

Robustifying Learnability

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Abstract

In recent years, the learnability of rational expectations equilibria (REE) and determinacy of economic structures have rightfully joined the usual performance criteria among the sought-after goals of policy design. Some contributions to the literature, including Bullard and Mitra (2001) and Evans and Honkapohja (2002), have made significant headway in establishing certain features of monetary policy rules that facilitate learning. However a treatment of policy design for learnability in worlds where agents have potentially misspecified their learning models has yet to surface. This paper provides such a treatment. We begin with the notion that because the profession has yet to settle on a consensus model of the economy, it is unreasonable to expect private agents to have collective rational expectations. We go further in assuming that agents have only an approximate understanding of the workings of the economy and that their task of learning true reduced forms of the economy is subject to potentially destabilizing errors. The issue is then whether a central bank can design policy to account for errors in learning and still assure the learnability of the model. Our test case is the standard New Keynesian business cycle model. For different parameterizations of a given policy rule, we use structured singular value analysis (from robust control theory) to find the largest ranges of misspecifications that can be tolerated in a learning model without compromising convergence to an REE.

In addition, we study the cost, in terms of performance in the steady state of a central bank that acts to robustify learnability on the transition path to REE.

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PRELIMINARY VERSION

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1 Introduction

It is now widely accepted that policy rules—and in particular, monetary policy rules—should not be chosen solely on the basis of their performance in a given model of the economy. There is simply too much uncertainty about the true structure of the economy to warrant taking the risk of so narrow a criterion for selection. Rather, policy should be designed to operate "well" in a wide range of models. There has been substantial progress in a relatively short period of time in the literature on robustifying policy. The first strand of the literature examines the performance of rules given the presence of measurement errors in either model parameters or unobserved state variables.¹ The second strand focuses on comparing rules in rival models to see if their performance spanned reasonable sets of alternative worlds.² The third considers robustifying policy against unknown alternative worlds, usually by invoking robust control methods.³

At roughly the same time, another literature was developing on the learnability (or E-stability) of models.⁴ The learnability literature takes a step back from rational expectations and asks whether the choices of uninformed private agents could be expected to lead to converge on a rational expectations equilibrium (REE) as the outcome of a process of learning. Important papers in this literature include Bray [5], Bray and Savin [6] and Marcet and Sargent [34]. Evans and Honkapohja summarize some of their many contributions to this literature in their book [18].

The question arises: could monetary policy help, or hurt, private agents learn the REE? The common features of the robust policy literature include, first, that it is the government that does not understand the true structure of the economy, and second, that the government's ignorance will not vanish simply with the collection of more data.⁵ By contrast, in the

¹ Brainard [4] is the seminal reference. Among the many, more recent references in this large literature are Sack [42], Orphanides et al. [39], Soderstrom [44] and Ehrmann and Smets [17].

² See, e.g., Levin *et al.* [29] and [30].

³ Hansen and Sargent [27] and [28], Tetlow and von zur Muehlen [47] and Coenen [13]. These strands of the robustness literature are named in the text in chronological order but the three methods should be seen as complementary rather than substitutes.

⁴ In this paper, as in most of the rest of the literature, the terms learnability, E-stability and stability under learning will all be used interchangeably. These terms are distinct from stable—without the "E-" or "under stability" added—which should be taken to mean saddle-point stable. The term saddle-point stable, determinate and regular are taken as equivalent adjectives describing equilibria.

⁵ The concept of "truth" is a slippery one in learning models. In some sense, the truth is jointly determined by the deep structural parameters of the economy and what people believe them to be. Only in steady state, and then only under some conditions, will this be solely a function of deep parameters and not of beliefs.

learning literature it is usually the private sector that is assumed not to have the information necessary to form rational expectations, but this situation has at least the prospect of being alleviated with the passage of time and the collection of more data. In this paper, we take the robust policy rules literature and marry it with the learnability literature.

Since the profession has been unable to agree on a generally acceptable workhorse model of the economy, it is unreasonable to expect private agents to have rational expectations. The most that one can expect is that agents have an approximate understanding of the workings of the economy and that they are on a transition path toward learning the true structure. This assumption we hold in common with those in the learning literature. From a policy perspective, it follows that the job of facilitating that transition is logically prior to the job of maximizing the performance of the economy once the transition is complete. In this paper, we consider two issues. The first is how a policy maker might choose policy to maximize the set of worlds occupied by private agents able to learn the REE. The second is an assessment of the welfare cost of assuring learnability in terms of forgone stability in equilibrium. Or, put differently, we measure the welfare cost of learnability insurance. Each of these questions is important. In worlds of model uncertainty, an ill-chosen policy rule—or policy maker—could lead to explosiveness or indeterminacy. At the same time, excessive concern for learnability will imply costs in terms of forgone welfare.

Ours is not the first paper to consider the issue of choosing monetary policies for their ability to deliver determinacy and learnability. Bernanke and Woodford [2] argue that inflation-forecast based policy rules—that is, those that feed back on forecasts of future output or inflation—can lead to indeterminacy in linear rational expectations (LRE) models. Clarida, Gali and Gertler [11] show that violation of the so-called Taylor principle in the context of an IFB rule may have been the source of the inflation of the 1970s.⁶ Bullard and Mitra [8] in an important paper show that higher persistence in instrument setting—meaning a large coefficient on the lagged instrument in a Taylor-type rule—can facilitate determinacy in the same class of models. Evans and Honkapohja [20] note similar problems in a wider class of rules and argue for feedback on structural shocks, although questions regarding the observ-

Nevertheless, in this paper, when we refer to a "true model" or "truth" we mean the REE upon which successful learning eventually converges.

⁶ Levin *et al.* [30] and Batini and Pearlman [1] study the robustness properties of different types of inflation-forecast based rules for their stability and determinacy properties.

ability of such shocks leave open the issue of whether such a policy is implementable. Evans and McGough [22] compute optimal simple rules conditional on their being determinate in rival models. Each of these papers makes an important contribution to the literature. But all consider special cases within broader sets of policy choices. In this paper, we follow a different approach and consider optimal policies to maximize learnability of the economy.

The remainder of the paper is organized as follows. The second section lays out the theory, beginning with a review of the literature on least-squares learnability and determinacy, and following with methods from the robust control literature. Section 3 blends the two together to establish the tools with which we work. Section 4 contains our results. It begins with our application of these methods to the case of the very simple Cagan model of money demand in hyperinflations and then moves on to the New Keynesian business cycle (NKB) model. For the NKB model, we study the design of time-invariant simple monetary policy rules to robustify learnability of three types: a lagged-information rule, a contemporaneous information rule and a forecast-based policy rule. And we close the section by covering the insurance cost of robustifying learnability. A fifth and final section sums up and concludes.

2 Theoretical overview

2.1 Expectational equilibrium under adaptive learning

The theory of *E-stability* or *learnability* in linear rational expectations models dates back more than 20 years to Bray [5] who showed that agents using recursive least squares would, if the arguments to their regressions were properly specified, eventually converge on the correct REE. This convergence property gave a considerable shot in the arm to rational expectations applications since proponents had an answer to the question "how could people come to have rational expectations?" The theory has been advanced by the work of Marcet and Sargent [34] and Evans and Honkapohja [various]. Our rendition follows Evans and Honkapohja [18], chapters 8-10.

Begin with the following linear rational expectations model:

$$Hy_t = a + FE_t y_{t+1} + Ly_{t-1} + Gv_t. \tag{1}$$

At this point, E_t represents a mathematical expectation conditioned on information available in period t . Later on, we shall distinguish between this rational expectation and an expectation based on adaptive learning. Assuming the inverse of H exists, we re-write this model as

$$y_t = A + ME_t y_{t+1} + Ny_{t-1} + Pv_t, \quad (2)$$

where y_t is a vector of n endogenous variables, including, possibly, policy instruments, and v_t comprises all m exogenous variables. Equation (2) is general in that both non-predetermined (or "jumper") variables, $E_t y_{t+1}$ and predetermined variables, y_{t-1} , are represented, and by defining auxiliary variables, e.g., $y_t^j = y_{t+j}$, $j \neq 0$, arbitrarily long (finite) lead or lag lengths can be accommodated. It can also be easily extended to allow lagged expectations formation; e.g., $E_{t-1} y_t$, and exogenous variables with some relatively minor changes in the results. Next, define the prediction error for y_{t+1} , to be $\epsilon_{t+1} = y_{t+1} - E_t y_{t+1}$. Under rational expectations, $E_t \epsilon_{t+1} = 0$, a martingale difference sequence. Evans and Honkapohja [18] show that for at least one rational expectations equilibrium to exist, the stochastic process, y_t , that solves (2), must also satisfy:

$$y_{t+1} = aM^{-1} + M^{-1}y_t - M^{-1}Ny_{t-1} - M^{-1}Pv_t + \epsilon_{t+1} \quad (3)$$

We can express (3) as a first-order system:

$$\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} aM^{-1} \\ 0 \end{bmatrix} + \begin{bmatrix} M^{-1} & -NM^{-1} \\ I_n & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} -M^{-1} \\ 0 \end{bmatrix} Pv_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \epsilon_{t+1} \quad (4)$$

or, rewriting:

$$Y_{t+1} = A + BY_t + Cv_t + D\epsilon_{t+1}, \quad (5)$$

where $Y = [y_t, y_{t-1}]'$. then we can easily show when (5) satisfies the Blanchard-Kahn [3] conditions for stability, namely, that the characteristic roots of the matrix B of norm less than unity equal the number of predetermined variables (taking y_t to be scalar, this is one), then the model is *determinate*, and there is just one martingale difference sequence, ϵ_{t+1} , that will render (3) stationary; if there are fewer roots inside the unit circle than there are predetermined variables, the model is *explosive* meaning that there is no martingale difference sequence that will satisfy the system; and if there are more roots inside the unit

circle than there are predetermined variables, the model is said to be *indeterminate*, and there are infinite numbers of martingale difference sequences that make (3) stable. The roots of B are determined by the solution to the characteristic equation: $\Omega\lambda^2 - \lambda + \delta = 0$. It follows that determinacy requires $|\delta + \Omega| < 1$.

Determinacy is one thing, learnability is quite another. As Bullard and Mitra [8] have emphasized, determinacy does not imply learnability, and indeterminacy does not imply a lack of learnability. We can address this question by postulating a representation for the REE that a learning agent might use. For the moment, we consider the *minimum state variable* (MSV) representation, advanced by McCallum [35]. Let us assume that v_t is observable and follows a first-order stochastic process,

$$v_t = \rho v_{t-1} + \epsilon_t, \quad (6)$$

where ϵ_t is an iid white noise process. The ρ matrix is assumed to be diagonal.

Under these assumptions, we can write the following *perceived law of motion* (PLM):

$$y_t = a + by_{t-1} + cv_t. \quad (7)$$

Rewrite equation (2) slightly, and designate expectations formed using adaptive learning with a superscripted asterisk on the expectations operator, E_t^* :

$$y_t = A + ME_t^* y_{t+1} + Ny_{t-1} + Pv_t. \quad (8)$$

Then, leading (7) one period, taking expectations, substituting (7) into the result, and finally into (8), we obtain the *actual law of motion*, (ALM), the model under the influence of the PLM :

$$y_t = A + M(I + b)a + (N + Mb^2)y_{t-1} + (M(bc + c\rho) + P)v_t. \quad (9)$$

So the MSV solution will satisfy the mapping from PLM to ALM:

$$A + M(I + b)a = a, \quad (10)$$

$$N + Mb^2 = b, \quad (11)$$

$$M(bc + c\rho) + P = c. \quad (12)$$

Learnability depends then on the mapping of the PLM on to the ALM, defined from (9):

$$T(a, b, c) = [A + M(I + b)a, N + Mb^2, M(bc + c\rho) + P] \quad (13)$$

The fixed point of this mapping is a MSV representation of a REE, and its convergence is given by the matrix differential equation:

$$\frac{d}{d\tau}(a, b, c) = T(a, b, c) - (a, b, c). \quad (14)$$

Convergence is assured if certain eigenvalue conditions for the following matrix differential equations are satisfied.

$$\begin{aligned} \frac{da}{d\tau} &= [A + M(I + b)]a - a, \\ \frac{db}{d\tau} &= Mb^2 + N - b, \\ \frac{dc}{d\tau} &= M(bc + c\rho) + P - c. \end{aligned} \quad (15)$$

As shown by Evans and Honkapohja (2001), the necessary and sufficient conditions for E-stability are that the eigenvalues of the following matrices have negative real parts:

$$DT_a - I = M(I + b) - I, \quad (16)$$

$$DT_b - I = b' \otimes M + I \otimes Mb - I, \quad (17)$$

$$DT_c - I = \rho' \otimes M + I \otimes Mb - I. \quad (18)$$

The important points to take from equations (15) are that the conditions are generally multivariate in nature—meaning that the coefficients constraining the intercept term, a , can be conflated with those of the slope term, b ; and that the coefficients of both the PLM and the ALM come into play. Learnability applications in the literature to date have been to very simple, small-scale models where these problems rarely come into play.⁷ In the kind of medium- to large-scale models that policy institutions use, these issues cannot be safely ignored.⁸ In "real-world models", to the question of whether private agents know the

⁷ A notable exception is Garratt and Hall [23], but even then the learning problem was constrained to exchange rate determination. The rest of the London Business School model that they used was taken as known.

⁸ At the Federal Reserve Board, for example, the staff use a wide range of models to analyze monetary policy issues, including a variety of reduced-form forecasting models, a calibrated multi-country DSGE model called SIGMA, a medium-scale DSGE U.S. model, and the FRB/US model, a larger-scale, partly micro-founded estimated model.

model (or have to learn it) one needs to add the issue of whether the policy maker himself knows the model. What properties should the selected policy rule have if the monetary authority is unsure of the model under control? Without taking away anything from the important contributions of Bullard and Mitra [8] and Evans and Honkapohja [20], the choice of monetary policy rules must not only consider how they foster learnability in a given model but whether they do so for the broader class of models within which the true model might be found. Similarly, taking as given the true model, the initial beliefs of private agents can affect learnability both through the inclusion and exclusion of states to the PLM and through the initial values attached to parameters. In the context of the above example, values of a , b , and c that are initially "too far" from equilibrium can block convergence. The choice of a particular policy can shrink or expand the range of values for a , b , and c that is consistent with E-stability.⁹ This is our concern in this paper: how can a policy maker deal with uncertainty in the choice of his or her policy rule—uncertainty on his or her part and on the part of the private sector—and maximize the probability that the economy will converge successfully on a rational expectations equilibrium? For this, we work with perturbations to the T-mapping described by equations (14) or systems like it. We take this up in the next subsection.

2.2 Structured robust control

In the preceding subsection, we outlined the theory of least-squares learning in a relatively general setting. In this subsection we review methods from robust control theory. Recall that our objective is to uncover the conditions under which monetary policy can maximize the prospect that the process of learning will converge on a REE—that is, to robustify learnability—so the integration of the theories of these two subsections is what will provide us with the tools we seek.

The argument that private agents might have to learn the true structure of the economy takes a useful step back from the assumption of known and certain linear rational expectations models. However, what the literature to date has usually taken as given is, first, that

⁹ In fact, in this example, the intercept coefficient, a , turns out to be irrelevant for the determination of learnability although this result is not general.

agents use least-squares learning to adapt their perceptions of the true economic structure, and second, that they know the correct linear or linearized form of the REE solution.¹⁰ Both of these assumptions can be questioned. It is a common-place convenience of macroeconomists to formulate a dynamic stochastic general equilibrium model and then linearize that model. It is certainly possible that ill-informed agents use only linear approximations of their true decision rules. But it is hard to argue that the linearized decision rule is any more valid than some other approximation. Similarly, least-squares learning is the subject of research more because of its analytical tractability than its veracity. The utility of tractable, linear formulations of economic forms is undeniable. At the same time, however, the risk in over-reliance on such forms should be just as apparent. There would appear to be a least a *prima facie* case for retaining the simplicity of linear state-space representations and linear rules, while taking seriously the consequences of such approximations.

With this in mind, we retain the assumption of a linear reference model, and least-squares learning on the part by agents, but assume that the process of learning is subject to uncertainty. This may be because agents take their decision rules as simplifications of truly optimal decision rules due to the complexity of such rules. Or it might be because our assumption of least-squares learning is untenable in worlds where agents pick and choose the information to which they respond in forming and updating beliefs. The point is that from the perspective of the monetary authority, there are good reasons to be wary of both the learning rules and the underlying models, and yet there is very little guidance on how to model those doubts. Accordingly, we analyze these doubts using only a minimum amount of structure, drawing on the literature on *structured model uncertainty* and robust control.

For the most part, treatments of robust control have regarded model uncertainty as unstructured; that is, as uncertainty not ascribed to particular features of the model but instead represented by one or more additional shock variables wielded by some "evil agent" bent on causing harm. The literature on *unstructured robust control* is relatively new but growing quickly¹¹. By contrast, in order to establish maximum allowable misspecifications

¹⁰ Evans and Honkapohja [18] survey variations on least-squares learning, including under- and over-parameterized learning models and discounted (or constant-gain) least squares learning. Still, in general, either least-squares learning or discounted least-squares (that is, constant gain learning) is assumed. An exception is Marcet and Nicolini [33].

¹¹ See, in particular, Sargent [43], Giannoni [25], Hansen and Sargent ([27], [28]), Tetlow and von zur

in agents' learning models that keep the economy just shy of becoming unlearnable, we need to consider *structured model uncertainty*. Structured model uncertainty shares with its unstructured sibling a concern for uncertainty in the sense of Knight—meaning that the uncertainty is assumed to be nonparametric—but adds the structure of ascribing the uncertainty to particular parts of the model. Work in this field was initiated by Doyle [16] and further developed in Dahleh and Diaz-Bobillo [54] and , Zhou *et al.* [53], among others. Recent applications of this strand of robust control to monetary policy can be found in Onatski and Stock [37], Onatski [36], and Tetlow and von zur Muehlen [46].

Unlike most contributions the literature on monetary policy, which are concerned with maximizing economic performance, our concern is maximizing the likelihood of learnability. Thus our metric of success is not the usual one of maximized utility or a quadratic approximation thereof.

Boiled down to its essence, the five steps to designing policies subject to the constraint that agents must adapt to those policies and the exogenously changing environment using data and a learning model that may be misspecified are (i) write down a *reference model* of the economy; (ii) formulate a learning model used by agents, taken to be the reduced form of that model possibly based on the minimum set of state variables comprising the fundamental driving forces of the economy, (iii) specify a set of perturbations to this model structured in such a way as to isolate the possible misspecifications to which the model is regarded to be most vulnerable; (iv) for a given policy, use *structured singular value analysis* to determine the maximum allowable perturbations to the PLM that will bring the economy up to, but not beyond, the point of E-instability; and finally, (v) compute the policy for which the maximum allowable range of misspecifications is the largest. The reference model is the best linearization of the true economy that the decision maker can come up with. However, notwithstanding the decision maker's best efforts, she understands that her model is an approximation of the true economy, and she retains doubts about its local accuracy.¹²

Muehlen ([46], [47]), Onatski and Stock [37], and Onatski and Williams [38].

¹² It is sometimes argued that robust control—by which people mean minmax approaches to model uncertainty—is unreasonable on the face of things. The argument is that the worst-case assumption is too extreme, that to quote a common phrase, "if I worried about the worst case outcome every day, I wouldn't get out of bed in the morning". Such remarks miss the point that the worst-case outcome should be thought of as local in nature. Decision makers are envisioned as wanting to protect against uncertainties that are empirically indistinguishable from the data generating process underlying their reference models.

When, in the agents' learning model—the MSV-based PLM described in (7)—the parameters a , b , and c or Π , have been correctly estimated by agents, this model should be considered to be the true reduced form of the structural model in (2). Note, however, that even if individuals manage to specify their learning model correctly in terms of included variables and lag structures, the expectations of future output and inflation they base on these estimates are (at best) in a period of transition towards being truly rational. The model that agents actually estimate may differ from (2) in various ways that may be persistent. We want to determine how far off the true model agents' learning model can become before it becomes in principle unlearnable.

To begin, we rewrite the ALM from (9) and vectorize the disturbance, ϵ_t , to emphasize the stochastic nature of the estimating problem faced by agents,

$$Y_{t+1} = \Pi Y_t + \tilde{\epsilon}_t,$$

where $Y_t = [1, y_{t-1}, v_t]'$ is of dimension $n + 1$, $\tilde{\epsilon}_t = [0 \ 0 \ \epsilon_t]'$ and

$$\Pi = \begin{bmatrix} 1 & 0 & 0 \\ A + M(I + b)a & (N + Mb^2) & M(bc + c\rho) + P \\ 0 & 0 & \rho \end{bmatrix}.$$

Notice that by using the ALM, we are modeling the problem from the policy authority's point of view. The authority is taken as knowing, up to the perturbations we are about to add to the model, the structure of private agents' learning problems. As a consequence, the authority is in a position to influence the resolution of that problem.

Potential errors in parameter estimation are then represented by a *perturbation block*, Δ . In principle, the Δ operator can be structured to implement a variety of misspecifications, including alternative dynamic features. Robust control theory is remarkably rich in how it allows one to consider omitted lag dynamics, inappropriate exogeneity restrictions, missing nonlinearities, and time variation. This being the first paper of its kind, we keep our goals modest: in the language of linear operator theory, we will confine our analysis to *linear time-invariant scalar* (LTI-scalar) perturbations. LTI-scalar perturbations represent such events as one-time shifts and structural breaks in model parameters, as agents perceive them. Such perturbations have been the subject of study of parametric model uncertainty;

see, e.g., Bullard and Euseppi [7].¹³ With this restriction, the perturbed model becomes:¹⁴

$$\begin{aligned}\tilde{\epsilon}_t &= Y_{t+1} - [\Pi + W_1\Delta W_2]Y_t, \\ &= [A - W_1\Delta W_2]Y_t,\end{aligned}\tag{19}$$

where $A = I_n L^{-1} - \Pi$, L is the lag operator, Δ is a $k \times k$ linear, time-invariant block-diagonal operator representing potentially destabilizing learning errors, and W_1 and W_2 are, respectively, $(n+1) \times k$ and $k \times (n+1)$ selector matrices of zeros and ones that select which parameters in which equations are deemed to be subject to such errors. Either W_1 or W_2 can, in addition, be chosen to attach scalar weights to the individual perturbations so as to reflect relative uncertainties with which model estimates are to be regarded. The second line is convenient for analyzing stability of the perturbed model under potentially destabilizing learning errors. Using this construction, the perturbation operator, Δ , and the weighting matrices can be structured so that misspecifications are focused on particular features of the model deemed especially susceptible to learning errors involving the model's variables for any chosen lag or lags.

The essence of this paper is to find out how large, in a sense to be defined presently, the misspecifications represented by the perturbations in (19)—called the *radius of allowable perturbations*—can become without eliciting a failure of convergence to rational expectations equilibrium. Any policy that expands the set of affordable perturbations is one that allows the widest room for misspecifications committed by agents and thus offers an improved chance that policy will not be destabilizing. To do this we bring the tools of *structured robust control* analysis mentioned earlier.

Let \mathcal{D} denote the class of allowable perturbations to the set of parameters of a model defined as those that carry with them the structure information of the perturbations. Let $r > 0$ be some finite scalar and define D_r as the set of perturbations in (19) that obey $\|\Delta\| < r$, where $\|\Delta\|$ is the *induced norm* of Δ considered as an operator acting in a normed space of random processes¹⁵. The scalar, r , can be considered a single measure of

¹³ See Evans and Honkapohja [18] for some treatment of learning with an over-parameterized PLM.

¹⁴ Multiplicative errors in specification would be modeled in a manner analogous to (19): $\epsilon_t = [A(1 - W_1\Delta W_2)]Y_t$.

¹⁵ Induced norms are defined as follows. Let X be a vector space. A real-valued function $\|\cdot\|$ defined on X is said to be a norm on X if it satisfies: (i) $\|x\| \geq 0$, (ii) $\|x\| = 0$ only if $x = 0$, (iii) $\|\alpha x\| = |\alpha| \|x\|$

the maximum size of errors in estimation. A policy authority wishing to operate with as wide a room to maneuver as possible will act to maximize this range. For the tools to be employed here, norms will be defined in complex space. In what follows, much use is made of the concept of *maximum singular value*, conventionally denoted by $\bar{\sigma}^{16}$. For reasons that will become clearer below, the norm of Δ that we shall use will be the L_∞ norm of the function $\Delta(e^{i\omega})$, defined as the largest singular value of $\Delta(e^{i\omega})$ on the frequency range $\omega \in [-\pi, \pi]$:

$$\|\Delta\|_\infty = \left\{ \sup_{\omega} \max \text{eig}[\Delta'(e^{-i\omega})\Delta(e^{i\omega})] \right\}^{1/2}, \quad (20)$$

where $\max \cdot \text{eig}$ denotes the maximum eigenvalue. The choice of $\|\Delta\|_\infty$ as a measure of the size of perturbations conveys a sense that the authority is concerned with worst-case outcomes.

Imagine two artificial vectors, $h_t = [h_{1t}, h_{2t}, \dots, h_{kt}]'$ and $p_t = [p_{1t}, p_{2t}, \dots, p_{kt}]'$, connected to each other and to Y_{t+1} via¹⁷

$$\begin{aligned} p_t &= W_2 Y_t \\ h_t &= \Delta \cdot p_t. \end{aligned}$$

Then we may recast the perturbed system (19) as the *augmented feedback loop*¹⁸

$$\begin{bmatrix} Y_{t+1} \\ p_t \end{bmatrix} = \begin{bmatrix} \Pi & W_1 \\ W_2 & 0 \end{bmatrix} \begin{bmatrix} Y_t \\ h_t \end{bmatrix}, \quad (21)$$

$$h_t = \Delta \cdot p_t. \quad (22)$$

A reduced-form representation of this loop (from h_t to Y_t and p_t) is the transfer function

$$\begin{bmatrix} Y_t \\ p_t \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} h_t,$$

where $G_1 = (I_n L^{-1} - \Pi)^{-1} W_1$, and $G_2 = W_2 (I_n L^{-1} - \Pi)^{-1} W_1$ is a $n \times k$ matrix, k being the number of diagonal elements in Δ . As we shall see, the stability of the interconnection

for any scalar α , (iv) $\|x + y\| \leq \|x\| + \|y\|$ for any $x \in X$ and $y \in X$. For $x \in C^n$, the \mathcal{L}_p vector p-norm on x is defined as $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, where $1 \leq p \leq \infty$. For $p = 2$, $\mathcal{L}_2 = \|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$, that is, the quadratic problem. Finally, let $A = [a_{ij}] \in C^{m \times n}$ in an equation $y_t = A_t x_t$, where x_t may be some random vector. The matrix norm *induced* by a vector p-norm, $\|x\|_p$, is $\|A\|_p \equiv \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$. More details are given in Tetlow and von zur Muehlen [46].

¹⁶ As is apparent from the expression in (20), the largest singular value, $\bar{\sigma}(X)$, of a matrix, X , is the largest eigenvalue of $X'X$.

¹⁷ See Dahleh and Bobillo [14], chapter 10.

¹⁸ Because the random errors in this model play no role in what follows, we leave out the ϵ vector.

between h_t and p_t , representing a feedforward $p_t = G_2 h_t$ and a feedback $h_t = \Delta \cdot p_t$, is critical. Note first that, together, these two relationships imply the homogenous matrix equation

$$0 = (I_k - G_2 \Delta) p_t. \quad (23)$$

An E-stable ALM is also dynamically stable, meaning that Π has all its eigenvalues inside the unit circle. This means that A , defined in (19), is invertible on the unit circle, allowing us to write¹⁹

$$\begin{aligned} \det(A) \det(I_k - G_2 \Delta) &= \det(A) \det(I_k - W_2 A^{-1} W_1 \Delta) \\ &= \det(A) \det(I_k - A^{-1} W_1 \Delta W_2) \\ &= \det(A - W_1 \Delta W_2). \end{aligned}$$

The preceding expressions establish the link between stability of the interconnection, G_2 , and stability of the perturbed model: if $\det(I_k - G_2 \Delta) = 0$, then the perturbed model (19) is no longer invertible on the unit circle, hence unstable, and vice versa.²⁰ Thus, any policy rule that stabilizes the G_2 also stabilizes the augmented system (21)-(22). The question to be asked then is how large, in the sense $\|\cdot\|_\infty$, can Δ become without destabilizing the feedback system (21)-(22).

The settings we consider involve linear time-invariant perturbations, where the object is to find the minimum of the largest singular value of the matrix, Δ , from the class of D_r such that $I - G_2 \Delta$ is not invertible. The inverse of this minimum, expressed in the frequency domain, is the *structured singular value* of G_2 with respect to D_r , defined at each frequency, $\omega \in [-\pi, \pi]$,

$$\mu[G_2(e^{i\omega})] = \frac{1}{\min\{\bar{\sigma}[\Delta(e^{i\omega})] : \Delta \in D_r, \det(I - G_2 \Delta)(e^{i\omega}) = 0\}}, \quad (24)$$

with the provision that if there is no Δ such that $\det(I - G_2 \Delta)(e^{i\omega}) = 0$, then $\mu[G_2(e^{i\omega})] = 0$.

Echoing the *small gain theorem*, an important result (see, Zames [52], Zhou *et al.*, [53]) states

¹⁹ The motivation behind the approach taken here is the important *small gain theorem* (see (Dahleh and Bobillo [14] and Zhou and Doyle [54]), linking the stability of the loop between p and h under perturbations to the full system subject to model uncertainty. It states that the interconnection in (21)-(22) is well posed and internally stable if for $\alpha > 0$, (a) $\|\Delta\|_\infty \leq \alpha$ if and only if $\|G_2\|_\infty < 1/\alpha$ and (b) $\|\Delta\|_\infty < \alpha$ if and only if $\|G_2\|_\infty 1/\alpha \leq \alpha$. This is the reason why the measure of robust stability will be the L_∞ norm, as indicated earlier. Clearly, for some sufficiently large number α , such that $\|\Delta\|_\infty < \alpha$, the determinant $\det(I_k - G_2 \Delta) \neq 0$. Now raise α to some value α_{max} such that $\det(I_k - G_2 \Delta) = \det(I_k - W_2 A^{-1} W_1 \Delta) = 0$.

²⁰ In essence, this is just another statement of the small gain theorem.

that, for some $r > 0$, the loop (21)-(22) is well posed and internally stable for all $\Delta(\cdot) \in \mathcal{D}_r$ with $\|\Delta\| < r$, if and only if $\sup_{\omega \in \mathcal{R}} \mu[G_2(e^{i\omega})] \leq 1/r$. Let ϕ denote a vector of policy parameters. Our goal in what follows is to seek a best $\phi = \phi^*$ by finding an maximum value of $\mu = \bar{\mu}$, satisfying

$$\bar{\mu}(\phi^*) = \inf_{\phi} \sup_{\omega \in \mathcal{R}} \mu[G_2(e^{i\omega})].$$

The solution to this problem is not amenable to analytical methods, except in special cases, an example of which we explore in the next section. Instead, we will employ efficient numerical techniques to find the lower bound on the structured singular value. The minimum of $\mu^{-1}(G_2)$ over $\omega \in [0, \pi]$ is exactly the maximal allowable range of misspecification for a given policy. A monetary authority wishing to give agents the widest latitude for learning errors that nevertheless allow the system to converge on REE selects those parameters in its policy rule that yield largest value of r .²¹

3 Two examples

We study two sample economies, one the very simple model of money demand in hyperinflations of Cagan [10], the other the linearized neo-Keynesian model originated by Woodford ([49], [51]), Rotemberg and Woodford [41] and Goodfriend and King [26]. Closed-form solutions for μ , being non-linear functions of the eigenvalues of models, are not generally feasible. However, some insight is possible through considering simple scalar example economies like the Cagan [10] model. The second has the virtue of having been studied extensively in the literature on monetary policy design. It thus provides some solid benchmarks for comparison.

3.1 The Cagan model

Consider a version of Cagan's [10] monetary model, cited in Evans and Honkapohja [18], although our rendition differs slightly. The model has two equations, one determining (the log of) the price level, p_t , and the other a simple monetary feedback rule determining the

²¹ In the Appendix, we describe how to use Matlab to determine μ .

(log of the) money supply, m_t :

$$m_t - p_t = -\kappa(E_t p_{t+1} - p_t)$$

$$m_t = \chi - \phi p_{t-1}.$$

All parameters should be greater than zero; for κ this means that money demand is inversely related to expected inflation. Combining the two equations, leads to:

$$p_t = \alpha + \beta E_t p_{t+1} - \gamma p_{t-1},$$

where $\alpha = \chi/(1 + \kappa)$, $\beta = \kappa/(1 + \kappa)$, and $\gamma = \phi/(1 + \kappa)$. To set the stage for what follows, let us consider the conditions for a unique rational expectations equilibrium:

For $\kappa > 0$, avoiding indeterminacy requires: $\phi > -1$.

For $\kappa > 0$, avoiding explosive solutions requires: $\phi < 1 + 2\kappa$.

The proofs are straightforward and are therefore omitted.

Now let us assume that agents form expectations employing adaptive learning and designate expectations formation in this way with the operator, E_t^* . The perceived law of motion for this model is assumed to be $p_t = a + b p_{t-1}$, implying $E_t^* p_{t+1} = (1 + b)a + b^2 p_{t-1}$. The actual law of motion is found by substituting the PLM into the structural model:

$$p_t = [\alpha + \beta a(1 + b)] + (\beta b^2 - \gamma)p_{t-1}. \quad (25)$$

Following the steps outlined earlier, the ALM is $p_t = T_a(a, b) + T_b(a, b)p_{t-1}$ where T defines the mapping $\begin{pmatrix} a \\ b \end{pmatrix} = T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right)$. This is

$$T_a: a = \alpha + \beta a(1 + b) \quad (26)$$

$$T_b: b = \beta b^2 - \gamma, \quad (27)$$

with the solutions

$$a = \alpha/[1 - \beta(1 + b)] \quad (28)$$

$$b = .5[1 \pm \sqrt{1 + 4\beta\gamma}]/\beta. \quad (29)$$

Equation (29) is quadratic with one root negative or equal to zero, and the other positive. Existence of the REE requires us to choose the smaller, negative root; otherwise, $b > 1$. The

ordinary differential equation system implied by this mapping is

$$\frac{d \begin{pmatrix} a \\ b \end{pmatrix}}{d\tau} = T \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix},$$

for which the associated DT matrix is derived by differentiating $[T_a \ T_b]'$ with respect to a and b :

$$DT = \begin{bmatrix} \beta(1+b) & a\beta \\ 0 & 2\beta b \end{bmatrix}.$$

The eigenvalues of $DT - I$ are,

$$\lambda_1 = \beta(1+b) - 1 \tag{30}$$

$$\lambda_2 = 2\beta b - 1. \tag{31}$$

Satisfaction of the weak E-stability condition requires that both eigenvalues be negative. Which of these two equations is critical for determining E-stability depends on whether $b \geq 1$. It is easy to show that if one uses the positive root for the solution to equation (29) then $\lambda_2 > 0$, and the model is not E-stable. Accordingly, we restrict our attention to the negative root. In this instance, $\lambda_2 > \lambda_1$ and so it will be equation (31) that will be instrumental in assuring E-stability, and for any feasible value of κ , it is necessary that $b \in (-\infty, 1)$.

Having outlined the connection between b and β (or κ) for E-stability, let us now consider unstructured perturbations to the ALM. Let $X_t = [1 \ p_t]'$. The reference ALM model is then written as $X_t = \Pi X_{t-1}$, where

$$\Pi = \begin{bmatrix} 1 & 0 \\ \alpha + \beta a(1+b) & \beta b^2 - \gamma \end{bmatrix}$$

is the model's transition matrix. For simplicity, let us focus on b as the object of concern to policy makers and let the policy maker apply structured perturbations to Π , scaled by the parameter, σ_b . The scaling parameter can be thought of as a standard deviation, but need not be. Letting $W_1 = [0 \ \sigma_b]'$ and $W_2 = [0 \ 1]$, write the perturbed matrix Π as:

$$\Pi_\Delta = \begin{bmatrix} 1 & 0 \\ \alpha + \beta a(1+b) & \beta b^2 - \gamma + \Delta \end{bmatrix}.$$

Define $z = e^{i\omega}$, $\omega \in [-\pi, \pi]$. To find the maximal allowable perturbation, write

$$G = \begin{bmatrix} z^{-1} - 1 & 0 \\ -\alpha - \beta a(1+b) & z^{-1} - \beta b^2 + \gamma \end{bmatrix},$$

which, defining $W_1 = [0 \ \sigma_b]'$ and $W_2 = [0 \ 1]$, is used to form G_2 :

$$\begin{aligned} G_2 &= W_2 G^{-1} W_1 \\ &= [0 \ 1] \begin{bmatrix} \frac{z}{1-z} & 0 \\ \frac{(\alpha - \beta a(1+b))z}{(1-z)(1-(\beta b^2 - \gamma)z)} & \frac{z}{1-(\beta b^2 - \gamma)z} \end{bmatrix} \begin{bmatrix} 0 \\ \sigma_b \end{bmatrix} \\ &= \frac{\sigma_b z}{1 - (\beta b^2 - \gamma)z}. \end{aligned}$$

In the multivariate case, the scaling parameter σ_b , can be parameterized as the standard deviation of b relative to a , although other methods of parameterization can be entertained. Doing so would reflect a concern for robustness of the decision maker and thus could also be thought of as a taste parameter. Since it is a *relative* term, it will turn out to be irrelevant in this scalar case, and so from here we set it to unity without loss of generality. The structured norm of G_2 —equal to the absolute value of this last expression—is μ . It is also easily established that the maximum of μ , let us call it $\bar{\mu}$ is $\bar{\mu} = \|\Delta\|_\infty = |G_2|$ at frequency π .²² Also, since at frequency π , $z = -1$, it follows that $|G_2| = \bar{\mu} = \frac{1}{1+b}$ or equivalently the *allowable perturbation* is:

$$\Delta = \frac{1}{\bar{\mu}} = 1 + \beta b^2 - \gamma \quad (32)$$

$$\begin{aligned} &= 1 + b \\ &= 1 + \frac{1}{2\beta} [1 - \sqrt{1 + 4\beta\phi/(1 + \kappa)}], \end{aligned} \quad (33)$$

which depends inversely on the policy parameter, ϕ .²³ Note that also that while we have derived this expression for Δ by applying perturbations to the ALM, we would have obtained exactly the same result by working with the PLM.

If equation Δ is the allowable perturbation, conditional on a given ϕ , then we can define a ϕ^* as the policy maker's optimal choice of ϕ , where optimality is defined in the sense of

²² The curious reader is invited to use the instructions in the Appendix to verify these assertions.

²³ Note that at frequency π , $1 - G_2\Delta = 1 - \frac{\sigma_b}{(1+b)} \frac{(1+b)}{\sigma_b} = 0$ as required by the definition of μ .

choosing the largest possible perturbation to b —call it Δ^* —such that the model will retain the property of E-stability. Let us call this the *maximum allowable perturbation*. It is the Δ^* and the associated ϕ^* that is at a boundary where Δ is just above -1 :

$$\phi^* = 1 + 2\kappa - \varepsilon, \tag{34}$$

where $\phi < 1 + 2\kappa$ maintains stable convergence toward an REE and ε is an arbitrarily small positive constant necessary to keep $b + \Delta$ off the unit circle. Note that this expression for ϕ^* indicates that the monetary authority will always respond more than one-for-one to deviations in lagged prices from steady state, with the extent of that over-response being a positive function of the slope of the money demand function. Substituting these expressions back into our perturbed transition matrix,

$$\begin{aligned} \Pi_{\Delta}^* &= \begin{bmatrix} 1 & 0 \\ \alpha - \beta a(1 + b) & \beta b^2 - \gamma + \Delta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \alpha - \beta a(1 + b) & 1 + 2(\beta b^2 - \gamma) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \alpha - \beta a(1 + b) & -1 + \eta \end{bmatrix}, \end{aligned} \tag{35}$$

where η is an arbitrarily small number, as determined by ε in (34). The preceding confirms that the authority’s policy is resilient to a perturbation in the learning model that pushes the transition matrix is to the borderline of instability. In other words, setting a ϕ that allows for the maximal stable misspecification of the learning model is one that permits convergence to the REE.

3.2 The canonical New Keynesian model

We now turn to an analysis of the canonical New Keynesian business cycle model of Rotemberg and Woodford [41], Goodfriend and King [41] and others. Clarida, Gali, and Gertler [12] used this model to derive optimal discretionary as well as optimal commitment rules. Their version includes a specified process for exogenous natural output. Evans and Honkapohja [19] study this model to explore issues of determinacy and learnability for several optimal commitment rules. Bullard and Mitra [9] likewise use the Woodford model to examine determinacy and learnability of variants of the Taylor rule.

The behavior of the private sector is described by two equations. The aggregate demand (IS) equation is a log-linearized Euler equation derived from optimal consumer behavior,

$$x_t = E_t^* x_{t+1} - \sigma[r_t - E_t^* \pi_{t+1} - r_t^n], \quad (36)$$

and the aggregate supply (AS) equation—indeed, the price setting rule for monopolistically competitive firms is,

$$\pi_t = \kappa x_t + \beta E_t^* \pi_{t+1}, \quad (37)$$

where x is the log deviation of output from potential output, π is inflation, r is a short-term interest rate controlled by the central bank, and r^n is the natural interest rate. For the application of Bullard and Mitra’s [8] (BM) example, we assume that r_t^n is driven by a first-order autoregressive process,

$$r_t^n = \rho_r r_{t-1}^n + \epsilon_{r,t}, \quad (38)$$

$0 \leq |\rho_r| < 1$, and $\epsilon_{r,t} \sim iid(0, \sigma_r^2)$. This is essentially Woodford’s [49] version of this model, which specifies that aggregate demand responds to the deviation of the real rate, $r_t - E_t \pi_{t+1}$ from the natural rate, r_t^n .

We need to close the model with an interest-rate feedback rule. We study three types of policy rules. In the first set of experiments described in Section 3.3, a central bank chooses an interest rate setting in each period as a reaction to observed events, such as inflation and the output gap, without explicitly attempting to improve some measure of welfare. Instead, the policy authority is mindful of the effect its policy has on the prospect of the economy reaching REE and designs its rule accordingly. Bullard and Mitra [8] study such rules for their properties in promoting learnable equilibria and consider that effort as prior to one of finding optimal policy rules consistent with REE. We take this analysis further by seeking to find policy rules that maximize learnability of agents’ models when policy influences the outcome.

The information protocols in these experiments is as follows. Economic agents and the central bank have the same information: the data and the perceived law of motion. Agents form expectations based on recursive (least squares) estimation of a reduced form. The data are regenerated each period, subject to the authority having implemented its policy and

agents' having made investment and consumption decisions based on their newly formed expectations. Since the central bank uses an arbitrary simple rule, it need not know the true structure of the economy, but, being aware of how individuals form their expectations, it designs policy that ensures the learnability of the PLM.

We assume that agents mistakenly specify a vector-autoregressive model in the endogenous and exogenous variables of the model. That means we assume the learning model to be overparameterized in comparison with the model implied by the MSV solution. The scaling factors used in W_1 to scale the perturbations to the PLM are the standard errors of the coefficients obtained from an initial run of a recursive least squares regression of such a VAR with data being updated by the true model, given an arbitrary but determinate parameterization of the policy rule being studied. As noted earlier, an alternative approach would be to revise the scalings with each trial policy, given that the VAR would likely change with each parameterization of policy. We leave this for a revision.

3.3 Simple interest rate feedback rules

This section describes two versions of the Taylor rule analyzed by Bullard and Mitra [8]. The complete system comprises equations (36)-(39), and the exogenous variable, r_t^n . The policy instrument is the nominal interest rate, r_t . The first policy rule specifies that the interest rate responds to lagged inflation and the lagged output gap. In their paper, BM study the role of interest-rate inertia and so include a lagged interest rate term.

$$r_t = \phi_\pi \pi_{t-1} + \phi_x x_{t-1} + \phi_r r_{t-1} \quad (39)$$

McCallum has advocated such a *lagged data* rule because of its implementability, given that contemporaneous real-time data are generally unavailable to policy makers.

Some research suggests that forward-looking rules perform well in theory (see, e.g., Evans and Honkapohja [19]) as well as in actual economies, such as Germany, Japan, and the US (see Clarida, Gali, and Gertler [11]). Accordingly, BM propose the rule

$$r_t = \phi_\pi E_t^* \pi_{t+1} + \phi_x E_t^* x_{t+1} + \phi_r r_{t-1}. \quad (40)$$

The expectations operator E^* has an asterisk to indicate that expectations need not be

rational. In specifying expected future inflation and output, we may allow for two alternatives, both of them altering the information protocol described earlier. We may assume that the central bank has superior information (as in the optimization case) and have it solve for the RE values of $E_t x_{t+1}$ and $E_t \pi_{t+1}$. Alternatively, we may posit that, more consistent with the previously described information protocol, the bank utilizes the same expectations that agents derive using their estimated PLM. A problem with this second specification may be that the problem becomes self-referential, thus possibly implying indeterminacy for all parameter values in the policy rule.

Finally, the most popular rules of this class are contemporaneous data rule, of which the following is our choice:

$$r_t = \phi_\pi \pi_t + \phi_x x_t + \phi_r r_{t-1} \quad (41)$$

where as before, we allow the lagged federal funds rate to appear to capture instrument-smoothing behavior by uncertainty averse decision makers.

We adopt BM's calibration for the model's parameters, $\sigma = 1/.157$, $\kappa = .024$, $\beta = .99$, and $\rho = .35$, the same calibration as in Woodford [51].

Let us consider the lagged-data rule first. BM find that determinacy of a unique rational expectations equilibrium and convergence toward it when agents learn adaptively is extremely sensitive to the policy parameters, ϕ_r , ϕ_x , and ϕ_π . Without some degree of monetary policy inertia, $\phi_r > 0$, this model is determinate and learnable only if, with the above calibrations, the Taylor principle, $\phi_\pi > 1$, holds **and** the response to the output gap is modest, $\phi_x \leq 0.5$. Insufficient or excessive responsiveness to either inflation or the output gap can, in some instances, lead to explosive instability or indeterminacy. Through simulation, BM establish the regions for the parameters that lead to determinacy as well as E-stability. We take this one step further by allowing the central bank to choose its policy with a view toward maximizing the prospect of convergence to REE under the greatest possible misspecification of the PLM model. Formally, the authority seeks values of ϕ_x , ϕ_π , and ϕ_r that make convergence to REE robust to agents' misspecification of their learning model. The optimal values are shown in the second line of Table 1.

The next-to-last column of the table gives a measure of the total uncertainty that the PLM can tolerate under the cited policy. It is a measure of the maximal allowable deviation

embodied in $1/\mu$. For comparison of the trials with each other and also to give a sense of natural units related to the scalings we employed, the radius is calculated as the H_∞ norm of the scaled perturbations to the PLM model: $\text{radius} = \|W_1 \Delta W_2\|_\infty$.

The last column provides a measure of the unconditional expected value of the loss function,

$$L_t = \frac{1000}{2} \sum_{j=0}^{\infty} \beta^j [(\pi_{t+j} - \pi^*)^2 + \lambda_x x_{t+j}^2 + \lambda_r (r_{t+j} - r^*)^2], \quad (42)$$

with the following assumed values, taken from Walsh [48]: $\lambda_x = .077$ and $\lambda_i = .027$.

Table 1 shows the results. The table is broken into three panels. The upper panel shows optimized Taylor-type rules of the sort described above. The coefficients to these rules are optimized using a standard hill-climbing algorithm using methods well described in the appendix to Tetlow and von zur Muehlen [46]. The second panel, contains some results for the generic Taylor rule. Finally, the third panel shows our robust learnable rules.

Let us concentrate, initially, on our optimized rules, including, along with the Taylor rule, to provide some context for the robust learnable rules. The lagged data rule, shown in the row numbered (1), and the contemporaneous data rule, (2), are essentially the same. They both feature very small feedback on the output gap, strong responses to inflation. Moreover, they also feature funds rate persistence that amounts to a first-difference rule. The forecast-based rule, in line (3), has much stronger feedback on the output gap, although proper interpretation of this requires noting that in equilibrium the expectation of future output gaps will always be smaller than actual gaps because of the absence of expected future shocks and the internalization of future policy in the formulation of that expectation. Thus, the response of the funds rate to the expected future gap will not be as large as the feedback coefficient alone would lead one to believe.

These three rules confirm the received wisdom of monetary control in New Keynesian models, to wit: strong feedback on inflation, comparatively little on output, and strong persistence in funds rate setting. We also note, in passing for the moment, that the losses for all three of these rules, shown in the far right column, are very similar, at a little over 3.6.

The results for the Taylor rule demonstrate, indirectly, the oft-discussed advantages of

persistence in funds rate setting for monetary control. The Taylor rule, without such persistence, produces losses that are substantially higher than those of the optimized rules.

Now let us turn to the robust learnable rules in the bottom panel of the table, concentrating for the moment, on the lagged data and contemporaneous data rules shown in lines (5) and (6). The first thing to note is that the results confirm the efficacy of persistence in instrument setting. The robust learnable rules are at least as persistent—if persistence greater than unity is a meaningful concept—than are the optimized rules. At the same time, while persistence is evidently useful for learnability, our results do not point to the hyper-persistence result, ($\phi_r \gg 1$), that they hint at. To understand this outcome, it is important to realize that while our results are related to the BM results, there are conceptual differences. BM describe the range of policy-rule coefficients for which the model is learnable, *taking as given the model*. We are describing the range of policy coefficients that maximizes the range of models that are still learnable. So while large values for ϕ_r are beneficial to learnability *holding constant the model and its associated ALM*, at some point, they come at a cost in terms of the perturbations that can be withstood in other dimensions.

Now let us look at the losses incurred as measured by the last column of these two rows. The results show that the cost of maximizing learnability measured in terms of foregone performance in the REE is very small. Evidently, learnability can be robustified without much concomitant loss in economic performance, at least in the canonical NKB model.

Now let us examine the results for the forecast-based policy shown in the seventh row. Here the prescribed robust learnable policy is much different from the optimized rule shown in line (3). The robust rule essentially removes the policy persistence that the optimized policy calls for. While this is superficially at odds with BM, the result really should not be surprising. The forecast-based rules are leveraging heavily the rational expectations aspects of the model—even more so than with the contemporaneous and lagged data rules since there is rational expectations in model itself and in the policy feedback—and there is risk in leverage. The learnability of the economy is highly susceptible to misspecification in this area. This is, of course, just a manifestation of the problem that Bernanke and Woodford [2] and others have warned about.

Table 1 : Standard and robust learnable rules

	column	ϕ_x	ϕ_π	ϕ_r	radius ¹	L^2
<i>optimized rules:</i>						
lagged data rule	(1)	0.052	0.993	1.13	1.07	3.679
contemporaneous data rule	(2)	0.053	0.995	1.12	1.06	3.626
forecast-based rule	(3)	0.286	0.999	1.32	0.88	3.628
<i>standard rules:</i>						
Taylor rule	(4)	0.500	1.500	0	0.85	5.690
<i>robust learnable rules:</i>						
lagged data rule	(5)	0.065	0.40	1.10	1.16	3.712
contemporaneous data rule	(6)	0.052	1.21	1.41	1.13	3.701
forecast-based rule	(7)	0.040	2.80	0.10	2.32	4.434
1. Magnitude of the largest allowable perturbation. $r = \ W_1 \Delta W_2\ _\infty$						
2. Asymptotic loss, calculated according to equation (42).						

We can obtain a deeper understanding of the effects of a concern for robust learnability on policy design by examining the properties of different calibrations of policy rules for their effects on the allowable perturbations. The magnitude of perturbations that a given model can tolerate, conditional on a policy rule, is given by the radius. The radii for the rules shown in Table 1 are in the column second from the right. We can, however, provide a visualization of radii mapped against policy-rule coefficients and judge how policy affects robust learnability.

Figure 1 provides one such visualization: contour maps of radii against the output-gap feedback coefficient, ϕ_x , and inflation feedback coefficient, ϕ_π , in this case for the contemporaneous data rule. The third dimension of policy, the feedback on the lagged fed funds rate, ϕ_r , is being held constant in these charts, at zero in the upper panel and at unity in the lower. The colors of the chart index the radii of allowable perturbations for each rule, with the bar at the right-hand side showing the tolerance for misspecification. The area in deep blue, for example, represents policies with no tolerance for misspecification of the model or learning whatsoever, either because the rule fails to deliver E-stability in the first place, or because it is very fragile. The sizable region of deep blue in the upper panel shows the area that violates the Taylor principle. The right of the deep blue region—where $\phi_\pi > 1$ —we enter regions of green, where there is modest tolerance for misspecification that allows learnability. In general, with no interest-rate smoothing, there is little scope for misspecification.

Now let us look at the case where $\phi_r = 1$ in the bottom panel. Now the region of deep

blue is relegated to the very south-west of the chart, as is the region of green. To the north-east of those are expansive areas of higher tolerance for misspecification. Evidently, at least some measure of persistence in policy is useful for robustifying learnability. Notice how there is a deep burgundy sliver of fairly strong robustness in the north-east part of the panel.

Figure 2 continues the analysis for the contemporaneous data rule by showing contour charts for two more levels of ϕ_r . The upper panel shows the value for the rule that allows the maximum allowable perturbation as shown in line (6) of the table. The burgundy region is now at its largest and the policy rule shown in line (6) of the table within. More generally, the area of significant robustness—the redder regions—are collectively quite large. Finally, we go to the bottom panel of the figure which shows the results for a relatively high level of ϕ_r . What has happened is that the regions shown in the top panel have rotated down and to the right as ϕ_r has risen. The burgundy region is now gone, and the red regions command much less space. Thus, while policy persistence is good for learnability, in terms of robustness of that result to misspecification, one can go too far.

Figures 1 and 2 covered the case of the contemporaneous data rule. We turn now to forecast-based rules. The results here look quite different, but the underlying message is very much the same. As before, Figure 3 shows the results for low levels of persistence in policy setting. The upper panel shows the static forecast-based rule. The deep blue areas to the left of $\phi_\pi = 1$ are areas of indeterminacy, as they were in Figure 1. There are, however, numerous blue "potholes" elsewhere in the panel. These are areas where equilibrium is feasible, but fragile. Even trivial perturbations to the model or the learning rule can overturn E-stability. Notice, however, that these blue regions border very closely to burgundy regions where the allowable perturbations exceed 2; that is, they are very large. The bottom panel shows contours covering the policy persistence level that is optimal, as shown in line (7) of table 1. There are fewer potholes. The optimal policy (in terms of robustness) is toward the top of this chart.

Finally, let us examine Figure 4. The top panel shows that a small increase in ϕ_r from 0.10 to 0.12, reduces the number of potholes to nearly zero. The radii shown in the rest of the chart remain high, but the optimal policy is not in this region.

The bottom panel of the chart shows the contours for a modest and conventional value

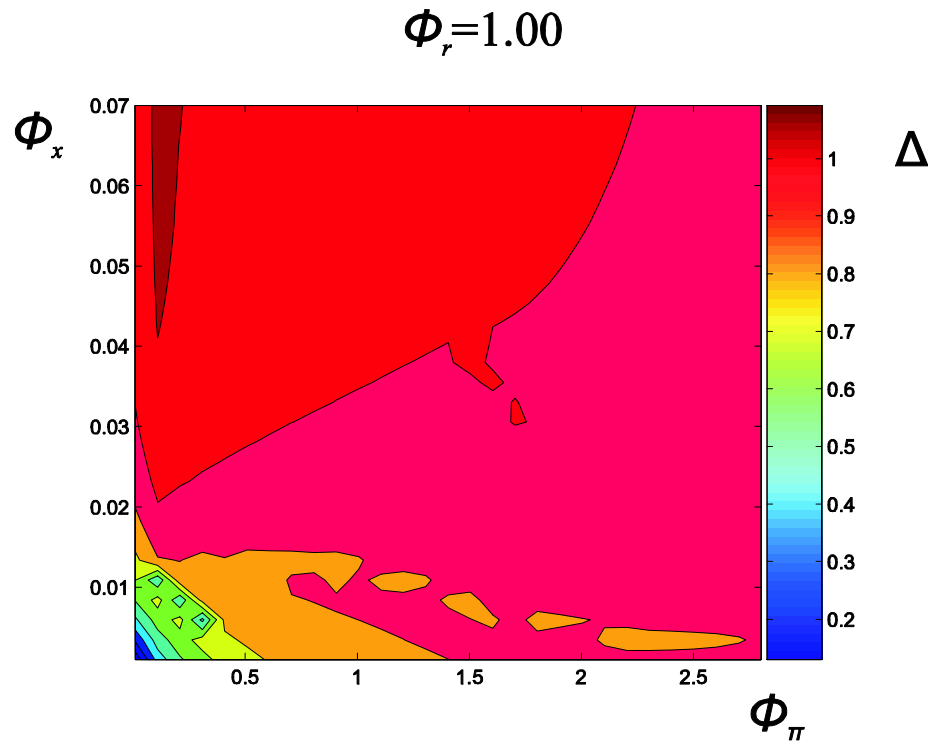
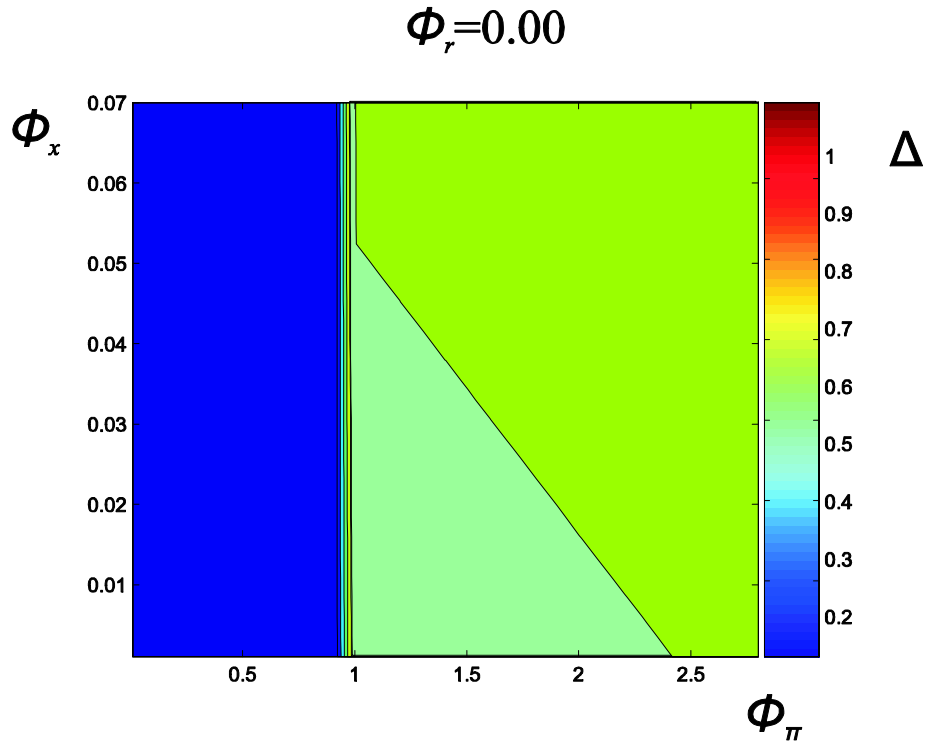


Figure 1: Contours of radii for the NKB model, contemporaneous data rule, selected ϕ_r

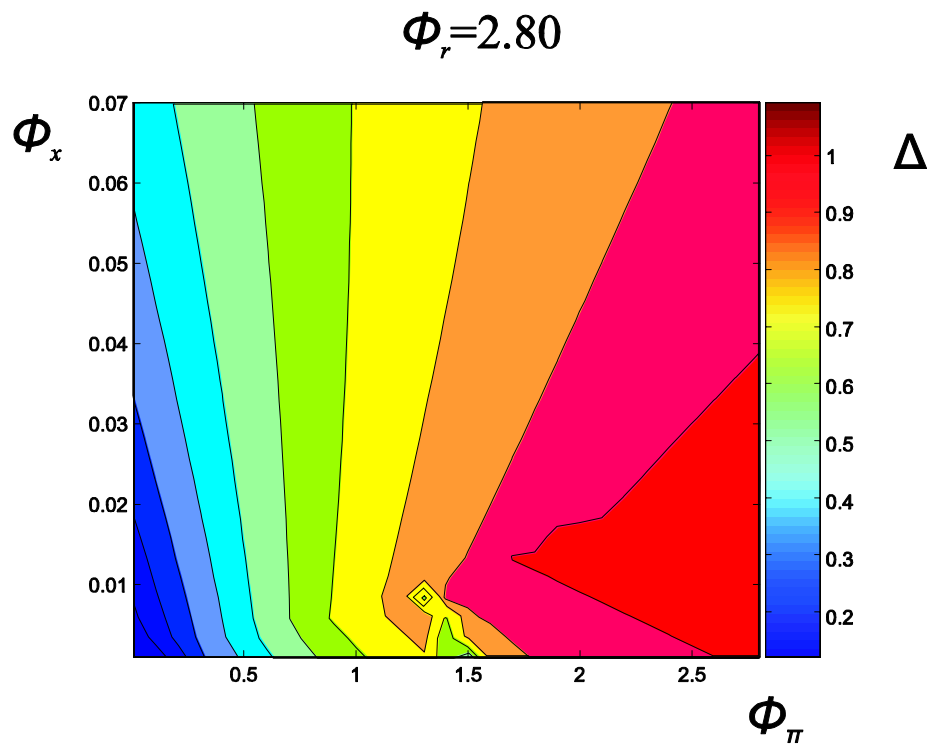
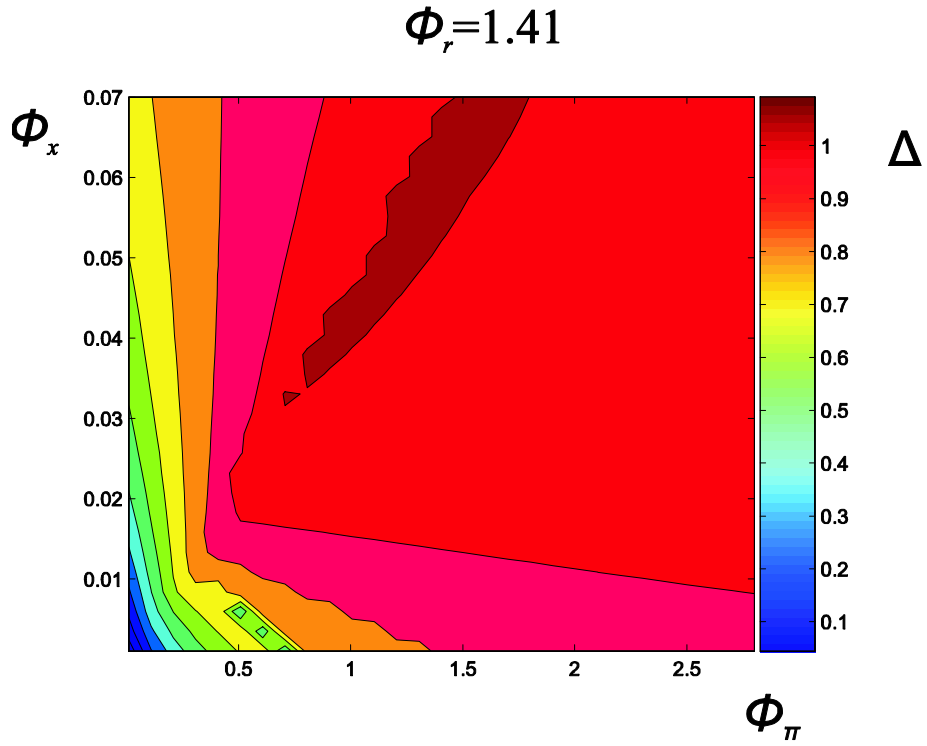


Figure 2: Contours of radii for the NKB model, contemporaneous data rule, selected ϕ_r

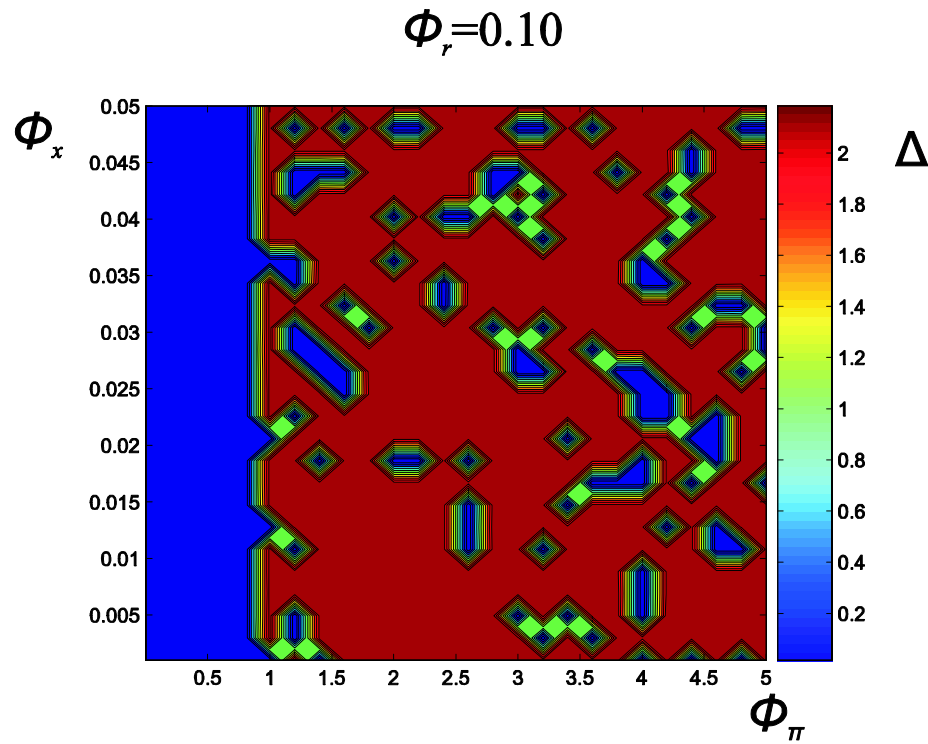
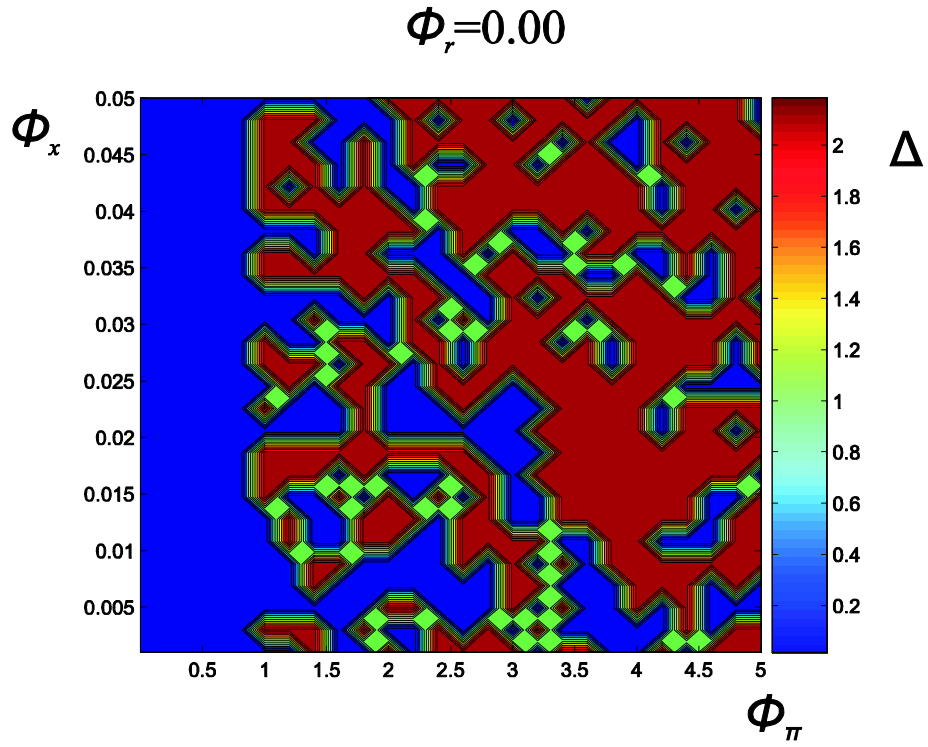


Figure 3: Contours of radii for the NKB model, forecast-based rule, selected ϕ_r .

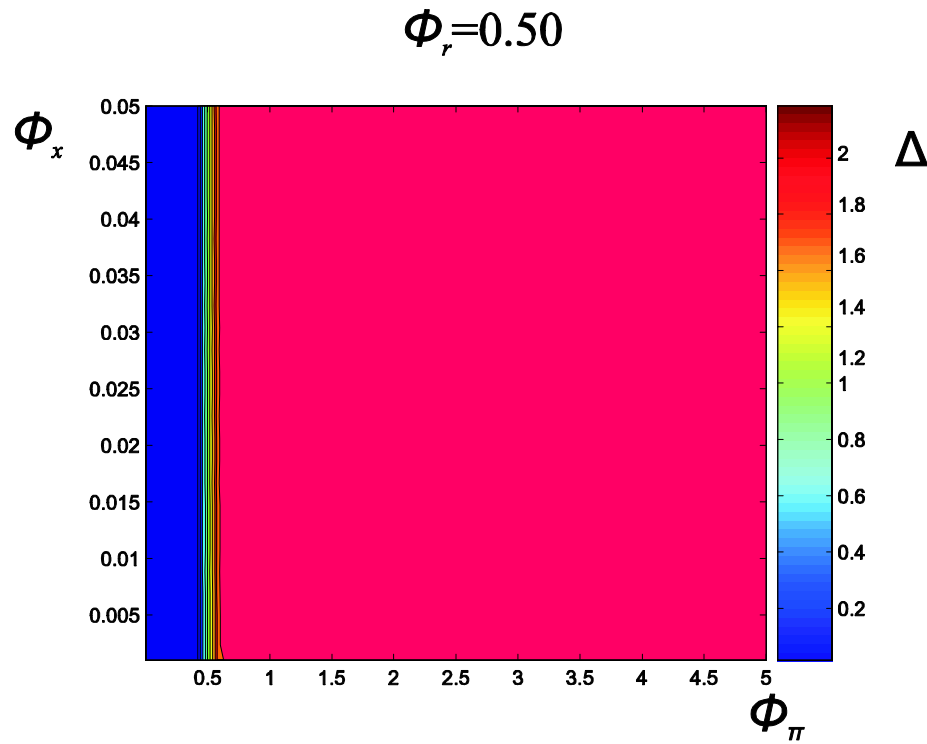
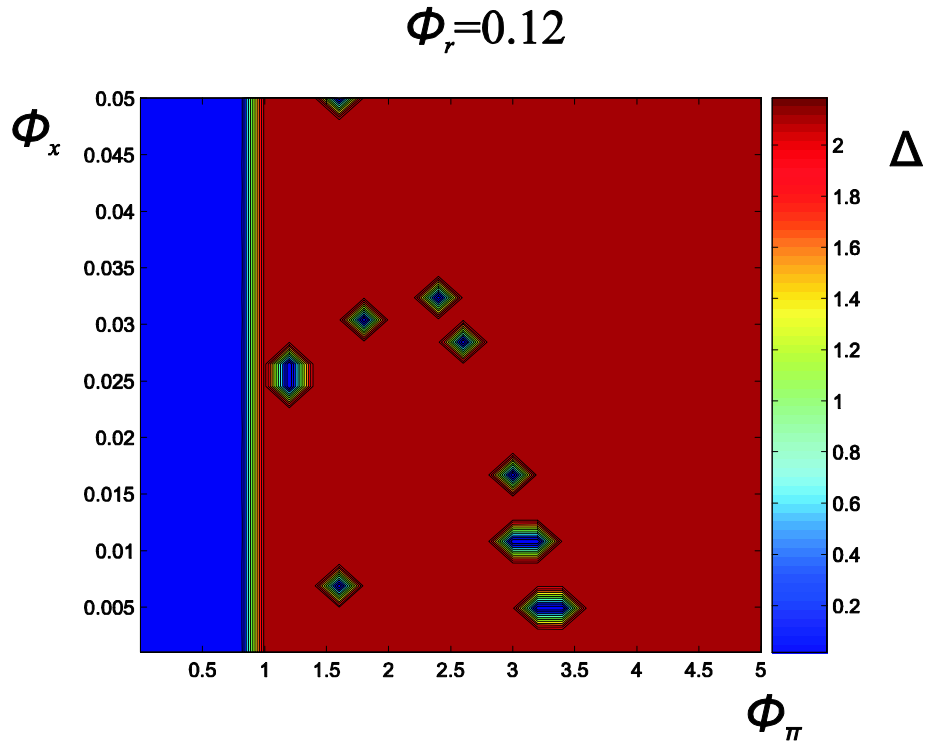


Figure 4: Contours of radii for the NKB model, forecast-based rule, selected ϕ_r .

of funds rate persistence, $\phi_r = 0.50$. The potholes have now completely disappeared, but the large red region is less robust than the burgundy regions in the previous charts. Not shown in these charts are still higher levels of persistence. These involve still lower levels of robustness, with radii for $\phi_r > 1$ associated with radii that are less than half the magnitude of the maximum allowable perturbation for this rule. Higher levels of persistence in policy setting are deleterious for robustification of model learnability.²⁴

Of course these particular results are contingent on the relative weightings for perturbations, captured in W_1 , and our selection is just one of many that could have been made. For the numerical experiments, the weightings were set equal to the standard deviations of the coefficients of a first-order VAR for the output gap, inflation, the interest rate, and the natural rate, estimated at the beginning of each experiment via recursive least squares. These should give a rough idea of the *relative* uncertainties associated with the coefficients of the PLM. Whether using estimated standard deviations to scale the relative impact of Knightian model uncertainty on the elements of the PLM is proper or desirable can be debated, of course. Further, since in a recursive world, the data are generated by the actual law of motion, which depends on the current setting of policy parameters and expectations based on the estimated PLM, the coefficients of the VAR model are not invariant to policy. Such issues we take up in a revision, when we will also deploy relative scalings obtained from a VAR that is re-estimated with each trial vector of policy parameters during the grid search. For now the salient point is that robustness of learning in the presence of model uncertainty is not the same thing as choosing the rule parameters for which the E-stable region of a given model is largest.

4 Concluding remarks

We have argued that model uncertainty is a serious issue in the design of monetary policy. On this score we are in good company. Many authors have advanced that minimizing a loss function subject to a given model presumed to be known with certainty is no longer

²⁴ We tested ϕ_r up to nearly 20. What we found is that the radii fell as ϕ_r rose for intermediate levels, and then rose slowly again for $\phi_r \gg 1$. However, for no level of ϕ_r could we find radii that came anywhere close to the maximum allowable perturbation shown in row (7) of the table.

best practice for monetary authorities. Central bankers must also take model uncertainty and learning into account. Where this paper differs from its predecessors is that we unify three considerations: uncertainty about the model, the learning mechanism used by private agents, and steps the monetary authority can take to address these issues. In particular, we examine a central bank that designs monetary policy to maximize the possible worlds in which ill-informed private agents need time to learn about their particular world and still allow convergence toward the rational expectations equilibrium (REE) of the true economy.

The motivation for this approach is straightforward: if economics as a profession cannot agree on what the true model of the economy is, it is a leap of faith to expect private agents to agree, coordinate, and find the REE themselves. Policy makers should do their part to facilitate the process of learning the REE through the design of policy, where the task of assuring convergence on REE is logically prior to the question of the design of policy, once that convergence has been achieved.

This paper has married the literature on adaptive learning to that of structured robust control to examine what policy makers can do to facilitate learning. We have introduced some tools with which the questions that Bullard and Mitra [8] are asking can be broadened and generalized.

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5 Appendix

Implementation of the techniques outlined in Section 2.2 requires some familiarity with Matlab's *μ -Analysis and Synthesis Toolbox*. Matlab's *User's Guide* and Zhou and Doyle's *Essentials of Robust Control* go a long way to aid the prospective user. The reader should be forewarned, however, that this technique and its literature have been developed in large part by aero-space engineers whose technical jargon is as peculiar and insular as any economist might dream of. From an economist's perspective, the examples in the *User's Guide* range from the exotic (design of F-14 fighter jets) to incomprehensible.

For illustrative purposes, let the perturbed system be one involving two state variables, and let the dynamic model be represented by A , where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Let us further assume that elements a_{12} and a_{21} are subject to scaled structured perturbations assumed to be scalar for our purposes:

$$A^P = \begin{bmatrix} a_{11} & a_{12} + \sigma_1 \Delta_{12} \\ a_{21} + \sigma_2 \Delta_{21} & a_{22} \end{bmatrix}$$

The σ s give relative scalings to the perturbations and might be reasonably related to known or estimated standard deviations of the elements in A . While calculations of μ are unaffected if all scaling factors are multiplied by the same factor, μ and its inverse, the maximal allowable range of perturbation, will be affected by the relative scalings imposed on Δ . So, it is of some consequence to give some thought to scaling. The perturbed system can be represented as the following product of matrices,

$$A^P = A + W_1 \Delta W_2,$$

where

$$\Delta = \begin{bmatrix} \Delta_{12} & 0 \\ 0 & \Delta_{21} \end{bmatrix} \quad W_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \quad W_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where σ_1 and σ_2 are scalar weights. To continue, we need to generate the frequency response, G_2 in (21)-(22). First create the "system"

$$G = \left[\begin{array}{c|c} A & W_2 \\ \hline W_1 & 0_{k,k} \end{array} \right]$$

using the Matlab command

```
G = pck(A,W1, W2,zeros(k, k));
```

where k is the number of diagonal elements in Δ . Next, create the frequency response over a range of frequencies, say 20 points on $\omega \in [0, 2\pi]$

```
 $\omega$  = linspace(0,2*pi,20);
```

and calculate

$$G_2(\omega) = W_2(e^{-i\omega} - A)^{-1}W_1$$

with

```
G2= frsp(G, $\omega$ ,1);
```

As shown in section 2.2, for a given specification of the policy rule, we seek the largest structured singular value of G_2 over the frequency range, ω . The Matlab call

```
opt = ['s','w']; [bnds,rowd,sens,rowp,rowg]=mu(G2,blk,opt);
```

where specifying "opt" as above eliminates annoying output messages. The matrix, *bnds*, contains the upper and lower bounds of μ . For reasons that are explained in Zhou and Doyle [54], *blk* is a $k \times 2$ input matrix that must be specified by the user. If the perturbations are assumed to be scalar in nature, representing one-time jumps, for example, *blk* has negative 'one's in the first column and zeros or 'one's in column two. In the above case, $\text{blk} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$. The relevant numbers are the lower bounds of μ , contained in the second column of *bnds*. For real, scalar perturbations, i.e., those not involving the complex domain, Matlab has trouble computing lower bounds on μ . The result is an annoying stream of Matlab warnings. To get around this, one can tweak the system slightly to introduce a second very small perturbation matrix of complex perturbations too small to affect the results, using the following commands:

```
alpha = .1;
```

```
fix = [eye(kblk);alpha*eye(kblk)];
```

```
blkr = [blk;abs(blk)];
```

```
G2r = mmult(fix,G2,fix');
```

```
opt = ['s','w']; [bnds,rowd,sens,rowp,rowg]=mu(G2r,blkr,opt);
```

The perturbation matrices over the frequencies, ω can be retrieved with the command

```
Delta=unwrapp(pvec,blk);
```

Lower bounds on μ are stored in the second column of *bnds*. The value of interest is the maximum in that column. At the frequency, for which μ is the maximum, the norm of Δ is the inverse of μ . This is the maximal allowable range of perturbations. The policy maker chooses parameters in the interest rate reaction function that cause this range to be the largest.