Endogenous growth, and time to build: the AK case*

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Abstract

In this paper an AK growth model is fully analyzed under the time to build assumption. The existence and uniqueness of the balance growth path and the oscillatory convergence of detrended capital while detrended consumption is constant over time is proved. Moreover the role of transversality conditions and the assumption of capital utilization, make these results hold for any value of the delay.
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1 Introduction

Recently Boucekkine & al. [9], have studied the dynamics of an AK-type endogenous growth model with vintage capital. They find that the inclusion of vintage capital leads to oscillatory dynamics governed by replacement echo. Thus, vintage capital is a possible source of aggregate fluctuations in the economy [8]. In this paper, we propose an AK endogenous growth model [4] when capital takes time to became productive. In the literature this assumption is often referred as "time to build". Jevons [16], was one of the first economists which underlined the important property of capital to be concerned with time: "A vineyard is unproductive for at least three years before it is thoroughly fit for use. In gold mining there is often a long delay, sometimes even of five or six years, before gold is reached"[1]. The time dimension of capital was further studied by Hayek [14], who identified in the structure of production and precisely in the time of production one of the possible sources of aggregate fluctuation. The Hayek's insight was formally confirmed for the first time by Kalecki [17], and afterward by Kydland and Prescott [18], who showed that it contributes to the persistence of business cycle. In this paper, the time to build assumption is introduced by a delay differential equation for capital. Delay differential equations, and in general functional differential equations are very interesting but, at the same time, quite complicated mathematical objects. Since the first contributions of Kalecki [17], Frisch and Holme [11], and Belz and James [6], very few authors have used this mathematical instrument for modelizing the time structure of capital. To our knowledge, the only two works in growth theory introducing time to build in this way, are Rustichini [21], and Asea and Zak [1]. Both of these contributions show that time to build is responsible of the oscillatory behavior of the main variables and that exists parametrization under which the dynamics of the economy can be a cycle. However, even if sometimes implicitly, it is also true that these models predict the

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[1] Jevons [16], Chapter VII: Theory of Capital, page 225.

existence of capital divergence regions, for different parametrizations, namely for values of the time to build coefficient sufficiently high. This result is very difficult to explain from an economic point of view and in our opinion has reduced the interest on this specific source of aggregate fluctuations. After introducing the so-called D-Subdivision method, we show in this paper how this implausible prediction can be eliminated by assuming a more complex structure of capital depreciation. The existence of a unique balance growth path and the dynamic behaviors of the detrendized variables is also fully analyzed and a comparison between the results obtained with the new structure of depreciation and the standard one are reported.

The paper is organized as follows. We first present the model setup in Section 2 and we derive the first order conditions by applying a variation of the Pontrjagin's maximum principle. In Section 3, we introduce some mathematical results on the theory of functional differential equations and the D-Subdivision method is fully described. Taking into account this theoretical background, the existence and uniqueness of the balance growth path is proved and the influence of a variation of the delay coefficient on the magnitude of the growth rate is fully analyzed and reported also in a picture. The transitional dynamics of the economy is reported in Section 5. The next section makes a comparison between the possible different results which can be obtained according to the choice of the structure of the depreciation of capital. An example showing the dynamic behavior of the economy is reported in Section 6. Finally, in Section 7 some concluding remarks are made.

2 Problem Setup

We analyse a standard one sector AK model with time to build. To be precise we assume from now on that capital takes d years to become productive. Then the social planner want to solve the following problem

$$\max \int_0^\infty \frac{c(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$$

subject to

$$\dot{k}(t) = \tilde{A}k(t-d) - c(t) \tag{1}$$

given initial condition $k(t) = k_0(t)$ for $t \in [-d, 0]$ with $d > 0$. All the variables are per capita. The variable $\tilde{A} = (A - \delta) e^{-\phi d} > 0$ depends on A which is the productivity level, δ which is the usual capital depreciation, and on ϕ which is the level of depreciation of capital before starting to become productive. A unit of capital at time t will be depreciated of $e^{-\phi d}$ with $\phi \geq 0$ at $t + d$; the term $e^{-\phi d}$ describes the so-called *capital utilization [12]*. Intuitively, the intensity of use depends inversely by the time to build parameter: the larger is d, the lower will be the utilization of capital in the interval of time $(t, t + d)$. Finally, observe that with no time to build the intensity of use will be maximum and the problem become a standard AK model. Following Kolmanovskii and Myshkis [19] it is possible to extend the *Pontrjagin's principle* to this optimal control problem. Then, the Hamiltonian for this system can be construct:

$$\mathcal{H}(t) = \frac{c(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} + \mu(t)\left[\tilde{A}k(t-d) - c(t)\right]$$

and its optimality conditions are

$$c(t)^{-\sigma} e^{-\rho t} = \mu(t) \tag{2}$$

$$\mu(t+d)\tilde{A} = -\dot{\mu}(t) \tag{3}$$

with the standard transversality conditions

$$\lim_{t\to\infty} \mu(t) \geq 0 \quad and \quad \lim_{t\to\infty} \mu(t)k(t) = 0$$

From equations (2) and (3) we can get the *forward looking Euler-type equation*

$$\frac{\dot{c}\,(t)}{c(t)} = \frac{1}{\sigma}\left[\tilde{A}\left(\frac{c(t)}{c(t+d)}\right)^{\sigma} e^{-\rho d} - \rho\right] \tag{4}$$

Exactly as in the standard AK model, consumption growth does not depend on the stock of capital per person. However in our context the positive constant growth rate is not explicitely given by the Euler equation which is a nonlinear advanced differential equation in consumption. This difference is due to the fact that the real interest rate $r = \tilde{A}\left(\frac{c(t)}{c(t+d)}\right)^{\sigma} e^{-\rho d}$, which the household gets investing in capital, is weighted by the marginal elasticity of substitution between consumption at time t and consumption at time $t+d$. Before proceeding with the analysis of the BGP of our economy, we present in the next section the mathematical instruments which will be used for proving the main results and characteristic of the economy under studying.

3 Some Preliminary Results

Before proceeding, let us evoke some theoretical results on functional differential analysis. Consider the general linear delay differential equation with forcing term $f(t)$:

$$a_0\dot{u}(t) + b_0 u(t) + b_1 u(t-d) = f(t) \tag{5}$$

subject to the initial or boundary condition

$$u(t) = \xi(t) \quad \text{with} \quad t \in [-d, 0]\,. \tag{6}$$

Theorem 1 ***(Existence and Uniqueness)*** *Suppose that f is of class C^1 on $[0,\infty)$ and that ξ is of class C^0 on $[-d,0]$. Then there exists one and only one continuous function $u(t)$ which satisfies (6), and (5) for $t \geq 0$. Moreover, this function u is of class C^1 on (d,∞) and of class C^2 on $(2d,\infty)$. If ξ is of class C^1 on $[-d,0]$, $\dot{u}$ is continuous at τ if and only if*

$$a_0\dot{\xi}(d) + b_0\xi(d) + b_1\xi(0) = f(d) \tag{7}$$

If ξ is of class C^2 on $[-d,0]$, $\ddot{u}$ is continuous at $2d$ if either (7) holds or else $b_1 = 0$, and only in these cases.

Proof. See Bellman and Cooke [5], , Theorem 3.1, page 50-51. ■

The function u singled out in this theorem is called the continuous solution of (5) and (6). Then in order to see the shape of this continuous solution the following theorem is useful:

Theorem 2 *Let $u(t)$ be the continuous solution of (5) which satisfies the boundary condition (6). If ξ is C^0 on $[-d,0]$ and f is C^0 on $[0,\infty)$, then for $t > 0$,*

$$u(t) = \sum_r p_r e^{z_r t} + \int_d^t f(s) \sum_r p_r e^{z_r (t-s)} ds \tag{8}$$

where $\{z_r\}_r$ and $\{p_r\}_r$ are respectively the roots and the residue coming from the characteristic equation, $h(z)$, of the homogeneous delay differential equation

$$a_0\dot{u}(t) + b_0 u(t) + b_1 u(t-d) = 0 \tag{9}$$

Note: $p_r = \frac{p(z_r)}{h'(z_r)}$ where

$$p(z_r) = a_0\xi(d)e^{-z_r d} + (a_0 z_r + b_0)\int_{-d}^{0} \xi(s) e^{-z_r s} ds$$

Proof. See Bellman and Cooke [5], Theorem 3.7, page 75, and Theorem 4.2 and Corollary 4.1, page 109-110. See also El'sgol'ts and Norkin [10],page 80-82. ■

Since in our context it shall be fundamental to have real continuous general solution, we present here other two theoretical results.

Theorem 3 *The unique general continuous solution of problem (5) with boundary condition $\xi : I \subset \mathbb{R} \to \mathbb{R}$ and forcing term $f : I \to \mathbb{R}$, has a unique representation of the form*

$$u(t) = \sum_{r=0}^{k} \varsigma_r e^{x_r t} + \sum_{r=k}^{\infty} \left(a_r e^{z_r t} + \bar{a}_r e^{\bar{z}_r t} \right) + \int_d^t f(s) \left[\sum_{r=0}^{k} \varsigma_r e^{x_r (t-s)} + \sum_{r=k}^{\infty} \left(a_r e^{z_r (t-s)} + \bar{a}_r e^{\bar{z}_r (t-s)} \right) \right] ds \tag{10}$$

where $\{x_r\}$ are real roots, $\{z_r\}$ are complex conjugate roots[2]*, $\{\varsigma_r\}$ are real constants, and $\{a_r\}$ are complex conjugate constants.*

Proof. See Appendix. ■

From this theorem follows immediately the Corollary

Corollary 1 *The general continuous solution (10) can also be written in the form*

$$\begin{aligned} u(t) \quad = \quad & \sum_{r=0}^{k} \varsigma_r e^{x_r t} + 2 \sum_{r=k}^{\infty} \left(\varsigma_r \cos y_r t - \omega_r \sin y_r t \right) e^{x_r t} + \\ & + \int_d^t f(s) \left[\sum_{r=0}^{k} \varsigma_r e^{x_r (t-s)} + 2 \sum_{r=k}^{\infty} \left(\varsigma_r \cos y_r (t-s) - \omega_r \sin y_r (t-s) \right) e^{x_r (t-s)} \right] ds \end{aligned} \tag{11}$$

where $\varsigma_r = \mathrm{Re}(a_r)$ and $\omega_r = \mathrm{Im}(a_r)$ with $\{a_r\}$ residues.

Proof. See Appendix. ■

Some considerations on these theorems are needed. We start with the last two results. The important message of Theorem 3 and Corollary 1 is the following: if we assume a boundary condition and a forcing term which are real functions then also the general continuous solution must be real. Other considerations regard the proofs of Theorem 1 and 2: Both of them are strictly related to the fact that all the roots of $h(z)$ lie in the complex z-plane to the left of some vertical line. That is, there is a real constant c such that all roots z have real part less then c. This consideration is in general no more true for advanced differential equation which are characterized by CE with zeros of arbitrarily large real part. However as explained by Bellman and Cooke[5],[3] it is possible to write the solution of any advanced differential equation as a sum of exponentials using the finite Laplace transformation technique. Moreover observe that the characteristic equation of (5),

$$h(z) \equiv z + a + be^{-zd} = 0 \tag{12}$$

with $a = \frac{b_0}{a_0}$ and $b = \frac{b_1}{a_0}$, is a trascendental function with infinitely many finite roots. Sometimes $h(z)$ is also called the characteristic quasi-polynomial. Asymptotic stability requires that all of these roots have negative real part. In order to help in the stability analysis we introduce two important mathematical results: the Hayes theorem and the *D-Subdivision method* or D-Partitions method. Hayes Theorem [15] in its more general formulation states the following:

[2] We have indicated the conjugate of a complex number a with $\bar{a}$.

[3] Look at Chapter 6 page 197-205.

Theorem 4 *The roots of equation $pe^z + q - ze^z = 0$ where $p, q \in \mathbb{R}$ lies to the left of* $\operatorname{Re}(z) = k$ *if and only if*

(a) $p - k < 1$

(b) $(p-k)\,e^k < -q < e^k\sqrt{a_1^2 + (p-k)^2}$

where a_1 is the root of $a = p\tan a$ such that $a \in (0,\pi)$. If $p = 0$, we take $a_1 = \frac{\pi}{2}$.

One root lies on $\operatorname{Re}(z) = k$ *and all the other roots on the left if and only if* $p - k < 1$ *and* $(p-k)\,e^k = -q$.

Two roots lies on $\operatorname{Re}(z) = k$ *and all the other roots on the left if and only if* $-q = e^k\sqrt{a_1^2 + (p-k)^2}$

Proof. See Hayes [15], page 230-231. ∎

However this Theorem doesn't tell anything about the sign of the real part of the roots of the trascendental function when the conditions *(a)* and *(b)* are not respected. For this reason the D-Subdivision method is now introduced (for more details on this method, El'sgol'ts and Norkin [10], or Kolmanovskii and Nosov [20]). Given a trascendental function like for example (12), this method is able to determine the number of roots having positive real part (for now on p-zeros) in accordance with the value of its parameters (a and b in our specific case). This is possible since the zeros of a trascendental function are continuous functions of their parameters. A D-Subdivision can be obtained by dividing the space of coefficients into regions by hypersurfaces, the points of which correspond to quasi-polynomials having at least one zero on the imaginary axis (the case $z = 0$ is not excluded). For continuous variation of the trascendental function parameters the number of p-zeros may change only by passage of some zeros through an imaginary axis, that is, if the point in the coefficient space passes across the boundary of a region of the D-Subdivision. Thus, to every region Γ_k of the D-Subdivision, it is possible to assign a number k which is the number of p-zeros of the trascendental function, defined by the points of this region. Among the regions of this decomposition are also found regions Γ_0 (if they exist) which are regions of asymptotic stability of solutions. Finally in order to clarify how the number of roots with positive real parts changes as some boundary of the D-Subdivision is crossed, the differential of the real part of the root is computed, and the decrease or increase of the number of p-zeros is determined from its algebraic sign. It turns very useful in the proceeding of the paper to study with the D-Subdivision method the trascendental function (12).

First of all, observe that this equation has a zero root for $a + b = 0$. This straight line (see Figure 1) is one of the lines forming the boundary of the D-Subdivision. It is also immediate to derive that the trascendental function (12) have purely imaginary root iy if and only if

$$a + b\cos dy = 0, \quad y - b\sin dy = 0 \tag{13}$$

or

$$b = \frac{y}{\sin dy}, \quad a = \frac{-y\cos dy}{\sin dy} \tag{14}$$

The equations in parametric form (13) or (14) identify all the other D-Subdivision boundaries. To be precise there is one boundary for any of the following interval of y: $\left(0, \frac{\pi}{d}\right), \left(\frac{\pi}{d}, \frac{2\pi}{d}\right), \left(\frac{2\pi}{d}, \frac{3\pi}{d}\right), \ldots$. Moreover it is possible (and useful) to find the values of b for which the boundaries intercept the b-axis. The sequence of such b is $\left\{\ldots, -\frac{7\pi}{2d}, -\frac{3\pi}{2d}, 0, \frac{\pi}{2d}, \frac{5\pi}{2d}, \ldots\right\}$. Finally we show how p-zeros rises. In particular we show how crossing C_l from Γ_0 to Γ_2 two p-zeros appear (that is, we focus on the interval $0 < y < \frac{\pi}{d}$). From (12) applying the implicit function theorem, we have that on

C_l

$$\begin{aligned} dx &= -\operatorname{Re}\frac{da}{1-bde^{-diy}} \\ &= -\operatorname{Re}\frac{da}{1-bd\left(\cos dy - i\sin dy\right)} \\ &= \frac{\left(1-bd\cos dy\right)da}{\left(1-bd\cos dy\right)^2 + b^2d^2\sin^2 dy} \end{aligned}$$

We find that $\cos yd < 0$ for $bd > 1$. Therefore, upon crossing the boundary C_l from region Γ_0 into Γ_2, the pair of complex conjugate roots gain positive real parts. The analysis on the other boundaries of the D-Subdivision is completely analogous. Taking into account all of these result, we are now ready to study completely our model.

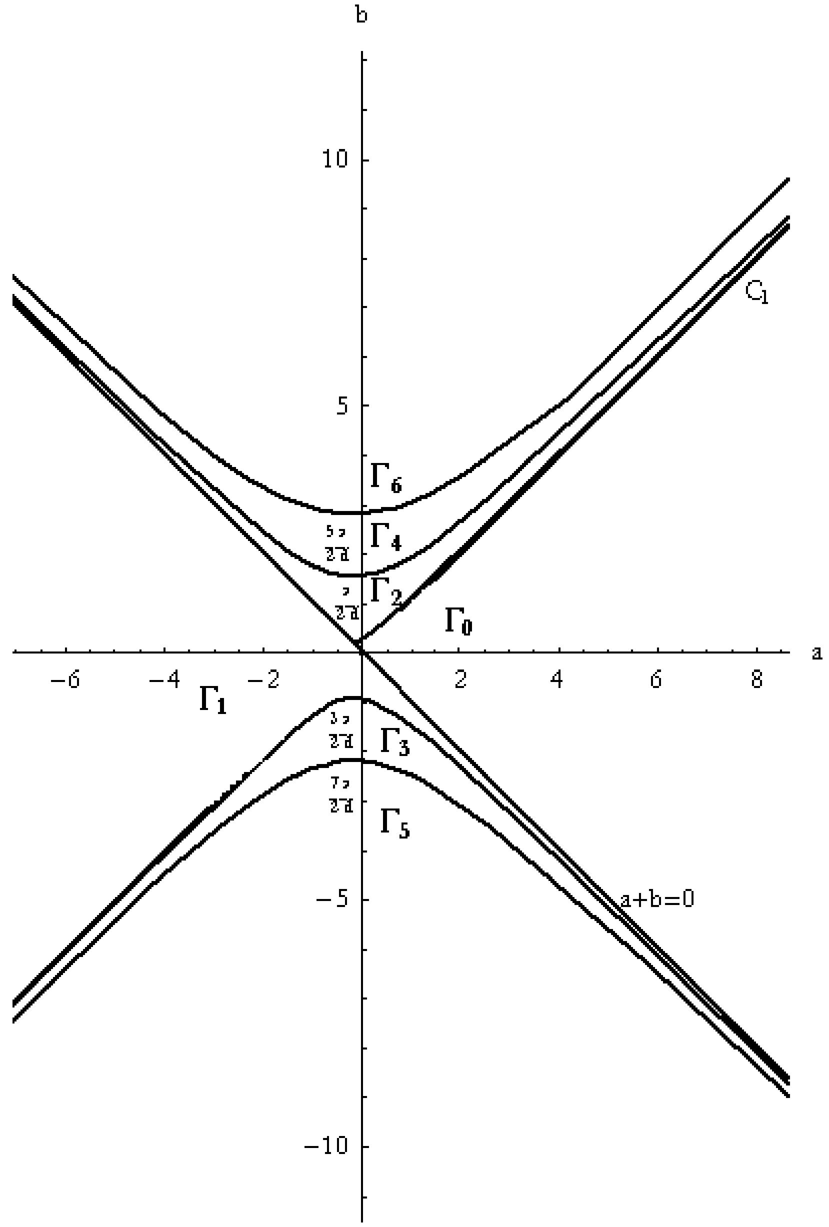

Figure 1: D-Subdivision for the trascendental function (12) assuming $d = 5$.

4 Balance Growth Path Analysis

In order to show the existence and uniqueness of the BGP we present before some results regarding the roots of the characteristic equation of the law of motion of capital, of its shadow price, and of consumption. These results are presented and proved in Lemma 1 and Lemma 2, respectively. Some pictures are also provided in order to help the reader in getting the main message behind the math. After that, the continuous solution of capital is rewritten as a sum of weighted exponential (Proposition 1) and then, following a very similar strategy as that used in the standard AK model, a unique balance growth path for consumption and capital is proved by checking the transversality condition. Very similar is also the requirement that for any exogenously given choice of the delay coefficient, the production function has to be sufficiently productive to ensure growth in consumption, but not so productive as to yield unbounded utility: $A \in (A_{\min}, A_{\max})$. On the other hand, it is possible to express the same requirement, given a certain level of technology, in term of the delay coefficient: $d \in (d_{\min}, d_{\max})$. Finally as in the standard case if $\sigma > 1$, then $A_{\max}$ is equal to plus infinity, while $d_{\min}$ to zero.

As anticipated in Lemma 1 we report some information on the roots of the CE of the law of motion of capital and its shadow price:

Lemma 1 *For any ϕ sufficiently high the following results hold:*
1) $\tilde{z}$ is the unique root with positive real part of the CE of the law of motion of capital;
2) $\tilde{s}$ is the unique root with negative real part of the CE of the law of motion of shadow price.

Proof. The characteristic equation of the law of motion of capital (1) is equal to the characteristic equation of its homogeneous part[4], namely

$$z - \tilde{A}e^{-zd} = 0 \tag{15}$$

It immediate to show that this equation has a unique positive real root $z_{\tilde{v}} = \tilde{z}$ which is also the highest among its roots. In particular, through the *D-Subdivision method* it is possible to prove that the trascendental equation (15) has an increasing number of p-zeros as d rises. On the other hand if we assume $\phi = \hat{\phi}$ sufficiently high,[5] it happens that $\tilde{A} < \frac{3\pi}{2d}$ for any choice of d and then a unique p-zero exists[6]. These facts can be easily observed in Figure 2. Finally, $\tilde{z} > \mathrm{Re}(z_v)$ for any $v \neq \tilde{v}$ since all the roots of the CE of (1) in the detrended variables $\hat{x}(t) = x(t)e^{-\tilde{z}t}$ are negative. This is sufficient to prove result 1). Now observe that the CE of the shadow price law of motion (3) is

$$-s - \tilde{A}e^{sd} = 0 \tag{16}$$

then we can put in correspondence the roots of (15) and (16) through the transformation $z = -s$. From this consideration follows immediately that $\mathrm{Re}(s) = -\mathrm{Re}(z)$ and $\tilde{s} = -\tilde{z}$ is the root with lowest real part of the characteristic equation of the law of motion of shadow price. ■

[4] The part of equation (1) not considering the forcing term $-bC(t)$.

[5] In the numerical simulation we may assume for example $\hat{\phi}$ close to δ.

[6] This is also a consequence of the fact that $\tilde{A}$ converges to zero faster than $\frac{3\pi}{2d}$ as $d \to \infty$.

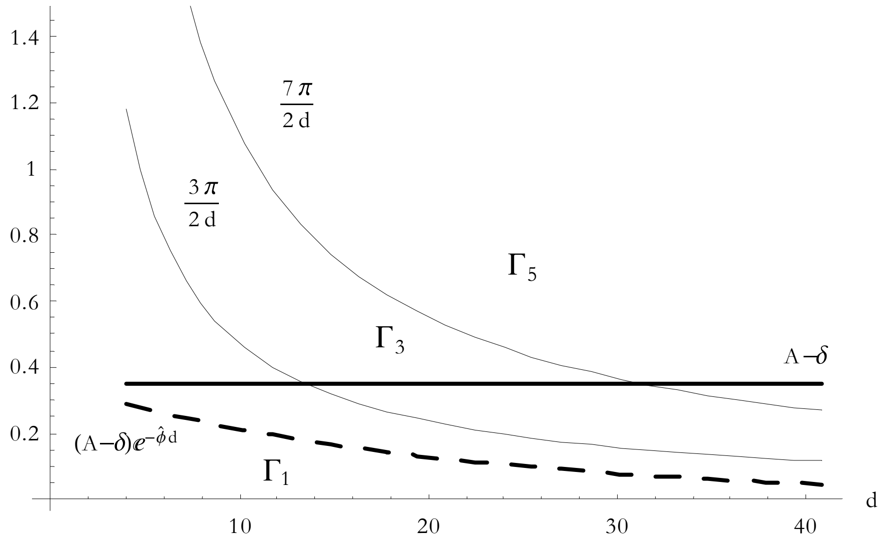

Figure 2: Number of p-zeros of (15) according to the choice of the delay coefficient.

Lemma 1 tells us that if we assume a sufficiently high depreciation $\hat{\phi}$, then $\tilde{z}$ is the constant growth rate of capital and the unique p-zero of (15). Now it will be useful for proving a common growth rate of consumption and capital to show the following Lemma:

Lemma 2 *A positive and constant growth rate of consumption, g_c, always exists for $A > A_{\min} = \delta + \rho e^{(\rho+\hat{\phi})d}$.*

Proof. First of all observe that since the Euler equation (4) is a nonlinear ADE we cannot write directly its continuous general solution (Theorem 2 doesn't apply). However it is possible to overcome this fact by observing that the general continuous solution of consumption can be obtained indirectly by the first order condition (2). Considering that the general solution of the shadow price of capital is $\mu(t) = \sum\limits_m a_m e^{-z_m t}$ we have that

$$c(t) = \frac{1}{\left(\sum\limits_m a_m e^{-\sigma\lambda_m t}\right)^{\frac{1}{\sigma}}} \tag{17}$$

where we have called

$$\lambda = \frac{1}{\sigma}(z - \rho) \tag{18}$$

From equation (17) we can derive that the basic solutions of (4) have exponential form, namely the basic solutions are $\left\{e^{\lambda_m}\right\}_m$; moreover taking into account (15) and (18) we can derive indirectly the characteristic equation of (4)

$$h(\lambda) = \sigma\lambda + \rho - \tilde{A}e^{-(\sigma\lambda+\rho)d} \tag{19}$$

Using the Hayes theorem or the D-Subdivision method, a unique positive real root, $\lambda_{\tilde{m}} = g_c$ exists for A sufficiently large, namely $A > A_{\min} = \delta + \rho e^{(\rho+\phi)d}$. This is exactly the same condition of the standard AK model when the assumption of time to build is introduced. Observe also that in this context the same requirement can be expressed in term of the delay, $d < d_{\max} = \frac{1}{\rho+\phi}\log\frac{A-\delta}{\rho}$. Exactly as before, a unique p-zero exists if $\tilde{A}e^{-\rho d} < \frac{3\pi}{2d}$. It is obvious that for $\phi = \hat{\phi}$ the inequality is always respected (see Figure 3) since $\hat{\phi}$ was sufficient to force $\tilde{A}$ to stay below $\frac{3\pi}{2d}$ and given that $(A-\delta)\, e^{-\hat{\phi} d} e^{-\rho d}$ is a product of functions which are positive and monotonic decreasing in d. As it will appear clear in Section 6 for any $\phi \in \left[0, \hat{\phi}\right)$, some economic implausible prediction may rise. Then from now on we focus on the case $\phi \geq \hat{\phi}$. Now endogenous growth implies that consumption and capital have to growth at a positive rate over time. This implies that $\lim_{t\to\infty} c(t) = +\infty$; then given (17), we have to impose that

$$\lim_{t\to\infty} \frac{1}{\left(a_{\tilde{m}} e^{-\sigma g_c t} + \sum_{m \notin \tilde{m}} a_m e^{-\sigma \lambda_m t}\right)^{\frac{1}{\sigma}}} = +\infty \tag{20}$$

Using the properties of the limits[7], it is possible to rewrite (20) as

$$\frac{1}{\left(\underbrace{\lim_{t\to\infty} a_{\tilde{m}} e^{-\sigma g_c t}}_{\to 0} + \underbrace{\sum_{m \notin \tilde{m}} \lim_{t\to\infty} a_m e^{-\sigma \lambda_m t}}_{\to \infty}\right)^{\frac{1}{\sigma}}} = +\infty$$

Then it results that the relation (20) is satisfied if and only if $a_m = 0$ for any $m \neq \tilde{m}$. Taking into account this fact, the general continuous solution of consumption is

$$c(t) = a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c t}$$

■

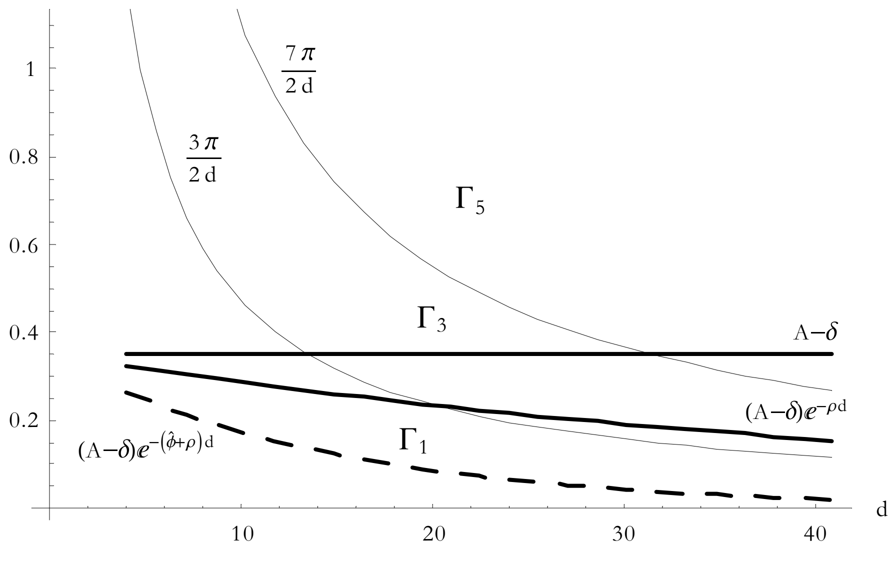

Figure 3: Number of p-zeros of (19) according to the choice of the delay coefficient.

Our objective is to prove that the growth rate of consumption and capital are the same $g = g_c$. However before proving it, we introduce the following Corollary of Theorem 2 which let us to rewrite the continuous solution of capital as a sum of weighted exponentials.

[7] The following properties have been used: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$, and $\lim_{x\to a} \left[\sum_i f_i(x)\right] = \sum_i \lim_{x\to a} f_i(x)$

Proposition 1 *The solution of the law of motion of capital can be written as*

$$k(t) = \sum_v P_{\tilde{m},v} e^{g_c t} + \sum_v N_{\tilde{m},v} e^{z_v t} \tag{21}$$

where $P_{\tilde{m},v} = \left(-\frac{a_{\tilde{m}}^{-\frac{1}{\sigma}} n_v}{g_c - z_v} \right)$ *and* $N_{\tilde{m},v} = n_v \left(\frac{a_{\tilde{m}}^{-\frac{1}{\sigma}}}{g_c - z_v} e^{(g_c - z_v)d} + 1 \right)$.

Proof. According to Theorem 2 and Lemma 2, the continuous general solution of consumption and capital are respectively

$$c(t) = a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c t} \tag{22}$$

$$k(t) = \sum_v n_v e^{z_v t} - \int_d^t c(s) \sum_v n_v e^{z_v (t-s)} ds \tag{23}$$

Now the integral part of equation (23) is equal to

$$\int_d^t a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c s} \sum_v n_v e^{z_v (t-s)} ds = \sum_v \left(\frac{n_v a_{\tilde{m}}^{-\frac{1}{\sigma}}}{g_c - z_v} \right) \left(e^{g_c t} - e^{z_v t + (g_c - z_v) d} \right)$$

and substituting in (23) after some algebra we get (21). ■

Some comments on equations (22) and (21) are needed. These equations are very close to the general solution form for consumption and capital in the usual framework, with ordinary differential equation; in particular $k(t)$ is a weighted sum of exponentials; however, this similarity can be found for system of mixed functional differential equations *only* in the particular case of a single equation with forced term. In the most general case it doesn't exist a theorem which let us to write the solution in this way[8]. Moreover, the continuous solution of the law of motion of consumption (22) and capital (21), are not the optimal solution exactly as it happens in the ordinary case. Before getting optimality, transversality conditions have to be checked. Using this corollary and TVC we prove now the existence of a unique balance growth path for consumption and capital.

Proposition 2 *Consumption and capital have the same balanced growth path* $g = g_c$. *This growth rate is positive and yield bounded utility if* $A \in (A_{\min}, A_{\max})$.

Proof. As shown in Lemma 2, the growth rate of consumption g_c is a positive constant if $A > A_{\min}$. Given that, we have to distinguish two cases: $\tilde{z} \leq g_c$ and $\tilde{z} > g_c$. The first case is never possible. In fact, assume that $\tilde{z} \leq g_c$ then g_c is also the growth rate of capital as follows immediately by looking at equation (21). Then we can rewrite the characteristic equation of capital, after the transformation $\hat{k}(t) = k(t)e^{-g_c t}$, as

$$-we^w - g_c d e^w + \tilde{A} d e^{-g_c d} = 0 \tag{24}$$

where $w = zd$. Since g_c is the root having greater positive real part all the roots of (24) must have negative real part which, from Hayes Theorem, it implies also that $g_c > \tilde{A}e^{-g_c d}$. However, this is never possible since it contradicts the positive sign of the consumption to output ratio at the balanced growth path

$$\frac{c(t)}{k(t)} = \tilde{A}e^{-g_c d} - g_c > 0$$

[8] Recently Asl and Ulsoy [2] have proved that a general solution form can be written for system of delay differential equations using Lambert function.

which can be obtained by dividing the law of motion of capital (1) by $k(t)$. Then the only possible case is $\tilde{z} = \sigma g_c + \rho > g_c$. This is exactly the requirement for having no unbounded utility: $(1-\sigma)\, g_c < \rho$. Then before passing to the TVC we observe that if $\sigma > 1$, the utility is always bounded; on the other hand if $0 < \sigma < 1$ we need a condition on A such that the utility is bounded. Taking into account the CE (19) after some algebra this condition is $A < A_{\max} = \delta + \frac{\rho}{1-\sigma} e^{\left(\frac{\rho+\phi(1-\sigma)}{1-\sigma}\right)d}$ which is exactly the same condition for the standard AK model when the time to build parameter is equal to zero. Observe also that such condition can be rewritten also in term of the delay, $d > d_{\min} = \frac{1-\sigma}{\rho+(1-\sigma)\phi} \log \frac{(A-\delta)(1-\sigma)}{\rho}$. Now we show that the TVC

$$\lim_{t\to\infty} \mu\,(t)\, k(t) = 0 \tag{25}$$

implies necessary a unique BGP which is g_c. In order to see this, we substitute the general continuous solutions of $\mu(t)$ and $k(t)$, into the TVC (25) and we get:

$$\lim_{t\to\infty} a_{\tilde{m}} e^{-\tilde{z}t} \left(\sum\nolimits_v P_{\tilde{m},v} e^{g_c t} + \sum\nolimits_v N_{\tilde{m},v} e^{z_v t}\right) = 0 \tag{26}$$

which is equal to

$$\lim_{t\to\infty} \left[a_{\tilde{m}} N_{\tilde{m},\tilde{v}} + \sum\nolimits_{v\neq\tilde{v}} N_{\tilde{m},\tilde{v}} e^{(z_v-\tilde{z})t} + \sum\nolimits_v P_{\tilde{m},v} e^{(g_c-\tilde{z})t}\right] = 0$$

now for $a_{\tilde{m}} \neq 0$, the second and third term in the parenthesis converge to zero since $z_v - \tilde{z} < 0$ for any v and $g_c - \tilde{z} < 0$. Then the TVC are respected if and only if

$$N_{\tilde{m},\tilde{v}} \equiv \frac{a_{\tilde{m}}^{-\frac{1}{\sigma}}}{g_c - \tilde{z}} e^{(g_c-\tilde{z})d} + 1 = 0 \tag{27}$$

which implies

$$a_{\tilde{m}} = \left(\frac{1}{\tilde{z} - g_c} e^{(g_c-\tilde{z})d}\right)^{\sigma} \tag{28}$$

Concluding TVC holds if and only if condition (27) is verified. Given this condition, g_c is also the growth rate of capital since the continuous general solution of capital (23) can be rewritten as follows

$$k(t) = \sum\nolimits_v P_{\tilde{m},v} e^{g_c t} + \sum\nolimits_{v\neq\tilde{v}} N_{\tilde{m},v} e^{z_v t} \tag{29}$$

Then the optimal solution of capital (29) is asymptotically driven by g_c which implies a common growth rate with consumption. ■

This proposition puts in evidence how a unique balance growth path for consumption and capital can be proved to exist also in the case of time to build by checking to the transversality conditions. In fact through the conditions (27) it is possible to rule out the eigenvalue coming from the characteristic equation of the law of motion of capital, having positive real part greater than g_c. Observe also that this fact and the assumption on the structure of capital depreciation make all of these results valid for any choice of the delay in the interval $(d_{\min}, d_{\max})$ which guarantees presence of endogenous growth and no unbounded utility.

Once we have shown that $g = g_c$ is the unique balanced growth path of consumption and capital it is also interesting to see the variation of it to different choices of the delay coefficient, d, and of the level of technology A. These considerations are reported in the the following corollary of Proposition 2:

Corollary 2 *Under* $A \in (A_{\min}, A_{\max})$, $\frac{\partial g}{\partial d}$ *is negative while* $\frac{\partial g}{\partial A}$ *is positive.*

Proof. Under $A \in (A_{\min}, A_{\max})$, we have shown that g is the unique positive balance growth path for consumption and capital. The effect of a variation of d or A on g can be easily computed by applying the Implicit Function Theorem on the trascendental equation (19) which is always satisfied for $\lambda = g$. After some algebra we obtain that

$$\begin{aligned} \frac{\partial g}{\partial d} &= -\frac{(A-\delta)(\sigma g+\rho+\phi)\, e^{-(\sigma g+\rho+\phi)d}}{\sigma+\sigma d\,(A-\delta)\, e^{-(\sigma g+\rho+\phi)d}} < 0 \\ \frac{\partial g}{\partial A} &= \frac{e^{-(\sigma g+\rho+\phi)d}}{\sigma+\sigma d\,(A-\delta)\, e^{-(\sigma g+\rho+\phi)d}} > 0 \end{aligned}$$

In Figure 5, we have reported the behavior of g as d rises (the decreasing curve) and the standard case with $d = 0$ for the following parametrization: $\sigma = 8$, $\rho = 0.02$, $A = 0.30$, $\delta = 0.04$, and $\hat{\phi} = \delta$. Given these values, d has to be in the interval $(0, 42.74)$ in order to have a positive balance growth path. ■

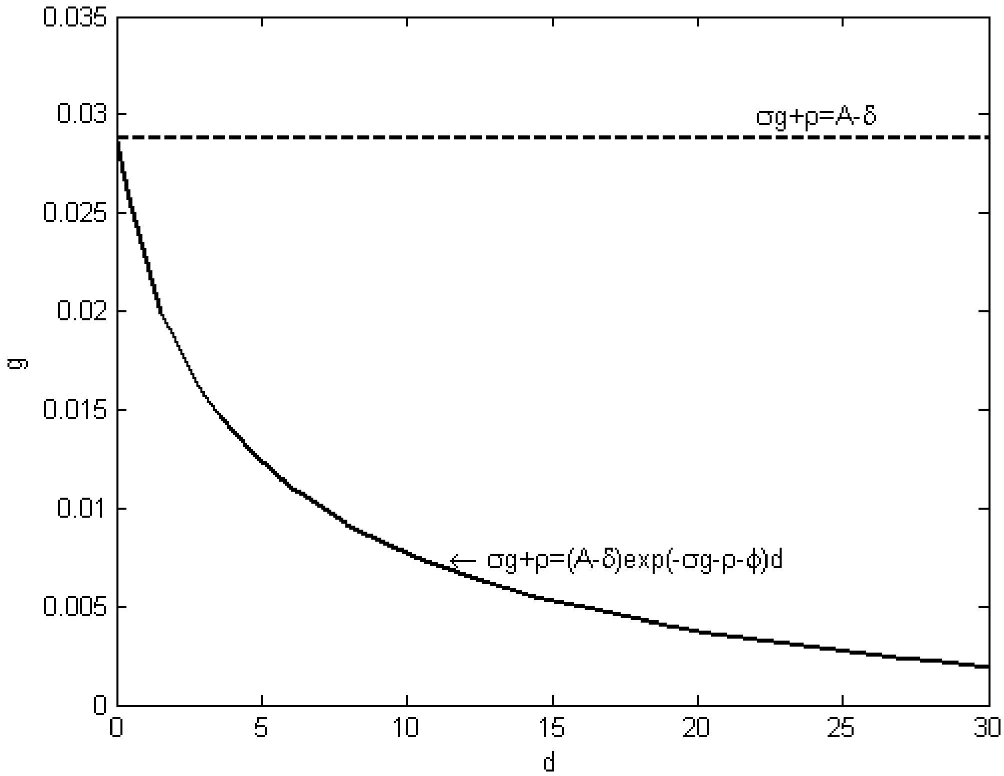

Figure 4: Behavior of the balance growth path, g, to variations of d.

5 Consumption and Capital Dynamics

In the previous section, we have proved the existence and uniqueness of the balance growth path. We have also shown the influence of the delay coefficient on the growth rate for a given level of technology. In this section, we focus on the dynamic behavior of the optimal detrended consumption and capital which let us to derive indirectly the behavior of detrended income and detrended investment.

Proposition 3 *Optimal detrended consumption is constant over time while optimal detrended capital path is unique and oscillatory converges to a constant.*

Proof. The optimal detrended solution of capital and consumption can be obtained by multiplying both sides of equations (29) and (22) by $e^{-g_c t}$ obtaining

$$\begin{aligned} \hat{c}(t) &= a_{\tilde{m}}^{-\frac{1}{\sigma}} && (30) \\ \hat{k}(t) &= \sum\nolimits_v P_{\tilde{m},v} + \sum\nolimits_{v\neq\tilde{v}} N_{\tilde{m},v} e^{(z_v - g_c)t} && (31) \end{aligned}$$

After calling $z = x + iy$ and $n = \alpha + i\beta$,.the detrended solution for capital can be rewritten, as shown in the Appendix, in the following way

$$\hat{k}(t) = \sum\nolimits_v \Psi_{0,v} + \sum\nolimits_v \left[(\alpha_v + \Psi_{1,v})\cos y_v t - (\beta_v + \Psi_{2,v})\sin y_v t\right] e^{(x_v - g_c)t} \tag{32}$$

where $\Psi_{0,v}, \Psi_{1,v}, \Psi_{2,v} \in \mathbb{R}$. Finally, consider the limit for t going to infinity we find that

$$\lim_{t\to\infty} \hat{k}(t) = \sum\nolimits_v \Psi_{0,v} \tag{33}$$

Expressions (32) and (33) tell us that the transition to the BGP is oscillatory since the presence of the cosine and sine term, and that the convergence is guarantee by the fact that $x_v = \mathrm{Re}\,(z_v) < g_c$ for any $v \neq \tilde{v}$. Finally the uniqueness of the path is due to the fact that the residue $\{n_v\}_v$ are fixed by the boundary condition of capital while the residue $a_{\tilde{m}}$ by the transversality condition through the expression (28). ■

In the following section, we introduce some technical reasons which in our opinion are convincing in the choice of a $\phi \geq \hat{\phi}$. However , as it appears clear soon, all the results obtained until now remain valid even for the extreme case $\phi = 0$ when an appropriate sub-interval of d is appropriately chosen.

6 Some considerations on the role of ϕ

(...)

7 Numerical Exercise

We show now very briefly a numerical exercise. Table

σ	ρ	δ	ϕ	d	A	$A_{\min}$	$A_{\max}$
0.8	0.02	0.05	0.03	15	0.3	0.092	0.753

Given this value we can report here the following two graphs

and

8 Conclusion

This paper fully analyzed a neoclassical endogenous growth model when the time to build assumption is introduced through a delay differential equation for capital. It has been proved that a unique BGP exists and that exists a unique optimal path of the detrended capital which oscillatory converge to the BGP while detrended consumption jumps directly on it as the usual case without delay. These results have been obtained through a careful analysis of the role of transversality conditions and the introduction of the assumption of a more complex structure of the depreciation of capital. This last assumption appears to be crucial in avoiding implausible economic predictions which always appears in this type of model for choice of the time to build coefficient sufficiently high. Finally the analysis of the model let us to confirm that time to build can be considered a source of aggregate fluctuation for capital and output exactly as the vintage capital assumption. An interesting topics for future research should be to measure how much of the cyclical behavior can be explained by the different hypothesis on the structure of capital.

A Appendix

A.1 Proof of Theorem 3.

From Theorem 2, the unique general solution of (5) with boundary condition (6) is

$$u(t)=\sum_r p_r e^{z_r t}+\int_d^t f(s)\sum_r p_r e^{z_r(t-s)}ds \tag{34}$$

where the roots $\{z_r\}$ and the residues $\{v_r\}$ come respectively by the characteristic equation of the homogeneous part of (5)

$$h\left(z\right)=a_0 z+b_0+b_1 e^{-zd} \tag{35}$$

and by the relation

$$p_r=\frac{p\left(z_r\right)}{h'\left(z_r\right)}=\frac{a_0\xi(d)e^{-z_r d}+(a_0 z_r+b_0)\int_{-d}^{0}\xi(s)e^{-z_r s}ds}{a_0-b_1 d e^{-z_r d}} \tag{36}$$

Moreover from the D-Subdivisions method we know that (35) has at most two real roots and an infinite number of complex conjugate roots. From (36), it appears also clear that the residues related to real roots are real while those related to complex roots are complex. Taking into account these results it is possible to split (34) as follows

$$u\left(t\right)=\sum_{r=0}^{k}\varsigma_r e^{x_r t}+\sum_{r=k}^{\infty}\left(a_r e^{z_r t}+c_r e^{\bar{z}_r t}\right)+\int_d^t f(s)\left[\sum_{r=0}^{k}\varsigma_r e^{x_r(t-s)}+\sum_{r=k}^{\infty}\left(a_r e^{z_r(t-s)}+c_r e^{\bar{z}_r(t-s)}\right)\right]ds$$

where $z=x+iy$ and $\bar{z}=x-iy$. We now show that $c_r=\bar{a}_r$ is always the case. This is equivalent to show that, given the expressions of a_r and c_r, the $\mathrm{Im}\left(c_r+a_r\right)=0$ and that the $\mathrm{Re}\left(c_r-a_r\right)=0$. We start showing the first relation. In order to simplify a bit the notation we omit the r:

$$\begin{aligned}
a+c &= \frac{a_0\xi(d)e^{-zd}+(a_0 z+b_0)\int_{-d}^{0}\xi(s)e^{-zs}ds}{a_0-b_1 d e^{-zd}}+\frac{a_0\xi(d)e^{-\bar{z}d}+(a_0\bar{z}+b_0)\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds}{a_0-b_1 d e^{-\bar{z}d}}\\
&= \frac{\left[\xi(d)e^{-zd}+\left(z+\tilde{b}_0\right)\int_{-d}^{0}\xi(s)e^{-zs}ds\right]\left[a_0-b_1 d e^{-\bar{z}d}\right]}{\frac{1}{a_0}\left[a_0-b_1 d e^{-zd}\right]\left[a_0-b_1 d e^{-\bar{z}d}\right]}+ \qquad (37)\\
&\quad +\frac{\left[\xi(d)e^{-\bar{z}d}+\left(\bar{z}+\tilde{b}_0\right)\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds\right]\left[a_0-b_1 d e^{-zd}\right]}{\frac{1}{a_0}\left[a_0-b_1 d e^{-zd}\right]\left[a_0-b_1 d e^{-\bar{z}d}\right]} \qquad (38)
\end{aligned}$$

where $\tilde{b}_0=\frac{b_0}{a_0}$. The denominator is always real since:

$$\begin{aligned}
\left[a_0-b_1 d e^{-zd}\right]\left[a_0-b_1 d e^{-\bar{z}d}\right] &= a_0^2-a_0 b_1 d\left(e^{-zd}+e^{-\bar{z}d}\right)+b_1^2 d^2 e^{-zd-\bar{z}d}=\\
&= a_0^2-a_0 b_1 d e^{-xd}\left(e^{-iyd}+e^{iyd}\right)+b_1^2 d^2 e^{-2xd}
\end{aligned}$$

and taking into account that $e^{iy}+e^{-iy}=2\cos y$ while $e^{iy}-e^{-iy}=2i\sin y$ we have that

$$\left[a_0-b_1de^{-zd}\right]\left[a_0-b_1de^{-\bar{z}d}\right]=a_0^2-2a_0b_1de^{-xd}\cos yd+b_1^2d^2e^{-2xd}$$

which is real. Then we have to show that also the numerator of relation (37) is real.

$$Num=A+B$$

where

$$\begin{aligned}
A &= \xi(d)e^{-zd}\left[a_0-b_1de^{-\bar{z}d}\right]+\xi(d)e^{-\bar{z}d}\left[a_0-b_1de^{-zd}\right]\\
&= a_0\xi(d)e^{-xd}\left(e^{iyd}+e^{-iyd}\right)+2b_1d\xi(d)e^{-2xd}\\
&= 2\xi(d)e^{-xd}\left[a_0\cos yd+b_1de^{-xd}\right]
\end{aligned}$$

which is real. On the other hand

$$B=\left[z+\tilde{b}_0\right]\left[a_0-b_1de^{-\bar{z}d}\right]\int_{-d}^{0}\xi(s)e^{-zs}ds+\left[\bar{z}+\tilde{b}_0\right]\left[a_0-b_1de^{-zd}\right]\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds \tag{39}$$

Now observe that $\int_{-d}^{0}\xi(s)e^{-zs}ds=\underbrace{\int_{-d}^{0}\xi(s)e^{-xs}\cos ys\; ds}_{\alpha}-i\underbrace{\int_{-d}^{0}\xi(s)e^{-xs}\sin ys\; ds}_{\beta}$ while $\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds=$ $\alpha+i\beta$. Taking into account this relation (39) is equivalent to

$$\begin{aligned}
B &= \left[z+\tilde{b}_0\right]\left[a_0-b_1de^{-\bar{z}d}\right][\alpha-i\beta]+\left[\bar{z}+\tilde{b}_0\right]\left[a_0-b_1de^{-zd}\right][\alpha+i\beta]\\
&= a_0\alpha\left(z+\bar{z}\right)-\alpha b_1d\left(ze^{-\bar{z}d}+\bar{z}e^{-zd}\right)-ia_0\beta\left(z-\bar{z}\right)+i\beta b_1d\left(ze^{-\bar{z}d}-\bar{z}e^{-zd}\right)+\\
&\quad +2\tilde{b}_0\alpha a_0-\alpha\tilde{b}_0b_1d\left(e^{-\bar{z}d}+e^{-zd}\right)-i\tilde{b}_0b_1\beta d\left(e^{-\bar{z}d}-e^{-zd}\right)\\
&= 2a_0\alpha x+2a_0\beta y+2\tilde{b}_0\alpha a_0-\alpha b_1de^{-xd}\left[x\left(e^{iyd}+e^{-iyd}\right)+iy\left(e^{iyd}-e^{-iyd}\right)\right]+\\
&\quad +i\beta b_1de^{-xd}\left[x\left(e^{iyd}-e^{-iyd}\right)+iy\left(e^{iyd}+e^{-iyd}\right)\right]-\alpha\tilde{b}_0b_1de^{-xd}\left(e^{iyd}+e^{-iyd}\right)+\\
&\quad -i\tilde{b}_0b_1\beta de^{-xd}\left(e^{iyd}-e^{-iyd}\right)\\
&= 2a_0\alpha x+2a_0\beta y+2\tilde{b}_0\alpha a_0-2\alpha b_1de^{-xd}\left[x\cos yd-y\sin yd\right]-2\beta b_1de^{-xd}\left[x\sin yd+y\cos yd\right]\\
&\quad -2\alpha\tilde{b}_0b_1de^{-xd}\cos yd+2\tilde{b}_0b_1\beta de^{-xd}\sin yd
\end{aligned}$$

which is real. This is sufficient to prove that $a+c$ is real given that is a ratio of real numbers. Now we have to show that $\mathrm{Re}\left(a_r-c_r\right)=0$.

$$a-c=\frac{\left[\xi(d)e^{-zd}+\left(z+\tilde{b}_0\right)\int_{-d}^{0}\xi(s)e^{-zs}ds\right]\left[a_0-b_1de^{-\bar{z}d}\right]-\left[\xi(d)e^{-\bar{z}d}+\left(\bar{z}+\tilde{b}_0\right)\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds\right]\left[a_0-b_1\right.}{\frac{1}{a_0}\left[a_0^2-2a_0b_1de^{-xd}\cos yd+b_1^2d^2e^{-2xd}\right]}$$

the denominator as before is real. Then we have to show that the numerator is purely imaginary. As before we split the numerator

$$Num=C+D$$

where

$$\begin{aligned}
C &= \xi(d)e^{-zd}\left[a_0 - b_1 d e^{-\bar{z}d}\right] - \xi(d)e^{-\bar{z}d}\left[a_0 - b_1 d e^{-zd}\right] \\
&= a_0\xi(d)e^{-xd}\left(e^{iyd} - e^{-iyd}\right) \\
&= 2ia_0\xi(d)e^{-xd}\sin yd
\end{aligned}$$

which is purely imaginary, and

$$\begin{aligned}
D &= \left[z+\tilde{b}_0\right]\left[a_0 - b_1 d e^{-\bar{z}d}\right]\int_{-d}^{0}\xi(s)e^{-zs}ds - \left[\bar{z}+\tilde{b}_0\right]\left[a_0 - b_1 d e^{-zd}\right]\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds \\
&= \left[z+\tilde{b}_0\right]\left[a_0 - b_1 d e^{-\bar{z}d}\right][\alpha - i\beta] - \left[\bar{z}+\tilde{b}_0\right]\left[a_0 - b_1 d e^{-zd}\right][\alpha + i\beta] \\
&= a_0\alpha\left(z-\bar{z}\right) - \alpha b_1 d\left(ze^{-\bar{z}d} - \bar{z}e^{-zd}\right) - ia_0\beta\left(z+\bar{z}\right) + i\beta b_1 d\left(ze^{-\bar{z}d} + \bar{z}e^{-zd}\right) - 2i\tilde{b}_0 a_0\beta + \\
&\quad -\alpha\tilde{b}_0 b_1 d\left(e^{-\bar{z}d} - e^{-zd}\right) - i\tilde{b}_0 b_1\beta d\left(e^{-\bar{z}d} + e^{-zd}\right) \\
&= 2i\left[a_0\alpha y - a_0\beta x - \tilde{b}_0 a_0\beta - \alpha b_1 d e^{-xd}\left(x\sin yd + y\cos yd\right) + \beta b_1 d e^{-xd}\left(x\cos yd - y\sin yd\right) + \right. \\
&\quad \left. -\alpha\tilde{b}_0 b_1 d e^{-xd}\sin yd - \tilde{b}_0 b_1\beta d e^{-xd}\cos yd\right]
\end{aligned}$$

which is purely imaginary too. Then $\mathrm{Re}\left(a_r - c_r\right) = 0$ since it is a ratio of a sum of purely imaginary numbers and of a real number. This is sufficient to prove expression (10).

A.2 Proof of Corollary 1.

Consider the general continuous solution of the homogeneous part of problem (5) with boundary condition (6)

$$u\left(t\right) = \sum_{r=0}^{k}\varsigma_r e^{x_r t} + \sum_{r=k}^{\infty}\left(a_r e^{z_r t} + \bar{a}_r e^{\bar{z}_r t}\right) \tag{40}$$

Calling $a = \varsigma + i\omega$ we have that

$$\begin{aligned}
a_r e^{z_r t} + \bar{a}_r e^{\bar{z}_r t} &= \left(\varsigma + i\omega\right)e^{xt}e^{iyt} + \left(\varsigma - i\omega\right)e^{xt}e^{-iyt} = \\
&= e^{xt}\left[\left(\varsigma + i\omega\right)\left(\cos yt + i\sin yt\right) + \left(\varsigma - i\omega\right)\left(\cos yt - i\sin yt\right)\right] \\
&= 2e^{xt}\left(\varsigma\cos yt - \omega\sin yt\right)
\end{aligned}$$

and then (40) becomes

$$u\left(t\right) = \sum_{r=0}^{k}\varsigma_r e^{x_r t} + 2\sum_{r=k}^{\infty}e^{xt}\left(\varsigma\cos yt - \omega\sin yt\right)$$

Taking into account the forcing term the expression (11) follows. For this reason the general solution (10) or equivalently (11) is a real function $u: I \to \mathbb{R}$.

A.3 How to get expression (32) from (31).

First of all observe that from Theorem 3 we can rewrite

$$\begin{aligned}
\sum\nolimits_v P_{\tilde{m},v} &= -a_{\tilde{m}}^{-\frac{1}{\sigma}}\sum\nolimits_v\left(\frac{n_v}{g_c - z_v} + \frac{\bar{n}_v}{g_c - \bar{z}_v}\right) \\
&= \sum\nolimits_v\left(-2a_{\tilde{m}}^{-\frac{1}{\sigma}}\frac{\alpha_v g_c - \alpha_v x_v - \beta_v y_v}{g_c^2 + x_v^2 + y_v^2 - 2g_c x_v}\right) \\
&= \sum\nolimits_v\Psi_{0,v}
\end{aligned}$$

and

$$\sum\nolimits_{v\neq\tilde{v}} N_{\tilde{m},v} e^{(z_v-g_c)t} = \sum\nolimits_v \left[(\alpha_v+\Psi_{1,v})\cos y_v t - (\beta_v+\Psi_{2,v})\sin y_v t\right] e^{(x_v-g_c)t}$$

where

$$\begin{aligned}
\Psi_{1,v} &= -a_{\tilde{m}}^{-\frac{1}{\sigma}} \left[\frac{n_v}{g_c-z_v} e^{(g_c-z_v)d} + \frac{\bar{n}_v}{g_c-\bar{z}_v} e^{(g_c-\bar{z}_v)d}\right] \\
&= -2a_{\tilde{m}}^{-\frac{1}{\sigma}} \frac{e^{g_c-x_v d}\left[(\alpha_v g_c - \alpha_v x_v - \beta_v y_v)\cos yd + (\beta_v g_c + \alpha_v y_v - \beta_v x_v)\sin yd\right]}{g_c^2+x_v^2+y_v^2-2g_c x_v}
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{2,v} &= -a_{\tilde{m}}^{-\frac{1}{\sigma}} i \left[\frac{n_v}{g_c-z_v} e^{(g_c-z_v)d} - \frac{\bar{n}_v}{g_c-\bar{z}_v} e^{(g_c-\bar{z}_v)d}\right] \\
&= 2a_{\tilde{m}}^{-\frac{1}{\sigma}} \frac{e^{g_c-x_v d}\left[(\alpha_v y_v - \beta_v x_v + \beta_v g_c)\cos yd + (\beta_v y_v + \alpha_v x_v - \alpha_v g_c)\sin yd\right]}{g_c^2+x_v^2+y_v^2-2g_c x_v}
\end{aligned}$$

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