Who Wants a Good Reputation?*

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Abstract

We examine a market in which firms confront a moral hazard problem in the provision of product quality, facing a short-term incentive to produce low quality but potentially earning higher profits from "committing" to high quality. There are two types of firms in the market, "inept" firms who can produce only low quality, and "competent" firms who have a choice between high and low quality. Firms occasionally leave the market, causing competent and inept potential entrants to compete for the right to inherit the departing firm's reputation. In a repeated interaction, with consumers receiving noisy signals of product quality, competent firms choose high quality in an attempt to distinguish themselves from inept firms. Competent firms find average reputations most valuable, in the sense that a competent firm is most likely to enter the market by purchasing an average reputation, in the hopes of building it into a good reputation, than either a very low reputation or a very high reputation. Inept firms, in contrast, find it profitable to either buy high reputations and deplete them or buy low reputations.

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Who Wants a Good Reputation?

by George J. Mailath and Larry Samuelson

1. Introduction

Describing an eventually unsuccessful joint venture between Time Inc. and a former advertising executive who was otherwise unconnected with Time, the *Wall Street Journal* commented, "More significantly, the company had given him the right to exploit one of its most prized assets: the formidable Time Life name."[1] This was the first time that Time Inc. had licensed the Time Life name to an outsider. "Countless copycat investors ... jumped in largely on the strength of the Time Life brand name ...."

Why doesn't a good reputation suffice to ensure success, as the copycat investors hoped it would? It may be simply that some firms are unlucky. But this paper suggests that more systematic forces may be at work. Relatively good reputations are especially valuable to relatively less competent firms, giving rise to a market in which good reputations are likely to be purchased and then depleted by inept firms.

We examine a market in which firms confront a moral hazard problem in the provision of product quality, facing a short-term incentive to produce low quality but potentially earning higher profits from "committing" to high quality. There are two types of firms in the market, "inept" firms who can produce only low quality, and "competent" firms who have a choice between high and low quality. In a repeated interaction, with consumers receiving noisy signals of product quality, competent firms choose high effort in an attempt to distinguish themselves from inept firms. We refer to the consumers' posterior expectation that a firm is competent as the firm's *reputation.*

Two aspects of the Time Life example play key roles in motivating our analysis: customers had no way of knowing that Time Life provided only its name to the venture, while leaving all operational decisions in the hands of the outsider, but customers also could not exclude such a possibility. We capture these features by assuming that the firms in our model occasionally leave the market, causing competent and inept potential entrants to compete for the right to inherit the departing firm's reputation, with consumers being sometimes unable to observe such replacements.

We show that competent firms find average reputations most valuable, in the sense that a competent firm is more likely to purchase an average reputation, in

[1]Both quotes are from "Tale of the Tapes: How One Media Deal Became Hazardous To Investors' Health," *Wall Street Journal* 136 (31), February 13, 1997, page 1.

the hopes of building it into a good reputation, than either a very low reputation or a very high reputation. Competent firms thus buy "fixer-uppers" that are not in too bad of shape. Inept firms, in contrast, find it profitable to either buy high reputations and deplete them or to buy very low reputations, which remain low.[2]

Our model of reputations is adapted from Mailath and Samuelson [7], and is designed to capture a setting in which reputation has the properties of an asset, being gradually built through a process of costly investment and gradually dissipated when neglected. As we discuss in [7], this contrasts with the use of the concept of a reputation in repeated games to motivate a selection from the vast set of possible equilibria.

The role of moral hazard in our model distinguishes it from a large literature that focuses on pure adverse selection problems. The paper from this literature that is closest to ours in motivation is Tadelis [8]. Tadelis [8] studies the market for firm names in a pure adverse selection environment. In his model, like ours, unobservable changes in ownership are crucial for names (or reputations) to have value. In every equilibrium of his model, some "bad types" must purchase "good names," a finding consistent with our result that whenever consumers assign a very high probability to a firm being competent, the possibility of an inept replacement causes this probability to fall. Finally, Gale and Rosenthal [5] examine a model in which exogenous events cause firms to leave the market, as in ours, though they to do not discuss the induced market for reputations.

The following section presents the model. Section 3 examines equilibria under the assumption that the types of entering firms are exogenously fixed. Sections 4-5 allow these types to be determined endogenously and examine which types of firms are most likely to buy which reputations. Section 6 discusses extensions of the analysis and Section 7 concludes. Most of the proofs are collected in the Appendix.

2. The Model

We consider an entrepreneur who owns a "name." This name may be a location for a business, a brand name for a product or service, the exclusive right to use a particular technology, or some similar means of identification. There may be close substitutes for this name, but the name itself is unique, and we will think of the entrepreneur as a monopolist.

Time is discrete and the horizon infinite. The entrepreneur can sell the right to use the name, for the duration of a firm's lifetime in the market, to a single firm.

[2] The *Wall Street Journal*'s description of the executive's use of Time Life's name suggests that it falls into the former category.

The lifetime of a firm is exponentially distributed, with a probability $\lambda \in (0,1)$ in each period that the firm exits. Upon departure of a firm, the entrepreneur sells the right to use the name to a new firm, who then retains the name until exiting.[3] The probability λ is constant and exogenously fixed. We think of firm exit as reflecting exogenously generated reasons to leave the market, such as a decision to retire.[4]

Consumers cannot observe firm exits nor the replacement of an exiting firm with a new firm. For example, the ownership of a restaurant might change without changing the restaurant's name and without consumers being aware of the change, or Time Life may commission an outsider to undertake a publishing venture that consumers cannot differentiate from previous publishing ventures.[5] At the same time, consumers know that such replacements are possible, and take this into account when forming their expectations.

Consumers repeatedly purchase an experience good or service from the firm. The experience good generates two possible utility levels, which we normalize to 0 and 1. We describe a utility of 1 as a good outcome, denoted by g, and a utility of 0 as a bad outcome, denoted by b. In each period, the firm exerts effort, which determines the probability of a good outcome. There are two possible effort levels, high (H) and low (L). There are also two possible types of firm, "inept" (I) and "competent" (C). An *inept* firm can only exert low effort, while a *competent* firm can exert either high or low effort. The prior probability that the firm in the market at time zero is competent is given by ϕ_0. The probability that a new firm replacing an exiting firm is competent is $\theta \in (0,1)$. We initially take θ to be exogenously fixed. Sections 4 and 5 focus on endogenous θ. When taking θ to be fixed, we assume $\phi_0 \in [\lambda\theta, 1-\lambda(1-\theta)]$. It would be natural to further assume that $\theta = \phi_0$, though nothing depends upon this equality.

[3] Rather than introducing an entrepeneur, we could assume that the current firm owns the name, and that an exiting firm sells its name to a new firm. This complicates the structure of the game by causing the value to a firm of changing consumer beliefs to have two components, the one in which we are interested, and the resale value.

This alternative assumption is exploited by Aoyagi [2], who shows, in a reputation game with Stackelberg and normal firms, that owners who live only one period value the normal firm more than do infinitely-lived owners. An important difference between Aoyagi [2] and our analysis is that in his model, the owners do not determine the type of the firm. Rather, the type of firm is determined once and for all at the beginning of time.

[4] When selling the name, the entrepreneur commits to allowing the firm to retain use of the name until the firm exits. This commitment is *ex ante* optimal, though there may be realizations in which the entrepreneur covets the ability to repossess and resell the name.

[5] In other cases where reputations are thought to be important, ownership changes can be observed. For example, ownership changes of private medical practices typically cannot be concealed. We return to this distinction below.

High effort yields a good outcome for consumers with probability $1-\rho > 1/2$. With probability $\rho > 0$, high effort results in a bad outcome. Conversely, low effort yields a bad outcome with probability $1-\rho$, and a good outcome with probability ρ.[6] Low effort is costless, while high effort incurs a cost of c, where $0 < c < 1 - 2\rho$. The latter inequality ensures that if consumers knew that the firm was competent, and could verify its effort before purchasing the good, then they would be willing to pay a premium for high effort sufficient to make high effort optimal for the competent firm.

There is a continuum of identical consumers, of unit mass. Since the product produced by the firm is an experience good, consumers can observe neither the effort expended in its production nor the utility it will yield before purchase. Moreover, it is not possible to write contracts conditional on these properties.

We assume that all consumers receive the *same* realization of utility outcomes, and that this realization is *public*. In particular, the firm and the market both observe the outcome at the end of the period. This is a strong assumption. In a more realistic model, each consumer receives an idiosyncratic, private utility outcome, so that some consumers receive a good utility outcome while others receive a bad utility outcome in each period. Since the utility outcomes are private in this more realistic model, no consumer knows the distribution of utility realizations (which would allow a consumer to infer the effort choice of the firm). In [7], we use such a model, for the case of exogenously fixed θ, to study the economics of building and maintaining a reputation. We begin with the viewpoint that a reputation is an asset and, like any other asset, requires investment to create and maintain. This leads us to examine reputation formation and maintenance under the assumption that the set of possible firm types does not include so-called Stackelberg types (see [7] for details). We show that, in the absence of Stackelberg types, the possibility of replacement by inept types plays a crucial role in the formation of reputation.

It is important in [7] that the entrant types are exogenously determined. The combination of idiosyncratic consumer realizations and endogenously-determined entrant types makes the model intractable. This paper accordingly resorts to a model in which consumers receive common utility realizations, which are observed by the firm (and market), in order to study endogenous entrant types. The cost of this simpler model is that the set of equilibria is large. In particular, it is possible to support high effort choices in equilibria in which reputation does not have the asset-like features that we consider essential to the study of the market

[6] We could dispense with the symmetry assumption that high effort produces a good outcome with the same probability that low effort produces a bad outcome without affecting the results, but at the cost of additional notation.

for reputations. We rule out these equilibria by imposing a Markov restriction on equilibrium behavior. We discuss this issue in more detail in the next section.

In each period, each consumer purchases the good at a price equal to her expected utility. For example, the firm may be able to post take-it-or-leave-it prices, or Bertrand competition among consumers may bid the price up to expected utility. The observed utility outcome provides a noisy signal of the effort choice of the firm, since a bad outcome can result from both low or high effort.

The sequence of events is as follows. At the beginning of period t, consumers have beliefs ϕ_t about the type of the firm and have an expected utility p_t from consuming the good. If the firm is competent, it makes its unobserved effort choice. The output is then produced, and the firm receives revenues of p_t, regardless of its type and regardless of the realized utility in that period. Consumers (and the firm) next observe the realized valuation of good and update their beliefs about the type of firm and hence their expected utility. Finally, with probability λ, the firm is replaced.

3. Exogenous Replacements

In this section, we assume that θ, the probability that a replacement is competent, is exogenously fixed.

A firm who owns the right to use the name maximizes the discounted sum of expected payoffs, with discount factor δ. Since there is a continuum of consumers, each consumer realizes that her own action will have no effect on the future play of the game. We accordingly treat consumers as being myopic, in the sense that the only issue for a consumer in period t is the probability she assigns to the product inducing a good outcome in that period, and the consumer pays her expected utility given that probability.

To motivate our restriction to Markov strategies, it is convenient to temporarily consider a model with no replacements and a firm known to be competent (i.e., $\lambda = 0$ and $\phi_0 = 1$). For δ close to 1 and ρ close to 0, it is possible to support high effort in an equilibrium in which the firm starts with high effort, and continues with high effort as long as consumers receive good utility realizations, switching to low effort forever as soon as consumers receive a bad realization.[7] In our view, such equilibria do not capture the asset-like features of reputations that we described in the previous section. Moreover, such equilibria depend upon an

[7] The restriction on ρ is only necessary for this particular profile to be an equilibrium. More complicated strategies can support payoffs close to the efficient frontier even if ρ is large (see Abreu, Pearce, and Stacchetti [1] and Fudenberg, Levine, and Maskin [4]).

implausible degree of coordination between firm behavior and consumer beliefs about firm behavior. This is the kind of behavior that the model we examine in [7], with consumers receiving idiosyncratic private utility realizations (discussed in the previous section), rules out. Since we have common public utility realizations in this paper, we must rule out such equilibria some other way, and the most convenient is to focus on Markov strategies.

We now return to the model with replacements and uncertainty concerning the firm's type. The state variable in our Markov formulation is the consumers' posterior probability that the current firm is competent, denoted by ϕ, with prior probability ϕ_0. A Markov strategy for the competent firm is a mapping

$$\tau : [0,1] \to [0,1],$$

where $\tau(\phi)$ is the probability of high effort when the consumers' posterior probability that the current firm is competent is ϕ. The inept firm makes no choices, and hence has a trivial strategy.

Note that with probability one, there will be an infinite number of replacement events, infinitely many of which will introduce new competent owners into the game. By restricting attention to strategies that only depend on consumers' posteriors, we are requiring that a new competent owner, entering when consumers have belief ϕ, behave in the same way as an existing competent owner when consumers have the same belief ϕ.[8] While restricting firms to such strategies rules out some equilibria, any equilibrium under this assumption will again be an equilibrium without it. We sometimes refer to a firm strategy as the competent owner's (or competent type's) strategy, although it describes the behavior of all new competent owners as well.

A Markov strategy for consumers is a function

$$p : [0,1] \to [0,1],$$

where $p(\phi)$ is the probability consumers assign to receiving a good outcome, given posterior ϕ.

We denote by $\varphi(\phi|x)$ or ϕ_x the posterior probability that the current firm is competent, given a realized utility $x \in \{g, b\}$ and a prior probability ϕ. If a competent firm always exerts high effort, then posterior beliefs are given by

$$\varphi(\phi \mid g) \equiv \phi_g = (1-\lambda)\frac{(1-\rho)\phi}{(1-\rho)\phi + \rho(1-\phi)} + \lambda\theta \tag{3.1}$$

[8]In equilibrium, we will thus be requiring different firms to behave identically in identical situations, yielding a *symmetric* Markov equilibrium.

and

$$\varphi(\phi \mid b) \equiv \phi_b = (1-\lambda)\frac{\rho\phi}{\rho\phi + (1-\rho)(1-\phi)} + \lambda\theta. \tag{3.2}$$

A Markov perfect equilibrium is a pair of Markov strategies with the properties that firms are maximizing profits, consumers' expectations are correct, and consumers use Bayes rule to update their posterior probabilities:

Definition 1. *A* Markov perfect equilibrium *is a pair of strategies and an updating rule* (τ, p, φ) *such that*
(a) $\tau(\phi)$ *is maximizing for all* ϕ,
(b) $p(\phi) = \{(1-\rho)\tau(\phi) + \rho(1-\tau(\phi))\}\,\phi + \rho(1-\phi)$,
(c) $\varphi(\phi|g) = (1-\lambda)\frac{[(1-\rho)\tau(\phi)+\rho(1-\tau(\phi))]\phi}{\{[(1-\rho)\tau(\phi)+\rho(1-\tau(\phi))]\phi+\rho(1-\phi)\}} + \lambda\theta$, *and*
(d) $\varphi(\phi|b) = (1-\lambda)\frac{[\rho\tau(\phi)+(1-\rho)(1-\tau(\phi))]\phi}{\{[\rho\tau(\phi)+(1-\rho)(1-\tau(\phi))]\phi+(1-\rho)(1-\phi)\}} + \lambda\theta$.

Notice that a strategy for the firm uniquely determines the updating rule that consumers must use if their beliefs are to be correct.

Proposition 1. *Let* $\lambda \in (0,1)$, $\theta \in (0,1)$, *and* $\phi_0 \in [\lambda\theta, 1-\lambda(1-\theta)]$. *Then there exists* $\bar{c} > 0$ *such that for all* $0 \leq c < \bar{c}$, *the pure strategy profile in which the competent firm always chooses high effort is a Markov perfect equilibrium.*

Proof. Consider the strategy

$$\tau(\phi) = \begin{cases} 1, & \text{if } \phi \geq \phi', \\ 0, & \text{if } \phi < \phi', \end{cases} \tag{3.3}$$

so that the competent firm exerts high effort if and only if the posterior attains a cutoff level ϕ', where the value of ϕ' is to be determined. If the posterior falls short of this cutoff level, so $\phi < \phi'$, then the firm chooses low effort and hence no further updating of beliefs occurs, leading to the continued choice of low effort, and a value function for the competent firm of $V_C(\phi) = \rho/(1-\delta(1-\lambda))$.

Now suppose that both ϕ and ϕ_b $(= \varphi(\phi|b))$ exceed ϕ'. Then $\varphi(\varphi(\phi|g)|g) \equiv \phi_{gg} > \phi_g > \phi > \phi_b \geq \phi_{bb}$ and $\phi_{gx} > \phi_{bx}$ for $x \in \{g, b\}$. The value function of the competent firm is given by

$$V_C(\phi) = p(\phi) - c + \delta(1-\lambda)\left[(1-\rho)V_C(\phi_g) + \rho V_C(\phi_b)\right].$$

The payoff from exerting low effort and thereafter adhering to the equilibrium strategy is

$$V_C(\phi; L) \equiv p(\phi) + \delta(1-\lambda)\left[\rho V_C(\phi_g) + (1-\rho)V_C(\phi_b)\right].$$

Thus,

$$\begin{aligned} V_C(\phi) - V_C(\phi; L) &= \delta(1-\lambda)(1-2\rho)(p(\phi_g) - p(\phi_b)) - c \qquad (3.4) \\ &\quad + \delta^2(1-\lambda)^2(1-2\rho)\left\{(1-\rho)\left[V_C(\phi_{gg}) - V_C(\phi_{bg})\right] \right. \\ &\qquad \left. + \rho\left[V_C(\phi_{gb}) - V_C(\phi_{bb})\right]\right\} \\ &\geq \delta(1-\lambda)(1-2\rho)\left[p(\phi_g) - p(\phi_b)\right] - c, \qquad (3.5) \end{aligned}$$

since V_C is monotonic in ϕ (given that τ is specified by (3.3)).

An equilibrium with the competent firm's strategy given by (3.3) requires $V_C(\phi) - V_C(\phi; L) \geq 0$ for any ϕ with both ϕ and ϕ_b exceeding ϕ'. From (3.5), a sufficient condition for this inequality is

$$p(\phi_g) - p(\phi_b) \geq \frac{c}{\delta(1-\lambda)(1-2\rho)}. \qquad (3.6)$$

Now suppose $\theta \in (0,1)$ and choose a $\phi_0 \in [\lambda\theta, 1-\lambda(1-\theta)]$. Posterior probabilities are then contained in the interval $[\lambda\theta, 1-\lambda(1-\theta)]$. In addition, the minimum of $p(\phi_g) - p(\phi_b) = p(\varphi(\phi|g)) - p(\varphi(\phi|b))$ over $\phi \in [\lambda\theta, 1-\lambda(1-\theta)]$ is strictly positive, because p and φ are continuous. We can thus find a value of c sufficiently small that (3.6) holds for all $\phi \in [\lambda\theta, 1-\lambda(1-\theta)]$. Setting $\phi' \leq \lambda\theta$ in (3.3) ensures that (3.6) is satisfied. Moreover, an argument analogous to that for the one-stage deviation principle for infinite horizon games shows that $V_C(\phi) - V_C(\phi; L) \geq 0$ for all ϕ implies that no deviation from always choosing high effort is profitable for the competent firm. ■

Proposition 1 indicates that an equilibrium in which the competent firm always exerts high effort exists as long as the cost of high effort is sufficiently low. The low-cost provision is to be expected. We can always ensure that high effort will not be produced by making its cost prohibitively high. Notice, however, that the efficiency condition $c < 1-2\rho$ alone does not suffice to ensure existence. The costs of high effort are borne immediately. The benefits are only partially recaptured, in the future, as a result of the favorable consumer belief revision induced by high effort. The existence of a high-effort equilibrium may then require values of c much smaller than $1-2\rho$.

The possibility that a competent firm may be replaced by an inept one ($\lambda > 0$ and $\theta < 1$) is necessary for this existence result.[9] We thus have the seemingly paradoxical result that it can be good news for the firm to have consumers constantly fearing that the firm might "go bad." However, the purpose of a reputation is to convince consumers that the firm is competent and hence will choose

[9] Holmström [6] earlier made an analogous observation in a signaling jamming model with imperfect public monitoring. We discuss replacements and the relationship of our work to Holmström [6] in Mailath and Samuelson [7].

high effort. The problem with maintaining a reputation in the absence of inept replacements ($\lambda = 0$ or $\theta = 1$) is that the firm does too good a job of convincing consumers it is competent. Consumers become so convinced the firm is competent (i.e., the posterior ϕ becomes so high), that subsequent evidence can only shake this belief very slowly. Once this happens, the incentive to choose high effort disappears, as the current cost savings of low effort overwhelm the distant adverse belief revision. But then the incentive to convince consumers the firm is competent also disappears, and the equilibrium collapses. If replacements continually introduce the possibility that the firm has become inept, then the firm cannot be "too successful" at convincing consumers it is competent. Formally, there is an upper bound, short of unity, on the posterior ϕ. The incentive to choose high effort in order to reassure consumers that the firm is still competent always remains, opening the door to an equilibrium in which the competent firm exerts high effort.

The closer θ is to one, the smaller must the cost of high effort be in order for an equilibrium to exist in which the firm exerts high effort. In equilibrium, the difference between the value of choosing high effort and the value of choosing low effort must exceed the cost of high effort. However, these value functions approach each other at $\phi = 1$, because the values diverge only through the effect of current outcomes on future posteriors, and current outcomes have very little effect on future posteriors when consumers are currently quite sure of the firm's type. The smaller is the probability of an inept replacement, the closer can the posterior expectation of a competent firm approach unity, and hence the smaller must be the cost in order to support high effort.

If replacements always introduced inept firms ($\theta = 0$), then there is no lower bound on the posterior probability of an inept firm. This again destroys the proposed equilibrium. For very low posteriors, the actions of the firm again have so little effect on consumer posteriors as to render high effort suboptimal. In this case, an alternative equilibrium exists in which competent firms sometimes (but not always) choose high effort. We need only choose ϕ' in (3.3) so that (3.6) holds for all feasible posteriors $\phi \geq \phi'$. The competent firm then exerts high effort as long as the posterior probability that they are competent remains sufficiently high ($\phi \geq \phi'$). The firm reverts to low effort whenever a string of bad luck reduces the posterior below the cutoff level ϕ', causing the competent firm to abandon the attempt to convince consumers of its type and resign itself to a life of low effort.

The alternative equilibrium constructed in the previous paragraph does not require $\theta = 0$, and a similar argument allows us to construct a collection of alternative pure-strategy equilibria for the case of $\theta > 0$, characterized by different choices of the value ϕ' below which a competent firm abandons a reputation.

Equilibria in mixed strategies also exist. We concentrate on the equilibrium described in Proposition 1, which is the most efficient at supporting high effort.

4. Endogenous replacements

In this section, the probability that a replacement firm is competent is determined by market forces rather than being exogenously fixed. In each period, there is again an exogenously fixed probability λ that the current firm leaves the market. If this occurs, then the entrepreneur sells the name to a new firm. There are always a large number of potential new firms who are inept. Formally, we only require two potential inept owners, but we think of this as an endeavor where it is easy to be inept, and hence where there is always an ample supply of inept owners. We normalize the opportunity cost of potential inept owners to 0.

We view this as a market in which competent firms are relatively scarce. Whether there is a potential competent owner, and the opportunity cost of that owner, is determined in each period by the outcome of a random variable, with these variables being independently, identically distributed across periods. With probability $\nu + \kappa D(d_C)$, there is a potential competent owner whose opportunity cost of participating in the market is less than or equal to d_C. We assume that $\nu \in (0,1)$, $\kappa \geq 0$, $\nu + \kappa \leq 1$, and that D is a strictly increasing, differentiable cumulative distribution function on $[0, \infty)$. Hence, ν is the probability that there is a competent firm with opportunity cost zero. With probability $\kappa D(d_C)$, there is a competent firm whose opportunity cost exceeds zero but falls short of d_C. With probability $1 - \nu - \kappa$, there is no potential competent firm. We assume there is at most one competent firm. If we set $\kappa = 0$, the model is formally identical to that of exogenous replacements, where the replacing firm is competent with probability ν.

When the entrepreneur has to sell the right to the name, then the right is sold by either a sealed-bid, second-price auction, or equivalently by an open, ascending-price auction.[10] Let V_C (V_I) denote the value function for the competent (inept) owner. The *price* of a name currently characterized by the posterior ϕ is then $V_I(\phi)$. The type C owner buys the name if

$$V_C(\phi) \geq d_C + V_I(\phi). \tag{4.1}$$

One of the type I owners buys the firm if the inequality is reversed. For completeness, we have imposed the tie-breaking rule that the competent owner wins the

[10] The reserve price in either case is zero, which is optimal as long as each auction attracts at most one good firm.

auctions in the case of a tie in (4.1). The precise specification of the tie breaking rule is irrelevant, since such a tie arises with zero probability (Lemma D in the Appendix). The probability that the replacement is competent is $D(V_C(\phi) - V_I(\phi))$, which depends upon the consumers' posterior ϕ.

We seek an equilibrium in which competent firms always exert high effort. In such an equilibrium, posterior beliefs of the consumers are given by

$$\varphi(\phi|g) = (1-\lambda)\frac{(1-\rho)\phi}{(1-\rho)\phi + \rho(1-\phi)} + \lambda\nu + \lambda\kappa D\left(V_C(\varphi(\phi|g)) - V_I(\varphi(\phi|g))\right), \tag{4.2}$$

and

$$\varphi(\phi|b) = (1-\lambda)\frac{\rho\phi}{\rho\phi + (1-\rho)(1-\phi)} + \lambda\nu + \lambda\kappa D\left(V_C(\varphi(\phi|b)) - V_I(\varphi(\phi|b))\right). \tag{4.3}$$

The belief functions $\varphi(\phi|g)$ and $\varphi(\phi|b)$ enter both sides of (4.2) and (4.3). This reflects the fact that beliefs depend upon whether entrants are likely to be competent or inept firms, which depends upon beliefs. The beliefs of the consumers are then a fixed point of (4.2) and (4.3).

The value function of the type I owner is

$$V_I(\phi) = (1-2\rho)\phi + \rho + \delta(1-\lambda)\left\{\rho V_I(\varphi(\phi|g)) + (1-\rho)V_I(\varphi(\phi|b))\right\}, \tag{4.4}$$

and the value function of the type C owner is

$$V_C(\phi) = (1-2\rho)\phi + \rho - c + \delta(1-\lambda)\left\{(1-\rho)V_C(\varphi(\phi|g)) + \rho V_C(\varphi(\phi|b))\right\}. \tag{4.5}$$

Finally, it must be optimal in equilibrium for the competent firm to choose high effort. From (4.5), a necessary condition is that for all possible posteriors ϕ,[11] a one period deviation to low effort not be profitable, i.e.,

$$\delta(1-\lambda)(1-2\rho)\left\{V_C(\varphi(\phi|g)) - V_C(\varphi(\phi|b))\right\} \geq c. \tag{4.6}$$

Moreover, as in the proof of Proposition 1, an argument analogous to that for the one-stage deviation principle for infinite horizon games shows that (4.6) is also sufficient.

Definition 2. *A* reputation equilibrium *is a triple (φ, V_C, V_I) satisfying (4.2)–(4.6), identifying the expectation updating rules and the value functions of the firms in a reputation equilibrium, coupled with strategies specifying that competent firms choose high effort for every possible posterior expectation.*

[11] Strictly speaking, we should only be requiring (4.6) for those posteriors that can be reached from the initial prior after some finite history $h^t \in \{g, b\}^t$. However, nothing is lost by requiring (4.6) for all posteriors and doing so avoids awkward statements.

Proposition 2. *Suppose*

$$\begin{aligned} \delta(1-\lambda) &< \rho(1-\rho)/(1-3\rho+3\rho^2), \\ \nu &> 0, \\ D((1-\rho)/(1-\delta)) &< 1-\nu, \end{aligned}$$

and D' is bounded. Then there exists $\kappa^* \geq 0$ and $c^* \geq 0$ such that a reputation equilibrium exists for all $\kappa \in [0, \kappa^*]$ and $c \in [0, c^*]$.

Proof. See the Appendix.

The difficulty in establishing the existence of an equilibrium in this case arises out of the linkage between the posterior updating rules and the firms' value functions. In the case of exogenous replacements, the updating rules were defined independently of the value functions. We could accordingly first calculate posterior beliefs, use these calculations to obtain value functions, and then confirm that the proposed strategies are optimal given the value functions. With endogenous replacements, the value functions enter the updating rules given by (4.2) and (4.3). As a result, we must now use a fixed-point argument to establish the existence of mutually consistent updating rules and value functions, and much of the proof is concerned with this fixed point argument. After concluding that consistent value functions and updating rules exist, we show that, as long as c and κ are not too large, the proposed strategies are optimal.

The first inequality restriction on δ, λ, and ρ in Proposition 2 ensures that the one-period discounted "average" derivative of the no-replacement updating rule is strictly less than one. Coupled with the requirement that D' is bounded, this ensures that the value functions have bounded derivatives. This in turn is required in order to have a well-behaved set of potential value functions to work with when constructing the fixed-point argument. The requirements that $\nu > 0$ and $D((1-\rho)/(1-\delta)) < 1-\nu$ ensure that for any allowable value of κ, there exist $\underline{\phi}$ and $\overline{\phi}$, with $0 < \underline{\phi} < \overline{\phi} < 1$, such that for any φ satisfying (4.2) and (4.3), $\varphi(\phi|x) \in [\underline{\phi}, \overline{\phi}]$ for all $\phi \in [0,1]$ and $x \in \{g, b\}$. As in the case of Proposition 1, this bounding of posterior probabilities away from the ends of the unit interval is necessary in order to preserve the incentive for competent firms to exert high effort.

We again require that c be sufficiently small that the potential future gains of maintaining a reputation can exceed the current cost. The requirement that κ is not too large plays a role in bounding the slopes of the value functions. Intuitively, if κ is too large, then we can introduce sharp variations in the value function $V_C(\phi)$ by using different values of ϕ to coordinate consumers and potential entrants on different posteriors concerning the types and hence value of entry. This can destroy the convention that higher reputations are good which lies at the heart of a reputation equilibrium.

5. Who Buys Good Reputations?

We now turn our attention to the market for reputations. In particular, which posteriors are most likely to attract competent firms as replacements, and which are most likely to attract inept firms? A competent firm is more likely to enter, the greater the difference $V_C(\phi) - V_I(\phi)$. The Appendix contains a proof of the following:

Proposition 3. *For any $\xi > 0$, there is a $\kappa^\dagger \leq \kappa^*$ such that for any $\kappa < \kappa^\dagger$, $V_C(\phi) - V_I(\phi)$ is strictly increasing for $\phi < \frac{1}{2} - \xi$ and strictly decreasing for $\phi > \frac{1}{2} + \xi$.*

Replacements are thus more likely to be competent firms for intermediate values of ϕ and less likely to be competent firms for extreme values of ϕ. Hence, firms with low reputations are relatively likely to be replaced by inept firms. Good firms find it too expensive to build up the reputation of such a name. On the other hand, firms with very good reputations are also relatively likely to be replaced by inept firms. These names are attractive to competent firms, who would prefer to inherit a good reputation to having to build up a reputation, and who would maintain the existing, good reputation. However, these names are even more attractive to inept entrants, who will enjoy the fruits of running down the existing high reputation (recall that if consumers are believe that the firm is almost certainly competent, then bad outcomes do not change consumer beliefs by a large amount).[12]

Replacements are relatively likely to be competent firms for intermediate reputations. These are relatively attractive to competent firms because less expenditure is required to build a reputation than is the case when the existing firm has a low reputation. At the same time, these reputations are less attractive than higher reputations to inept entrants, because the intermediate reputation offers a smaller stock that can be profitably depleted.

[12] In a pure adverse selection environment, Tadelis [8] similarly identifies two effects: a "reputation maintenance effect" (reflecting the higher value that competent firms assign to owning a name for any posterior) and a "reputation start-up effect" (which roughly reflects the change in the difference $V_C - V_I$ as ϕ becomes large).

As a result, we can expect reputations to exhibit two features. First, there will be churning: high reputations will be depleted while intermediate reputations will be enhanced. Secondly, low reputations are likely to remain low.

6. Extensions

The strongest assumption in our model is the restriction that consumers observe neither the type of a newly entering firm, nor even the fact that the existing firm has been replaced. It clearly stretches credibility that consumers will never be able to observe the replacements of firms. However, our results require only that firm changes be *sometimes* unobserved. We would get equivalent results if entries were observed with some probability $v \in (0,1)$ and unobserved with probability $1 - v$.

Left to their own devices, consumers may well remain unaware of some changes of ownership. But a firm that has newly purchased the right to use the name in our model may want consumers to know of the ownership change. This is especially likely to be the case if the current reputation attached to the name is lower than the prior expectation attached to entrants. New firms concerned that consumers may not have observed their entry may then seek ways to announce their presence. Signs proclaiming "under new management" and "grand opening" are commonplace. This section explores some of the strategies that entrants might adopt to announce their presence to consumers.

We study the case of exogenously-determined entrant types throughout, with θ being the probability that an entrant is competent, in order to focus on the effects of the announcement strategies. Allowing entrant types to be endogenously determined retains the basic properties of the analysis with exogenous entrant types, but would require an involved existence argument.

In each of the following models, there remains an equilibrium in which all announcements are ignored and our previous results hold. However, we concentrate on the possible existence and properties of equilibria in which announcements are effective.

Under new management. We first suppose firms can make public announcements that they have newly purchased the right to use the name. However, these announcements are cheap talk in two respects: they are costless, and consumers cannot verify whether claims to have newly purchased the name are correct. It is immediately apparent that such announcements cannot convey information. In particular, such an announcement will be valuable only if it increases the posterior belief ϕ that the firm is competent. But both competent and inept firms would like ϕ to be higher, as would both existing firms and entrants. Any announcement that had an effect on ϕ would then be made regardless of the firm's identity, ensuring that the announcement is uninformative.

Meet our new chef. Announcements of changes in the characteristics of a firm are often accompanied by invitations to verify these changes. The chefs in good restaurants sometimes mingle with the customers, especially early in their tenure, as do mechanics and other service technicians. Firms call press conferences to display new members of their management staff, as do professional sports teams with new athletes. Firms sometimes invite customers to tour their new facilities. To the extent that the personnel or facilities in question are responsible for the quality of the firm's product, these activities make ownership changes verifiable.[13]

Let us suppose that whenever the right to use the name has changed owners, the new owner can make a costless, verifiable announcement of this change before the beginning of the next period, though no information concerning the quality of the new owner can be verified. Upon hearing such an announcement, consumers will expect the new firm to be competent with probability θ. As a result, no firm purchasing a name whose current reputation satisfies $\phi > \theta$ will announce the change of ownership, while a firm for whom $\phi < \theta$ earns an instant increase in reputation from such an entry announcement. A candidate for an equilibrium then calls for announcements if and only if $\phi < \theta$. As a result, the reputation updating for posteriors $\phi > \theta$ will proceed according to (3.1)–(3.2), as is the case without announcements. For posteriors $\phi < \theta$, the lack of an announcement indicates that no change of ownership has occurred. The reputation updating then proceeds according to (3.1)–(3.2) with $\lambda = 0$.

Because the lack of an announcement when $\phi < \theta$ ensures the absence of an ownership change, the smallest possible posterior is $\underline{\phi} = 0$. As we discussed in connection with Proposition 1, there will then be a posterior ϕ' with the property that a competent firm chooses low effort when $\phi < \phi'$ and high effort if $\phi > \phi'$. If $\theta > \phi'$, then new owners announce their presence if and only if they acquire a reputation less than θ.[14] Consumers thus observe changes in the ownership

[13] Throughout the analysis, we have viewed a change in the identify of the firm as a change in the owner of the right to use the name. More generally, the relevant change involves a change in the input that is responsible for the ability of the firm to produce high quality. Hence, a change of chef in a restaurant, for a given proprieter, is the equivalent of a change of ownership in our model if the chef is the primary determinant of the restaurant's quality, but not if quality depends mostly upon a manager who chooses what dishes to offer, what ingredients to order, and what level of service to offer.

[14] If $\theta < \phi'$, then there is never a strict incentive to announce a replacement, though it is a weak best response to do so.

of names whose reputation falls short of that of a randomly drawn entrant, and observe no ownership changes of higher reputations. The inability to observe the latter creates an incentive for the competent firm to build its reputation by choosing high effort.

Newly opened after remodeling. Changes of ownership are often accompanied by the remodeling of a firm's facilities, especially in the service industry. Walls are moved and repainted. Artwork, carpeting and furniture is replaced. The result is often a facility that is different, but not obviously superior to the previous one. We view such remodeling as a costly signal. If signals are not verifiable, and hence costless signals are uninformative, owners may still convey information through the use of costly signals.

We are interested in the ability to use costly but nonverifiable signals to convey information about the owner of a name. We assume the owner can choose how much to spend on sending a costly signal after the realization of the consumers' utility, but before the replacement event is realized.[15] The signal and its cost is observed by consumers before they purchase.

It is again clear that a nonverifiable signal, however costly, cannot usefully convey information about ownership changes. Any reputation revision prompted by such an signal is equally valuable to a new or existing firm. However, a competent owner may be able to reveal his *type* by sending a costly signal.

We accordingly describe an equilibrium in which costly signals are sent only by competent owners. Upon sending such a signal in a period, at the beginning of the next period, the posterior probability consumers assign to the firm being competent is the maximum possible, $\bar{\phi} \equiv 1 - \lambda(1 - \theta)$. We focus on equilibria of the following form: there is a critical posterior ϕ^* such that if (and only if) $\phi < \phi^*$ any competent firm spends k to send a signal, yielding a posterior of $\bar{\phi}$. An inept firm for whom $\phi < \phi^*$ sends no signal, yielding a posterior in the beginning of the next period of $\lambda\theta$. A firm who is believed to be inept retains this reputation until sending a signal at cost k, which will happen when a competent firm replaces the existing inept firm. No signals are sent for posteriors $\phi \geq \phi^*$. The mechanism by which the firm contrives to spend the cost k is immaterial, as

[15] An alternative assumption would be to have the costly signal sent after the replacement event, so that a new competent firm could attempt to signal its arrival. The difficulty is that in a separating equilibrium, such a signal leads to a posterior of 1, at which point the competent firm must choose low effort, disrupting the putative equilibrium. In order to support an equilibrium in which the competent firm chooses high effort immediately after the signal, a bad realization must have some information content. This will be the case if the competent firm may have been replaced by an inept firm after the signal.

long as consumers can verify that it has been spent. Remodeling facilities is one possibility, but publicly burning the money would serve as well.

In this equilibrium, competent firms choose high effort and inept firms choose low effort. Both competent and inept firms refrain from costly signals as long as their reputations are sufficiently high. Eventually, however, a firm's reputation will slip below the critical reputation ϕ^*. Good firms then reset their reputations by sending a costly signal, while inept firms resign themselves to a future reputation of $\lambda\theta$. The latter persists until a new, competent owner appears, who enhances his reputation by the only means possible, sending the costly signal. In order to support these strategies as an equilibrium, it must be profitable for the competent, but not the inept, firm to send the costly signals, for any posterior $\phi' < \phi^*$. This requires that the cost k be such that

$$\delta V_I(\bar{\phi}, \phi^*, k) - \delta \frac{\rho}{1-(1-\lambda)\delta} \leq k \leq \delta V_C(\bar{\phi}, \phi^*, k) - \delta \frac{\rho}{1-(1-\lambda)\delta}, \qquad (6.1)$$

where $\delta\rho/(1-(1-\lambda)\delta)$ is the expected value, to both a competent and inept firm, of failing to send the signal and hence inducing a reputation of $\lambda\theta$, and where we write the value functions as $V_I(\bar{\phi}, \phi^*, k)$ and $V_C(\bar{\phi}, \phi^*, k)$ to emphasize that continuation values depend upon the posterior at which signals are send and the cost of the signals. Note that the role of ϕ^* is similar to that of ϕ' in the proof of Proposition 1.

In general, there will be multiple solutions to (6.1), and hence multiple equilibria in which firms use costly signals to identify their types. Refinements such as the intuitive criterion can be applied to the out-of-equilibrium beliefs supporting these equilibria to select values of k that satisfy the first relationship in (6.1) with equality, so that the competent owner sends the least costly signal consistent with separating from the inept owner. However, there will still be multiple values of ϕ^* consistent with equilibrium. The various equilibria can be supported by out-of-equilibrium beliefs that a firm who sends a signal when their posterior satisfies $\phi > \phi^*$ is likely to be competent with probability ϕ, which are not precluded by standard equilibrium refinements. Higher values of ϕ^* correspond to cases in which firms use frequent signals to constantly nudge their reputations upwards. Lower values of ϕ^* correspond to cases in which signals are infrequently used but accomplish large increases in reputations.

Limited-time introductory offer. We have assumed that the firm prices at the consumer's reservation price in each period, given the firm's current reputation. However, a common way for firms to send signals to consumers is by appropriately choosing their prices, with low prices potentially serving as a costly signal of a competent firm. In our model, setting a price less than the consumers' reservation price is equivalent to burning money, and hence equivalent to the costly signaling possibilities we have just considered. Expanding our model to allow the firm more discretion in setting prices then leads to equilibria equivalent to those in which the firm can send costly signals.

We have assumed that consumers are homogeneous, and each consumer purchases the good in each period. The scope for introductory pricing may be expanded if consumers are heterogeneous. In particular, consumers who have formed relatively pessimistic posterior expectations may then cease patronizing the firm altogether, and hence cease collecting information about the firm. If the right to use the name passes to a competent firm, then the latter may find it profitable to incur costs to bring these consumers back to the firm, and may find introductory pricing specials more effective than burning money in doing so.

I'm your new doctor. Our model of reputations depends upon the assumption that customers cannot *always* tell who currently owns the right to use the name. This is likely to be the case in many markets, including services industries such as restaurants, auto repair, and the Time Life example with which we opened. In other cases in which reputations are commonly said to be traded, any such uncertainty is highly unlikely. For example, private medical and dental practices often command high prices. Because very little in the way of physical assets typically changes hands in such a transaction, much of the price is ascribed to reputation. But patients cannot help but notice that their doctor or dentist has changed, making both our model and the extension to voluntary, verifiable announcements of ownership changes inapplicable. Our suspicion is that the bulk of the price for such practices represents compensation for in-place physical assets, which may dwarf the replacement cost of these assets. The key is to notice that patients have no reason to believe that the purchasing doctor is better than the expected outcome they could obtain by returning to the market, despite the selling doctor's glowing recommendation, nor do they have reason to believe that the purchasing doctor is worse. The presence of even extremely small costs of returning to the market to seek a new doctor will thus suffice to retain them at the current practice, making the mere collection of patient records a valuable asset.[16]

[16] An interesting prediction of this suspicion is that medical practices which include physical facilities and support staff should exceed the value of those sold without staff and facilities, by an amount exceeding the physical value of the facilities and costs of obtaining new staff, since patient familiarity with existing facilities will increase the costs of switching.

7. Conclusion

Does a good reputation ensure a firm's success? Obviously not, as any firm can be sufficiently unlucky as to squander a superb reputation. Just as obviously, a good reputation is better than a bad one. But our analysis shows that these considerations can interact in unexpected ways. In a market with potentially unobserved firm turnover, the current reputation of a firm may not be a good predictor of its future success. Instead, a selection bias arises in the process by which firms acquire reputations, with relatively capable firms tending to buy medium reputations, leaving high reputations to be acquired and spent by less capable firms.

This acquisition pattern arises because the advantage enjoyed by a competent firm is the ability to boost consumers' posterior expectations of the firm's quality, by exerting high effort. But the consumer posteriors that are most easily influenced are intermediate posteriors, giving competent firms a comparative advantage in medium reputations.

The more general implication of our analysis is that embedding reputation considerations in a market can produce new insights into the economics of building and maintaining a reputation. We have taken only the first step in modeling the market, with many factors still being exogenously fixed that future work might usefully bring within the purview of the model.

A. Appendix

A.1. Proof of Proposition 2

The proof is involved and requires some lemmas. For the reader's convenience, we have collected the lemmas at the end of this subsection.

[Step 1] We first show that any admissible posterior updating rule implies a unique pair of value functions. Fix $\eta \in \big(\delta(1-\lambda)(1-3\rho+3\rho^2)/(\rho-\rho^2), 1\big)$, and let $\mathfrak{X} \equiv \{f \in \mathcal{C}^1([0,1],[0,1]^2) : \delta\,|\rho f_1'(\phi) + (1-\rho) f_2'(\phi)| \leq \eta, \delta\,|(1-\rho) f_1'(\phi) + \rho f_2'(\phi)| \leq \eta,\ |f_1'(\phi)| \leq 2/\rho,\ |f_2'(\phi)| \leq 2/\rho$ for all $\phi \in [0,1]\}$. Any function $\varphi(\phi) \equiv (\varphi(\phi|g), \varphi(\phi|b)) \in \mathfrak{X}$ is a potential posterior updating rule, giving, for any prior probability ϕ, the posteriors $(\varphi(\phi|g), \varphi(\phi|b))$ that follow a good and bad utility realization. The set $\mathfrak{X}$ is a convex, compact subset of the normed space $\mathcal{C}^1([0,1],[0,1]^2$, with the $\mathcal{C}^1$-norm,

$$\| f \|_1 = \max\{\sup_\phi |f_1(\phi)|, \sup_\phi |f_1'(\phi)|, \sup_\phi |f_2(\phi)|, \sup_\phi |f_2'(\phi)|\}.$$

The set $\mathfrak{X}$ is nonempty. In particular, let φ_0 denote the exogenous updating rules, i.e., κ is set equal to zero in (4.2) and (4.3). It is straightforward to verify that, for $x \neq y \in \{g,b\}$, $0 \leq \rho\varphi_0'(\phi|x)+(1-\rho)\varphi_0'(\phi|y) \leq (1-\lambda)(1-3\rho+3\rho^2)/(\rho-\rho^2) < \eta/\delta$, and so $\varphi_0 \equiv (\varphi_0(\cdot|g), \varphi_0(\cdot|b)) \in \mathfrak{X}$.

Let $Y \equiv (1-\rho)/(1-\delta)$ and $\mathfrak{Y} \equiv \{f \in \mathcal{C}^1([0,1],[-Y,Y]^2) : \sup_x |f_1'(x)| \leq (1-2\rho)/(1-\eta),\ \sup_x |f_2'(x)| \leq (1-2\rho)/(1-\eta)\}$. Interpret an element of $\mathfrak{Y}$ as a pair of possible value functions, one for the inept firm and one for the competent firm. Fix an updating rule $\varphi \equiv (\varphi(\cdot|g), \varphi(\cdot|b)) \in \mathfrak{X}$, and let $\Psi^\varphi : \mathfrak{Y} \to \mathcal{C}^1([0,1],\Re^2)$ denote the mapping whose coordinates are the functions:

$$\Psi_1^\varphi(V_I,V_C)(\phi) = (1-2\rho)\phi + \rho + \delta(1-\lambda)\left\{\rho V_I(\varphi(\phi|g)) + (1-\rho)V_I(\varphi(\phi|b))\right\},$$

and

$$\Psi_2^\varphi(V_I,V_C)(\phi) = (1-2\rho)\phi + \rho - c + \delta(1-\lambda)\left\{(1-\rho)V_C(\varphi(\phi|g)) + \rho V_C(\varphi(\phi|b))\right\}.$$

The mapping Ψ^φ is a contraction on $\mathfrak{Y}$ (Lemmas A and B), and so has a unique fixed point. For any updating rule φ, this fixed point identifies the unique value functions that are consistent with φ, in the sense of satisfying (4.4) and (4.5). Let $\Phi : \mathfrak{X} \to \mathfrak{Y}$ denote the mapping that associates, for any updating rule φ in $\mathfrak{X}$, the fixed point of Ψ^φ. The mapping Φ is continuous (Lemma C).

[Step 2] We now show that there exist updating rules and value functions that are consistent, in the sense that using Φ to obtain value functions from an updating rule and then applying (4.2)–(4.3) returns the original updating rules. Let $\hat{\varphi}$ denote the updating rule obtained from φ and $\Phi(\varphi) = (V_I, V_C)$ by using (4.2)–(4.3):

$$\hat{\varphi}(\phi|g) = \varphi_0(\phi|g) + \lambda\kappa D\left(V_C(\varphi(\phi|g)) - V_I(\varphi(\phi|g))\right),$$

$$\hat{\varphi}(\phi|b) = \varphi_0(\phi|b) + \lambda\kappa D\left(V_C(\varphi(\phi|b)) - V_I(\varphi(\phi|b))\right).$$

Since

$$\hat{\varphi}'(\phi|x) = \varphi_0'(\phi|x) + \lambda\kappa D'(V_C(\varphi(\phi|x) - V_I(\varphi(\phi|x)) \cdot (V_C'(\varphi(\phi|x) - V_I'(\varphi(\phi|x)) \cdot \varphi'(\phi|x),$$

we have (for $x \neq y \in \{g,b\}$)

$$\begin{aligned} \left|\rho\hat{\varphi}'(\phi|x) + (1-\rho)\hat{\varphi}'(\phi|y)\right| \leq\ & \left|\rho\varphi_0'(\phi|x) + (1-\rho)\varphi_0'(\phi|y)\right| \\ & + \lambda\kappa \sup_{d_C} D'(d_C) \cdot \sup_{\phi} \left|V_C'(\phi) + V_I'(\phi)\right| \\ & \cdot \sup_{\phi} \left|\rho\varphi'(\phi|x) + (1-\rho)\varphi'(\phi|y)\right|, \end{aligned}$$

which is less than or equal to η/δ for κ small (but nonzero).

Hence, for sufficiently small values of κ (say $\kappa \leq \kappa^{**}$), the mapping $\Upsilon^\kappa(\varphi) = \hat{\varphi}$ is a mapping from $\mathfrak{X}$ into $\mathfrak{X}$. Moreover, it is clearly continuous, and hence, by the Schauder-Tychonoff theorem (Dunford and Schwartz [3, p. 456]), has a fixed point. For each value of κ, we let φ_κ denote a posterior updating function which is a fixed point of the mapping Υ^κ and we let $(V_I^\kappa, V_C^\kappa) = \Phi(\varphi_\kappa)$ denote the corresponding value functions. Together, φ_κ and (V_I^κ, V_C^κ) satisfy (4.2)–(4.5).

[Step 3] We now verify (4.6). Since $\nu > 0$ and $\nu + D((1-\rho)/(1-\delta)) < 1$, there exist $\underline{\phi}$ and $\overline{\phi}$, $0 < \underline{\phi} < \overline{\phi} < 1$, such that for all $\varphi \in \mathfrak{X}$ satisfying (4.2) and (4.3) with $\kappa \leq 1$, $\varphi(\phi|x) \in [\underline{\phi}, \overline{\phi}]$ for all $\phi \in [0,1]$ and $x \in \{g, b\}$. Thus, there exist κ' and $\zeta > 0$ such that for all $\phi \in [\underline{\phi}, \overline{\phi}]$ and all $\kappa < \kappa'$,

$$\varphi_\kappa(\phi|g) - \varphi_\kappa(\phi|b) > \zeta. \tag{A.1}$$

There is a unique triple $(\varphi_0, V_I^0, V_C^0)$ satisfying (4.2), (4.3), (4.4), and (4.5) when $\kappa = 0$. Moreover, there exists c_0^* such that $\delta(1-\lambda)(1-2\rho)\left\{V_C^0(\varphi_0(\phi|g)) - V_C^0(\varphi_0(\phi|b))\right\} > c$ for all $\phi \in [\underline{\phi}, \overline{\phi}]$ and $c < c_0^*$ (Lemma D).

Fix $c^* < c_0^*$. The sequential compactness of $\mathfrak{X}$ and $\mathfrak{Y}$ then implies the existence of κ^* ($\leq \kappa^{**}$) such that for all $\kappa \leq \kappa^*$, $\delta(1-\lambda)(1-2\rho)\left\{V_C^\kappa(\varphi_\kappa(\phi|g)) - V_C^\kappa(\varphi_\kappa(\phi|b))\right\} > c^*$ for all $\phi \in [\underline{\phi}, \overline{\phi}]$.

■

Lemma A. $\Psi^\varphi(\mathfrak{Y}) \subset \mathfrak{Y}$.

Proof. Denote the image of (V_I, V_C) under Ψ^φ by $(\hat{V}_I, \hat{V}_C)$. We verify that $(\hat{V}_I, \hat{V}_C) \in \mathfrak{Y}$ for all $(V_I, V_C) \in \mathfrak{Y}$. Clearly, both $\hat{V}_I$ and $\hat{V}_C$ are C^1, and it is straightforward that $\left|\hat{V}_I(\phi)\right|, \left|\hat{V}_C(\phi)\right| \leq Y$. Now,

$$\begin{aligned}\left|\hat{V}_I'(\phi)\right| &\leq (1-2\rho) + \delta(1-\lambda)\left|\rho\varphi'(\phi|g) + (1-\rho)\varphi'(\phi|b)\right|(1-2\rho)/(1-\eta) \\ &\leq (1-2\rho) + \eta(1-2\rho)/(1-\eta) = (1-2\rho)/(1-\eta),\end{aligned}$$

while a similar calculation shows that $\left|\hat{V}_C'(\phi)\right| \leq (1-2\rho) + \eta(1-2\rho)/(1-\eta) = (1-2\rho)/(1-\eta)$, and so Ψ^φ maps $\mathfrak{Y}$ into $\mathfrak{Y}$.

■

Lemma B. Ψ^φ *is a contraction.*

Proof. First note that

$$
\begin{aligned}
&\sup_\phi \left|\Psi_1^\varphi(V_I, V_C)(\phi) - \Psi_1^\varphi(\hat{V}_I, \hat{V}_C)(\phi)\right| \\
\leq\ & \delta(1-\lambda)\left\{\sup_\phi \rho \left|V_I(\varphi(\phi|g)) - \hat{V}_I(\varphi(\phi|g))\right| + \sup_\phi (1-\rho)\left|V_I(\varphi(\phi|b)) - \hat{V}_I(\varphi(\phi|b))\right|\right\} \\
\leq\ & \delta(1-\lambda)\sup_\phi \left|V_I(\phi) - \hat{V}_I(\phi)\right|
\end{aligned}
$$

and similarly that $\sup_\phi \left|\Psi_2^\varphi(V_I, V_C)(\phi) - \Psi_2^\varphi(\hat{V}_I, \hat{V}_C)(\phi)\right| \leq \delta(1-\lambda)\sup_\phi \left|V_C(\phi) - \hat{V}_C(\phi)\right|$.

Turning to the derivatives,

$$
\begin{aligned}
&\sup_\phi \left|\left(\Psi_1^\varphi(V_I, V_C)\right)'(\phi) - \left(\Psi_1^\varphi(\hat{V}_I, \hat{V}_C)\right)'(\phi)\right| \\
\leq\ & \delta(1-\lambda)\sup_\phi \Big\{\rho\varphi'(\phi|g)\left|V_I'(\varphi(\phi|g)) - \hat{V}_I'(\varphi(\phi|g))\right| \\
& +(1-\rho)\varphi'(\phi|b)\left|V_I'(\varphi(\phi|b)) - \hat{V}_I'(\varphi(\phi|b))\right|\Big\} \\
\leq\ & \delta(1-\lambda)\sup_\phi\{\rho\varphi'(\phi|g) + (1-\rho)\varphi'(\phi|b)\}\sup_\phi \left|V_I'(\phi) - \hat{V}_I'(\phi)\right| \\
\leq\ & \eta(1-\lambda)\sup_\phi \left|V_I'(\phi) - \hat{V}_I'(\phi)\right|,
\end{aligned}
$$

while a similar calculation shows that

$$
\sup_\phi \left|\left(\Psi_2^\varphi(V_I, V_C)\right)'(\phi) - \left(\Psi_2^\varphi(\hat{V}_I, \hat{V}_C)\right)'(\phi)\right| \leq \eta(1-\lambda)\sup_\phi \left|V_C'(\phi) - \hat{V}_C'(\phi)\right|.
$$

Thus,

$$
\left\|\Psi^\varphi(V_I, V_C) - \Psi^\varphi(\hat{V}_I, \hat{V}_C)\right\|_1 \leq \max\{\delta(1-\lambda), \eta(1-\lambda)\}\left\|(V_I, V_C) - (\hat{V}_I, \hat{V}_C)\right\|_1
$$

and, as claimed, Ψ^φ is a contraction.

■

Lemma C. Φ *is continuous.*

Proof. Suppose $\varphi_n \to \varphi_\infty$. Since $\mathfrak{Y}$ is sequentially compact (it is an equicontinuous collection of uniformly bounded functions on a compact space), there is a subsequence, denoted $\{\varphi_m\}$, with $(V_I^m, V_C^m) \equiv \Phi(\varphi_m)$ uniformly converging to some $(V_I, V_C) \in \mathfrak{Y}$. To see that V_I satisfies (4.4), note that

$$\begin{aligned}
& |V_I(\phi) - (1-2\rho)\phi + \rho + \delta(1-\lambda)\{\rho V_I(\varphi_\infty(\phi|g)) + (1-\rho)V_I(\varphi_\infty(\phi|b))\}| \\
\leq\ & |V_I(\phi) - V_I^m(\phi)| + |V_I^m(\phi) - (1-2\rho)\phi + \rho \\
& \quad + \delta(1-\lambda)\{\rho V_I^m(\varphi_m(\phi|g)) + (1-\rho)V_I^m(\varphi_m(\phi|b))\}| \\
& + \delta(1-\lambda)\rho\,|V_I(\varphi_\infty(\phi|g)) - V_I^m(\varphi_m(\phi|g))| \\
& + \delta(1-\lambda)(1-\rho)\,|V_I(\varphi_\infty(\phi|b)) - V_I^m(\varphi_m(\phi|b))| \\
=\ & |V_I(\phi) - V_I^m(\phi)| + \delta(1-\lambda)\rho\,|V_I(\varphi_\infty(\phi|g)) - V_I^m(\varphi_m(\phi|g))| \\
& + \delta(1-\lambda)(1-\rho)\,|V_I(\varphi_\infty(\phi|b)) - V_I^m(\varphi_m(\phi|b))|, \qquad (A.2)
\end{aligned}$$

where the equality holds because $(V_I^m, V_C^m) \equiv \Phi(\varphi_m)$.

Now, fix $\epsilon > 0$. There exists m_ϵ such that for all $m \geq m_\epsilon$ and all ϕ, $|V_I(\phi) - V_I^m(\phi)| < \epsilon/3$. Moreover, since V_I is uniformly continuous and $\Phi(\varphi_m)$ converges uniformly to φ_∞, m_ϵ can be chosen such that $|V_I(\varphi_\infty(\phi|x)) - V_I^m(\varphi_m(\phi|x))| \leq |V_I(\varphi_\infty(\phi|x)) - V_I^m(\varphi_\infty(\phi|x))| + |V_I^m(\varphi_\infty(\phi|x)) - V_I^m(\varphi_m(\phi|x))| \leq \epsilon/3$ for $x \in \{g, b\}$. Thus, (A.2) is less than or equal to ϵ, for all $\epsilon > 0$, and so V_I satisfies (4.4) for the updating rule φ_∞. Because there is a unique solution to (4.4) given φ_∞, it must then be that V_I^m converges to V_I. A similar argument shows that $V_C^m(\varphi_m)$ converges to $V_C(\varphi_\infty)$, giving the result.

■

Lemma D. *The slopes of V_I^0 and V_C^0 are bounded below by $(1-2\rho)$. There exists a cost c_0^{**}, such that $V_C^0(\phi) > V_I^0(\phi)$ for all $\phi \in [\underline{\phi}, \overline{\phi}]$ and all $c < c_0^{**}$.*

Proof. Given $\varphi \equiv (\varphi(\cdot|g), \varphi(\cdot|b)) \in \mathfrak{X}$ and $h^t \in \{g, b\}^t$, denote the consumers' posterior after observing the sequence $h^t = (x_1, \ldots, x_t)$ by $\varphi(\phi|h^t) \equiv \varphi(\cdots\varphi(\varphi(\phi \mid x_1) \mid x_2)\cdots \mid x_t)$. The value functions $(V_I, V_C) = \Phi(\varphi)$ can be written as, by recursively substituting,

$$V_I(\phi) = \frac{\rho}{1-(1-\lambda)\delta} + (1-2\rho)\phi + (1-2\rho)\sum_{t=1}^{\infty}\delta^t(1-\lambda)^t \sum_{h^t\in\{g,b\}^t} \varphi(\phi|h^t)\Pr(h^t|L), \tag{A.3}$$

and

$$V_C(\phi) = \frac{\rho - c}{1-(1-\lambda)\delta} + (1-2\rho)\phi + (1-2\rho)\sum_{t=1}^{\infty}\delta^t(1-\lambda)^t \sum_{h^t\in\{g,b\}^t} \varphi(\phi|h^t)\Pr(h^t|H), \tag{A.4}$$

where $\Pr(h^t|H)$ is the probability of realizing the sequence of outcomes h^t given that the firm chooses high effort in every period, and $\Pr(h^t|L)$ is the probability of realizing h^t given low effort in every period. Since $\varphi_0(\phi|g)$ and $\varphi_0(\phi|b)$ are strictly increasing in ϕ for $\phi \neq 0,1$, it is immediate from (A.3) and (A.4) that V_I^0 and V_C^0 have slopes of at least $(1-2\rho)$.

In addition,

$$\begin{aligned}
V_C^0(\phi) - V_I^0(\phi) &= \frac{-c}{1-(1-\lambda)\delta} \\
&\quad +(1-2\rho)\sum_{t=1}^{\infty}\delta^t(1-\lambda)^t \sum_{h^t\in\{g,b\}^t} \varphi_0(\phi|h^t)\left\{\Pr(h^t|H)-\Pr(h^t|L)\right\} \\
&= \frac{-c}{1-(1-\lambda)\delta} \\
&\quad +\sum_{t=1}^{\infty}\delta^t(1-\lambda)^{t-1} \sum_{h^t\in\{g,b\}^t} K(\varphi_0(\phi|h^t))\left\{\Pr(h^t|H)-\Pr(h^t|L)\right\},
\end{aligned}$$

where $K(\phi) = (1-\lambda)(1-2\rho)\phi$ is strictly increasing in ϕ. Let $F_{H,0}^t$ denote the distribution function of the t-period posterior induced by φ_0 and high effort in every period, i.e., $F_{H,0}^t(p) = \Pr\{\varphi_0(\phi|h^t) \leq p|H\}$. Similarly, $F_{L,0}^t$ is the distribution function induced by low effort in every period. Clearly, $F_{H,0}^t$ first order stochastically strictly dominates $F_{L,0}^t$, so that (where $\mathcal{E}_{H,0}^t$ and $\mathcal{E}_{L,0}^t$ denote expectation with respect to the distribution functions $F_{H,0}^t$ and $F_{L,0}^t$, respectively)

$$\begin{aligned}
\sum_{h^t\in\{g,b\}^t} K(\varphi_0(\phi|h^t))\Pr(h^t|H) &= \mathcal{E}_{H,0}^t[K] \\
&> \mathcal{E}_{L,0}^t[K] = \sum_{h^t\in\{g,b\}^t} K(\varphi_0(\phi|h^t))\Pr(h^t|L).
\end{aligned}$$

Thus, the terms in the summation for $t \geq 2$ are strictly positive and, using (A.1),

$$\begin{aligned}
V_C^0(\phi) - V_I^0(\phi) &> -c/(1-(1-\lambda)\delta) + \delta(1-2\rho)^2(1-\lambda)\left(\varphi_0(\phi|g)-\varphi_0(\phi|b)\right) \\
&> -c/(1-(1-\lambda)\delta) + \delta(1-2\rho)^2\zeta,
\end{aligned}$$

and so a lower bound on c is

$$c_0^{**} \equiv \frac{\delta^2(1-2\rho)^3\zeta(1-\lambda)(1-(1-\lambda)\delta)}{1+2\rho\delta(1-\lambda)}.$$

Note that this not a tight bound, since we only used only the inequalities that pertain to the first period of the value-function calculations.

■

A.2. Proof of Proposition 3

Consider first the case of exogenous entry, $\kappa = 0$. From (A.3) and (A.4),

$$\begin{aligned} V_C^0(\phi) - V_I^0(\phi) &= (1-2\rho)\sum_{t=1}^{\infty}\left\{\sum_{h^t}\delta^t(1-\lambda)^t \Pr(h^t|H)\varphi_0(\phi|h^t)\right. \\ &\quad \left. -\sum_{h^t}\delta^t(1-\lambda)^t \Pr(h^t|L)\varphi_0(\phi|h^t)\right\} + k, \qquad \text{(A.5)} \end{aligned}$$

where k is independent of ϕ. The set of histories $\{g, b\}^t$can be partitioned into sets of "mirror images," $\{h^t, \hat{h}^t\}$, where h^t specifies g in period $\tau \leq t$ if and only if $\hat{h}^t$ specifies b in period $\tau \leq t$. It suffices to show that

$$\beta(\phi) \equiv \varphi_0(\phi|h^t)\Pr(h^t|H) + \varphi_0(\phi|\hat{h}^t)\Pr(\hat{h}^t|H) - \varphi_0(\phi|h^t)\Pr(h_t|L) - \varphi_0(\phi|\hat{h}^t)\Pr(\hat{h}^t|L)$$

is convex and maximized at $\phi = \frac{1}{2}$, since (A.5) is a weighted sum of such terms. Now notice that

$$\begin{aligned} \Pr(h^t|H) &= \Pr(\hat{h}^t|L) \equiv x, \text{ and} \\ \Pr(\hat{h}^t|H) &= \Pr(h^t|L) \equiv y, \end{aligned}$$

which implies

$$\varphi_0(\phi|h^t) = (1-\lambda)^t \frac{x\phi}{x\phi + y(1-\phi)} + (1-(1-\lambda)^t)\gamma$$

and

$$\varphi_0(\phi|\hat{h}^t) = (1-\lambda)^t \frac{y\phi}{y\phi + x(1-\phi)} + (1-(1-\lambda)^t)\gamma,$$

where γ does not depend on ϕ. Letting $x\phi + y(1-\phi) \equiv Z_x$ and $y\phi + x(1-\phi) \equiv Z_y$, we can then calculate

$$\beta' = (1-\lambda)^t \left[\frac{x^2y}{Z_x^2} + \frac{xy^2}{Z_y^2} - \frac{xy^2}{Z_x^2} - \frac{x^2y}{Z_y^2}\right],$$

which equals zero when $\phi = \frac{1}{2}$. We can then calculate

$$\beta'' = -2xy(1-\lambda)^t \left[\frac{(x-y)^2}{Z_x^3} + \frac{(y-x)^2}{Z_y^3}\right] < 0,$$

so that $V_C^0 - V_I^0$ is strictly concave and maximized at $\phi = \frac{1}{2}$.

Moreover, since $d\left\{V_C^0(\phi) - V_I^0(\phi)\right\}/d\phi$ is strictly decreasing, it is bounded away from zero from below for $\phi \leq \frac{1}{2} - \xi$ and it is bounded away from zero from above for $\phi \geq \frac{1}{2} + \xi$.

The extension to κ small but nonzero is now an immediate implication of the sequential compactness of $\mathfrak{X}$ and $\mathfrak{Y}$ in the $\mathcal{C}^1$norm (since convergence in this norm implies uniform convergence of the first derivative). ■

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