliefs of the form in assumption B1. For example if the $\{y_t\}$ process is IID, the fundamental price is a constant, and the conditional variance of each belief of the form posited in assumption B1 is just the variance of y_t. This argument can be generalized to the case where $\{y_t\}$ is an autoregression of finite order driven by IID shocks. See Brock (1997) for discussion.

where E_t denotes the common conditional expectation of all traders at the beginning of period t, based on a publically available information set I_t. This arbitrage market equilibrium equation (6) states that today's price of the risky asset must be equal to the sum of tomorrow's expected price and expected dividend, discounted by the risk free interest rate. It is well known that, using the arbitrage equation (6) repeatedly and assuming that the *transversality condition*

$$\lim_{t\to\infty} \frac{E_t[\tilde{p}_{t+k}]}{R^k} = 0 \tag{7}$$

holds, the price of the risky asset is uniquely determined by

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[\tilde{y}_{t+k}] - a\sigma^2 z^s}{R^k}. \tag{8}$$

The price p_t^* in (8) is called the EMH fundamental rational expectations (RE) price, or the *fundamental price* for short. The fundamental price is completely determined by economic fundamentals and given by the discounted sum of expected future dividends minus the risk premium. In general, the properties of the fundamental price p_t^* depend upon the stochastic dividend process y_t. We will mainly focus on the case of an IID dividend process y_t, with constant mean $E[y_t] = \bar{y}$. We note however that any other random dividend process y_t may be substituted in what follows[3]. For an IID dividend process y_t with constant mean, the fundamental price is constant and given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{R^k} = \frac{\bar{y} - a\sigma^2 z^s}{r}. \tag{9}$$

There are two crucial assumptions underlying the derivation of the RE fundamental price. The first is that expectations are *homogeneous*, all traders are *rational* and it is *common knowledge* that all traders are rational. Only in such an ideal, perfectly rational world the fundamental price can be derived from economic fundamentals. In contrast, in a world with heterogeneous traders having different beliefs or expectations about future prices and dividends, derivation of a RE fundamental price requires perfect knowledge about the beliefs of *all* other traders. In a real market understanding the beliefs and strategies of all other, competing traders is virtually impossible, and therefore in a heterogeneous world derivation of the RE-fundamental price becomes impossible. The second crucial assumption underlying the derivation of the fundamental price is the transversality condition (7), requiring that the long run growth rate of prices (and risk adjusted dividends) is smaller than the risk free growth rate r. In fact, in addition to the fundamental solution (8) so-called *speculative bubble solutions* of the form

$$p_t = p_t^* + R^t(p_0 - p_0^*) \tag{10}$$

[3]Brock and Hommes (1997b) for example discuss a non-stationary example, where the dividend process is a geometric random walk .

also satisfy the arbitrage equation (6). It is important to note that along the speculative bubble solution (10), traders have rational expectations. Solutions of the form (10) are therefore called *rational bubbles*. These rational bubble solutions are unbounded and do *not* satisfy the transversality condition. In a perfectly rational world, traders realize that speculative bubbles can not last forever and therefore they will never get started and the finite fundamental price p_t^* is uniquely determined. In a perfectly rational world, all traders thus believe that the value of a risky asset equals its fundamental price forever. Changes in asset prices are solely driven by unexpected changes in dividends and random 'news' about economic fundamentals. In a heterogeneous evolutionary world however, the situation will be quite different.

2.2 Heterogeneous beliefs

In the asset pricing model with heterogeneous beliefs, market equilibrium in (5) states that the price p_t of the risky asset equals the discounted value of tomorrow's expected price plus tomorrow's expected dividend, *averaged* over all different trader types. In such a *heterogeneous* world temporary upward or downward bubbles with prices deviating from the fundamental may arise, when the fractions of traders believing in those bubbles is large enough. Once a (temporary) bubble has started, evolutionary forces may reinforce deviations from the benchmark fundamental. We shall now be more precise about traders' expectations (forecasts) about future prices and dividends. It will be convenient to work with

$$x_t = p_t - p_t^*, \tag{11}$$

the *deviation* from the fundamental price. We make the following assumptions about the beliefs of trader type h:

B1 $V_{ht}[\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t] = V_t[\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t] = \sigma^2$, for all h, t.

B2 $E_{ht}[\tilde{y}_{t+1}] = E_t[\tilde{y}_{t+1}]$, for all h, t.

B3 All beliefs $E_{ht}[\tilde{p}_{t+1}]$ are of the form

$$E_{ht}[\tilde{p}_{t+1}] = E_t[\tilde{p}^*_{t+1}] + E_{ht}[\tilde{x}_{t+1}] = E_t[\tilde{p}^*_{t+1}] + f_h(x_{t-1}, ..., x_{t-L}), \qquad \text{for all } h, t. \tag{12}$$

According to assumption B1 beliefs about conditional variance are equal and constant for all types, as discussed above already. Assumption B2 states that expectations about future dividends $\tilde{y}_{t+1}$ are the same for all trader types and equal to the conditional expectation. According to assumption B3, beliefs about future prices consist of two parts: a common belief about the fundamental plus a heterogeneous part for each type h. The benchmark fundamental represents the traders' response to common knowledge exogenous news, such as interest rate movements, growth of the firms profits, earnings or dividends, etc.. All traders are able to use this information to derive the fundamental price p_t^* in (8) that would

prevail in a perfectly rational world. According to assumption B3, traders nevertheless believe that in a heterogeneous world prices may *deviate* from their fundamental value p_t^* by some function f_h depending upon past deviations from the fundamental. Each forecasting rule f_h represents the *model of the market* according to which type h believes that prices will deviate from the commonly shared fundamental price. For example, a forecasting strategy f_h may correspond to a technical trading rule, based upon short run or long run moving averages, of the type used in real markets. We will use the short hand notation

$$f_{ht} = f_h(x_{t-1}, ..., x_{t-L}) \tag{13}$$

for the forecasting strategy employed by trader type h. Brock and Hommes (1998) have investigated evolutionary competition between the simplest linear trading rules with only one lag, i.e.

$$f_{ht} = g_h x_{t-1} + b_h. \tag{14}$$

Simple forecasting rules may be relevant in real markets, because for a complicated forecasting rule it seems unlikely that enough traders will coordinate on that particular rule so that it affects market equilibrium prices. Although the linear forecasting rule (14) is extremely simple, it does in fact represent a number of important cases. For example, when both the trend parameter and the bias parameter $g_h = b_h = 0$ the rule reduces to the forecast of *fundamentalists*, i.e.

$$f_{ht} \equiv 0, \tag{15}$$

believing that the market price will be equal to the fundamental price p^*, or equivalently that the deviation x from the fundamental will be 0. Other important cases covered by the linear forecasting rule (14) are the pure *trend followers*

$$f_{ht} = g_h x_{t-1}, \tag{16}$$

and the pure *biased belief*

$$f_{ht} = b_h. \tag{17}$$

Notice that the simple pure bias (17) rule represents any positively or negatively biased price forecast that traders might have. Instead of these extremely simple habitual rule of thumb forecasting rules, some economists might prefer the rational, *perfect foresight* forecasting rule

$$f_{ht} = x_{t+1}. \tag{18}$$

We emphasize however, that the perfect foresight forecasting rule (18) assumes perfect knowledge of the heterogeneous market equilibrium equation (5), and in particular perfect knowledge about the beliefs of *all* other traders. Although the case with perfect foresight certainly has theoretical appeal, its practical relevance in a complex heterogeneous world should not be overstated since this underlying assumption seems highly unrealistic. However, in the manner of Brock and Hommes (1997) and de Fontnouvelle (2000), one could make predictor (18) (or a signal of high precision on x_{t+1}) available at a cost, and add it to the set of predictors we consider. We leave this to future research but remark that it may generate interesting dynamics.

An important and convenient consequence of the assumptions B1-B3 concerning traders' beliefs is that the heterogeneous agent market equilibrium equation (5) can be reformulated in *deviations* from the benchmark fundamental. In particular substituting the price forecast (12) in the market equilibrium equation (5) and using the facts that the fundamental price p_t^* satisfies $Rp_t^* = E_t[p_{t+1}^* + y_{t+1}] - a\sigma^2 z^s$ and the price $p_t = x_t + p_t^*$ yields the equilibrium equation in deviations from the fundamental:

$$Rx_t = \sum_{h=1}^{H} n_{ht} E_{ht}[\tilde{x}_{t+1}] \equiv \sum_{h=1}^{H} n_{ht} f_{ht}, \tag{19}$$

with $f_{ht} = f_h(x_{t-1}, ..., x_{t-L})$. An important reason for our model formulation in terms of deviations from a benchmark fundamental is that in this general setup, the benchmark rational expectations asset pricing model is *nested* as a special case, with all forecasting strategies $f_h \equiv 0$. In this way, the adaptive belief systems can be used in empirical and experimental testing whether asset prices deviate significantly from anyone's favorite benchmark fundamental.

2.3 Evolutionary dynamics

The evolutionary part of the model describes how beliefs are updated over time, that is, how the fractions n_{ht} of trader types in the market equilibrium equation (19) evolve over time. Fractions are updated according to an *evolutionary fitness* or performance measure. The fitness measures of all trading strategies are publically available, but subject to noise. Fitness is derived from a random utility model and given by

$$\tilde{U}_{ht} = U_{ht} + \epsilon_{ht}, \tag{20}$$

where U_{ht} is the *deterministic part* of the fitness measure and ϵ_{ht} represents IID noise across $h = 1, ...H$. In order to obtain analytical expressions for the probabilities or fractions, it will be assumed that the noise ϵ_{ht} is drawn from a double exponential distribution. In that case, in the limit as the number of agents goes to infinity, the probability that an agent chooses strategy h is given by the well known *discrete choice model* or 'Gibbs' probabilities[4]

$$n_{ht} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}. \tag{21}$$

Note that the fractions n_{ht} add up to 1. The crucial feature of (21) is that the higher the fitness of trading strategy h, the more traders will select strategy h. The parameter β in (21) is called the *intensity of choice*, measuring how sensitive the mass of traders is to selecting the optimal prediction strategy. The intensity of choice β is inversely related to

[4]See Manski and McFadden (1981) and Anderson, de Palma and Thisse (1993) for extensive discussion of discrete choice models and their applications in economics.

the variance of the noise terms ϵ_{ht}. The extreme case $\beta = 0$ corresponds to the case of infinite variance noise, so that differences in fitness can not be observed and all fractions (21) will be fixed over time and equal to $1/H$. The other extreme case $\beta = +\infty$ corresponds to the case without noise, so that the deterministic part of the fitness can be observed perfectly and in each period, *all* traders choose the optimal forecast. An increase in the intensity of choice β represents an increase in the degree of rationality w.r.t. evolutionary selection of trading strategies. The timing of the coupling between the market equilibrium equation (5) or (19) and the evolutionary selection of strategies (21) is crucial. The market equilibrium price p_t in (5) depends upon the fractions n_{ht}. The notation in (21) stresses the fact that these fractions n_{ht} depend upon *past* and most recently observed fitnesses $U_{h,t-1}$, which in turn depend upon past prices p_{t-1} and dividends y_{t-1} in periods $t-1$ and further in the past as will be seen below. After the equilibrium price p_t has been revealed by the market, it will be used in evolutionary updating of beliefs and determining the new fractions $n_{h,t+1}$. These new fractions $n_{h,t+1}$ will then determine a new equilibrium price p_{t+1}, etc.. In the ABS, market equilibrium prices and fractions of different trading strategies thus co-evolve over time.

A natural candidate for evolutionary fitness is accumulated *realized profits*, as given by[5]

$$U_{ht} = (\tilde{p}_t + \tilde{y}_t - Rp_{t-1})\frac{E_{h,t-1}[\tilde{p}_t + \tilde{y}_t - Rp_{t-1}]}{a\sigma^2} + wU_{h,t-1}, \tag{22}$$

where $R = 1+r$ is the gross risk free rate of return, and $0 \leq w \leq 1$ is a *memory* parameter measuring how fast past realized fitness is discounted for strategy selection. The first term in (22) represents last period's realized profit of type h given by the realized excess return of the risky asset over the risk free asset times the demand for the risky asset by traders of type h. In the extreme case with no memory, i.e. $w = 0$, fitness U_{ht} equals net realized profit in the previous period, whereas in the other extreme case with infinite memory, i.e. $w = 1$, fitness U_{ht} equals total wealth as given by accumulated realized profits over the entire past. In the intermediate case, the weight given to past realized profits decreases exponentially with time. It will be useful to compute the realized excess return R_t in

[5]Given that investors are risk averse mean-variance maximizers, maximizing their expected utility from wealth another natural candidate for fitness are the *risk adjusted profits*. In fact, the fitness measure (22) based upon realized profits does not take into account the variance term in (2) capturing the investors' risk taken before obtaining that profit. On the other hand, in real markets realized net profits or accumulated wealth may be what investors care about most, and the non-risk adjusted fitness measure (22) may thus be practically important. See BH 1998 and Hommes (2001) for a discussion of this point. Gaunersdorfer and Hommes (2000) investigate an evolutionary model with fitness given by risk adjusted profits. In any event the methodology of LTL developed in this paper can be developed for alternative fitness functions to (22). The general point that taking the LTL drastically reduces the number of "parameters" in heterogeneous belief models with large numbers of such beliefs to the small set of parameters that determine "underlying characteristics" of underlying "belief distributions" remains independently of what choice the modeler makes for the fitness function.

deviations from the fundamental to obtain

$$\begin{aligned} R_t &= \tilde{p}_t + \tilde{y}_t - Rp_{t-1} = \tilde{x}_t + \tilde{p}^*_t + \tilde{y}_t - Rx_{t-1} - Rp^*_{t-1} \\ &= \tilde{x}_t - Rx_{t-1} + \tilde{p}^*_t + \tilde{y}_t - E_{t-1}[\tilde{p}^*_t + \tilde{y}_t] + E_{t-1}[\tilde{p}^*_t + \tilde{y}_t] - Rp^*_{t-1} \\ &\equiv \tilde{x}_t - Rx_{t-1} + a\sigma^2 z^s + \delta_t, \end{aligned} \tag{23}$$

where we used that $E_{t-1}[\tilde{p}^*_t + \tilde{y}_t] - Rp^*_{t-1} = a\sigma^2 z^s$ since the fundamental p^*_t satisfies the market equilibrium equation (6), and $\delta_t \equiv \tilde{p}^*_t + \tilde{y}_t - E_{t-1}[\tilde{p}^*_t + \tilde{y}_t]$ is a martingale difference sequence. The random term δ_t enters because the dividend process is stochastic, and thus represents intrinsic uncertainty about economic fundamentals[6]. According to the decomposition (23) excess return consists of a conventional EMH term δ_t plus a risk premium $a\sigma^2 z^s$ and an additional *speculative* term $x_t - Rx_{t-1}$ of the ABS theory. Our ABS theory thus allows for the possibility of *excess volatility*. The extra term is zero if either $x_t \equiv 0$, that is prices equal their fundamental value, or if $x_t = Rx_{t-1}$, that is when prices follow a RE bubble solution. The ABS theory predicts excess volatility in periods when asset prices grow faster or slower than the risk free rate of return, or when prices switch between a temporary bubble solution and the fundamental.

Fitness can now be rewritten in deviations from the fundamental as

$$U_{ht} = (\tilde{x}_t - Rx_{t-1} + a\sigma^2 z^s + \delta_t)(\frac{f_{h,t-1} - Rx_{t-1} + a\sigma^2 z^s}{a\sigma^2}) + wU_{h,t-1}. \tag{24}$$

3 Large Type Limits

This section introduces the notion of a *Large Type Limit* (LTL), which is obtained as a 'limit approximation' of an ABS with belief types drawn randomly from a fixed distribution, as the number of types tends to infinity.

3.1 Evolution for finitely many traders

The starting point is the equilibrium equation for a market with heterogeneous beliefs:

$$Rx_t = \sum_{h=1}^{H} n_{ht} f_{ht}\,, \tag{25}$$

where x_t is the *deviation* from the fundamental price and n_{ht} the fraction of type h traders, as before. The function f_{ht}, expressing the belief of type h on the price of the risky asset at time $t+1$, is assumed to have the following general form:

$$f_{ht} = f(x_{t-1}, ..., x_{t-d}, \lambda, \vartheta_h)\ .$$

[6] In the special case of an IID dividend process $y_t = \bar{y} + \epsilon_t$ we simply have $\delta_t = \epsilon_t$.

The x_{t-j}, $j = 1, ..., d$, are the past price deviations up to the maximal time delay d, λ is a multidimensional *structural parameter* (containing all economic parameters, such as the interest rate r or the mean dividend $\bar{y}$, that may be used in the forecasting functions), and the *belief variable* ϑ_h is a multidimensional stochastic variable which characterizes the belief h. This belief is sampled from a general distribution of beliefs. To fix ideas, consider as an example the case where beliefs are linear with the stochastic variable ϑ_h drawn from a multivariate normal distribution:

$$f_{ht} = f_t(\vartheta_h) = \vartheta_{0h} + \vartheta_{1h} x_{t-1} + ... + \vartheta_{dh} x_{t-d}.$$

Recall that the fractions n_{ht} are given by the *discrete choice model* probabilities

$$n_{ht} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}. \tag{26}$$

Since the *fitness function* $U_{h,t-1}$ of type h depends upon x_{t-1}, x_{t-2} and upon the forecasting function $f_{h,t-2}$ their general form is:

$$U_{h,t-1} = U(x_{t-1}, x_{t-2}, ..., x_{t-d-2}, \lambda, \vartheta_h)\,,$$

where the parameter λ is the structural parameter as before, including for example the risk aversion coefficient a appearing in the traders' demand function for the risky asset.

The equilibrium equation (25) can be rewritten to:

$$x_t = \frac{1}{R} \frac{\sum_{h=1}^{H} e^{\beta U_{h,t-1}} f_{ht}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}\,. \tag{27}$$

It gives rise to a *dynamical system* in the following way. The *state variables* $\mathbf{x}_t = (x_{1t}, ..., x_{d+2,t})$ are introduced by $x_{jt} = x_{t-j}$, $1 \leq j \leq d+2$. The *state space* X is taken to be an open subset of $\mathbf{R}^{d+2}$.

The equilibrium equation (27) determines the evolution of the system with H trader types - this information is coded in the *evolution map* $\varphi_H(\mathbf{x}, \lambda, \vartheta)$:

$$\varphi_H(\mathbf{x}, \lambda, \vartheta) = \frac{1}{R} \frac{\sum_{h=1}^{H} e^{\beta U(\mathbf{x},\, \lambda, \vartheta_h)} f(\mathbf{x}, \lambda, \vartheta_h)}{\sum_{h=1}^{H} e^{\beta U(\mathbf{x},\, \lambda, \vartheta_h)}}. \tag{28}$$

The structural parameter $\lambda = (\beta, R, \lambda_3, ..., \lambda_q)$ takes values in a bounded open subset Λ of $\mathbf{R}^q$, and contains all structural parameters of the evolutionary, heterogeneous agents economy, such as the intensity of choice β, the interest rate r (or gross rate of return $R = 1+r$), the risk aversion coefficient a, etc. When the number of trader types H is large, the evolution map φ_H contains a large number of stochastic variables $\vartheta = (\vartheta_1, ..., \vartheta_H)$,

with the ϑ_j independently identically distributed (henceforth: IID), with *distribution function* F_μ. The distribution function of the stochastic belief variable ϑ_h depends on a multi-dimensional parameter μ, called the *belief parameter*, which takes values in a bounded open subset M of $\mathbf{R}^r$. This setup allows to vary the population out of which the individual beliefs are sampled. The explicit dependence of a probability on μ is denoted by $P = P_\mu$.

If the state $\mathbf{x}_t$ at time t is known, the state $\mathbf{x}_{t+1}$ at time $t+1$ is given by $\mathbf{x}_{t+1} = \Phi_H(\mathbf{x}_t, \lambda, \vartheta)$, where:

$$\Phi_H(\mathbf{x}, \lambda, \vartheta) = (\varphi_H(\mathbf{x}, \lambda, \vartheta), x_1, \ldots, x_{d+1}) \ . \tag{29}$$

The map Φ_H defines the *dynamical system* associated to the evolution map φ_H corresponding to a market with H different belief types.

3.2 The limit evolution

The main contribution of this paper is to show that the time evolution of a market with a large, but finite, number of beliefs H, randomly drawn from a fixed distribution, is well described, in the sense to be made precise below, by a much simpler system called the *large type limit (LTL)*. The LTL represents in a way the 'average' dynamical behavior of all interacting traders. Observe that both the denominator and the nominator of the evolution map φ_H in (28) may be divided by the number of trader types H and thus may be seen as sample means. The evolution map ψ of the large type limit is then simply obtained by *replacing the sample means in the evolution map φ_H by population means*, to obtain:

$$\psi(\mathbf{x}, \lambda, \mu) = \frac{1}{R} \frac{\mathrm{E}_\mu \left[e^{\beta U(\mathbf{x}, \lambda, \vartheta_0)} \, f(\mathbf{x}, \lambda, \vartheta_0) \right]}{\mathrm{E}_\mu \left[e^{\beta U(\mathbf{x}, \lambda, \vartheta_0)} \right]} \, . \tag{30}$$

Here ϑ_0 is a stochastic variable which is distributed in the same way as the ϑ_h, with distribution function F_μ. The *structural* parameter vector λ of the evolution map φ_H and the LTL evolution map ψ coincide. However, whereas the evolution map φ_H in (28) of the heterogeneous agent system contains H randomly drawn multi-dimensional stochastic variables ϑ_h, the LTL evolution map ψ in (30) only contains the *belief parameter* vector μ describing the joint probability distribution. Taking a large type limit thus leads to a huge reduction in stochastic belief variables. In section 6 we will consider an example where the randomly drawn beliefs are linear, with multi-variate normally distributed belief variables, and the LTL contains as belief parameters the means and the variances of the corresponding multi-variate distribution.

The dynamical system corresponding to the LTL evolution map ψ is denoted by $\Psi(\mathbf{x}, \lambda, \mu)$ and given by:

$$\Psi(\mathbf{x}, \lambda, \mu) = (\psi(\mathbf{x}, \lambda, \mu), x_1, \ldots, x_{d+1}) \ . \tag{31}$$

Section 4 will state and prove a theorem saying that the LTL is a good approximation of an ABS with many belief types, and section 5 will show that all 'generic' and 'persistent' dynamic properties will be preserved with high probability.

4 Convergence to large type limits

This section states and proves the theorem about convergence to the large type limit. More precisely, it will be shown that the map Φ_H, describing the evolutionary dynamics in a market with H randomly drawn beliefs, converges almost surely to the LTL map Ψ as the number of trader types H tends to infinity.

4.1 Notation

In order to formulate the theorem below succinctly, the following notation is introduced. If the natural numbers are denoted by $\mathbf{N} = \{0, 1, 2, ...\}$, then let $\alpha \in \mathbf{N}^{d+q}$ be a multi-index:

$$\alpha = (\alpha_1, ..., \alpha_{d+q}) \ .$$

Let $|\alpha| = \sum |\alpha_j|$, and let D_α denote differentiation with respect to $\mathbf{x}$ and λ as follows:

$$D_\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} ... \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \frac{\partial^{\alpha_{d+1}}}{\partial \lambda_1^{\alpha_{d+1}}} ... \frac{\partial^{\alpha_{d+q}}}{\partial \lambda_q^{\alpha_{d+q}}} \ .$$

It is assumed that $\mathbf{x}$ and λ take values in bounded open sets $V \subset \mathbf{R}^d$ and $\Lambda \in \mathbf{R}^q$ respectively. For functions that are k times differentiable on $V \times \Lambda$, and that have together with their derivatives, continuous extenstions to the closure $\overline{V \times \Lambda}$, the following C^k norm is introduced:

$$\|f\|_k = \max_{|\alpha| \leq k} \sup_{V \times \Lambda} |D_\alpha f(\mathbf{x}, \lambda)|$$

The norm of vector valued functions is defined analogously, by taking appropriate vector norms on the right hand side.

4.2 The convergence theorem

Note that the stochastic variables of the previous section are all of the general form:

$$X = g(\Theta, s) \, ,$$

where Θ is another (multi-dimensional) stochastic variable, and where s is an ordinary variable; for instance, $s = (\mathbf{x}, \lambda)$ consists of a vector $\mathbf{x}$ of state variables and a vector λ of structural parameters. We call X a *parameter dependent stochastic variable*. Let F_μ

be the distribution function of Θ, depending on a vector μ of belief parameters. If g is differentiable with respect to s, the parameter dependent stochastic variable X is said to be *differentiable with respect to s* as well. If g is differentiable up to order k, the derivative $D_\alpha X$ with $|\alpha| \leq k$ of X with respect to s is defined as

$$D_\alpha X = (D_\alpha g)(\Theta, s).$$

In the present context, let

$$X_h = \begin{pmatrix} X_{1h} \\ X_{2h} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\beta U(s,\vartheta_h)} \\ \mathrm{e}^{\beta U(s,\vartheta_h)} f(s,\vartheta_h) \end{pmatrix},$$

where $s = (\mathbf{x}, \lambda)$ and where U and f are as in the previous section. If f and U are differentiable up to order k in $\mathbf{x}$ and λ, then X_h is differentiable up to the same order in s.

Theorem (C^k-convergence almost surely to the LTL)
Let $k > 0$ be fixed, and assume that for all α with $|\alpha| \leq k$, there exists a nonnegative random variable $Y_\alpha \geq 0$, independent of s, such that

$$|D_\alpha X_h| < Y_\alpha \quad \textit{and} \quad EY_\alpha < \infty.$$

Then for every belief parameter $\mu \in M$, and for any $\varepsilon > 0$ and $\delta > 0$, there is a $H_0 > 0$ such that:

$$P_\mu \left\{ \text{For any } H \geq H_0 \,:\, \|\Psi - \Phi_H\|_k > \varepsilon \right\} < \delta \,.$$

That is, for $H \geq H_0$, Ψ and Φ_H are, with probability $1-\delta$, ε–close in the C^k topology.

This theorem states that the H-belief types system Φ_H (29) converges almost surely to its large type limit Ψ in (31) as H tends to infinity. In other words, if H is large enough, with high probability the dynamical systems Φ_H, Φ_{H+1}, $\cdots$, and the Large Type Limit Ψ, are ε–close in the C^k topology. Note that the theorem holds *pointwise* for μ, that is, for each $\mu \in M$, as H becomes large the H-type system $\Phi_H(\mathbf{x}, \lambda, \vartheta)$ converges almost surely to its LTL $\Psi(\mathbf{x}, \lambda, \mu)$.

4.3 Applicability

To what kind of stochastic variables is this theorem applicable? Recall that $X_{1h} = \mathrm{e}^{\beta U_h}$, and that $X_{2h} = \mathrm{e}^{\beta U_h} f_h$. Moreover, above it has been assumed throughout that the forecasting rules f_h and the fitness measures U_h are linear in the stochastic variables ϑ_j.

Consider a typical ϑ, and assume that its probability distribution $\mathrm{d}F(\vartheta)$ is of the form $F'(\vartheta)\,\mathrm{d}\vartheta$, with $F'(\vartheta)$ a measurable function. Note that

$$X_1, X_2 \sim Y = \mathrm{e}^{c\vartheta}(a + b\vartheta),$$

where a, b and c can take any values. In order that $\mathrm{E}Y$ exists, we have the condition:

$$\int \mathrm{e}^{c|\vartheta|}|\vartheta|F'(\vartheta)\,\mathrm{d}\vartheta < \infty$$

for all c. Hence $F'(\vartheta)$ has to fall off faster than $\mathrm{e}^{-|c||\vartheta|}$ as $|\vartheta| \to \infty$ for all positive values of $|c|$. On the other hand, if

$$F'(\vartheta) \sim \mathrm{e}^{-k|\vartheta|^{1+\gamma}},$$

for some $\gamma, k > 0$, then the conditions of the theorem are satisfied. Note that normally distributed variables, as well as variables whose distribution has compact support, satisfy these conditions.

The rest of this section is devoted to the proof of this theorem.

4.4 Preliminaries

It clearly suffices to show that:

$$P_\mu\left\{\text{For all } H \geq H_0 \,:\, \|\psi - \varphi_H\|_k > \varepsilon\right) < \delta\,,$$

where the evolution maps ψ in (30) and φ_H in (28) take the place of the dynamical systems Ψ in (31) and Φ_H in (29).

Let $\bar{X}$ denote the 'average' over the $X_h = (X_{1h}, X_{2h})$:

$$\bar{X} = \frac{1}{H}\sum_{h=1}^{H} X_h\,.$$

Finally, note that $\mathrm{E}\bar{X} = (\mathrm{E}X_1, \mathrm{E}X_2) = \mathrm{E}\left(\mathrm{e}^{\beta U_0}, \mathrm{e}^{\beta U_0} f_0\right)$. With this notation, compare:

$$\varphi_H(s,\vartheta) = \frac{1}{R}\frac{\bar{X}_2}{\bar{X}_1} = \frac{1}{R}\frac{\mathrm{E}X_2 + \left(\bar{X}_2 - \mathrm{E}X_2\right)}{\mathrm{E}X_1 + \left(\bar{X}_1 - \mathrm{E}X_1\right)} \qquad \text{and} \qquad \psi(s) = \frac{1}{R}\frac{\mathrm{E}X_2}{\mathrm{E}X_1}\,.$$

Noting that $\mathrm{E}X_1 \geq c > 0$, the strong law of large numbers shows that as $H \to \infty$, the terms between brackets in the expression of φ_H tend to 0. It follows that $\varphi_H(s,\vartheta) \to \psi(s)$ almost surely, *pointwise* in s. However, this is not enough for our purposes: we need $\varphi_H(s,\vartheta)$, *together with its derivatives* up to k'th order, to converge *uniformly* in s.

In the next subsection, a uniform law of large numbers is quoted, which helps us to establish precisely what we want.

4.5 Uniform law of large numbers

The following theorem, with changed notation, is quoted from Jennrich (1969, p.636, theorem 2).

Theorem
Let $g(s,\vartheta)$ be defined on $\mathcal{S}\times\mathcal{E}$, where $\mathcal{E}$ is a Euclidean space, and where $\mathcal{S}$ is a compact subset of an Euclidean space. Let g be continuous in s for each ϑ, and measurable in ϑ for each s. Assume that $|g(s,\vartheta)|\leq h(\vartheta)$ for all s and ϑ, where h is integrable with respect to a probability distribution F on $\mathcal{E}$. If ϑ_1, ϑ_2, $\cdots$, is a random sample from F, then for almost every sequence $\{\vartheta_h\}$,

$$\frac{1}{H}\sum_{j=1}^{H} g(\vartheta_h,s)\to\int g(\vartheta,s)\,\mathrm{d}F(\vartheta)\qquad \textit{almost surely,}$$

uniformly in s.

Note that if $|\alpha|\leq k$, the hypothesis of this theorem is satisfied for $D_\alpha X_h$ since these stochastic variables are dominated by Y_α. This shows already that $\varphi_H\to\psi$ almost surely in the C^0 topology.

We have that

$$D_\alpha\bar{X}\to \mathrm{E}D_\alpha X\qquad a.s., \tag{32}$$

uniformly in s. Moreover, again because of domination

$$\mathrm{E}D_\alpha X = D_\alpha \mathrm{E}X.$$

The derivative $D_\alpha\varphi_H$ is computed for $|\alpha|=1$:

$$\begin{aligned} D_\alpha\varphi_H \;=\; & \frac{\begin{array}{r}\left[D_\alpha\mathrm{E}X_2+\left(D_\alpha\bar{X}_2-\mathrm{E}D_\alpha X_2\right)\right]\left[\mathrm{E}X_1+\left(\bar{X}_1-\mathrm{E}X_1\right)\right]\\ -\left[\mathrm{E}X_2+\left(\bar{X}_2-\mathrm{E}X_2\right)\right]\left[D_\alpha\mathrm{E}X_1+\left(D_\alpha\bar{X}_1-\mathrm{E}D_\alpha X_1\right)\right]\end{array}}{\left[\mathrm{E}X_1+\left(\bar{X}_1-\mathrm{E}X_1\right)\right]^2}. \end{aligned}$$

Because of the uniform convergence of the derivatives (32), it follows that if $|\alpha|\leq 1$,

$$D_\alpha\varphi_H(\vartheta,s)\to D_\alpha\psi(s)\qquad a.s.,$$

uniformly in s. The case of higher order derivatives is left to the reader. This finishes the proof of the theorem.

5 Dynamical consequences

In what sense is the large type limit a 'good' approximation of the system with a large but finite number of beliefs? This question leads to two key concepts in the theory of dynamical systems, *structural stability* and *persistence*. These are discussed in the next subsection, and applied to our evolutionary system in subsection 5.2. It will be argued that all generic and persistent properties of the LTL evolution map Ψ, with probability arbitrarily close to 1 also occur for the evolution map Φ_H if the number of trader types H is sufficiently large.

5.1 Structural stability and persistence

In this subsection Ψ and Φ denote general dynamical systems. A dynamical system Ψ is called *C^k-structurally stable*, if every system Φ sufficiently close to Ψ in the C^k-norm $\|.\|_k$ is conjugated to Ψ; that is, by a suitable change of variables and parameters, Φ can be transformed into Ψ.

Structural stability of a system is usually hard to prove. However, if the concept is restricted to certain properties of the system - like having an attracting fixed point, or undergoing a saddle-node bifurcation - sometimes it can be shown that all systems Φ sufficiently C^k-close also have the same property, which is then called a *C^k-persistent* property. Usually the degree of differentiability k is not explicitly mentioned: 'persistent' means C^k-persistent with k large enough for the purpose at hand. See e.g. Guckenheimer and Holmes (1986) for a general discussion of structural stability and persistence.

Examples of persistent properties

The simplest example of a persistent property is a hyperbolic fixed point. That is, if Ψ has a fixed point x_0, and all eigenvalues of $D_x\Psi(x_0)$ are off the unit circle in the complex plane, then x_0 is called a *hyperbolic fixed point*. Every Φ sufficiently close to Ψ in the $\|.\|_1$-norm also has a hyperbolic fixed point: they are C^1-persistent.

An important class of examples is furnished by so-called *generic bifurcations*, like the saddle-node, Hopf, and period-doubling bifurcations of fixed points. Every generic bifurcation has a positive integer ℓ associated to it, the *co-dimension* of the bifurcation; see e.g. Kuznetsov (1998) for an introduction to bifurcation theory and a detailed mathematical treatment of bifurcations of co-dimension 1 and 2. If Ψ depends on a q-dimensional parameter λ, the co-dimension ℓ of the bifurcation considered is less than or equal to q, and Ψ has for $\lambda = \lambda_0$ a generic co-dimension–ℓ bifurcation, then any q–parameter system Φ sufficiently close to Ψ in the C^k-norm (the required degree k is determined by the specific bifurcation) has the same bifurcation, possibly for a different value of the parameter $\lambda = \lambda_1$. In short: for a dynamical system depending on q parameters, all generic bifurcations with co-dimension up to and including q are persistent.

In particular, generic saddle-node, Hopf, and period doubling bifurcations are persistent for 1-parameter families of dynamical systems. In 2-parameter systems, cusp and Bogdanov-Takens bifurcations (to name a few) persist.

These are all examples of persistent *local* properties. An important example of a persistent *global* property is a so–called transversal homoclinic point, that is, a transversal intersection p of stable and unstable manifolds of a (saddle) fixed point. Homoclinic points are characterized by the fact that its iterates $\varphi^n(p)$ approach the fixed point for $n \to \infty$ *as well as* for $n \to -\infty$. Transversal homoclinic points are persistent under small perturbations. Note that the existence of a transversal homoclinic point implies the occurrence of horseshoe dynamics in the system.

The genesis of transversal homoclinic intersections by a *homoclinic tangency bifurcation* in one–parameter families is another global example of a persistent property. In systems with two–dimensional phase space, such a bifurcation implies (under some mild conditions) the existence of strange attractors for a set of parameters of positive measure. See Palis and Takens (1993) for a recent mathematical treatment of homoclinic bifurcation theory. De Vilder (1996) contains a stimulating introduction of phenomena associated to homoclinic bifurcations and application to a two-dimensional version of the overlapping generations model.

Another example is furnished by the Newhouse–Ruelle–Takens phenomenon: they showed that for a family of mappings, having an invariant quasi–periodic circle bifurcating to an invariant 2–torus (a so–called quasi–periodic Hopf bifurcation), by a C^2–small perturbation, a system with a strange attractor instead of a quasi–periodic 2–torus can be obtained (Ruelle and Takens (1971) and Newhouse, Ruelle and Takens (1978)). Hence every system displaying a quasi–periodic Hopf bifurcation is C^2–near to a system with a strange attractor. Note that quasi–periodic Hopf bifurcations occur generically in two–parameter families (though only on parameter sets of positive measure, but empty interior, see Broer et al. (1990)).

Examples of non-persistent properties

A non-persisting example is furnished by the pitchfork bifurcation. It will be useful to discuss this example here, since it will play some role in the simple LTL studied in section 6. A simple example exhibiting a pitchfork bifurcation is the 1-parameter family dynamical system $\Psi(x, \lambda)$ for $\lambda = 0$, where Ψ is given by:

$$\Psi(x, \lambda) = x + \lambda x - x^3\,, \quad |x| < \frac{1}{3}\,,\ |\lambda| < \frac{2}{3}\,. \tag{33}$$

For $\lambda > 0$ there are two stable fixed points $x = \pm\sqrt{\lambda}$ and one unstable fixed point $x = 0$, which coalesce at $\lambda = 0$. For $\lambda < 0$ there is only one stable fixed point $x = 0$.

By adding a small constant $\varepsilon > 0$ to Ψ, the picture disintegrates. The system Ψ_ε, with:

$$\Psi_\varepsilon(x, \lambda) = x + \lambda x - x^3 + \varepsilon\,, \tag{34}$$

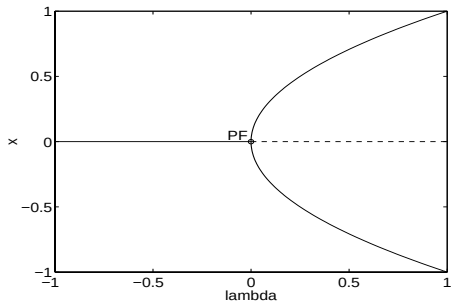


Figure 1: *Bifurcation diagram for the pitchfork bifurcation of* Ψ *in (33).* In this (μ, x)-plot, the phase space is plotted vertically for every parameter μ. Drawn lines indicate the position of fixed points. The dot (PF) indicates the pitchfork bifurcation point.

has a generic saddle node bifurcation of a stable and an unstable fixed point, together with a stable (hyperbolic) fixed point which exists for all λ. Hence the pitchfork bifurcation is not generic. In fact, the pitchfork bifurcation only occurs in systems with a reflectional symmetry. For example, the system (33) is symmetric w.r.t. to $x = 0$, since $\Psi(-x, \lambda) = -\Psi(x, \lambda)$. Adding a perturbation parameter ϵ in (34) means a *breaking of the symmetry* in the system and the pitchfork bifurcation disappears. It should be noted however that, for small ϵ the bifurcation diagram of the perturbed pitchfork in figure 2 is close to the bifurcation diagram of the pitchfork in figure 1

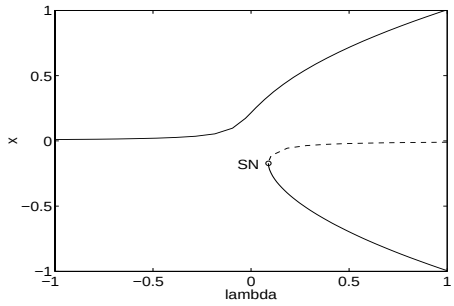


Figure 2: *Bifurcation diagram for the perturbed pitchfork bifurcation in (34).* Legend as in figure 1. The pitchfork bifurcation does not persist. Note the occurrence of a saddle-node bifurcation (SN), which is in contrast to the pitchfork a generic co-dimension-1 bifurcation.

5.2 Application to large type limits

The above considerations can be applied to the case of finite belief systems and their corresponding large type limits.

Persistence of bifurcations 'in probability'

Let $\Phi_H(x, \lambda, \vartheta)$ for $H = 1, 2, ...$ be a H-belief types system as in (29), depending on a q-dimensional structural parameter λ. Let $\Psi(x, \lambda, \mu)$ be the large type limit as in (31), which depends on λ as well as on the belief parameter μ. The theorem of section 4 states that the systems Φ_H, together with their derivatives up to order k, converge in probability to their large type limit as H tends to infinity.

Let $\varepsilon \in (0, 1)$ and $\mu \in M$ be fixed. Since generic co-dimension-ℓ bifurcations persist if $\ell \leq q$, it follows that any generic co-dimension ℓ bifurcation with respect to the structural parameter λ for Ψ (μ is fixed) occurs with probability at least $1 - \varepsilon$ for Φ_H, for H large.

The role of belief parameters

Note the distinction between structural parameters λ and belief parameters μ. This is roughly due to the fact that for Φ_H and Ψ, only bifurcations in λ can be compared with each other, since Φ_H does not depend in a direct way on μ. The following example may illustrate this point further. Suppose $\Phi_H(x, \vartheta)$ denotes the following dynamical system:

$$\Phi_H(x, \vartheta) = \frac{1}{H} \sum_{h=1}^{H} \vartheta_h + x - x^2 \,,$$

with ϑ_h IID, $\mathrm{E}_\mu \vartheta_h = \mu$ and $\mathrm{Var}\, \vartheta_h = \sigma^2$. The parameters μ and σ are belief parameters. The large type limit Ψ of Φ_H is given by:

$$\Psi(x, \mu) = \mu + x - x^2 \,.$$

For $\mu > 0$, this system has two hyperbolic fixed points; for $\mu < 0$, there are none. At $\mu = 0$, there is one non-hyperbolic fixed point.

Now, let again $\varepsilon \in (0, 1)$ and $\mu > 0$ be fixed. Set $\delta = \frac{\mu}{2}$. Then by the theorem of section 4, there is a $H_0 > 0$, such that for all $H \geq H_0$:

$$\|\Psi - \Phi_H\|_k = \left| \sum_{h=1}^{H} \vartheta_h - \mu \right| \leq \delta \,,$$

with probability at least $1 - \varepsilon$. (Of course, the same result can be found directly by Tschebycheff's inequality). Hence, with this probability:

$$\sum_{h=1}^{H} \vartheta_h \geq \frac{\mu}{2} > 0 \,,$$

and the system $\Phi_H(x, \vartheta)$ has two fixed points.

On the other hand, if μ is set to 0, then asymptotically:

$$\frac{1}{H}\sum \vartheta_h \sim N(0, \sigma^2/H)\,,$$

where $N(0, \sigma^2/H)$ denotes the normal distribution around 0. Hence the probabilities of $\frac{1}{H}\sum \vartheta_h > 0$ and $\frac{1}{H}\sum \vartheta_h < 0$ are both approximately $\frac{1}{2}$. The large type limit furnishes no information about the number of steady states of the finite belief systems in this case.

We summarize this as follows: bifurcations in the large type limit system may be found using belief parameters, but they have to be analyzed in terms of structural parameters.

Corollary of large type limit convergence theorem

The discussion in the present section is summarized in the following corollary, listing some important persistent properties which, if they occur for the LTL, with high probability also occur for the H belief type system when H is large.

Corollary of LTL convergence theorem
Assume that Φ_H in (29) is a system with H belief types and q structural parameters, and Ψ is its Large Type Limit *in (31) as H tends to infinity. Then for every k and every C^k persistent property of Ψ and for every ε, there exists a number H_0 of belief types, such that for every $H \geq H_0$, with probability $\geq 1 - \varepsilon$ the system Φ_H generically has the same property.*

In particular, this means that if Ψ has

1. *hyperbolic fixed or periodic points,*
2. *transversal homoclinic intersections of stable and unstable manifolds of a periodic point,*
3. *horseshoes,*
4. *local or global bifurcations up to co-dimension q,*
5. *open neighborhoods in function space with strange attractors (Newhouse–Ruelle–Takens phenomenon),*

then Φ_H will have these too with probability $\geq 1 - \varepsilon$ if $H \geq H_0$.

6 Evolutionary dynamics in a simple LTL

In this section we investigate the dynamical behavior, focusing on generic and persistent properties, of a simple example of an LTL, where traders' forecasting rules are linear functions of past price deviations, i.e.

$$f_{ht} = f_t(\vartheta_h) = \vartheta_{h0} + \vartheta_{h1}x_{t-1} + \cdots + \vartheta_{hd}x_{t-d}, \tag{35}$$

with stochastic belief variables ϑ_{jh} distributed multivariate normal. We calculate LTL's for the simple and tractable case where fitness is last period's realized profits, that is,

$$U_{ht} = (x_t - Rx_{t-1} + a\sigma^2 z^s + \delta_t)(\frac{f_{h,t-1} - Rx_{t-1} + a\sigma^2 z^s}{a\sigma^2}). \tag{36}$$

Including more memory in the fitness measure is straightforward, but leads to a high dimensional LTL[7]. Recall that $\{\delta_t\}$ is the Martingale Difference Sequence component in the decomposition of the excess returns process $\{R_t\}$ in (23). We set $\delta_t = 0$, so that fitness becomes

$$U_{ht} = (x_t - Rx_{t-1} + a\sigma^2 z^s)(\frac{f_{h,t-1} - Rx_{t-1} + a\sigma^2 z^s}{a\sigma^2}), \tag{37}$$

to get what we shall call "the deterministic skeleton" of the stochastic dynamical system. We wish to uncover stabilizing and destabilizing economic forces by studying the skeleton. The deterministic skeleton corresponds to the case where the fitness measure equals the conditional expectation of profits.

The fractions n_{ht} of trader type h, as given by the discrete choice probabilities (21), can be simplified by noting that they are *not* affected when subtracting the same term, independent of h, from all fitnesses $U_{h,t-1}$. Therefore, we can ignore the term $(-Rx_{t-1} + a\sigma^2 z^s)(x_t - Rx_{t-1} + a\sigma^2 z^s)$ in the nominator of the second part of (37) to obtain the equivalent fitness

$$U_{h,t-1} = (x_{t-1} - Rx_{t-2} + a\sigma^2 z^s)\frac{f_{h,t-2}}{a\sigma^2}. \tag{38}$$

Using (30), the LTL for linear forecasting rules (35) is then given by

$$x_t = \frac{1}{R}\frac{E\left[e^{\eta(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s)f_{t-2}(\vartheta_0)}f_t(\vartheta_0)\right]}{E\left[e^{\eta(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s)f_{t-2}(\vartheta_0)}\right]}, \tag{39}$$

where $\eta \equiv \beta/(a\sigma^2)$ and

$$f_t(\vartheta_0) = \vartheta_{00} + \vartheta_{01}x_{t-1} + \cdots + \vartheta_{0d}x_{t-d}, \tag{40}$$

[7] In particular, with fitness as in (22) given by a weighted average of all past profits with exponentially decreasing weights, the LTL becomes infinite dimensional with infinitely many terms with exponentially decreasing weights added. When fitness is given by a weighted average of L past profits, the dimension of the LTL becomes $L + d + 1$.

with $\vartheta_0 = (\vartheta_{00}, \vartheta_{01}, \cdots, \vartheta_{0d})$ multivariate normal. Introduce $s_0 = \eta(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s)$, $s_j = \eta(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s)x_{t-2-j}$, $1 \leq j \leq d$ and put

$$e^{\eta(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s) f_{t-2}(\vartheta_0)} = e^{\sum s_j \vartheta_{0j}}.$$

In order to calculate a closed form expression for the LTL (39) we use moment generating function formulae from normal distribution theory. Note that

$$E[e^N] = e^{E[N] + \frac{1}{2} Var[N]},$$

for a normal random variable $N = \sum s_j \vartheta_j$. Note also that

$$E[e^{\sum s_j \vartheta_{0j}} \vartheta_{0k}] = \frac{d}{ds_k} E[e^{\sum s_j \vartheta_{0j}}].$$

Assume that the ϑ's are uncorrelated (i.e. independent for this multivariate normal case) for simplicity. It is straightforward to extend the method to correlated ϑ's. Using these moment generating function formulae for multivariate normals the following closed form expression for the LTL (39) is obtained:

$$\begin{aligned} Rx_t &= \mu_0 + \mu_1 x_{t-1} + ... + \mu_d x_{t-d} \\ &\quad + \eta(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s)(\sigma_0^2 + \sigma_1^2 x_{t-1} x_{t-3} + \sigma_2^2 x_{t-2} x_{t-4} \cdots + \sigma_d^2 x_{t-d} x_{t-d-2}), \end{aligned} \tag{41}$$

where $\mu_k = E[\vartheta_{0k}]$, $\sigma_k^2 = Var[\vartheta_{0k}]$, $0 \leq k \leq d$ and $\eta = \beta/(a\sigma^2)$.

The simplest special case of (41) that still possesses dynamics is obtained when all $\vartheta_{0k} = 0$, $1 \leq k \leq d$, that is, when the forecasting function is purely biased, i.e. $f_t(\vartheta_0) = \vartheta_{00}$. In this simplest case, the LTL reduces to the linear system

$$Rx_t = \mu_0 + \eta\sigma_0^2 \left(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s\right). \tag{42}$$

This simplest case already provides important economic intuition about the (in)stability of the (fundamental) steady state in an evolutionary system with many trader types. When there is no intrinsic mean bias, that is when $\mu_0 = 0$, and the risk premium is zero, i.e. $z^s = 0$, the steady state of the LTL (42) coincides exactly with the fundamental, i.e. $x^* = 0$. When the mean bias and risk premium are both positive (negative) the steady state deviation x^* will be positive (negative) so that the steady state will be above (below) the fundamental. The natural bifurcation parameter tuning the (in)stability of the system is $\alpha = \eta\sigma_0^2 = \beta\sigma_0^2/a\sigma^2$. We see immediately that instability occurs if and only if α increases beyond the bifurcation point $\alpha_c = 1$. Hence this simple case already suggests forces that may destabilize the evolutionary system: an increase in choice intensity β for evolutionary selection, a decrease in risk aversion "a", a decrease in conditional variance of excess returns σ^2, or an increase in the diversity of purely biased beliefs σ_0^2 can push α beyond α_c and set off instability of the (fundamental) steady state.

The LTL (42), obtained for the simplest case where all trader types are purely biased, is a *linear* system, which is either globally stable or globally unstable. Except for the hairline case at the border of stability, $\alpha = 1$, bounded solutions *not* converging to the steady state do not exist in this simplest case. When lags are included in the linear forecasting rules, the LTL (41) becomes a *nonlinear* system exhibiting much richer dynamical behavior. We consider a simple but typical example, with linear forecasting rules with three lags, i.e.

$$f_t(\vartheta_0) = \vartheta_{00} + \vartheta_{01}x_{t-1} + \vartheta_{02}x_{t-2} + \vartheta_{03}x_{t-3}. \tag{43}$$

With three lags in the linear forecasting rule, the corresponding LTL becomes a 5-D nonlinear system:

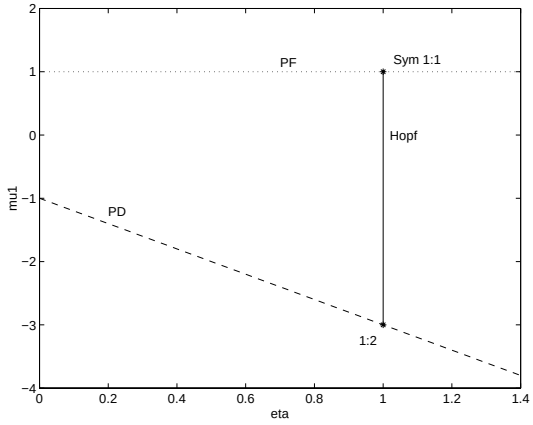
$$\begin{aligned} Rx_t &= \mu_0 + \mu_1 x_{t-1} + \mu_2 x_{t-2} + \mu_3 x_{t-3} && (44)\\ &\quad +\eta(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s)(\sigma_0^2 + \sigma_1^2 x_{t-1}x_{t-3} + \sigma_2^2 x_{t-2}x_{t-4} + \sigma_3^2 x_{t-3}x_{t-5}). && (45)\end{aligned}$$

We will discuss the most important generic and persistent features of the dynamical behavior of this 5-D LTL; higher dimensional versions of the LTL exhibit similar dynamical behavior.

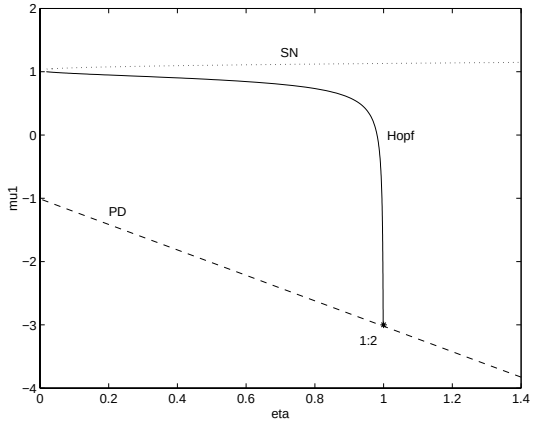
First we investigate steady states and their stability. Figure 3 shows a 2-D bifurcation diagram in the (η, μ_1) parameter plane, where μ_1 represents the mean of the first order stochastic trend variable ϑ_{01} in the forecasting rule (43). When the mean bias $\mu_0 = 0$, with μ_0 the mean of the constant ϑ_{00} in the forecasting rule (43), and the risk premium is also zero, i.e. $a\sigma^2 z^s = 0$, the LTL is symmetric w.r.t. the fundamental steady state and the LTL is therefore non-generic. In the symmetric case (figure 3a), between the Hopf, period doubling (PD) and pitchfork (PF) bifurcation curves the fundamental steady state is unique and stable. As the parameters cross the PF curve, two additional non-fundamental steady states are created. Other routes to instability occur when crossing the PD curve, where the fundamental steady state becomes unstable and a (stable) 2-cycle is created, or when crossing the Hopf curve, where the fundamental steady state becomes unstable and a (stable) invariant circle with periodic or quasi-periodic dynamics is created. The pitchfork bifurcation curve is *non-generic* and only occurs in the symmetric case. When the symmetry is broken by a non-zero mean bias $\mu_0 \neq 0.1$, as illustrated in figure 3b for $\mu_0 = 0.1$, the PF curve disappears and breaks up into two generic co-dimension one bifurcation curves, a Hopf and a saddle-node (SN) bifurcation curve. When crossing the SN curve from below, two additional steady states are created, one stable and one unstable. Notice that, as illustrated in figure 3c, when the perturbation is small as for $\mu_0 = 0.1$, the SN and the Hopf curves are close to the PF and the Hopf curves in the symmetric case. In this sense the bifurcation diagram depends continuously on the parameters, and it is useful to consider the symmetric LTL as an "organizing" center to study bifurcation phenomena in the generic, non-symmetric LTL.

The most relevant case seems to be the case when the mean μ_1 of the first order coefficient ϑ_{01} in the forecasting rule (43) satisfies $0 \leq \mu_1 \leq 1$. In that case, as the structural parameter η increases, the (fundamental) steady state loses stability through a Hopf bifurcation.

(a) symmetric case ($\mu_0 = 0$)



(b) generic, non-symmetric case ($\mu_0 = -0.1$)



(c) non-symmetric and symmetric cases

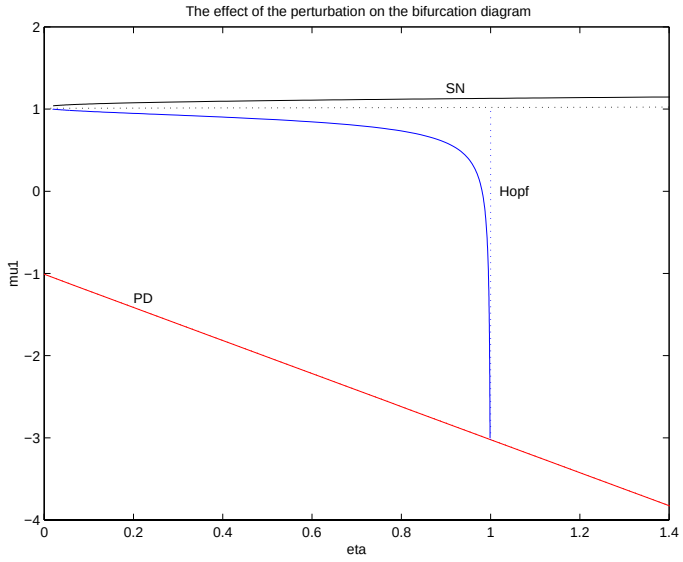
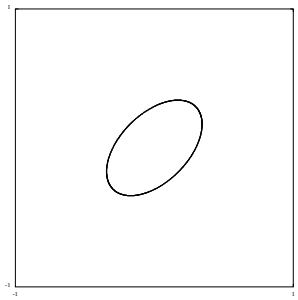
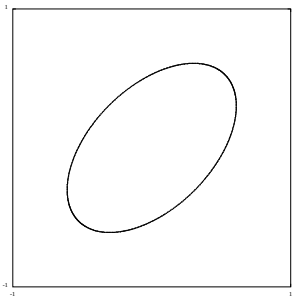


Figure 3: *Bifurcation diagrams in the* (η, μ_1) *parameter plane for LTL (44), where* μ_1 *represents the mean of the first order stochastic trend variable* ϑ_{01} *in the forecasting rule (43). For* $\mu_0 = a\sigma^2 = 0$*, with* μ_0 *the mean of the constant* ϑ_{00} *in the forecasting rule (43), the LTL is symmetric and thus non-generic; when* $\mu_0 \neq 0$ *the LTL is non-symmetric and generic. The diagrams show Hopf (H), period doubling (PD), pitchfork (PF) and saddle-node (SN) bifurcation curves in* (η, μ_1) *parameter plane, with other parameters fixed at* $R = 1.01$*,* $z^s = 0$*,* $\mu_2 = \mu_3 = 0$*,* $\sigma_0 = \sigma_1 = \sigma_2 = 1$ *and* $\sigma_3 = 0$*. Between the Hopf and PD curves (and the PF curve when* $\mu_0 = 0$*) there is a unique, stable steady state. This steady state becomes unstable when crossing the Hopf or the PD curve. Above the PF curve or the SN curve the system has three steady states. The PF curve is non-generic and only arises in the symmetric case with mean bias* $\mu_0 = 0$*. When the symmetry is broken by perturbing the mean bias to* $\mu_0 = -0.1$*, the PF curve 'breaks' into generic Hopf and SN curves.*

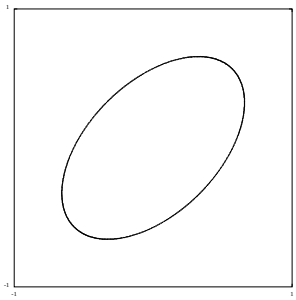
Figure 4 illustrates the dynamical behavior of the LTL as the parameter η further increases. Immediately after the Hopf bifurcation, for $1.1 \leq \eta \leq 1.5$ (figures 4a-c), periodic or quasi-periodic dynamics on a stable invariant circle occurs. After a quasi-periodic Hopf bifurcation, for $\eta = 1.51$ and $\eta = 1.52$, (quasi-)periodic dynamics on a stable invariant torus occurs. Recall from the Newhouse–Ruelle–Takens phenomenon that C^2-close to a system with a quasi-periodic Hopf bifurcation systems with strange attractors exist. The numerical observation that a quasi-periodic Hopf bifurcation arises in our LTL together with the LTL convergence theorem therefore suggests that our evolutionary systems with many trader types can exhibit strange attractors. Figures 4f-g show the unstable manifold of a periodic saddle point; the curling shape of the unstable manifold suggest near homoclinic bifurcations and breaking up of the invariant torus into a strange attractor. Figures 4h-m show that increasing η further leads to a bifurcation route to chaos and strange attractors culminating into a 'large' strange attractor for $\eta = 1.6$. In particular, figure 4 presents numerical evidence of the occurrence of what Brock and Hommes (1997) called a *rational route to randomness*, that is, a bifurcation route to strange attractors as the intensity of choice to switch forecasting strategies increases. If such rational routes to randomness occur for the LTL, the LTL convergence theorem implies that in evolutionary systems with many trader types rational routes to randomness occur with high probability.



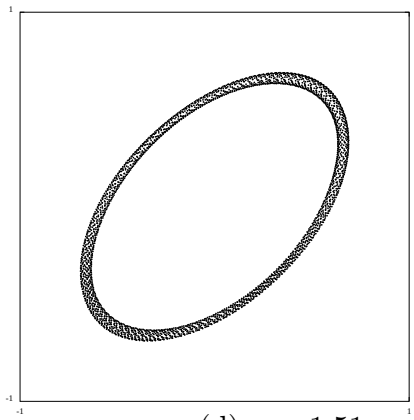
(a) $\eta = 1.1$



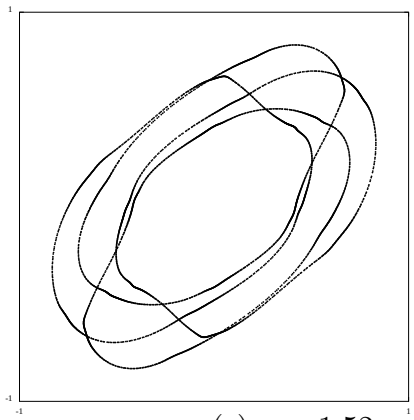
(b) $\eta = 1.4$



(c) $\eta = 1.5$

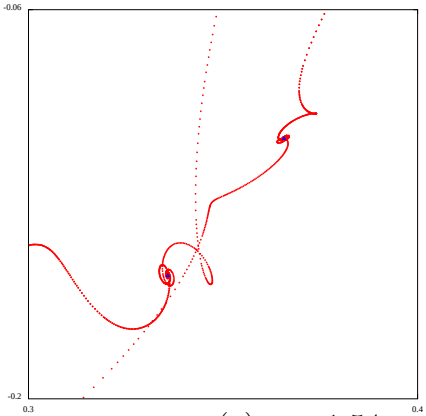


(d) $\eta = 1.51$

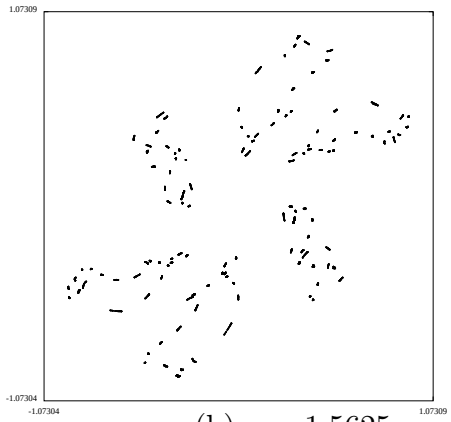


(e) $\eta = 1.52$

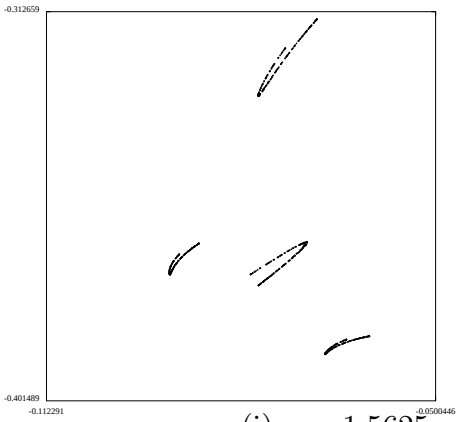
(f) $\eta = 1.54$



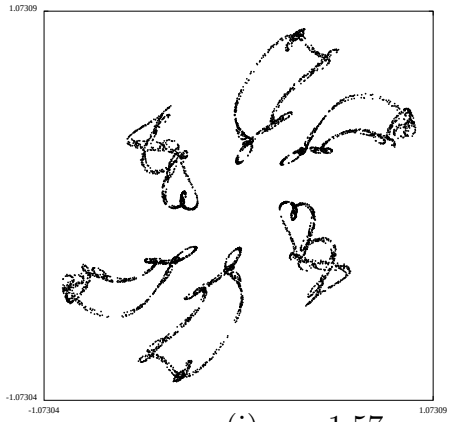
(g) $\eta = 1.54$



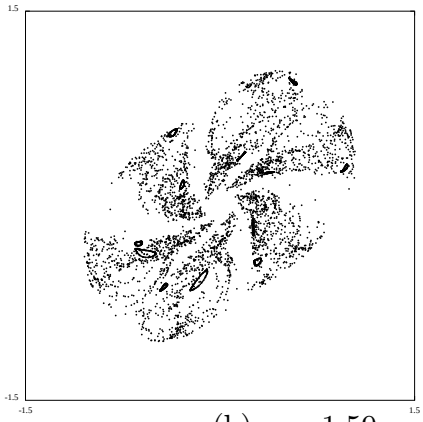
(h) $\eta = 1.5625$



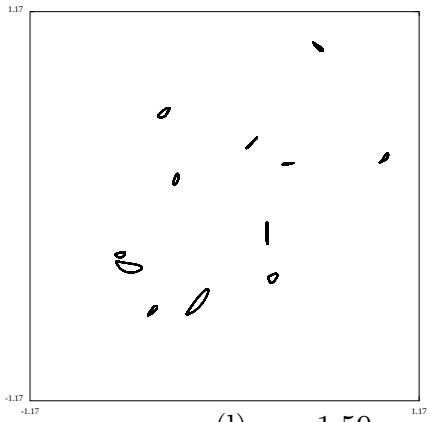
(i) $\eta = 1.5625$



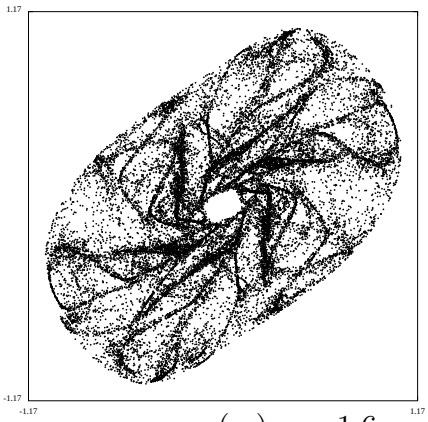
(j) $\eta = 1.57$



(k) $\eta = 1.59$



(l) $\eta = 1.59$



(m) $\eta = 1.6$

Figure 4: *Attractors in the phase space for the 5-D LTL with parameters* $R = 1.01$, $z^s = 0$, $\mu_0 = 0$, $\mu_1 = 1$, $\mu_2 = \mu_3 = 0$, $\sigma_0 = \sigma_1 = \sigma_2 = 1$ *and* $\sigma_3 = 0$*: (a-c) immediately after the Hopf bifurcation periodic or quasi-periodic dynamics on a stable invariant circle occurs; (d-e) after a quasi-periodic Hopf bifurcation (quasi-)periodic dynamics on a stable invariant torus occurs; (f-g) the unstable manifold of a periodic saddle point; the curling shape of the unstable manifold suggest near homoclinic bifurcations and breaking up of the invariant torus into a strange attractor; (h-m) bifurcation route to chaos and strange attractors culminating into a 'large' strange attractor for* $\eta = 1.6$.

7 A LTL with noise

This section investigates some of the time series properties of a LTL buffeted with dynamic noise. The purpose of this section is to investigate whether an LTL with noise can generate some of the stylized facts observed in financial returns data, in particular volatility clustering and long memory. We emphasize that our 'calibration' exercise here is just intended to be suggestive that there is enough promise here to suggest potential payoff to doing a more serious study in future research. That more serious study must deal with the issue of the number of "free parameters" relative to the number of moments being matched. See Kurz and Motolese (2001, pp. 526-532) for a discussion of this problem in their context. A more serious study must also deal with the issues raised by Hansen and Heckman (1996). All this is beyond the scope of the current paper.

Adding noise to the LTL (41) derived from linear forecasting rules (40) may sometimes lead to explosive time paths. Intuitively this is to be expected, since for linear forecasting rules predictions are always proportional to the deviation from the fundamental price and there is no 'stabilizing force' keeping asset prices bounded. Therefore, we consider a simple example of an LTL derived from a class of *nonlinear* forecasting rules with the property that for large deviations from the fundamental price the nonlinear forecasting rule will predict that price will return close to the fundamental price, i.e. for $|x_{t-1}|$ large the prediction will be close to 0. Including such a stabilizing force in the forecasting rule will ensure that even in the presence of noise all simulated time paths remain bounded. The idea of a stabilizing force in heterogeneous agents modeling is not new. For example, De Grauwe et al. (1993) use a stabilizing force in their exchange rate models with chartists and fundamentalists. Arthur et al. (1997) and LeBaron et al. (1999) introduce a stabilizing force in their artificial SFI stock market by assuming that chartists condition their technical trading rules upon market fundamentals such as the price/dividend ratio. Gaunersdorfer and Hommes (2000) have recently investigated a simple adaptive belief system with a stabilizing force, namely an adaptive belief system with fundamentalists versus trend followers who condition their rule upon the price deviation from the fundamental price. It should be emphasized that in all these examples, the stabilizing force only becomes important when prices move far away from the fundamental price. As long as prices are close to the fundamental, dynamics are driven by noise and evolutionary forces. Hommes (2001) interprets the stabilizing force due to conditioning of technical trading rules upon market fundamentals as a *transversality condition* in a heterogeneous world, allowing for temporary bubbles but not for indefinite and unbounded price bubbles.

Consider the following nonlinear prediction rule:

$$f_t(\vartheta_0) = f(x_{t-1}, \vartheta_{01}, \vartheta_{02}, \vartheta_{03}) = \frac{\vartheta_{01} x_{t-1} + \vartheta_{02} x_{t-1}^3 + \vartheta_{03} x_{t-1}^5}{1 + x_{t-1}^6}. \tag{46}$$

Here ϑ_{0j} is normally distributed, with mean μ_j and standard deviation σ_j, $j = 1, .., 3$. Notice that for small deviations from the fundamental, i.e. for $x_{t-1} \approx 0$, the nonlinear forecasting rule (46) approximates the linear pure trend following forecasting rule

$f(x_{t-1}, \vartheta_{01}, \vartheta_{02}, \vartheta_{03}) \approx \vartheta_{01} x_{t-1}$, whereas for large deviations $|x_{t-1}|$ from the fundamental $f(x_{t-1}, \vartheta_{01}, \vartheta_{02}, \vartheta_{03}) \approx 0$.

With beliefs randomly drawn from the class of forecasting rules (46), the Large Type Limit evolution map (with noise) becomes

$$\psi(x_1, x_2, x_3, \lambda, \mu) = \frac{\mu_1 x_1 + \mu_2 x_1^3 + \mu_3 x_1^5}{R(1+x_1^6)} + \frac{\eta}{R} \frac{(x_1 - R x_2)\,(\sigma_1 x_1 x_3 + \sigma_2 x_1^3 x_3^3 + \sigma_3 x_1^5 x_3^5)}{(1+x_1^6)\,(1+x_3^6)} + \varepsilon. \tag{47}$$

Since the nonlinear forecasting rule (46) has only one lag, the LTL is a 3-D system. Here λ denotes the totality of structural parameters, in this case $\lambda = (R, \eta)$, while μ denotes the belief parameters: $\mu = (\mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3)$. The noise ε is normally distributed around 0, with standard deviation σ_ε.

In all time series simulations below, the following parameters have been fixed:[8]

$$\begin{aligned} &\mu_1 = 0.999, \mu_2 = 0.1, \mu_3 = -1, \sigma_1 = 1, \sigma_2 = 5, \sigma_3 = 10 \\ &\eta = 5, R = 1.00038, z^s = 0, \sigma_\epsilon = 0.008. \end{aligned} \tag{48}$$

Figure 5 compares time series properties of 20 years of daily S&P 500 market index data, January 1, 1980 – May 10, 2000 (5312 observations)[9] to noisy LTL generated time series. Recall from (11) that x_t is the deviation from a benchmark fundamental $p_t^* = p^* = \bar{y}/r$. A LTL returns series r_t is thus given by

$$r_t = \frac{p_t - p_{t-1}}{p_{t-1}} = \frac{x_t - x_{t-1}}{x_{t-1} + p^*}, \tag{49}$$

where p^* is the fundamental price and the deviation x_t from the fundamental was generated by the noisy LTL (47). In the simulations we have normalized the fundamental price $p^* = 1$.

The simulated time series of returns clearly exhibits volatility clustering, similar to that observed in the S&P 500 returns. The distributions of the simulated returns are also comparable to the empirical distribution of S&P 500 returns. Simulated price deviations x_{t-1} show persistent deviations from the benchmark fundamental price p^*. Autocorrelation plots of simulated returns and squared returns (bottom left) as well as autocorrelation plots of simulated returns and absolute returns closely match the autocorrelation patterns in the S&P 500 data.

[8] R=1+r=1.00038 has been chosen such that, with 250 trading days per year, the yearly compounded return of the risk free asset will be 10%.

[9] The October 1987 crash on October 19, 1987, and the day before and the day after, with returns of -5.2%, -20.4% and +5.3% have been excluded from the data analysis since our aim is not so much to explain the extremes but rather some typical stylized facts. In particular, the autocorrelation pattern of the squared returns changes somewhat and is less persistent when the crash is included.

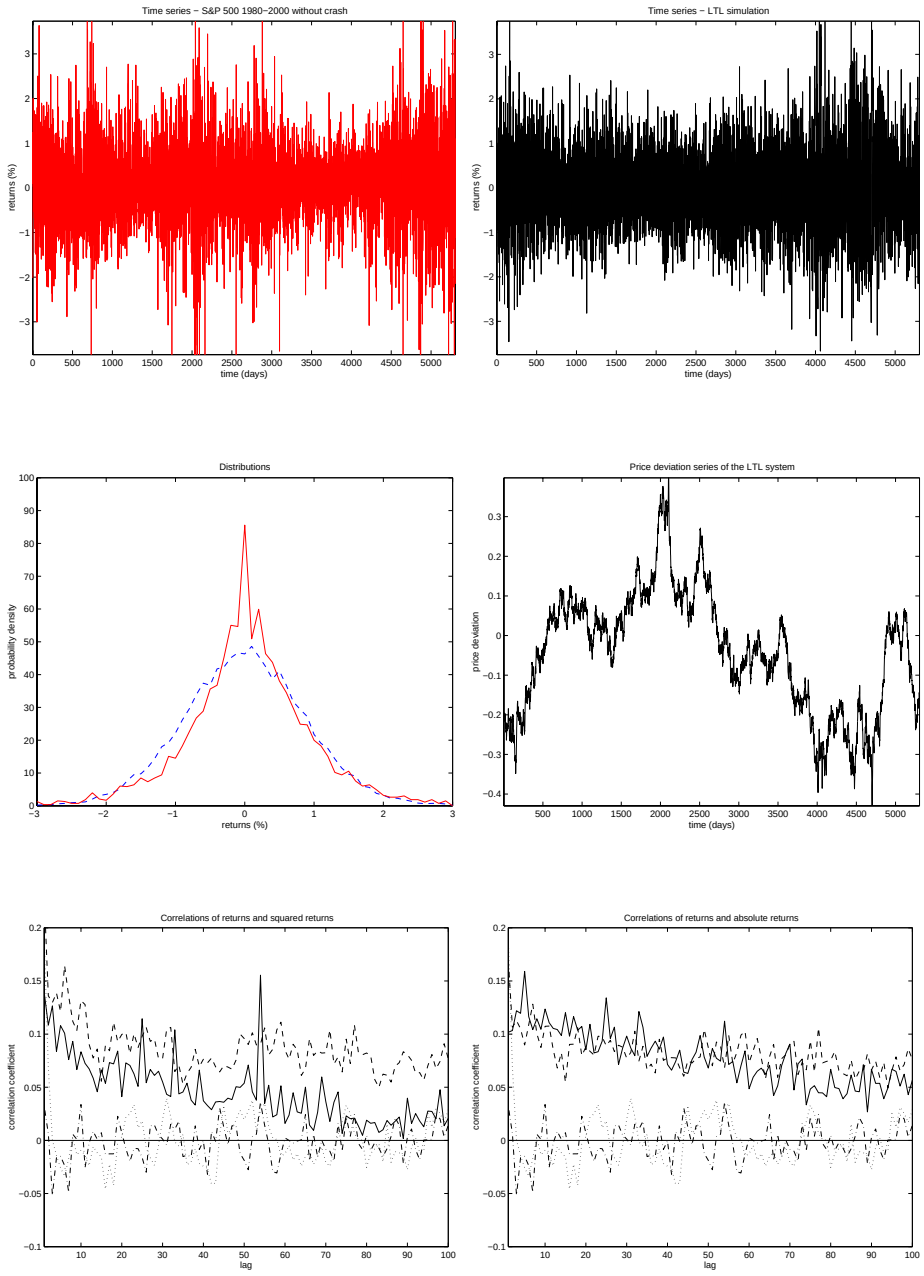


Figure 5: *Time series of simulated returns (top right) matching observed volatility clustering in the daily S&P 500 returns (top left), January 1, 1980 – May 10, 2000 (5312 observations; crash excluded). The distributions (middle left) of the simulated returns together with the empirical distribution of S&P 500 returns. Simulated price deviations* x_{t-1} *(middle right) show persistent deviations from the benchmark fundamental price* p^* *(corresponding to price deviation* $x = 0$*). Autocorrelation plots of simulated returns and squared returns (bottom left) as well as autocorrelation plots of simulated returns and absolute returns (bottom right) closely match the autocorrelation patterns in the S&P 500 data.*

In order to understand the time series properties of the noisy LTL it is useful to study the properties of the nonlinear forecasting function (46), with coefficients ϑ_{oj} equal to their sample means μ_j, $j = 1, 2, 3$ as in (48). Figure 6 shows the graph of the nonlinear forecasting function $f(x_{t-1})$ in (46) as well as the graph of $f(x_{t-1}) - x_{t-1}$. The latter graph shows that the nonlinear forecasting function has multiple steady states, namely three stable steady states at $x = 0$, $x \approx -0.29$ and $x \approx 0.29$ and two unstable steady states at $x \approx -0.11$ and $x \approx 0.11$. Recall that in the time series simulations in figure 5, the fundamental price was normalized to $p^* = 1$. Hence, the nonlinear forecasting function with coefficients at the sample means predicts that there are three stable price levels, one at the fundamental price level, one about 30% overvalued and one about 30% undervalued. These three stable price levels can still be recognized from the simulated price deviation series of the LTL with noise in figure 5 (middle right). Notice that the noise level in these simulations is small ($\sigma_\epsilon = 0.008$). The changes in price levels are triggered by noise, but reinforced by trend following trading rules. The occurrence of volatility clustering may now be understood as coming from two sources: (i) large price changes when prices move from one locally stable price level to another; and (ii) large (small) relative price changes when the price level is low (high). Multiple stable steady states of the nonlinear LTL thus lead to persistent price deviations from the fundamental and clustered volatility and long memory in asset returns.

Our LTL with noise is an extremely simple 3-D nonlinear system matching the autocorrelation patterns of returns, squared returns and absolute returns of S&P 500 daily data fairly well. In particular, this simple LTL has little or no *linear* predictability, since autocorrelations of returns are close to 0 at all lags. But *if* our simple LTL would be an accurate description of a real financial market, wouldn't there be *nonlinear* structure left that could be exploited by smart arbitrage traders? Stated differently, is a financial market described by our LTL informationally efficient or do prices and asset returns exhibit forecastable and exploitable nonlinear structure?

In order to shed some light on this question, figure 7 illustrates the application of one of the standard nonlinear predictions methods, *nearest neighbor forecasting*, to the returns of both the S&P 500 series and our simulated LTL with noise[10]. Nearest neighbor forecasting looks for past patterns in the data that are close to the most recent pattern, and then yields as the prediction the average of the next value following all nearby past patterns. It follows essentially from Takens' embedding theorem (Takens (1981)) that this method yields good forecasts for deterministic chaotic systems. Here we employ a forecasting algorithm that has strong power to distinguishing between low dimensional deterministic structure and high dimensional stochastic models, as proposed in Casdagli (1991). This algorithm is used to investigate whether our simple LTL buffeted with small noise still exhibits low dimensional structure that could be exploited for out of sample forecasting. We refer to Kantz and Schreiber (1997) for an excellent recent introduction and extensive discussion of nonlinear time series analysis and nonlinear forecasting methods. The plots in figure 7 show the average one step ahead forecasting error as a function of the number

[10]We would like to thank Sebastiano Manzan for providing these figures.

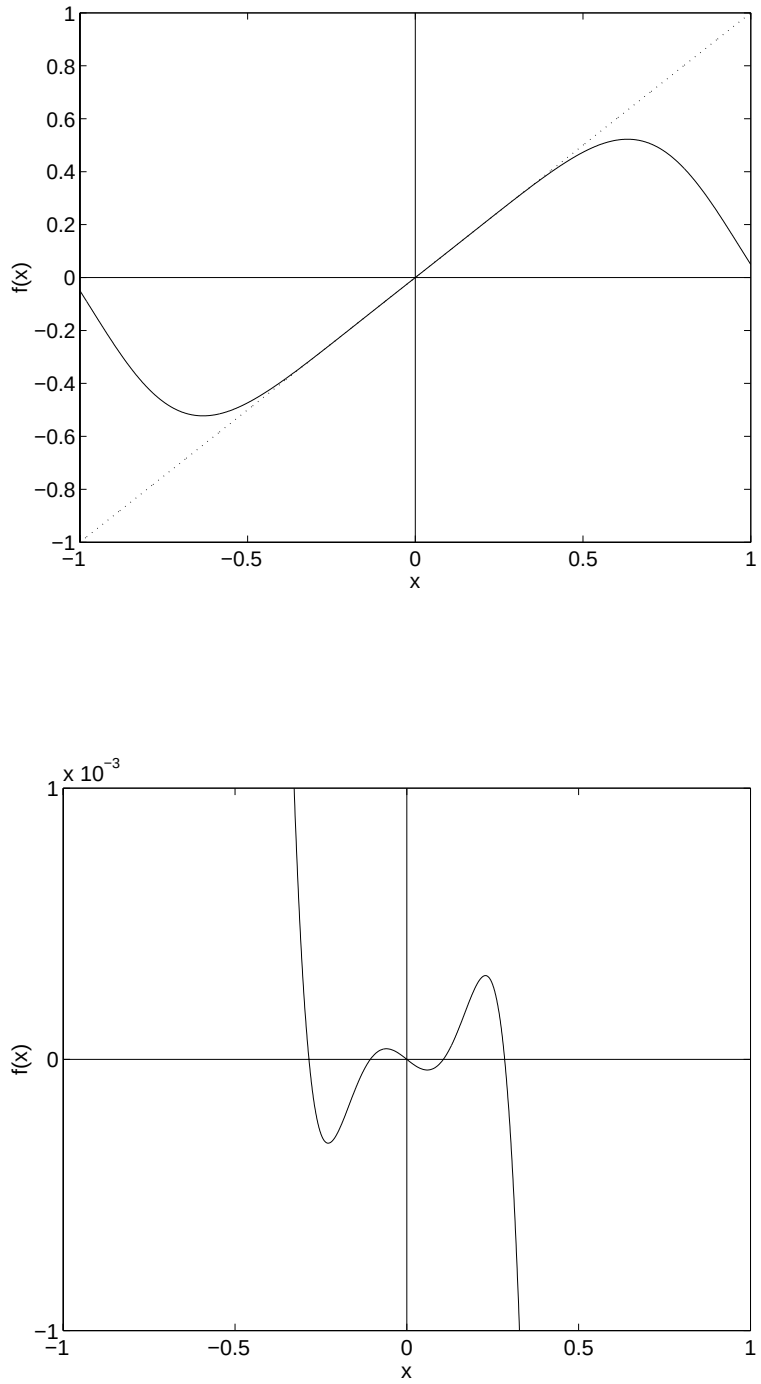


Figure 6: *Top: graph of the nonlinear forecasting function* $f(x_{t-1})$ *in (46); Bottom: graph of* $f(x_{t-1}) - x_{t-1}$*. The bottom graph shows that the nonlinear forecasting function has five steady states, namely three stable steady states at* $x = 0$*,* $x \approx -0.29$ *and* $x \approx 0.29$ *and two unstable steady states at* $x \approx -0.11$ *and* $x \approx 0.11$*.*

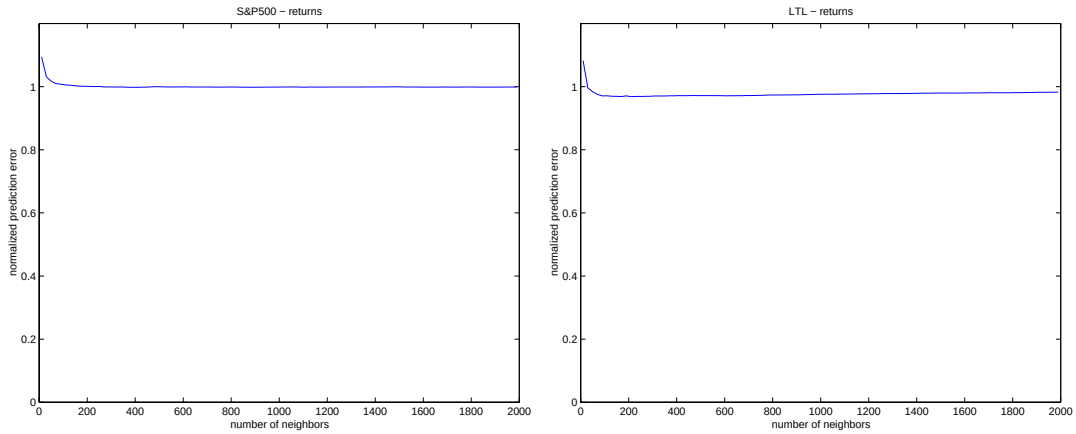


Figure 7: *Nearest neighbors forecasting method applied to S&P 500 returns (left panel) and simulated LTL returns (right panel). Both plots show the one step ahead forecasting errors as a function of the number of neighbors used for forecasting, averaged over* 3000 *out of sample forecasts, with embedding dimension 3. A horizontal line at height* 1 *corresponds to the forecasting error from the mean predictor. The plots show that there is hardly any nonlinear forecastability compared to the mean predictor. The simulated returns series from our low dimensional LTL buffeted with dynamic noise thus contains little or no nonlinear structure that could easily be exploited for forecasting.*

of neighbors used. The embedding dimension, i.e. the number of time lags, was 3 for both series. The time series were divided into a fitting set of 2300 observations and a testing set of the remaining 3000 observations, so that the one step ahead forecasting errors were averaged over 3000 out of sample forecasts. The horizontal line at height 1 represents the forecasting error from the mean predictor, which would be the optimal forecast for an IID stochastic time series. For a deterministic chaotic system, much smaller forecasting errors typically occur when the number of neighbors is small. As the number of neighbors becomes large, the forecasting method converges to the mean predictor and the forecasting error approaches 1. A low dimensional chaotic time series is therefore characterized by an increasing plot with small forecasting errors when the number of neighbors is low and larger forecasting errors, possibly approaching 1, when the number of neighbors increases; see for example Casdagli (1991, p.309 figure 1). In contrast, both plots in figure 7 are essentially horizontal and close to 1, thus showing little forecastability compared to the mean predictor. We therefore conclude that the S&P 500 returns as well as the simulated returns of our simple 3-D LTL system with noise show little or no evidence of nonlinear predictability. Our simple LTL system thus captures the inherent *unpredictability* typically observed in real financial returns series. In a LTL world nonlinear structure would be difficult to exploit and the market is close to being efficient.

8 Concluding Remarks

We have presented a theoretical framework for an evolutionary system with many different trader types. Our notion of Large Type Limit (LTL) describes the average dynamical behavior in an evolutionary system with many competing different trader types. All generic and persistent dynamical features occurring in an LTL also occur, with probability arbitrarily close to 1, in the corresponding evolutionary model with many trader types. The LTL framework can be applied to any class of forecasting rules indexed by a finite dimensional vector designating the type.

Within this theoretical framework, conditions can be obtained for which asset prices will reflect economic fundamentals as well as conditions leading to deviations from a RE benchmark fundamental and excess volatility. In particular, an increase in the "intensity of adaptation" to switch prediction strategies and an increase in the dispersion of potential belief types can lead to emergence of complicated dynamics for the trajectory of deviation from RE for asset returns. These dynamics are suggestive of complicated dynamics for volatility, and volume. Our many trader type evolutionary system buffeted with noise, are consistent with important observed stylized facts such as unpredictability of returns, fat tails, clustered volatility and long memory. In particular, a simple version of our LTL buffeted with noise is able to match the temporal correlation structure of returns, absolute returns and squared returns of 20 years of daily S&P 500 data.

Our theoretical framework may be useful to address these questions also empirically. The RE fundamental benchmark is nested as a special case within the general model, and our framework may thus be used to test whether the extra parameters are empirically relevant. See Baak (1999) and Chavas (2000) for empirical evidence of heterogeneity and the presence of non-rational traders in the hog and beef markets. Our theoretical framework is also useful for experimental laboratory testing of the expectations hypothesis. By controlling the RE benchmark fundamental in the laboratory, it can be tested whether experimental markets will or will not deviate from the RE benchmark fundamental; see Hommes (2001) for some first experimental results in this direction. Experimental as well as empirical testing of our LTL theory will be left for future work.

At the risk of repeating ourselves, we explain why we have chosen to work in the space of deviations about the fundamental rather than some other space. The recent article by Sobel (2000) reviews the large literature in economics that attempts to model how people learn. He states that "Models necessarily must specify what agents initially know and how they build on this knowledge" (Sobel (2000, p. 256)). He points out in a footnote that much modeling in economics of the "inductive" type that builds upon patterns extractible from accumulating data that the system co-creates is way too slow relative to what real humans glean from living in their system. Our work attempts to capture some of this power of real humans by positing that our humans understand their system well enough to compute the fundamental equilibrium, but they are wary about dynamical deviations from that fundamental, caused, perhaps, by deviations of other agents in the economy,

and, hence, the dynamics of our systems are in units of deviations about the fundamental. In particular, when in our evolutionary setting the intensity of choice to switch strategies is high, quick changes of beliefs and large changes of asset prices co-evolve. Although we believe that our choice of state space for the dynamics of learning helps remove some of the charges of "ad hoccery" and "too mechanical" against theories of learning in economics, there are still major problems.

For example, a powerful route to learning in real financial markets which is ignored in this article is the learning that occurs while trading what Kurz and Wu (cf. Kurz (1997), Chapter 7) call Price Contingent Claims (PCC's). If enough of these instruments are present, trading them can place bounds on the amount of belief diversity that can exist and, perhaps, even eliminate it in some cases. Of course a tension may arise if too much diversity is eliminated (or to put it more accurately if too much of any type of reasons for trading is eliminated) because it costs real resources to set up and operate an organized trading market for a PCC. One can think of a PCC as a generalized derivative security. Put options and call options are examples of derivative securities. Hence if there are "too many" PCC's in the sense that trading volume in some of the PCC's dries up, then some PCC trading markets may be forced to go out of business. So there may well be a type of "meta equilibrium" that structures the number of trading markets that could moderate belief diversity and the amount of belief diversity and other sources of demand for trading that is needed to support the operating costs of such markets.

Hence, at this stage, we stress that the asset markets used in this paper should be viewed as primarily a way of illustrating the technique of LTL analysis. Much more attention to institutional modeling such as modeling the incentives to form PCC trading markets, the cost structure of such markets, the instruments to raise revenue for operating expenses of each such market, and so on, before we can lay any claim to realism of the asset markets treated in this paper. Indeed, a promising research program that we are undertaking is the investigation of the impact on dynamics studied in this paper when PCC's are added.

The concept of LTL developed in this paper is a general tool that can be used to enhance the tractability of models with large numbers of heterogeneous learning agents in many different contexts. While we have developed LTL theory in this paper in an illustrative financial model where beliefs differ only across conditional means of returns, the same idea can be applied to other modeling contexts. For example, consider models with heterogeneity in other dimensions besides beliefs on conditional means. An example of such work is Chiarella and He (1999). Their Assumptions A1 and A2 place a structure of beliefs on both conditional mean and conditional variance heterogeneity which is expressed in deviations form from the fundamental baseline. Hence their model appears especially amenable to LTL type analysis.

Another source of examples, is models with a hierarchy of dynamics from fast to slow where the heterogeneity of beliefs and learning is placed upon the slow dynamics. For an example of fast/slow dynamics, consider the paper of de Fontnouvelle (2000). Here the context is a standard noisy rational expectations asset pricing framework at the fast dynamical level.

But agents in the model have a choice of purchasing an expensive more accurate signal (predictor) on the future earnings of the asset or using publically available information that costs nothing. Belief heterogeneity is placed upon the "slow hyperdynamics." Here agents must predict what fraction of their rivals purchase the expensive signal . LTL modeling as in this paper could be adapted to do that and help produce a model that is analytically tractable as we have shown here. We leave all such extensions and experimental and empirical testing of the evolutionary LTL framework for future work.

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