REGIME SHIFTS, ENVIRONMENTAL SIGNALS, UNCERTAINTY, AND POLICY CHOICE

by

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Contribution to
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1. INTRODUCTION

Regime shifts, substantial reorganizations of complex systems with prolonged consequences, have been described for many natural and social systems relevant to environmental science (Steele 1998, Scheffer et al. 2001, National Research Council 2002, Carpenter 2003, Brock 2004). In environmental policy, regime shifts raise the prospect that incremental stresses may evoke large, unexpected changes in ecosystem services and human livelihoods. There is considerable interest in understanding regime shifts and developing early warning indicators of impending regime shifts. This paper addresses these issues using minimal models. We focus particularly on the problem of early warning indicators, and the related issue of using data to discipline the range of model uncertainties in practical policy making.

A major goal of the paper is to contribute to building a theory of burden of proof in contentious debates that involve environmental regime shifts. A prominent example is human-induced climate change and what, if anything, should be done about it. In this case, inference is difficult because there is only one system (the global climate system) undergoing unprecedented changes. There is no opportunity to learn from replicates and the relevance of historical experience is debatable. By contrast, we consider the case of ecosystem management in which a number of similar systems are to be managed. Water quality of lakes and reservoirs provide a well-studied example (Scheffer 1998, Carpenter et al. 1999, Ludwig et al. 2003, Carpenter 2003). In this case, information obtained from changes in similar systems may greatly reduce uncertainties in managing a particular system of interest.

A secondary goal is to relate the primary part of this paper to recent work in social science that uses threshold and tipping point models to try to understand the causes of rapid shifts in social phenomena ranging from relatively trivial things such as fads, fashions, and the like to dynamics of public opinion on critical issues such as global climate change.

Section two sketches a minimal modeling framework that will be used throughout the paper. The model represents a two-level hierarchy of dynamics where the fast dynamics is a stochastic differential equation and the slow dynamics of a bifurcation parameter is deterministic. Section three discusses basic identification problems in using data to infer what dynamical process is generating the system behavior. The section also discusses model uncertainty and offers some suggestions on how to deal with it. Section four contains a brief discussion of related issues in social science. Section five contains a brief application to lake systems and their management. Section six concludes.

2. Models of Regime Shifts

Consider the following stochastic differential equation (SDE) system adapted from Berglund and Gentz (2002), and Kleinen et al. (2003),

(2.1a) $dx/dr= f(x,a) + s\, dW/dr$, $x(0)$ given,

(2.1b) $da/dr = e$, $0 < e << 1$, $a(0)$ given, $e(0)=0$

where x is a 1-dimensional system state variable, a is an 1-dimensional slow moving parameter, dW is a Wiener process (i.e. dW is uncorrelated over time and dW is normally distributed with mean zero and variance dt), and s measures the standard deviation of the disturbances dW. Here f is called the instantaneous mean and s^2 is called the instantaneous variance, respectively. Since a=er, for precise work it is standard in the mathematical literature to follow Berglund and Gentz (2002a,b) and rescale time by putting slow time, t=er. This results in the stochastic differential equation,

(2.1'a) $dx(t)=(1/e)f(x(t),t)dt+(s/e^{(1/2)})dW(t)$.

We proceed informally here by working with (2.1a) for each fixed value of parameter a and examining what happens as we step through values of parameter a, but remind the reader that much of the mathematically precise literature on slow time/fast time systems works with (2.1'a).

This framework can be generalized to the case where x is an n-dimensional vector and a is an m-dimensional vector, but we shall work with scalar cases here. The SDE (2.1) will always be an Ito SDE here, but the Stratonovich case can also be treated. See Horsthemke and Lefever (1984) for treatment of both cases.

If there is a function F(x,a) such that

(2.2a) $dF(x,a)/dx = f(x,a)$,

then the system (2.1a) is a simple hill climbing system for s=0 and parameter vector a fixed. For example, for an evolutionary dynamical system, one can think of F(x,a) as a "fitness landscape" and "a" as a slow moving bifurcation parameter where (2.1a) is the "fast" dynamics and (2.1b) is the "slow" dynamics. In the scalar case considered here, there is always such an F(x,a) for each fixed a. One constructs F(x,a) by integrating f(z,a) over z from z=0 to z=x.

A solution {X(t,x(0),a(0)),A(t,x(0),a(0))}, given initial conditions, x(0), a(0) of (2.1) is a stochastic process. When e=0, i.e. a is fixed, we shorten the notation to {X(t,x(0);a)}. Keep a fixed for now. The steady state density, P(X=x;a) plays the same role in the stochastic case as does a steady state of the deterministic system when s=0. We write P(X=x;a) for the precise expression, P{X in [x,x+dx];a}=P(X=a;a)dx+o(dx), where o(dx)/dx→0 as dx→0. P(X=x;a) denotes the density function of X which depends upon the "shift" parameter a. It is well known that under regularity conditions on F (Bhattacharya and Majumdar (1980),Horsthemke and Lefever (1984)), we have,

(2.2b) $P\{X=x;a\}=\exp((2/s^2)\, F(x,a))/Z$, where

Z is a normalization factor so that P{X=x;a} integrates to one. There is a vector version of (2.2b) (Bhattacharya and Majumdar (1980)). See Bhattacharya and Majumdar (1980) for a discussion of existence and uniqueness of invariant measures for multivariate diffusions as well as a careful discussion of the regularity conditions needed. Their treatment does not require the existence of a "potential function" F(x,a) which is highly restrictive because, at the minimum, this requires symmetry of the cross partial derivatives of f(x,a) in x as well as the more modest requirement of connectedness of the domain of f(x,a) in x for each value of a.

Regularity conditions used by Bhattacharya and Majumdar include connectedness of the domain and thrice differentiability of the instantaneous mean function, f(x,a) and instantaneous standard deviation function (s in our case, s(x,a) in Bhattacharya and Majumdar's case). Existence and uniqueness of invariant measures do not require the restrictive sufficient conditions for existence of a "potential function" F(x,a) such that F'(x,a)=f(x,a). Hence it should be possible to generalize some of the results discussed in this paper to general f(x,a). More will be said about this below.

Indeed under modest regularity conditions F(x,a) always exists for the case where x is one dimensional. Just set F(x,a) equal to the integral of f(z,a) between x_0 and x for a fixed x_0. One can apply Bhattacharya and Majumdar to locate sufficient conditions for existence and uniqueness of an invariant measure for general f(x,a) when x is one dimensional. Berglund and Gentz (2002a,b) provide the sufficient conditions for the stochastic bifurcation theory that we discuss below.

The disturbances of dx/dt cause transient changes in x. These changes are inversely proportional to the slope of f(x,a) near the stable equilibrium point (Figure 1). In rough terms, then, the variance of x will be larger the smaller the slope of f(x,a) near the stable equilibrium point. If the system is near a stable point and slow changes in a are causing the slope of f(x,a) to decrease, then the variance of x should increase.

Others have noted that changes in variance over time may provide a clue to impending bifurcation in environmental systems (Kleinen et al. 2003; See Box 1). One approach used in economics depends on the fact that if one has a noise free time series on x over any interval (t,t+dt), then s can be estimated to any degree of accuracy within (t,t+dt) by sampling at higher and higher frequencies within (t,t+dt). This is called "continuous record asymptotics" and it plays a big role in the estimation of conditional variances in finance (Foster and Nelson 1996, Campbell, Lo, and MacKinlay 1997).

If one has structural information on how s relates to other features of the system, e.g, f(x,a), one can exploit this information using continous record asymptotics (Brock and Evans (1996)) and get better estimates of f(x,a). There are some applications in social science where f(x,a) depends upon s, e.g. one component of the vector a is s itself. Although there is a limit on how much continuous record asymptotics can improve the estimate of the conditional mean (unlike the improvements available for estimation of the conditional variance), obviously improvement can be had for cases where f explicitly depends upon s.

Environmental systems have some important complications, such as observation errors in the time series of x. Also, some environmental data are most sensibly collected at discrete time intervals (e.g. daily or annually) and this may limit the range of frequencies that can be studied. Nevertheless, the method of continuous record asymptotics offers an interesting approach for empirical study of environmental systems governed by equations like 2.1.

Here are some examples of systems that can be put into the framework (2.1) besides the ones mentioned by Berglund and Gentz (2002a,b), and Kleinen et al. (2003).

EXAMPLE I

Consider the social system

(2.3) dx/dt = tanh(h+Jx) – x + s dW/dt, a := (h,J),

studied, for the deterministic case s=0, by Brock and Durlauf (2001a,b), Scheffer et al. (2000,2003), and Brock (2004). In this case x(t) is the difference, at date t, between the fraction of a population that chooses a choice -1 (e.g. do not regulate emissions of greenhouse pollutants) and the fraction that chooses a choice +1 (e.g. regulate emissions of greenhouse pollutants), where h stands for the utility difference between the two choices, J is a measure of the strength of social interactions or peer effects, the hyperbolic tangent function, tanh, emerges from a discrete choice model, and dW/dt represents outside shocks to the system (e.g. climate events that influence the public's attitudes toward climate change.). System (2.3) will be discussed in Section 4 below.

Brock (2004) reviews work on "punctuated policy changes" in economics, political science, and other areas of social sciences using a system like (2.3) as a central organizing expository vehicle. We will go beyond Brock (2004) in this article by beginning a study of the impact of noise induced transitions that adapts work of Berglund and Gentz (2002a,b), Kleinen et al. (2003), and, especially, Horsthemke and Lefever (1984).

EXAMPLE II

For another example consider the dynamical system of the state of a lake studied by Scheffer (1997), Carpenter et al. (1999), Ludwig, et al. (2003), and Carpenter (2003). A minimal model of this system is

(2.4) $dx/dt = c + f(x,a) + s\, dW/dt$,

where x is phosphorus sequestered in algae, c is loading from outside activities such as agriculture, developments, etc., a is a slow moving parameter (e.g. sedimented phosphorus) and dW/dt is outside shocks to the system. Here the payoff is U(x,c) which decreases in x and increases in c. The control design problem is to design a control c=C(x) to optimize the payoff U. Carpenter et al. (1999) and Ludwig et al. (2003) study the problem of designing a control sequence to optimize the discounted sum of payoffs.

It is clear that a large class of examples can be subsumed under the framework of (2.1). Suppose we have measurements of x and, maybe, but not always, a, which may be corrupted with measurement noise. We are concerned here with the following questions:

(i) If we have a time series of observations on our measurements, what patterns in such a time series would give us an early warning signal of an impending bifurcation?.

(ii) If there is a payoff U(x,a) of the state of the system, how should we design a control C(x,a) to optimize the payoff?

(iii) If the true system is {F*(x*,a*),s*;U*(x*,a*)} but we specify it as {F(x,a),s;U(x,a)} how should we design the control C(x,a) to be robust against such misspecification? How can we use our observational time series data on our measurements of x and a to discipline the set of specifications we must consider in order to make our control more robust?

(iv) Suppose instead of just one system under observation we have a collection of systems {F(x(i),a(i),i),s(i), U(x(i),a(i),i), i=1,2,...,I} under observation. How may we use information on a subset of such systems that have passed through a regime change to forecast impending regime change on other systems that have not passed through a regime change?

(v) How might one use data to estimate features of the system (2.1)?

We do not pretend to answer all these questions in this paper. Instead we consider two polar cases. The first is the global climate system, where there is only one system under scrutiny and our main problem is to glean from patterns in the time series of observations, evidence of impending regime change (Box 1). This is extremely difficult. The second leading case is one in where there exists a collection of systems 2.1 with observations on each, such as the problem of managing water quality for a collection of lakes. This second problem is more tractable because of the multiplicity of systems available for study.

3. A General Identification Problem, and Policy Action

We extend the mathematical models sketched above to put forth a simple framework that we hope sheds light on how one might usefully deal with heated debates in policy making. There are two basic issues raised by this problem. First there is a basic empirical identification problem in using time series data or any data to adduce evidence for or against alternative stable states. Second, there is model uncertainty. We discuss the identification problem first.

An important problem of model identification is the separation of spurious evidence, caused by dynamics of unobserved state variables, from true evidence of alternative stable states. This problem looms especially large in empirical work that attempts to use data to separate endogenous dynamics from exogenous dynamics. This turns out to be very difficult in social science (e.g. Brock and Durlauf (2001)). There is no reason to suspect that this problem would be any easier in other sciences.

Here we will suggest some approaches for addressing this problem. To focus discussion, we consider a highly simplified model of the identification problem (Box 2).

First we can attempt to use continuous record asymptotics to build estimators of $s_1(x_1,x_2)$ based upon data on x_1 even though we can not observe x_2, if x_2 moves slowly enough relative to x_1. The intuition is that if x_2 is constant over an interval (t,t+dt) then the continuous record estimator in (3.1) below, call it $S_1^2(dt,n)$, converges to the true value of $s_1(x_1,x_2)$ on (t,t+dt). This idea can be made more precise by using the work of Foster and Nelson (1996). Construct n equally spaced subintervals from j=1 to j=n of the interval (t,t+dt) and construct the estimator

(3.1) $S_1^2(dt,n) := (1/dt)\ \Sigma_j\{[x_1(t-(j-1)dt/n) - x_1(t-j\ dt/n)]^2\}$.

Foster and Nelson (1996) show that, under modest regularity conditions, the increments in (3.1) are approximately Independently and Identically Distributed Normal with mean zero and variance $s_1(x_1,x_2)^2$. Hence, if we are able to sample within (t,t+dt) at any frequency we like, we can essentially assume that s_1^2 is observable as well as x_1.

One could also consider empirical methods that have been used in areas like finance to estimate diffusion processes of the form,

(3.2) $dx = f(x,a)dt + s(x,a)\ dW$,

Estimators of systems like (3.2) can be constructed using time series data on x over a finite interval t in [0,T] (e.g. Hansen and Scheinkman 1995, Conley et al. 1997 and their references). We explain the basic idea here. Fix a over

[0,T]. Assume conditions on f and s so that there is a unique stationary density P{X=x;a}. See Bhattacharya and Majumdar (1980), Horsthemke and Lefevre (1984), Hansen and Scheinkman (1995), Conley et al. (1997) for precise conditions.

Now let H(x) be any smooth function of x. On the stationary distribution EH is constant through time and, hence, its time derivative is zero. Therefore we compute by expanding H(X(t+dt))=H(x+X(t+dt)-x) in a Taylor series around x,

(3.3) $0 = E[E\{H(X(t+dt))|X(t)=x\} - H(x)\}/dt] \rightarrow E[H'f+(1/2)H''s^2]$, $dt \rightarrow 0$.

Since the expectation E can be replaced with a sample average, and the function H was arbitrary, we can construct an infinite number of moment conditions using (3.3). Conley et al. (1997) show how to choose a collection of H's to do the best job of estimating f under regularity conditions by adapting the Generalized Method of Moments (GMM) following the theory of Hansen and Scheinkman (1995). This method might be adapted to give us an early warning signal of impending bifurcation in f(x,a) by constructing good estimators of f over moving intervals [T,T+N].

The GMM can also be adapted to address the problem of observation error. Consider the discrete-time model of Carpenter (2003), which uses Bayesian methods to estimate both linear and nonlinear models to detect possible regime shifts in the presence of measurement error. We discuss here how GMM might be used. Consider the system,

(3.4) x(t+1) = f(x(t),b) + e(t+1), y(t) = x(t) + m(t),

where {e},{m} are mutually independent, identically distributed mean zero finite variance processes. Here {y(t)} is observed but {x(t)} is not and the task is to estimate the parameter vector b. Consider a test function H(y) and examine the quantity

(3.5) H(y(t)) [y(t+1) - m(t+1) - f(y(t) - m(t),b)] = H(y(t)) e(t+1).

Under our assumptions of mutual independence and independence over time of {e}, and {m}, we have

(3.6) $0 = E\{H(y(t))\ e(t+1)\} = E\{H(y(t)\ [y(t+1) - \int(f(y(t)-m,b)\ g_m(m)\ dm]\}$.

Hence, if we assume a known density g_m for the measurement error, m, e.g. Normal with mean zero and finite variance, we have a GMM system. Hence we can apply standard GMM, under regularity conditions following Hansen (1982) and choose a set of "test functions" {H} that give us good estimates of b. It would be interesting to compare the performance of GMM methods with the methods used by Carpenter (2003).

The problem of model uncertainty is fundamental to science-based disagreements about environmental policy (Brock, Durlauf and West 2003, Carpenter 2003). Consider climate change as an example (Box 2). Suppose there is only one stable state in the observed variable x1, which is the observed state of the climate. Further, suppose the x2 variable (an unobserved dynamic variable which impacts the climate) operates on a slow scale of time in such a way that the x1 dynamics generates time series data that resembles a system with alternative stable states. Suppose there are two plausible types of social actions on management of potential climate change. Type one is cautious, with low b and low emission of greenhouse gas, type two is the opposite. Hence, the policy maker faces four possible outcomes, depending upon her choice of actions and also the resolution of the state of model uncertainty represented by the x2 variable. The most benign outcome is type one policy combined with the "good" value of x2, and the worst outcome is the type two policy combined with the "bad" value of x2. The other two pairs are intermediate.

Similar model uncertainty problems were addressed by Brock, Durlauf, and West (2003) and Carpenter (2003). The former paper addresses model uncertainties related to monetary and economic growth policy, and the latter paper addresses model uncertainties in fishery management. Both papers employ Bayesian Model Averaging (BMA) methods. Brock, Durlauf and West(2003) use methods from the robustness literature and literature on ambiguity aversion to frame a data-based approach to policy action. They conduct two empirical exercises, one in monetary policy, the other in economic growth policy, using linear models. Carpenter (2003) applies BMA to both linear and nonlinear models. The regime shift problem could be addressed using a nonlinear version of Brock, Durlauf and West(2003) using models similar to those of Carpenter (2003) and supplemented with methods from robust analysis and methods to deal with ambiguity aversion. Our discussion of the model of Box 2 suggests the use of an action dispersion plot with two choices of action and two choices (by nature) of the state of model uncertainty. Brock, Durlauf and West (2004) argue for the presentation of empirical action dispersion plots in advisement of policymakers by scientists, and present such plots for monetary policy. Such plots are useful in environmental policy disputes for exactly the same reasons.

We believe our review of new literature on estimation techniques and dealing with model uncertainty should be useful for both scientists who must report to policymakers and to policymakers who must make demands on scientists to present their results in an understandable manner to the policymakers together with an honest reporting of the true level of uncertainty. However, there is tremendous need for empirical exploration of concrete, data-driven examples in environmental science. We turn now to a brief discussion of related methods in social science.

5. Social Systems and Coupled Natural and Social Systems.

Wood and Doan (2003) build a regime change theory to argue that "whenever there is a preexisting condition that many find privately costly, but with widespread public acceptance, the system is ripe for change." They conduct an empirical exercise on public attitudes to sexual harassment and found evidence consistent with a regime shift around the time of the Clarence Thomas hearings. Brock (2004) provides a broad review of similar types of regime change in social science, many of which could be addressed by the methods described above.

A main interest in social interactions studies (modeled by 2.3 above) is the dynamics of polarization due to non negative social interactions effects $J>0$. Since $f(x,a) := \tanh(h+Jx)-x$ is scalar, it is trivial to find $F(x,a)$ such that $dF(x,a)/dx=f(x,a)$. It is easy to show that system (2.3) has only one stable state for $J<1$ and that if J slowly increases, a pitchfork bifurcation appears if J becomes positive and large enough (compare panels A and B of Figure 3). We first discuss the deterministic case $s=0$, then we discuss what happens when s becomes positive.

First consider the deterministic case $s=0$ with $h=0$, and note that the only stable state is $x=0$ for small J (Figure 3A). As J increases beyond unity, two new stable states appear close to x=zero and depart further away from x=0 (which becomes unstable) as J continues to increase (Figure 3B). Another bifurcation, called a saddle node bifurcation can be produced at, for example, the negative stable state for J>1. Do this by slowly increasing h from its initial value of zero. As h is slowly increased a critical value of h will be reached where the negative stable state and the steady state at zero come together and both vanish as h continuues to increase leaving only the positive stable state (Figure 3D).

A good example is to think of the state x=0 as representing an evenly divided electorate where 1/2 are for an incumbent candidate and 1/2 are against. The quantity J measures the cost of deviating from the average view of the whole population. Think of it as a measure of conformity pressure. As J increases beyond unity, two new stable states appear, one "anti-incumbent" the other "pro-incumbent" while the zero stable state loses its stability (panel A to panel B, Figure 3). As J becomes very large the two new stable states are pushed close to -1 and +1 (Figure 3B).

Now suppose that h has been negative in the past, i.e. the incumbent has been preferred to alternatives (Figure 3C). Also suppose that social pressure to conform to the majority opinion is very high so that J is very large. So the system is at the negative stable state and it is close to −1. But as news appears that is unfavorable to the incumbent, h slowly rises. In the deterministic case, s=0, the system remains stuck at the pro-incumbent position, but becomes unstuck when h becomes large enough that the negative stable state vanishes (panel C to panel D of Figure 3). At this point the system moves rapidly towards the positive stable state. This is an example of what is sometimes called a "macro-punctuated change" in political science.

Now let us see what happens when s is positive. Let a := (h,J) and let F(x,a) denote the integral of tanh(h+Jz)-z up to x. Then the steady state density of x is given by

(4.1) $\Pr\{X=x;a\} = \exp[(2/s^2)\ F(x,a)] / Z,$

where Z is a normalization factor so (4.1) integrates to one. Obviously as s converges to zero, (4.1) spikes at the global maximum of F and the global maximum of F shifts abruptly for the case J>1 from a large in absolute value negative x to a positive x as h passes through zero. In other words, slight tilt of h against the incumbent, i.e. h>0 can cause a highly polarized electorate (a high value of J) to cluster at a new position, i.e. a large and positive (almost symmetrically opposite) value of x.

The reader may ask what happened to the hysteretic stickiness that was present in the case s=0. It is easiest for many to think in terms of a cup and ball metaphor. Think of -F as a potential by writing dx=fdt+sdW as dx=-(-dF/dx)dt+sdW. Then we now have a "rattling" ball in a cup. If s is small a ball lying in a negative well of a two-welled potential will have a hard time getting out, although it will get out with probability one. Once the ball drops into a deeper well it is relatively harder to get out, but it will with probability one escape the deeper well and fall back into the more shallow one. As s becomes smaller the relative probabilities shift more towards staying in the deeper well and in the limit as s goes to zero, the ball will end up in the deeper well. This idea is closely related to the method of simulated annealing in numerical optimization (Kirkpatrick, Gelatt, and Vecchi 1983).

Berglund and Gentz (2002a) point out that, in the "small noise limit", the expected time between transitions can be approximated by Kramer's formula which is proportional to exp[(2/s^2)G] where G is the barrier height between the two wells. Hence when G is high and s is low we expect it to take a long time to move from a pro-incumbent position, i.e. the negative x stable state to an anti-incumbent position, i.e. the positive x stable state. An increase in J increases G. This is a stochastic analog of hysteresis in the deterministic case. Berglund and Gentz (2002a) and Norberg et al. (2001) study stochastic systems with periodic forcing. These studies give us some insight in what to expect when the system is forced by a slow moving, but non periodic and possibly random force. Turn now to an ensemble type of social interactions model where the interaction between the increasing size of the system and the outside shocking process produces new non-hysteretic behavior as the size of the system approaches infinity.

There is an interesting model that builds on Dawson (1983) that helps us understand why tipping points occur in Brock (2004) as well as in Wood and Doan (2003). It does not appear in those papers. We exposit it here.

Consider the stochastic differential equation,

(4.2) $dx = (-x^3 + a\ x)dt + s\ dW.$

As s goes to zero the mass of the stationary density, P{X=x;a} clumps onto a Dirac delta distribution spike at

(4.3) $x^*(a)=\text{argmax}\{-(1/4)x^4 + a\ x^2/2\}.$

For a=1, x*(1) is -1 and +1 and as a becomes greater than one, the global maximum is the larger local maximum. This sets the stage for the coupled system, where dWdW'=Idt,

(4.4) $dx_j=[-x_j + a\ x_j]dt + s\ dW_j - J(x_j - \text{xbar}),\ \text{xbar} := (1/N)\Sigma_i\{x_i\},$

where the Sum runs from i=1,2,...,N. Note that dW is an Nx1 vector here. Put $x=(x_1,...,x_N)$. Following Bhattacharya and Majumdar (1980) we find the invariant measure P(X=x)=exp(bU(x))/Z, $b:=2/s^2$, by constructing U(x) such that,

(4.5) $dx = dU(x)/dx + s\ dW$

for an appropriate "potential function" U(x). It is easy to check that the cross partial symmetry conditions needed for existence of U are satisfied. One may check that U given by

(4.6) $U(x) = \Sigma_j[u(x_j)-Jx_j^2/2]+(J/2)[\text{xbar}]^2,\ u(x\_j) := -(1/4)x_j^4 + a\ x_j^2/2,$

is appropriate.

Dawson (1983) studies the limiting behavior of a system very close to this one. It is basically the same as systems of coupled oscillators in statistical physics. The thrust of this type of work is to locate sufficient conditions for the average $\Sigma\{x_j\}/N$ to converge in an appropriate probabilistic sense to the global maximum of some deterministic function, call it S. Brock and Durlauf (2001a,b) show that there is a close relationship between this literature and that on discrete choice modeling in econometrics. The deterministic function, S, in the discrete choice case turns out to be a measure of limiting expectation of maximal social welfare per capita and the average, $\Sigma\{x_j\}/N$ converges to the global maximum of that function. It turns out that small changes in the individual payoffs coupled with large enough values of the conformity index J cause large movements in the limiting value of the average behavior as N becomes large. Brock (2004) attempts to explain this type of behavior in intuitive language and apply it to predicting apparent phase transitions in social systems. The main finding is this. The hysteretic sticky movement across different stable states as payoffs slowly change gets replaced by a non hysteretic jump to the global maximum of a function S, wherever that global maximum may move. In other words, the social system becomes more smoothly adaptive to changing conditions as s increases. Turn now to a brief discussion of dependence of our methods on the assumption that a "potential" function F(x,a) exists such that F'(x,a)=f(x,a) for each a.

If one sets the shocks to zero, there is a classification theory for bifurcations for the differential equation,dx/dt=f(x,a) (Kuznetsov (1995)). For example for the case where parameter a is one-dimensional, there are two "primary" bifurcation types, one is when the "largest" eigenvalue of the Jacobian matrix, f'(x*(a),a) is real and passes from negative to positive as parameter a increases and the other is when the "largest" eigenvalue is complex and the pair consisting of this eigenvalue and its complex conjugate share a real part that passes from negative to positive. Here x*(a) denotes a solution of the steady state deterministic equation, 0=f(x*(a),a) for each value of parameter a.

There is also a classification theory for the next "level" of bifurcation which concerns the induced Poincare' map when a limit cycle appears. There is also a theory of "global" bifurcations, for example the homoclinic bifurcation. All this is covered by Kuznetsov (1995) and none of it is dependent upon existence of a "potential" F(x,a) such that F'(x,a)=f(x,a). The corresponding theory of closed form analytic expressions for invariant measures is not as well developed in the case of general f(x,a) as it is for the "integrable" case where F'(x,a)=f(x,a) for some "potential" function F(x,a)that maps x-space to the real line for each value of a. The symmetry of cross partials of f(x,a) plays a big role in producing analytic closed form expressions for invariant measures.

However, one can still obtain computational results for stochastic bifurcations quite easily. This is a centerpiece of the research program of Cars Hommes and his CeNDEF group in Amsterdam. See Hommes's review (2005) for example. In Hommes's research strategy one uses bifurcation classification theory to classify the primary (and in some cases the more refined secondary bifurcations) for the analytical part of the research strategy. Then one adds the noise, sdW, for small s and turns to the computer. Using the computer one can produce the analogs of bifurcation diagrams for the small noise stochastic case and study what happens. We believe that such a study might give useful insight into observable signals of impending bifurcations as one moves parameter a slowly towards and through a bifurcation point.

If the noise is very small one can produce a local linear approximation to the solution of (2.1a) for general vector cases and use "small noise asymptotics" (e.g. Magill (1977)) to produce closed form analytical expressions for objects such as the spectral density matrix of the local linear approximation. This possibility suggests that it might be useful to step parameter a through a bifurcation point and study what happens to the spectral density matrix of the local linear approximation as one steps parameter a through a bifurcation point. This would be a generalization of the Kleinen et al. (2003) approach for vector cases where, additional types of bifurcations, for example, the Hopf bifurcation, can occur. Local linearization "small noise" approximation theory does not depend upon the existence of a potential. Neither does the use of GMM-based methods to estimate f(x,a) from continuous record data. It is beyond the scope of this paper to pursue these potentially promising research avenues and generalizations further.

Section 5. Collections of Many Similar Systems

The ecosystem regime changes of Scheffer (1997), Carpenter et al. (1999) and Carpenter (2003) are situations where a large number of similar systems can be studied, some of which have passed across thresholds. Although the systems are not identical replicates, they are similar enough that information from one set of systems is transferable to another set of systems. Analogous situations occur in panel data studied by social scientists.

By studying many shallow lakes, for example, it has become clear that lakes with a total phosphorus level of more than 0.1 mg l^{-1} are at risk of collapsing to a turbid state (Jeppesen et al. 1990, Scheffer 1998). Importantly, a single threshold level will never apply to all systems. In terms of bifurcation theory, the bifurcation point will always be dependent on various parameters. In the case of lakes, for instance, differences in size will affect their sensitivity to collapse at increasing phosphorus levels. A recent study of 240 shallow floodplain lakes which are similar in nutrient level due to annual inundation by the river Rhine showed that the probability of being in a clear state is considerably higher for smaller lakes (Van Geest et al. 2003). Thus, if one has the possibility to study many similar systems, empirical indicators for the risk of regime shifts may be obtained.

Carpenter (2003) shows how measurements from many lakes can be used to construct informative priors for Bayesian estimation of systems like 3.4. Such informative priors reduce the uncertainty of threshold estimates. In addition, the informative priors increase the posterior weight of the correct model in simulation studies that use Bayesian model averaging to compute policies under model uncertainty.

The tendency of variance to increase near thresholds has not been widely exploited by ecosystem scientists. The intuitive reason for the increase in variance can be seen by noticing that as parameter a moves from Figure 3A to Figure 3B very slowly, parameter a "flattens" the slope of f(x,a) towards unity. This action magnifies the impact of an outside shock, sdW. This suggests that if one could follow a cross section of lakes, some nearer to bifurcation than others that it might be possible to use such data to forecast impending bifurcations.

For example, it would be interesting to explore indicator variates that can be measured at high frequency with low observation error. One possibility is the volume of anoxic water in a lake subject to eutrophication. In deep thermally-stratified lakes, the volume of anoxic water is directly related to the proximity of a threshold for eutrophication (Carpenter et al. 1999, Carpenter 2003) and formation of marine hypoxic zones (Stow et al. 2005) Technology exists for accurate and rapid monitoring of oxygen in lakes. It may be possible to devise monitoring schemes using continuous record asymptotics, as described above, to create leading indicators of breakdowns in water quality for lakes and reservoirs (Stow et al. 2005).

In using indicators, one must be cautious about the identification problems described in Box 2 and related material from Section 3. For example, variance of fish populations and their prey might be expected to increase near a depensation point, a type of threshold that occurs in population dynamics (Carpenter 2003). However, there is also evidence that MSY-type management schemes cause increases in the variance of fish populations and entire pelagic food webs through trophic cascades (Carpenter and Kitchell 1993). The ecological conditions near a MSY target can be quite different from those near a depensation point. Thus increases in the variance of fish populations may have ambiguous implications for ecosystem dynamics.

On the other hand, there may be specific subtle changes in a particular system that signal deterioration of the resilience of its current state. Such changes are not generic (such as colouring of noise) but rather unique to the type of system under study. For instance, in shallow lakes, a shift to a turbid state is typically preceded by an increase of the periphyton-layer covering the macrophytes and a reduction in the proportion of piscivorous fish (Meijer et al. 1994, Scheffer 1998). Knowing such clues of proximate regime shifts requires a mechanistic insight into the functioning of the system, which is more likely if one has the possibility to study many systems.

The multiplicity of lakes on the landscape offers the possibility of mosaic management (Carpenter and Brock 2004). In mosaic management, different ecosystems are managed for different objectives, including deliberate experimentation to reduce uncertainty. Mosaic management is also an approach for addressing strongly divergent social goals for ecosystems (for example, lakes for water supply or recreation versus lakes for dilution of pollutants). On the other hand, mosaic management leads to the possibility of complex spatial dynamics of ecosystem users, which may create new types of thresholds on complex landscapes (Carpenter and Brock 2004).

6. CONCLUSIONS

Can early warnings of regime shifts can come soon enough for people to act to avert an unwanted regime shift? In rare cases, in which numerous similar systems have been intensively studied, such as in the case of shallow lakes, empirical rough rules of the thumb of where the threshold lies may sometimes be found. However, this situation seems the exception rather than the rule. Simulation studies using phosphorus recycling as an indicator of impending lake eutrophication suggest that decision makers will receive 1 to 3 years advance warning of breakdowns in lake water quality (Carpenter 2003). This lead time is not sufficient for effective action by any existing lake management system. Kleinen et al. (2003) suggest that the time scales for detecting climate regime shifts are about the same as the time scales for effective action to prevent the regime shifts (100 to 1000 years). Thus it seems unlikely that an impending regime shift could be detected in time to evoke effective action, unless the management system allowed for rapid and massive response. In practice there are many difficulties with implementing early warning and rapid response systems (Sarewitz et al. 2000). More conclusive evidence for the possibility and social costs of crossing thresholds would contribute to the general problem of designing adaptive strategies for environmental regime shifts.

This chapter has centered on an identification problem and a policy design problem. The identification problem is to determine whether threshold dynamics can occur, and whether a threshold is near. The identification problem is complicated by hidden variables with return times near those of key state variables, observation error, and the multiplicity of plausible models for many social-ecological systems. We suggest some approaches for addressing these challenges. Our review of the literature shows how dynamics of the variance of state variables near a threshold could provide some advance warning of impending regime shifts.

The policy design problem is to identify patterns of evidence that should prompt us to choose actions to avert unwanted and impending regime shifts. Growing evidence for environmental thresholds and regime shifts suggests that this policy design problem will become more prominent in coming decades.

This policy design problem is complicated by the presence of more than one model that gains substantial support from data and basic scientific understanding. Brock, Durlauf, and West (2003) discuss this problem in detail in the contexts of monetary policy and growth policy. Their discussion is pertinent to the regime shifts discussed here.

Bayesian model averaging is one way to report the fundamental uncertainty about which model is appropriate. A closely related approach is to report the entire posterior distribution of payoffs to the policy maker over the set of models, weighted by the posterior probability of each of these models when fitted to the data available. The policymaker, when presented with this posterior distribution, can then choose an action by whatever preferences she has. Evidence that dates back to Ellsberg suggests that decision makers include some aspects of avoidance of worst case scenarios in their preferences. In other words, they do not act like a Bayesian using Bayesian model averaging of the payoffs when they choose their optimal action. To put it another way, they might act like they put some weight on the worst case scenario and some weight on the Bayesian model average. This observation suggests the possibility of developing a theory of how much of the burden of proof each side should bear in scientific disputes.

In our context, it is especially important to discount past observations as we collect more and more current information as the system moves forward through time. Contrasting views may converge as more evidence comes in. Discounting past observations is important to avoid Bayesian posteriors being frozen by history when the system may be approaching a bifurcation point. Discounting is important because the distant past contains less information about an impending bifurcation than the recent past.

The use of techniques like continuous record asymptotics to estimate conditional variance has the potential to sharpen development of early warning indicators. The approach may also sharpen the development of theories of the burden of proof in scientific disputes in situations where impending regime change is possible and has significant implications for human welfare.

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Box 1. Global Climate: Changing Variance as a Clue to Bifurcation?

There is considerable interest in the possibility that gradual increases in the emission of greenhouse gases could lead to large changes in the climate system (National Research Council 2002). This general problem was considered in an abstract way by Kleinen, Held, and Petschel-Held (2003). Consider their model

(B1.1) $dx = (x^2 - x + a)dt + s\ dW := f(x,a)dt + sdW,$

$$F(x,a) = (1/3)x^3 - (1/2)\ x^2 + ax.$$

It is easy to see that steady states for the deterministic system are given by

(B1.2) $x_1 = ½ - [(¼) - a]^{½},\ x_2 = ½ + [(¼) - a]^{½},$

where, for $a< a_c := ¼$, x_1 is locally stable, x_2 is locally unstable, and a saddle node bifurcation takes place at a = ¼. The goal of environmental monitoring is to provide a warning when the system approaches the bifurcation.

Kleinen et al. (2003) propose to estimate the distance da from bifurcation by looking at the spectrum of x and study how it shifts as da slowly changes. If we linearize the Kleinen et al. (2003) system at the locally asymptotically stable steady state x_1, we find that for

(B1.3) $B = 0,\ A = 2x_1 - 1= - [(¼) - a]^{½},$

we have

(B1.4) $Ez^* = 0,\ V(z^*) = s^2 / [(¼) - a]^{½},$

for a < ¼.

Kleinen et al. (2003) suggest estimating the spectrum generated by (B.1) over segments of time [T,T+N] where N is large enough so that the steady state distribution is a good approximation, and monitoring the change in shape of the spectrum. A more direct method is to use continuous record asymptotics. As Kleinen et al. point out, (¼) – a is the distance from bifurcation for a < ¼ and we wish to use time series records to estimate this quantity. One way to do it is to use a rolling window estimate of variance that uses data from t to t+N. Omitting data before t prevents the estimator from being overwhelmed by history and thereby becoming insensitive to more recent data which contains information on more recent values of a. Once we have an estimator of $V = = s^2 / [(¼) - a]^{½}$, we can estimate the trend in the distance from bifurcation (¼) – a because s is assumed constant.

Box 2. Minimal Model of the Identification Problem for Regime Shift

A highly simplified heuristic model of the identification problem for regime shift of a pollutant-driven regime shift such as climate change is presented here.

(B2.1) $dx_1/dt = b\ c + f_1(x_1,a_1;x_2) + s_1\ dW_1,\ s_1 = s_1(x_1,x_2)$

(B2.2) $da_1/dt = e_1\ A_1(x_1,x_2)$

(B2.3) $dx_2/dt = e_2\ f_2(x_1,x_2,a_2) + s_2\ dW_2,\ s_2=s_2(x_1,x_2,e_2)$

(B2.4) $da_2/dt = e_3\ A_2(x_1,x_2)$

(B2.5) $U(c,x_1) = c-(½)Px_1^2$,

where x_1 is the state of the climate with higher x_1 corresponding to worse climate and U is the payoff to humans of human consumptive activities and the state of the climate x_1. The slope b represents pollutant emissions per unit of consumptive activity. The scalar x_2 and the dynamics (B2.3) represent other factors that shift the climate system and make it difficult for an analyst to sort out whether the climate is shifting because $b>0$ or whether it is shifting because of these other factors.

There are four scales of time with (B2.1) having the reference scale of one. (B2.2) has scale e_1, (B2.3) the scale e_2, and (B2.4) the scale e_3. In the Kleinen et al. (2003) example the "confounding" variable x_2 was not present and e_1 was a slow scale, i.e. $0 < e_1 << 1$. This is the situation analyzed by Berglund and Gentz (2002). The most difficult identification case is where $e_3 = e_1 := e,\ e_2=1$.

The dynamics of x_2 impacting the dynamics of x_1 make it difficult to adduce evidence for or against the presence of alternative stable states in the dynamics of x_1. In order to bring out these empirical issues clearly we assume that x_1 is observable but x_2 is not. We allow the standard deviation functions to depend upon x_1 and x_2.

We assume for each fixed value of a that there is a landscape function $H(x_1,x_2,a)$, sometimes called a "potential function", such that

(B2.6) $f_1 = \partial H/\partial x_1,\ f_2 = \partial H/\partial x_2$,

Of course there is no reason why the symmetry conditions for cross partials needed for existence of a landscape function should hold. But we focus on this case so we can use the useful expository devices of a landscape diagram (for the metaphor of climbing up) and cup and ball diagram (for the metaphor of seeking the lowest point in a valley). Under the assumption of existence of a landscape, if one is more comfortable with cup and ball diagrams, the deterministic system with c=0 moves to local minima of the function $V(x_1,x_2,a) := -H(x_1,x_2,a)$.

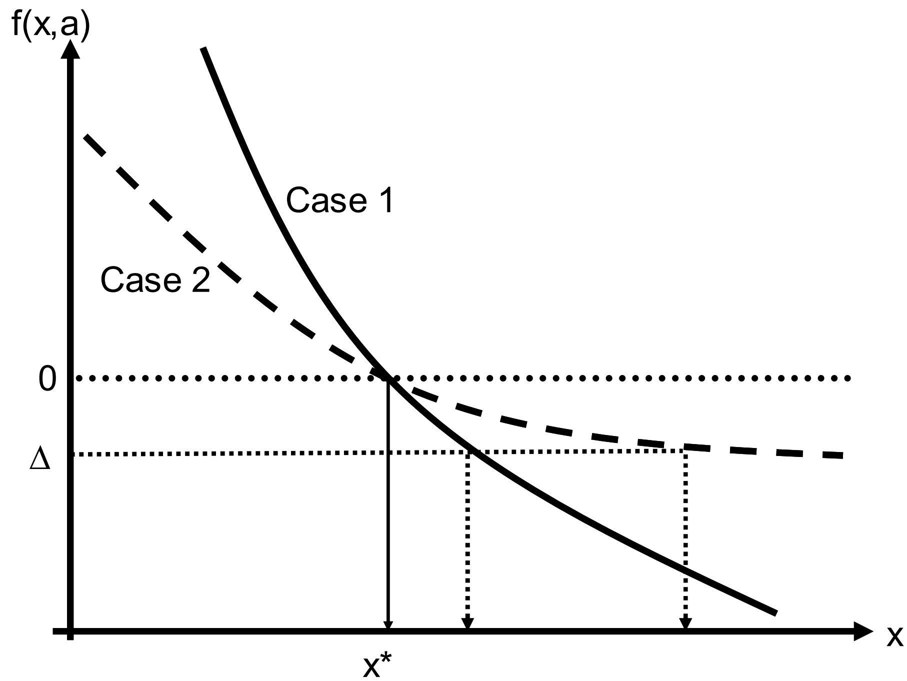

Figure 1. Plot of f(x,a) versus x, illustrating the effect of the slope of f(x,a) near steady state (x*) on the value of x after a small perturbation Δ. In Case 1, the slope of f(x,a) near x* is relatively large, and the change in x immediately following the perturbation is relatively small. In Case 2, the slope of f(x,a) near x* is smaller, and the change in x following the perturbation is relatively large.

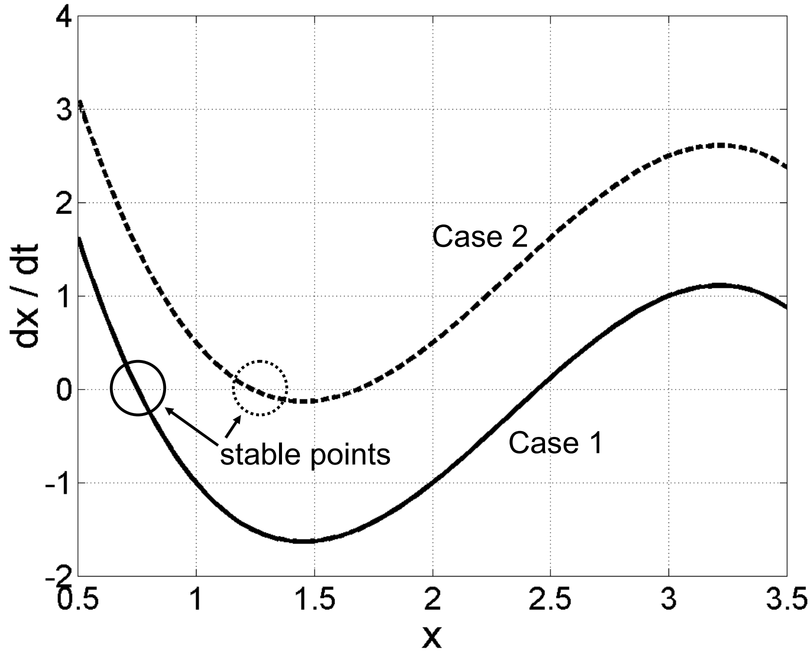

Figure 2. Plot of dx/dt versus x for a system with multiple stable points, illustrating how the slope of f(x,a) near a stable point changes as the system approaches a bifurcation. Case 1: system relatively far from bifurcation. Case 2: system closer to bifurcation.

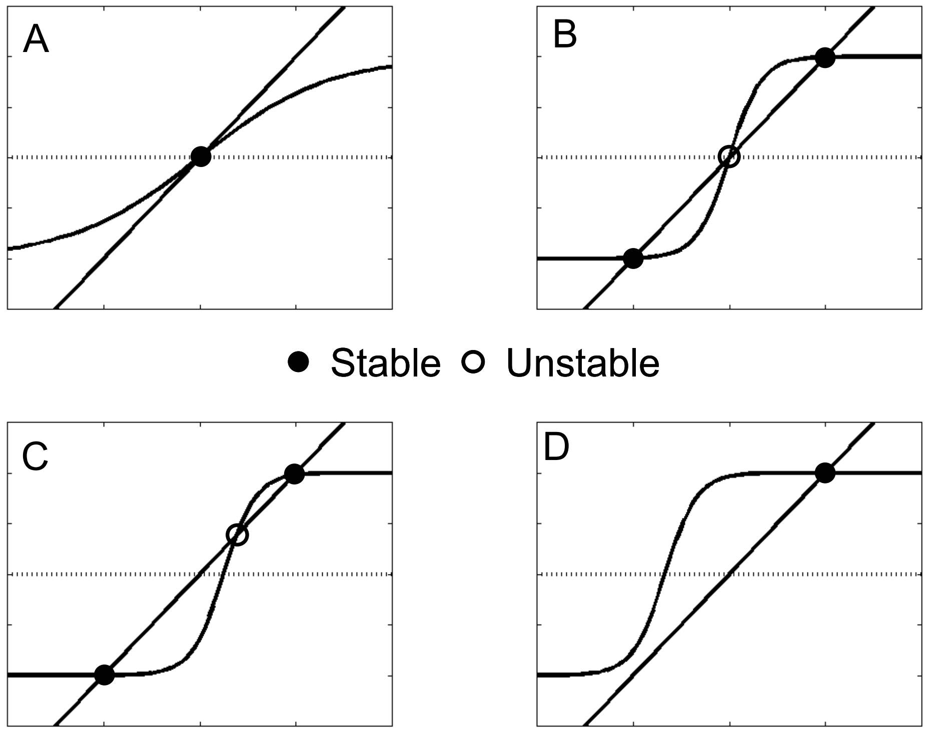

Figure 3. Illustrations of Example 1. (A) Split population (h = 1) with low social interactions (low J). There is a single equilibrium. (B) Split population with high social interactions (high J). There are three equilibria, two of which are stable. (C) Biased population (h < 0) with high social interactions. The unstable threshold has moved to the right. (D) News spreads among a population with high social interaction, causing h to rise above zero. Two of the equilibria vanish, leaving only one stable equilibrium.