# Employment Fluctuations with Downward Wage Rigidity[1]

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**Abstract:** This paper considers a dynamic matching model with imperfectly observable worker effort, combining the matching technology of Mortensen and Pissarides (1994) with the shirking problem of Shapiro and Stiglitz (1984). We characterize the equilibrium and propose a quick algorithm for computing it to arbitrary accuracy.

In our economy, the no-shirking condition endogenously imposes real downward wage rigidity on the matching market. This type of wage rigidity implies that inefficient separations occur, as in Ramey and Watson (1997). Nonetheless, our main numerical finding is that imperfectly observable effort smoothes job destruction over the cycle, because firms are forced, in good states, to terminate some marginal jobs which they cannot commit to maintain in bad states. This time-inconsistency problem casts doubt on the importance of inefficient churning ("contractual fragility") as an explanation of observed employment fluctuation. On the other hand, the no-shirking condition implies that firms' share of surplus is procyclical, which can amplify fluctuations in job creation. Thus, our model is consistent with recent evidence that job creation is more important than job destruction in driving labor market fluctuations, and it therefore also tends to generate a robust Beveridge curve. However, the overall impact on unemployment and vacancy volatility is ambiguous.

**JEL classification:** C78, E24, E32, J64

**Keywords:** Job matching, wage rigidity, efficiency wages, contractual fragility

# 1 Introduction

1. Matching models are now standard model of labor market dynamics; but lots of recent controversy about their empirical success. One key issue: matching models have a hard time generating as much cyclical volatility in unemployment and vacancies as is observed in the data (Shimer 2004, 2005; Costain and Reiter2003; Hall 2003).

2. Broadly speaking, two general classes of mechanisms have been identified which might help generate cyclical labor market volatility.

   (a) One line of research, stemming from the papers mentioned above, suggests that rigid wages may generate volatility. In these models rigid wages create large fluctuations in profits which amplify the fluctuations in job creation and unemployment.

      i. However, only few papers have attempted to incorporate microfoundations for wage stickiness into the model (Shimer 2004; Hall 2005; Shimer and Wright 2005), and it remains an open question whether the quantitative effects of wage stickiness are large when a consistent model of wage stickiness is used.

   (b) A second (but earlier) line of research, initiated by work of Ramey and Watson (1997), shows that incentive problems may amplify fluctuations in job destruction. In these models of "contractual fragility" or "inefficient churning", the effort of agents is imperfectly observable and small perturbations of productivity may eliminate incentive compatibility, causing a wave of job destruction.

      i. The merit of this second line of research is that it is based on a coherent theory for the rigidity of wages. However, so far the existing papers have focused on steady states, or the transition path after a single shock from exogenously-imposed initial conditions.

3. (a) In this paper, we study a matching model with incentive constraints that lead to downward wage rigidity.
      i. Like Mortensen and Pissarides (1994), we study job creation and job destruction in a matching model with aggregate and match-specific productivity shocks.
      ii. But like Shapiro and Stiglitz (1988), we assume that wages must satisfy an incentive compatibility constraint that prevents shirking.
   (b) Thus our paper incorporates a microfounded form of real downward wage rigidity.
      i. Interesting to see how labor market fluctuations change when a microfounded wage is used instead of an ad hoc sticky wage.
      ii. In particular, since we assume labor productivity is cyclical but the disutility of effort is not, our model implies worker's surplus share may be higher in recessions. This suggests that it could amplify the variation in firms' hiring incentives.
   (c) The incentive compatibility constraint also means that job matches exhibit "contractual fragility".
      i. Unlike Ramey and Watson (1997), Mortensen and Pissarides (2001), and Jansen (2001), we will characterize the equilibrium labor market dynamics.
      ii. In particular, need to see whether large waves of firing can occur along the equilibrium path.
4. Combining the matching structure of Mortensen and Pissarides (1994) with the "efficiency wages" of Shapiro and Stiglitz (1984) is not only interesting for the light it sheds on the question of unemployment volatility. Our model has potential to address several other prominent issues in recent literature.

(a) Using new data, Shimer and Hall have recently argued that fluctuations in job creation are more important for explaining the movement of unemployment, and fluctuations in job destruction are less important, than was previously thought. The effects of downward wage rigidity on job creation and the implications of “contractual fragility” for job destruction means that our model has interesting implications for both margins.

(b) One of the most robust stylized facts about the labor market is the negative correlation between unemployment and vacancies (“Beveridge curve”). But previous papers with time-varying job destruction (Cole and Rogerson, Mortensen and Pissarides, Costain and Reiter, den Haan et.al.) have often found that the Beveridge curve is delicate in the model.

(c) On the theoretical side, our paper helps to correct a misconception about the dynamic properties of efficiency wage models. In models without matching frictions (Kimball, Kiley), it has been argued that efficiency wages serve to smooth the flow of profits to the firm, by driving down the wage in periods when unemployment is high. In a matching context, too, wages fall in recessions. But more importantly, a negative aggregate shock makes it more likely that the incentive compatibility constraint will bind, decreasing firms' share of surplus, and thus amplifying changes in firms' profits.

5. Previewing our simulation results, the surprising lesson of our paper is that this problem tends to smooth the cyclical fluctuations in job destruction.

   (a) Job fragility means that continuation value of marginal jobs is low. Therefore it is very expensive for firms to provide incentives in marginal jobs during good times.

   (b) Result may be that such marginal jobs are never formed.

(c) Therefore economy never reaches state with large number of "fragile jobs". Waves of firing fail to occur. On the equilibrium path, "contractual fratility" arguments fail; job destruction rate is constant.

6. On the other hand, fact that firms' share of surplus rises in recessions may increase volatility of hiring, and therefore of job creation.

   (a) Overall effect of no-shirking constraint on unemployment volatility is ambiguous: it tends to smooth job destruction, but amplify fluctuations in job creation.

   (b) On the other hand, this result is consistent with recent claims that unemployment variability is driven mostly by job creation, not by job destruction.

   (c) Fact that our model generates more fluctuations in job creation, and less in job destruction, also helps our model generate a Beveridge curve.

# 2 Model

This section presents a continuous-time, infinite horizon matching model with imperfectly observable worker effort.

## 2.1 Preferences and production technology

Our economy is populated by a continuum of workers with measure normalized to one. There is also a continuum of firms; the number of firms is infinitesimal compared with the number of workers. All agents are risk-neutral and discount the future at the common rate $r$.

Workers are identical and derive utility from consumption and leisure. The instantaneous utility function of a worker is given by [1]:

[1] This specification yields the same payoffs as in Mortensen and Pissarides (1999). Moreover, Marimon and Zilibotti (1999) and Rocheteau (2000) use similar utility specifications to study the effects of working time reduction. For an analysis of labour supply decisions with non-separable utility see Marimon and Zilibotti (1999).

$$U(c, n) = c + (1 - n)b, \tag{1}$$

where $c$ denotes consumption, $n \in \{0, 1\}$ is the fraction of time devoted to work and $b$ is the imputed value of leisure. Without loss of generality we assume that workers consume their entire income at any moment. During employment $c$ is therefore equal to the worker's wage $w$. In addition, workers can obtain a private gain from shirking that is assumed to be equal to the leisure a worker would get from not going to work, $b$ (this normalization is also without loss of generality). Accordingly, we can write the flow utility of a worker who exerts effort as $U(w, 0) = w$, while the utility of a worker who shirks is $U(w, 1) = w + b$. Unemployed workers, on the contrary, receive no income and just enjoy leisure $U(0, 1) = b$. Finally, the discounted lifetime utility of a worker with income and working time paths $\{z(t); t \in \mathbf{R}^+\}$ and $\{n(t); t \in \mathbf{R}^+\}$ equals

$$\int_{\mathbf{R}^+} \exp(-rt)U[z(t), n(t)]dt. \tag{2}$$

All firms are identical and have a continuum of jobs that are either filled with a worker or vacant. Besides effort, the productivity of a firm-worker pair depends on two components: a match-specific shock $x$ and an aggregate shock $X$ that affects all firms in the economy. Formally, the flow output of a match, denoted by $y(x, X; n)$, satisfies

$$y(x, X; n) = \begin{cases} y(x, X) & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

According to the above specification, firms are a collection of independent jobs. We assume that new idiosyncratic productivity shocks $x$ arrive at Poisson rate $\lambda$. These shocks are *i.i.d.* draws from a distribution $F$ with support $[\underline{x}, \overline{x}]$. Shocks to the aggregate state of productivity $X$ arrive at rate $\mu$, and are drawn from $\{1, 2, ...N\}$. The conditional probability that the current state changes from $X$ to $Z$ is denoted by $G_{ZX}$, and we write the matrix of Markov transition rates as

$$G \equiv \begin{pmatrix} G_{11} & & G_{1N} \\ & ... & \\ G_{N1} & & G_{NN} \end{pmatrix}$$

Here, column $j$ represents the probabilities of the $N$ possible states that could follow state $j$, so each column sums to one.

## 2.2 Moral hazard

To introduce a shirking motive, we assume that firms cannot perfectly monitor individual effort. At any moment in time, the firm observes total output, but given that the firm has a continuum of workers this does not reveal information about the effort of individual workers.

Faced with this moral hazard problem, firms offer incentives by promising to fire workers caught shirking. We assume that the firm's participation in the match causes it to observe worker's effort at the Poisson rate $\phi$. Firing observed shirkers (off the equilibrium path) is an equilibrium strategy for the firm if failing to do so would cause all workers to shirk. Shirking by all workers (off the equilibrium path) is an equilibrium strategy for the workers since individual workers cannot demonstrate to the firm that they are not shirking.[2] In other words, an equilibrium within the firm involving effort by all workers, under a threat of firing, is sustained by trigger strategies involving a jump to a new equilibrium at that firm involving shirking by all workers, and therefore separation of all that firm's matches.

As in Shapiro and Stiglitz (1984), equilibria of this form must satisfy an incentive compatibility constraint. This constraint, referred to as the no-shirking condition ($NSC$), will act as a lower-bound on the outcomes during the wage negotiations. In the remainder of this section we embed our version of the shirking model into a matching model of unemployment.

[2] Assuming that the firm is capable of monitoring more often, at a cost, equilibria with monitoring rates higher than the exogenous rate $\phi$ might be sustainable. Such equilibria would depend on workers' ability to observe or infer the firm's monitoring rate. We will not enter into these complications here.

### 2.3 Matching

Unemployed workers and firms are matched together in pairs through an imperfect matching technology (*e.g.* Pissarides 2000). The gross rate of formation of new matches $m_t$ is given by

$$m_t = M(u_t, v_t) \tag{4}$$

where $u_t$ is the number of unemployed workers, and $v_t$ is the number of vacancies open, at time $t$. We assume $M$ exhibits constant returns to scale. Therefore, the worker's probability of finding a match, per unit of time, can be written in terms of tightness $\theta_t \equiv v_t/u_t$ as

$$p(\theta_t) = \frac{M(u_t, v_t)}{u_t} = M\left(1, \frac{v_t}{u_t}\right) \tag{5}$$

Similarly, the probability that an open vacancy finds a match is

$$q(\theta_t) = \frac{M(u_t, v_t)}{v_t} = M\left(\frac{u_t}{v_t}, 1\right) \tag{6}$$

so that $p(\theta) = \theta q(\theta)$.

### 2.4 The value of matching

Before stating the Bellman equations for workers' and firms' value functions, we assume two restrictions on the equilibrium which are known to be valid for related models (Mortensen and Pissarides 1994, Cole and Rogerson 1999). First, we assume that aggregate jump variables may depend on the aggregate productivity state $X$, and that match-specific jump variables may depend on $x$ and $X$, but that neither may depend on other state variables, like the unemployment rate or the distribution of idiosyncratic productivities across existing jobs. We will see that the Bellman equations can be written in terms of $x$ and $X$ only, so it is not unreasonable to conjecture that such a minimum-state equilibrium exists. Second, we impose the reservation property. That is, we assume there exists a vector of reservation productivities $R(X)$ such that matches with idiosyncratic productivity $x$ continue in state $X$ if and only if $x \geq R(X)$. In our numerical work, we prove by construction that equilibria of this form exist, although this does not rule out other types of equilibria. For notational convenience, we will refer to the vector of reservation productivities as $R$, and the continuation region as $\mathcal{C}(R)$. That is, a match continues if productivity lies in the set $\mathcal{C}(R) \equiv \{(x, X) : x \geq R(X)\}$.

We can now spell out the Bellman equations. Call the wage $w(x, X)$, and let the value functions of employed and unemployed workers be $W(x, X)$ and $U(X)$, respectively. For any state $(x, X) \in \mathcal{C}(R)$, function $W$ must satisfy:

$$
\begin{aligned}
rW(x,X) \;=\;& w(x,X) + \delta\left(U(X) - W(x,X)\right) + \lambda\left[\int_{R(X)}^{\overline{x}} W(z,X)dF(z) + F(R(X))U(X) - W(x,X)\right] \\
&+ \mu\left\{\sum_{Z:x\geq R(Z)} G_{ZX}\left[W(x,Z) - W(x,X)\right] + \sum_{Z:x<R(Z)} G_{ZX}\left[U(Z) - W(x,X)\right]\right\}
\end{aligned}
\tag{7}
$$

This equation states that the flow of returns to matched worker includes the wage, plus three flows of expected capital losses and gains: the loss from exogenous separation, which occurs at rate $\delta$; the gains from drawing a new idiosyncratic shock $z$, at rate $\lambda$; and the gains from switching to a new aggregate state $Z$, drawn with conditional probability $G_{ZX}$, at rate $\mu$. Conditional on an idiosyncratic shock, the separation probability is $F\left(R(X)\right)$, and conditional on an aggregate shock, separation occurs if the current idiosyncratic $x$ is less than the new reservation productivity, $R(Z)$.

The unemployed obtain a constant flow payoff of $b$ from leisure and search for jobs. Let $\theta(X)$ be labor market tightness, and suppose the rate of job finding is $p\left(\theta(X)\right)$. Then for any $X$, the value of unemployment satisfies:

$$
rU(X) = b + p\left(\theta(X)\right) N^W(X) + \mu \sum_Z G_{ZX}\left[U(Z) - U(X)\right] \tag{8}
$$

where $N^W(X)$ is the worker's expected increase in value from a new job offer (which need not necessarily be accepted), conditional on aggregate state $X$. We will consider two cases. On one hand, we consider the case where all new jobs are drawn from the top of the distribution, so that $N^W(X) =$

$[W(\overline{x}, X) - U(X)]$, which guarantees acceptance. On the other hand, we also consider the case where new jobs are drawn from the same productivity distribution that governs the idiosyncratic shocks, so that $N^W(X) = \int_{R(X)}^{\overline{x}} (W(z,X) - U(X))\, dF(z)$. In this latter case, some new jobs are rejected, and the value $N^W(X)$ reflects this.

Now consider the value functions associated with vacancies, $V(X)$, and filled jobs, $J(x, X)$. For any state $(x, X)$ in the continuation region $\mathcal{C}(R)$, the value of a filled vacancy satisfies:

$$rJ(x,X) = y(x,X) - w(x,X) + \delta (V(X) - J(x,X)) + \lambda \left[ \int_{R(X)}^{\overline{x}} J(z,X)\, dF(z) + F(R(X))V(X) - J(x,X) \right]$$
$$+ \mu \left\{ \sum_{Z:x \geq R(Z)} G_{ZX} [J(x,Z) - J(x,X)] + \sum_{Z:x<R(Z)} G_{ZX} [V(Z) - J(x,X)] \right\} \tag{9}$$

Thus the flow of profits to the matched firm consists of output minus wages, plus three flows of expected losses and gains analogous to those of the worker.

Next, suppose that maintaining a vacancy costs $c$ per period, and that vacancies are filled at rate $q(\theta(X))$. Then for each $X$, the value of a vacancy must satisfy:

$$rV(X) = -c + q(\theta(X))\, N^F(X) + \mu \sum_Z G_{ZX} [V(Z) - V(X)] \tag{10}$$

where $N^F(X)$ is a firm's expected increase in value resulting from finding a possible match. If all new jobs come from the top of the productivity distribution, then $N^F(X) = [J(\overline{x}, X) - V(X)]$. On the other hand, if new jobs are drawn from the same distribution $F$ that governs idiosyncratic shocks, then $N^F(X) = \int_{R(X)}^{\overline{x}} (J(z,X) - V(X))\, dF(z)$, which includes the value of rejected jobs.

Finally, we assume that firms are free to open any number of vacancies. Thus, in equilibrium, the value of a vacancy is zero in any aggregate state $X$:

$$V(X) = 0 \tag{11}$$

but for the time being we prefer to clarify the structure of the equations by showing $V$ where it appears, rather than eliminating it.

## 2.5 Incentive compatibility

We are now in a position to derive the $NSC$. A worker will never shirk if the gain from shirking during a short interval $dt$ is less than the expected cost of a disciplinary layoff in case the worker is detected. The logic also works in the opposite direction. If it pays to shirk during a short period $dt$, then workers will always choose this option.

Formally, let $W^s(x,X)$ denote the value function for a worker who shirks during the interval $dt$. Assuming that the worker exerts effort during the rest of the time the firm-worker pair remains together, we obtain

$$\begin{aligned} rW^s(x,X)\,dt &= w(x,X)\,dt + bdt + \phi dt\left[U(X) - W(x,X)\right] + \delta dt\left(U(X) - W(x,X)\right) \\ &\quad + \lambda dt\left[\int_{R(X)}^{\overline{x}} W(z,X)\,dF(z) + F(R(X))U(X) - W(x,X)\right] \\ &\quad + \mu dt\left\{\sum_{Z:x\geq R(Z)} G_{ZX}\left[W(x,Z) - W(x,X)\right] + \sum_{Z:x<R(Z)} G_{ZX}\left[U(Z) - W(x,X)\right]\right\} + o(dt) \end{aligned} \tag{12}$$

where $o(dt)$ signifies a quantity which becomes negligible compared to $dt$ as $dt \to 0$.

Comparing this equation to (7), dividing by $dt$ and taking the limit as $dt \to 0$, we find that the only difference between shirking and not shirking is

$$rW^s(x,X) - rW(x,X) = b + \phi\left(U(X) - W(x,X)\right).$$

Hence, workers (weakly) prefer not to shirk as long as their match surplus exceeds $b/\phi$:

$$W(x,X) - U(X) \geq \frac{b}{\phi}, \tag{13}$$

where $b/\phi$ is the expected gain in leisure (loss of effort) before the worker is caught shirking.[3] The above inequality acts as an incentive-compatibility constraint that must be satisfied at all states $(x,X) \in \mathcal{C}(R)$ since we rule out temporary layoffs.

[3]Below we will use the variable $\phi$ to generate different values for $b/\phi$. This is why assuming that the gain from shirking equals the value of unemployed leisure is just a notational simplification that implies no loss of generality.

## 2.6 Wages and turnover

The contract of a worker stipulates a wage flow $w(x, X)$ that can be renegotiated after any shock. Other transfers that could alleviate the moral hazard problem of workers, such as shirking penalties or bond payments, are ruled out.

As is standard in the matching literature, we assume that the flow wage is determined through Nash bargaining. For any state $(x, X)$, we define the total surplus relative to the threat point of separation, as follows:

$$S(x, X) = W(x, X) - U(X) + J(x, X) - V(X) \tag{14}$$

We assume that the worker receives fraction $\beta$ of this total surplus, unless the incentive compatibility constraint binds, in which case the wage must rise until the constraint is satisfied. (In the appendix, we derive these conditions from a Nash bargaining game that determines the wage over a short interval $dt$.) Thus, for states $(x, X)$ in the continuation region $\mathcal{C}(R)$, the worker's surplus is given by:

$$W(x, X) - U(X) = \max\{\beta S(x, X), B/\phi\} \tag{15}$$

and the firm's surplus is

$$J(x, X) - V(X) = \min\{(1-\beta)S(x, X), S(x, X) - B/\phi\} \tag{16}$$

Of course, the firm also has the possibility of separating from the match. So for any $(x, X) \in \mathcal{C}(R)$, the firm's surplus must satisfy

$$J(x, X) - V(X) \ \geq \ 0 \tag{17}$$

which, together with (13), (14), and (16) implies that

$$S(x, X) \ \geq \ B/\Phi \tag{18}$$

for $(x, X)$ in the continuation region $\mathcal{C}(R)$.

Given that surplus is split according to the rules (15) and (16), (18) is both a necessary and sufficient condition on total surplus $S$ for the match to continue. Since workers and firms are better off separated outside the continuation region, for $(x, X)$ outside $\mathcal{C}(R)$ we can define

$$W(x, X) - U(X) \ = \ J(x, X) - V(X) \ = \ S(x, X) \ = \ 0 \tag{19}$$

## 2.7 Privately optimal outcomes

In our economy separations correspond to layoffs. The firms sever a relationship when it is no longer profitable to pay the worker an incentive compatible wage. Workers, on the contrary, base their effort decisions on their beliefs about the duration of their job. From existing studies we know that this non-cooperative choice of effort and reservation strategies may lead to multiple Pareto rankable outcomes (Den Haan *et al.* 1999; Mortensen and Pissarides 1999). This multiplicity is due to a positive feedback between the reservation productivities and the minimum incentive-compatible wage chosen inside a particular match, taking as given aggregate tightness. Intuitively, suppose a worker anticipates an increase in the reservation productivity for the current state. Given the shorter job duration, the worker needs a higher flow wage in order to exert effort, and given this increase in the wage floor the firm may find it profitable to fire at the higher reservation productivity.

Since a mutually beneficial deviation by the worker and firm alone— without any change in aggregate conditions— suffices to eliminate multiplicity of this kind, we think it makes sense to focus on contracts that are constrained optimal for the firm-worker pair. Thus, to rule out spurious job destruction, we assume that the firm and the worker have perfect and symmetric information about the current state $(x, X)$ and about the stochastic process over $x$ and $X$. This information allows them to choose the vector of reservation productivities and effort decisions that maximize their joint value, subject only to the $NSC$.[4] In the next section we will characterize these privately optimal outcomes.

# 3 Analysis

## 3.1 The match surplus equation

In this section, we will define equilibrium and propose an algorithm to compute it. But first, we show how the model can be simplified in order to define

[4]For a thorough analysis of this problem, see Jansen (2001).

the equilibrium concisely. Note that in the continuation region $\mathcal{C}(R)$, we can simplify (7) by rewriting it as:

$$(r+\lambda+\mu+\delta)\,W(x,X) = w(x,X)+\delta U(X)+\lambda\left[\int_{R(X)}^{\overline{x}} W(z,X)\,dF(z)+F(R(X))U(X)\right]$$

$$+\mu\left[\sum_{Z:x\geq R(Z)} G_{ZX}W(x,Z)+\sum_{Z:x<R(Z)} G_{ZX}U(Z)\right]$$

Therefore, an employed worker's surplus $W(x,X)-U(X)$ satisfies

$$\begin{aligned}(r+\lambda+\mu+\delta)\,(W(x,X)-U(X)) &= w(x,X)+\lambda\int_{R(X)}^{\overline{x}}(W(z,X)-U(X))\,dF(z)\\ &-b-p(\theta(X))N^{W}(X)+\mu\sum_{Z:x\geq R(Z)} G_{ZX}\,(W(x,Z)-U(Z)) \qquad (20)\end{aligned}$$

where we have used (8) to eliminate $rU(X)$ on the right hand side. The surplus of a filled job is similar, but can be simplified further by setting $V(X)=0$ for all $X$:

$$(r+\lambda+\mu+\delta)\,J(x,X) = y(x,X)-w(x,X)+\lambda\int_{R(X)}^{\overline{x}} J(z,X)dF(z)+\mu\sum_{Z:x\geq R(Z)} G_{ZX}J(x,Z) \qquad (21)$$

Summing equations (20) and (21), we obtain:

$$(r+\lambda+\mu+\delta)\,S(x,X) = y(x,X)-b-p(\theta(X))\,N^{W}(X)+\lambda\int_{R(X)}^{\bar{x}} S(z,X)\,dF(z)+\mu\sum_{Z:x\geq R(Z)} G_{ZX}S(x,Z) \qquad (22)$$

This expression is fairly intuitive: we see that the surplus includes the flow payoff $y$ minus the flow payoff $b$ associated with unemployment and minus the gains that accrue to unemployed workers from finding new jobs, plus capital gains due to individual and aggregate shocks. Solving (22) is the main challenge to characterizing our model. We now characterize $S$ and explain how (22) can be solved.

## 3.2 Optimal time-consistent continuation

As we mentioned earlier, we assume that each worker-firm pair follows a continuation strategy that maximizes its joint surplus, subject to incentive compatibility. However, we must look at this issue in greater detail to be sure that this assumption is well-defined.

Since we are looking for a solution based on a reservation strategy, the $N$ aggregate states imply the existence of $N$ reservation productivities $\{R_1, ...R_N\}$. However, these need not all be distinct: some aggregate states could have the same reservation productivity. For notational convenience, we will number the reservation productivities, in backwards order, as

$$R_{N+1} \leq R_N \leq \quad \ldots \quad \leq R_1 \leq R_0$$

where we have also defined the notation $R_{N+1} \equiv \underline{x}$ and $R_0 \equiv \overline{x}$. We can then divide up the support $[\underline{x}, \overline{x}]$ of the idiosyncratic shock into $N+1$ intervals of the form $I_i \equiv [R_i, R_{i-1})$. (If some of the reservation productivities are equal, then some of the segments are empty. If we assume that all new jobs have the best productivity, then the upper bound of the support should also be thought of as a separate interval $I_0 \equiv \{\overline{x}\}$. Otherwise, this point should be included in the first interval, defining $I_1 \equiv [R_1, \overline{x}]$).

If the worker-firm pair maximizes surplus, then this means they must never separate when incentive compatibility is satisfied. This condition determines the reservation productivities. Given aggregate economic conditions, which from the pair's perspective are summarized by the vectors of tightness, $\theta$, and expected new job values, $N^W$, guaranteeing incentive compatibility means guaranteeing a sufficiently high surplus if the match continues. Thus, suppose that the pair's surplus function is $S$ and suppose that the pair expects to play the reservation strategy $R$ in the future. Then, using equation (22), the surplus associated with continuation at any state $(x, X)$ can be defined as

$$T(x, X; S, R, \theta, N^W) \quad \equiv$$

$$(r+\lambda+\mu+\delta)^{-1}\left\{y(x,X)-b-p\left(\theta(X)\right)N^W(X)+\lambda\int_{R(X)}^{\bar{x}}S\left(z,X\right)dF(z)+\mu\sum_{Z:x\geq R(Z)}G_{ZX}S(x,Z)\right\} \tag{23}$$

Joint efficiency requires that the pair continue in state $(x, X)$ as long as $T(x, X; S, R, \theta, N^W)$ is at least equal to $b/\phi$. Therefore the reservation productivities must satisfy a fixed-point relation of the following form:

$$R(X) \ = \ \min\{x : T(x, X; S, R, \theta, N^W) \geq b/\phi\} \tag{24}$$

Surplus inside the continuation region must satisfy (22); outside, by definition, it is zero. This condition can also be written as a fixed-point relation in terms of the function $T$:

$$S(x,X) \ = \ \begin{cases} T(x,X;S,R,\theta,N^W) & \text{for} \quad x \geq R(X) \\ 0 & for \quad x < R(X) \end{cases} \tag{25}$$

Thus, given aggregate conditions, the matched pair's strategies must satisfy the two fixed point relations (24) and (25). However, we have not yet shown that our assumption— that there exist strategies which maximize the surplus— makes sense. In other words, we have not ruled out the possibility that there might be two fixed points of (24) and (25), one involving a higher surplus at some states $(x, X)$, while the other involves higher surplus at other states. This would not only be a mathematical problem. Economically, it would make it impossible to find a time-consistent vector of reservation productivities: for some states $(x, X)$ one reservation strategy would be preferred, and for other states, a different one. The following proposition shows that this problem does not arise: in other words, given any aggregate conditions $\theta$ and $N^W$, there always exists a unique surplus-maximizing reservation strategy for the pair.

**Proposition.** For any aggregate conditions $\theta$ and $N^W$, such that $y(\bar{x}, N) - b > p(\theta(N))N^W(N)$, there exists a unique pair $\bar{S}$ and $\underline{R}$ such that:

1. $\underline{R}$ solves (24) given surplus function $\bar{S}$
2. $\bar{S}$ solves (25) given reservation vector $\underline{R}$

3. If there exists another pair $(S', R')$ that solve (24) and (25), then $\underline{R}(X) \leq R'(X)$ and $\bar{S}(x, X) \geq S'(x, X)$ for all $x$ and $X$.

**Proof.** This proof is based on the methods of Rustichini (1998). Note that $S^0(x, X) = y(\bar{x}, N)/r$ is an upper bound to the true surplus function, and that the vector $R_0$ which has each element equal to $\underline{x}$ is a lower bound to the true reservation productivity vector.

Note that if we define mappings from the right-hand sides to the left-hand sides of (24) and (25), then these mappings are monotonic both in $R$ and $S$. That is, plugging a higher $S$ into the right-hand sides of (24) and (25), we obtain a lower new $R$ and a higher new $S$. Likewise, plugging a higher $R$ into the right-hand sides of (24) and (25), we obtain a higher new $R$ and a lower new $S$.

Now define a sequence of functions $S^i$ and $R_i$ by iterating on the right hand sides of (24) and (25). Since the coefficients on the integral and sum terms in $T$ are positive and less than one, we find immediately that $S^1(x, X) \leq S^0(x, X)$ at all $(x, X)$, and therefore $R_1(X) \geq R_0(X)$ for all $X$. Since the mapping is monotonic with respect to $S$ and $R$, it furthermore follows that $S^{i+1}(x, X) \leq S^i(x, X)$ at all $(x, X)$ and $R_{i+1}(X) \geq R_i(X)$ for all $X$. However, the sequence $S^i$ is bounded below by the constant function equal to zero, and the sequence $R_i$ is bounded above by the vector with all elements equal to $\bar{x}$; therefore the $S$ and $R$ sequences must converge. Call the limits of these sequences $\bar{S}$ and $\underline{R}$; by definition, they are fixed points of (24) and (25).

Now suppose there exists another fixed point pair $(S', R')$. Since $S^0$ and $R_0$ are upper and lower bounds to all fixed points of (24) and (25), and since the mapping is monotonic, we have

$$S^1(x, X) \equiv T(x, X; S^0, R, \theta, N^W) \;\geq\; T(x, X; S', R, \theta, N^W) = S'(x, X)$$

and likewise, $R_1$ is a lower bound for $R'$. Now by induction, $S^i$ and $R_i$ bound $S'$ and $R'$ for all $i$, and thus in the limit we have $\bar{S}(x, X) \geq S'(x, X)$ for all $x$ and $X$ and $\underline{R}(X) \leq R'(X)$ for all $X$.

**Q.E.D.**

The preceding proposition also helps us characterize the surplus function and reservation productivities. Note that function $S^0$ is weakly increasing in $x$, and that the mapping (25) preserves this property. In fact, it maps weakly increasing functions into functions that are strictly increasing in the continuation interval. Function $S^0$ is also weakly increasing in $X$, and it preserves this property too under two conditions: first, the probabilities $G$ exhibit first-order stochastic dominance; and second:

$$y(x, X+1) - y(x, X) > p(\theta(X+1))N^W(X+1) - p(\theta(X))N^W(X) \qquad (26)$$

for all $X \in \{1, 2, ...N-1\}$. Since these properties are preserved by each step of mapping (25), they also hold in the limit. Therefore we have proved the following corollary.

**Corollary.** Suppose $\theta$ and $N^W$ satisfy $y(\bar{x}, N) - b > p(\theta(N))N^W(N)$ and also satisfy (26), and that $G$ exhibits first-order stochastic dominance. Then the fixed point pair $(S', R')$ of (24) and (25) has the following properties:

1. Function $\bar{S}$ is strictly increasing in $x$ for $x \geq R(X)$
2. Function $\bar{S}$ is weakly increasing in $X$
3. The vector of reservation productivities $\underline{R}$ is weakly decreasing in $X$

From now on, we will assume that $G$ exhibits first-order stochastic dominance, and we will restrict attention to equilibria satisfying $y(\bar{x}, N) - b > p(\theta(N))N^W(N)$ and(26), so that in equilibrium the reservation productivities are monotonic. Therefore the surplus function will be increasing in both arguments, which immediately implies that the reservation productivities are decreasing. Hence, the $N$ reservation productivities which we called $R_N \leq R_{N-1} \leq ... \leq R_1$ correspond, in order, to the $N$ aggregate states: $R(N) \leq R(N_{N-1}) \leq ... \leq R(1)$. Thus we can use the notation $R_X$ interchangeably with $R(X)$, and we know that on the (possibly empty) interval $I_i \equiv [R_i, R_{i-1})$, all states $X \geq i$ will continue.

## 3.3 Characterizing the surplus function

We are now ready to describe in more detail what the solution to the surplus equation (22) looks like.

### 3.3.1 Calculating the slope of the surplus function

Within the segments $I_i$, we can differentiate equation (22) to calculate the slope of the surplus function (for each $X$) on that segment; we obtain

$$(r + \lambda + \mu + \delta) \frac{\partial S(x, X)}{\partial x} = \frac{\partial y(x, X)}{\partial x} + \mu \sum_{Z: R_Z \leq x} G_{ZX} \frac{\partial S(x, Z)}{\partial x} \qquad (27)$$

Notice that this equation contains just one value of $x$. Therefore, the equations on any segment $I_i$ can be solved independently from those on all other segments, and the possible existence of empty segments is irrelevant for the solution. Since the reservation productivities are monotonic in $X$, on any non-empty segment $I_i$ (27) constitutes a system of $N+1-i$ differential equations in the $N+1-i$ unknowns $\frac{\partial S(x,X)}{\partial x}$, for $X \geq i$. The equations for segment $I_i$ can be simplified as follows:

$$\begin{pmatrix} \frac{\partial S(x,i)}{\partial x} \\ \dots \\ \frac{\partial S(x,N)}{\partial x} \end{pmatrix} = ((r+\lambda+\mu+\delta)I - \mu M_i)^{-1} \begin{pmatrix} \frac{\partial y(x,i)}{\partial x} \\ \dots \\ \frac{\partial y(x,N)}{\partial x} \end{pmatrix} \tag{28}$$

where $I$ is an identity matrix of order $N+1-i$ and $M_i$ is the matrix

$$M_i = \begin{pmatrix} G_{ii} & & G_{Ni} \\ & \dots & \\ G_{iN} & & G_{NN} \end{pmatrix}$$

($M_i$ is the transpose of the last $N+1-i$ rows and columns of the matrix $G$.)

Thus, changes in $S$ can be calculated explicitly on each segment $I_i$ as long as we choose a production function $y(x,X)$ that can be integrated explicitly with respect to the distribution of idiosyncratic shocks $F$. Similarly, we can integrate the surplus functions $W-U$ and $J$ segment by segment, with one additional caveat: workers must receive surplus $B/\Phi$ when the incentive compatibility constraint binds. Given that $S$ is strictly increasing, we can uniquely define the cutoff point $\hat{x}(X)$ below which incentive compatibility is binding, by

$$\beta S(x,X) < \frac{B}{\phi} \text{ for } x < \hat{x}(X) \quad \text{and } \beta S(x,X) > \frac{B}{\phi} \text{ for } x > \hat{x}(X) \tag{29}$$

Thus the formula for the worker's surplus, and likewise that for the firm's surplus, will differ depending on whether $x$ is less or greater than $\hat{x}(X)$. Note that, as with our definition of the reservation productivity $R(X)$, the threshold $\hat{x}(X)$ must be defined by an inequality instead of an equation because the surplus $S$ has discontinuities.

### 3.3.2 Calculating the discontinuities in the surplus function

Now that we know how to integrate the surplus inside the segments $[R_i, R_{i-1})$, we must next ask what happens to the surplus at the endpoints of these segments. As the incentive compatibility constraint (18) shows, the surplus function need not be continuous at the reservation productivities. To be precise, the jump in $S(x, X)$ at $x = R_i$ can be defined as $j(R_i, X) \equiv \lim_{dx\to 0} S(R_i + dx, X) - S(R_i - dx, X)$. If there is continuation in state $X$ on both sides of $R_i$, then equation (22) must hold in a neighborhood around $R_i$, and therefore the jumps at $R_i$ must satisfy

$$(r + \lambda + \mu + \delta)\, j(R_i, X) \; = \; \mu \sum_{Z:R_Z \leq R_i} G_{ZX}\, j(R_i, Z) \tag{30}$$

Thus the jumps at $R_i$ are nonzero except in two possible cases. If there is no incentive problem, so that $S(R_i, i) = 0$, then equation (30) is solved by $j(R_i, Z) = 0$ for all $Z$. The jump would also be zero if $G_{ZX}$ were zero for all $Z$ satisfying $R_Z \leq R_i$.

So far, we have characterized the jumps in $S(x, X)$ at points $x$ strictly inside the continuation interval $[R_X, \bar{x}]$; these points must be reservation productivities $R_i$ for other states $i > X$. However, since $S(x, X)$ is zero outside of $[R_X, \bar{x}]$ and satisfies (**??**) inside it, there must also be a jump of at least $B/\phi$ at $R_X$ in state $X$.

As we saw earlier, the jump in $S(x, X)$ at $x = R_X$ could in fact be strictly greater than $B/\phi$. Our analysis in section 3.2 shows that inequality (18) can be seen as the definition of a set of complementary slackness conditions governing the reservation productivities $R(X)$ and the corresponding surpluses $S(R(X), X)$. For any $i \in \{2, 3, ...N\}$, monotonicity of the surplus implies:

$$R_i \leq R_{i-1}$$

and incentive compatibility implies:

$$S(R_i, i) \geq \frac{B}{\Phi}$$

The fact that the surplus is differentiable away from the reservation productivities implies that if $dR_i$ is strictly negative, then $dS_i$ must be zero. Therefore at least one of the inequalities must hold with equality:

$$(R_i - R_{i-1})\left(S(R_i, i) - \frac{B}{\Phi}\right) \equiv dR_i dS_i = 0$$

Notice therefore that we can now summarize the entire surplus function by a vector of $N$ numbers: first $R_1$, and then for each $i \in \{2, 3, ...N\}$, either $dR_i$ or $dS_i$. The two possible cases for these last $N-1$ numbers can be easily distinguished, since $dR_i$ is necessarily nonpositive, while $dS_i$ is nonnegative.

## 3.4 Equilibrium

As we have seen, the surplus functions can be defined in terms of the productivity pair $(x, X)$ without reference to the current distribution of employment and unemployment. Therefore, it suffices to define (and calculate) an equilibrium in terms of the minimum state variable $(x, X)$ before considering other state variables. We therefore postpone for later the discussion of the dynamics of unemployment.

Obviously this model has trivial equilibria in which workers always shirk, and therefore firms never hire them. But we are interested in **no-shirking equilibria** in which the worker's surplus is sufficiently large to provide incentives not to shirk. Summarizing the relationships discussed so far, such an equilibrium can be defined in terms of just four objects, $S$, $R$, $\theta$, and $N^W$.

**Definition.** A no-shirking equilibrium is a surplus function $S(x, X)$, a vector of reservation productivities $R$, a labor market tightness vector $\theta$, and a vector of new job values $N^W$ that satisfy the following conditions:

1. For each $X$, the surplus function satisfies the system of differential equations (22) for all $x \in [R(X), \bar{x}]$, and is zero for $x \in [\underline{x}, R(X))$.
2. For each $X$, the surplus function satisfies the boundary condition (24) at the reservation productivity $R(X)$.

3. If new jobs have productivity $\bar{x}$, then labor market tightness $\theta(X)$ and the new job value $N^W(X)$ are given by

$$c = q(\theta(X)) \min\{S(\bar{x}, X) - B/\phi, (1-\beta)S(\bar{x}, X)\} \quad (31)$$

$$N^W(X) = \max\{B/\phi, \beta(\bar{x}, X)\} \quad (32)$$

4. Alternatively, if the productivity of new jobs is drawn from distribution $F$, then labor market tightness $\theta(X)$ and the new job value $N^W(X)$ are given by

$$c = q(\theta(X)) \int_{R(X)}^{\bar{x}} \min[S(z, X) - B/\phi, (1-\beta)S(z, X)] dF(z) \quad (33)$$

$$N^W(X) = \int_{R(X)}^{\bar{x}} \max[B/\phi, \beta S(z, X)] dF(z) \quad (34)$$

## 3.5 An algorithm to calculate equilibrium

By now it should be clear that the main challenge in solving our model is solving equation (22) to find the surplus function $S$. We only need to be sure that this surplus function takes as given a tightness vector consistent with zero profits on vacancy creation. We will now outline an algorithm for calculating $S$, using the formulas for the slopes and discontinuities given in subsections 3.3.1 and 3.3.2. Once we find an $S$ consistent with zero profits, the simulation of employment and productivity dynamics is straightforward.

One approach to solving for $S$ would be to use backwards induction, conditional on a given $\theta$. The results of Rustichini (1998) guarantee that this converges to the correct solution of the worker and firm's optimal reservation strategy. But this could be extremely slow. Therefore, we propose a faster algorithm, based on the fact that the entire surplus function can be summarized by a single $N$-dimensional vector which we will call $Q$. We define

$$Q_1 \equiv R_1 \quad (35)$$

$$Q_i \equiv dR_i \equiv R_i - R_{i-1} \text{ if } R_i < R_{i-1} \quad (36)$$

$$Q_i \equiv dS_i \equiv S(R_i), i) - B/\Phi \text{ if } R_i = R_{i-1} \quad (37)$$

This definition takes advantage of the complementary slackness relations that govern the surplus at the reservation productivities. If (for $i > 1$) $Q_i$ is negative, then this indicates that $R_i$ is strictly less than $R_{i-1}$, and therefore that $S(R_i, i) = B/\Phi$. In this case, $Q_i \equiv R_i - R_{i-1}$. If (for $i > 1$) $Q_i$ is positive, then this indicates that $R_i = R_{i-1}$, and in this case $Q_i$ equals the excess jump $S(R_i, i) - B/\Phi$ of the surplus function in state $i$. $Q_i = 0$ indicates the knife-edge case in which $R_i = R_{i-1}$ and $S(R_i, i) = B/\Phi$.

All equilibrium quantities can be constructed from $Q$. Given a candidate value of $Q$, we can construct the surplus $S$ and related objects, and check whether the equilibrium relationships hold. Thus, instead of repeatedly solving a dynamic programming problem for each value of $\theta$, we solve a single $N$-dimensional root-finding problem to calculate $S$, $R$, and $\theta$ simultaneously. The steps are as follows.

1. Loop over aggregate states $X$ from 1 to $N$, using the information in $Q$ to calculate $R_X$ and $S(R_X, X)$.
2. For each $X$ from 1 to $N$, loop over other aggregate states $Z$ from $X$ to 1. If $R_{Z-1}$ differs from $R_Z$, solve the differential equations (28) to calculate the increase in $S$ on the interval $I_Z = [R_Z, R_{Z-1})$, and use the equations (30) to calculate the jump in $S(x, X)$ at $x = R_{Z-1}$.[5]

Given these two steps, we have constructed the surplus function $S$ implied by the vector $Q$. Note that it will be a strictly increasing function. Therefore we can calculate the intervals over which the incentive compatibility constraint binds:

3. Use equation (29) to calculate the cutoffs $\hat{x}(X)$ for all $X$.

Next we calculate tightness in each state $X$:

4. Use equation (16) to calculate the firm's value $N^F(X)$ of a new job in state $X$, given the surplus function $S$.[6]

[5]If on the other hand $R_{Z-1} = R_Z$, it is superfluous but may be numerically helpful to solve the differential equations (28) on the empty interval $I_Z$ and to set $S(R_{Z-1}, X) = S(R_Z, X)$.

[6]If new jobs have random productivity, this involves integrating the surplus function $J(x, X)$. Like $S$, it can be explicitly integrated, piecewise, given the function $S$, the reservation productivities $R$, and the cutoffs $\hat{x}$.

5. Use equation (10), which reduces to the zero-profit condition $c = q(\theta(X))N^F(X)$, to calculate the firm's probability of job finding $q$.
6. Use (6) to calculate labor market tightness $\theta(X)$.
7. Use equation (5) to calculate the worker's job finding probability.

At this point, we know all the objects that appear in the surplus equation (22). We can now check whether the complementary slackness conditions on the surplus function are satisfied at the reservation productivities, given the conjectured vector $Q$. Note that $Q$ tells us directly the value of $S(R_X, X)$:

$$S(R_X, X) \;=\; \begin{cases} B/\Phi \;\text{ if }\; Q(X) < 0 \\ B/\Phi + Q(X) \;\text{ if }\; Q(X) \geq 0 \end{cases} \tag{38}$$

We can now check (for each $X$) whether (22) is satisfied at $x = R_X$:

$$(r+\lambda+\mu+\delta)\, S(R_X, X) \;=\; y\,(R_X, X) - b + \lambda \int_{R_X}^{\overline{x}} S\,(z, X)\, dF(z) + \mu \sum_{Z: R_X \geq R_Z} G_{ZX} S\,(R_X, Z) - p\,(\theta\,(X))\, N^W\,(X) \tag{39}$$

(Checking this equation involves integrating $S(x, X)$, and will also involve integrating $W(x, X)$ if new jobs have random productivity. But this is no problem since we know how to integrate them piecewise.)

If we find a vector $Q$ that satisfies (39), then we have found the equilibrium surplus function. With it, we have also found the reservation productivities. Given the reservation productivities, we can next simulate the dynamics of the distribution of employment and productivity.

## 3.6 Employment dynamics

This is a heterogeneous agent model in which the state variable of the economy includes the full distribution of idiosyncratic productivities. Nonetheless, the model can be explicitly solved in two steps. First, we have seen (as in Mortensen and Pissarides 1994) that the equations defining values, surpluses, and labor market tightness can be written without reference to the unemployment rate or the distribution of idiosyncratic productivities. The characterization of the surplus function in subsection 3.3 gives us sufficient information to solve for the surplus and all jump variables, including the reservation productivities, independently of employment. Once the reservation productivities are known, we can then simulate the dynamics of employment and productivity.

To define these dynamics, note that new matches $m_t$ may or may not result in employment. If all new matches offer productivity $\overline{x}$, then in equilibrium such matches will never be rejected, so new jobs formed will equal $m_t$. On the other hand, if we assume that new matches are drawn from the same productivity distribution as continuing matches, then when the current aggregate state is $X_t$, only fraction $1 - F(R(X_t))$ of the new matches $m_t$ will result in employment.

Next, to describe the dynamics of the distribution of employment productivity, we keep track of the mass of employment in each interval $I_i \equiv [R_i, R_{i-1})$ separately. Jobs with productivity in $[R_1, \overline{x}]$ are *stable*: they will not be destroyed under any value of aggregate productivity. But when the current aggregate shock is $X \geq 2$, then there will be some other continuing jobs that are *fragile*, because they can be destroyed when aggregate productivity decreases; these jobs have individual productivity less than $R_1$. Finally, any job that receives an individual shock $x < R_N$ will be immediately destroyed under all circumstances.

Suppose new jobs have the highest possible productivity, $\bar{x}$. Let $e_t(\mathcal{X})$ be the measure of employed workers whose productivities lie in the set $\mathcal{X}$. Thus we can write the mass of new jobs as $e_t(\{\overline{x}\})$, and the mass of jobs in any other interval $I_i$, $i \in \{1, 2, ...N\}$ as $e_t([R_i, R_{i-1}))$. (The following notation is correct even for empty intervals $R_i = R_{i-1}$.) Let unemployment be $u_t$, and total employment be $e_t \equiv e_t([R_N, \overline{x}]) \equiv e_t(\{\overline{x}\}) + \sum_{i=1}^{N} e_t(I_i)$. Then the change in the mass of individuals in each of these employment states over a short time interval $dt$ can be written as follows, dropping terms of order $o(dt)$:

$$de_t(\{\overline{x}\}) \;=\; p(\theta(X_t))u_t dt - (\lambda + \delta)\, e_t(\{\overline{x}\})\, dt \tag{40}$$

$$de_t([R_i, R_{i-1})) \;=\; \mathbf{1}(X_{t+dt} \geq i)\left[\lambda(F(R_i) - F(R_{i-1}))e_t - (\lambda + \delta)\, e_t([R_i, R_{i-1}))\right] dt - \mathbf{1}(X_{t+dt} < i)\, e_t([R_i, R_{i-1})) \tag{41}$$

$$du_t = (\delta + \lambda F(R(X_{t+dt})))\, e_t\left([R(X_{t+dt}), \overline{x}]\right) dt - p(\theta(X_t))u_t dt + \mathbf{1}\left(X_{t+dt} < X_t\right) e_t\left([R(X_t), R(X_{t+dt}))\right) \tag{42}$$

It can be verified that these flows sum to zero. Note that the terms $\mathbf{1}\left(X_{t+dt} < i\right) e_t\left([R_i, R_{i-1})\right)$ and $\mathbf{1}\left(X_{t+dt} < X_t\right) e_t\left([R(X_t), R(X_{t+dt}))\right)$ which appear as outflows from fragile employment and an inflow to unemployment, are not of order $dt$. These terms represent the spike of destruction of fragile jobs that occurs any time the aggregate state $X$ decreases.

Next, suppose new jobs are drawn from distribution $F$. Let $e_t(\mathcal{X})$ be the measure of employed workers whose productivities lie in the set $\mathcal{X}$. Thus we can write the mass of jobs in any other interval $I_i$, $i \in \{1, 2, ...N\}$ as $e_t\left([R_i, R_{i-1})\right)$, defining $R_0 \equiv \overline{x}$. Let unemployment be $u_t$, and total employment be $e_t \equiv e_t\left([R_N, \overline{x}]\right) \equiv \sum_{i=1}^{N} e_t(I_i)$. Then the change in the mass of individuals in each of these employment states over a short time interval $dt$ can be written as follows, dropping terms of order $o(dt)$:

$$\begin{aligned} de_t\left([R_i, R_{i-1})\right) &= \mathbf{1}\left(X_{t+dt} \geq i\right)\left[\left(F(R_i) - F(R_{i-1})\right)\left(\lambda e_t + p(\theta(X_t))u_t\right) - (\lambda + \delta)\, e_t\left([R_i, R_{i-1})\right)\right] dt \\ &\quad - \mathbf{1}\left(X_{t+dt} < i\right) e_t\left([R_i, R_{i-1})\right) \qquad (43) \\ du_t &= (\delta + \lambda F(R(X_{t+dt})))\, e_t\left([R(X_{t+dt}), \overline{x}]\right) dt - (1 - F(R(X_{t+dt})))p(\theta(X_t))u_t dt \\ &\quad + \mathbf{1}\left(X_{t+dt} < X_t\right) e_t\left([R(X_t), R(X_{t+dt}))\right) \qquad (44) \end{aligned}$$

# 4 Intuition: two aggregate states

To illustrate the main features of the model we will solve the asset value equations for a simple example with two aggregate states, called 1 and 2, that correspond to recessions and booms, respectively. Suppose, for concreteness, that $F$ is uniform and that output is given by $y(x, X) = x + \zeta_X$. Thus, the surplus functions are linear upper semi-continuous functions. For moderate values of $b/\phi$ (or equivalently large aggregate shocks), this example generates counter-cyclical job destruction, *i.e.* $R_1 > R_2$. In booms some jobs survive that are destroyed when the economy enters into a recession. On the contrary, when the moral hazard problem is severe, it is too costly for firms to provide incentives in fragile jobs and $R_1 = R_2$, resulting in acyclical job destruction.

For further simplification, we will assume that $G_{12} = G_{21} = 1$: that is, any aggregate shock takes us to the opposite state, which means that the two aggregate states each occur 50% of the time, on average. Finally, suppose that new jobs start are the most productive jobs and that the support of $F$ is wide enough to ensure that job destruction takes place in both states.

## 4.1 Counter-cyclical job destruction

For the case in which $R_1 > R_2$, our solution for the surplus slopes, (28), gives us

$$\frac{\partial S(x, X)}{\partial x} = \frac{1}{r + \lambda + \delta} \qquad \text{for both } X \text{, if } \; x > R\,(1)$$

and

$$\frac{\partial S(x, 2)}{\partial x} = \frac{1}{r + \lambda + \mu + \delta} \qquad \text{if } \; x < R\,(1)$$

Since job destruction condition occurs at a surplus value of $b/\phi$ in both states, we can write the surplus functions as follows:

$$S(x, 1) = \frac{x - R_1}{r + \lambda + \delta} + \frac{b}{\phi} \tag{45}$$

or

$$S(x, 2) = \frac{x - R_2}{r + \lambda + \mu + \delta} + \frac{b}{\phi} \qquad \text{for} \quad x < R_1 \tag{46}$$

and

$$S(x, 2) = \frac{x - R_1}{r + \lambda + \delta} + \frac{\mu b/\phi + R_1 - R_2}{r + \lambda + \mu + \delta} + \frac{B}{\phi} \qquad \text{for} \quad x \geq R_1 \tag{47}$$

Inspection of (46) shows that output of fragile jobs is discounted at a higher rate than the output of robust jobs that will survive during a recession. The slope of the surplus function $S\,(x, 2)$ is therefore lower to the left of $R_1$ than to the right of this point. Finally, at $R_1$ the surplus function jumps up by an amount

$$j\,(R_1, 2) = \frac{\mu}{r + \lambda + \mu + \delta} \frac{b}{\phi} \tag{48}$$

.

The above surplus functions are illustrated in figure 1. Finally, using (11), (9), and (16), we can write the free-entry condition on vacancies in terms of the surplus of new jobs ($x = 1$) as:

$$c = q(\theta(X)) \min \left\{ (1-\beta) S(\overline{x}, X), S(\overline{x}, X) - \frac{B}{\phi} \right\} \tag{49}$$

Similarly, evaluating surplus equation (14) at the reservation productivity and substituting $R_i = b/\phi$ for $i = 1, 2$, we find that job destruction in booms is governed by

$$(r + \lambda + \mu + \delta) \frac{b}{\phi} = R_2 + \zeta_2 - b + \lambda \int_{R_2}^{\overline{x}} S(z, 2)\, dF(z) - p(\theta(2))\, N^W(2) \tag{50}$$

because the marginal job does not survive if the cycle changes. In recessions, on the contrary, the job destruction condition satisfies:

$$(r + \lambda + \mu + \delta) \frac{b}{\phi} = R_1 + \zeta_1 - b + \lambda \int_{R_1}^{\overline{x}} S(z, 1)\, dF(z) + \mu S(R_1, 2) - p(\theta(1))\, N^W(1) \tag{51}$$

where $\mu S(R_1, 2)$ denotes the option value from the improvement in the productivity of the marginal job if the economy inters into a boom.

The above conditions for job creation and job destruction define a set of four equations in four unknowns, namely $(\theta_1, \theta_2, R_1, R_2)$. Moreover, these conditions confirm that job destruction decisions are driven by the $NSC$ while job creation decisions are either driven by the bargained wages $w(\overline{x}, X)$ or by the minimum incentive compatible wages. In the latter case, the wage distribution is degenerate since the $NSC$ would bind on all jobs. Finally, if the $NSC$ binds on new jobs in recessions but not in booms, the surplus share of firms is clearly pro-cyclical. [7]

[7] With stochastic initial match values this is always the case. Since $S(x, X_1) < S(x, X_2)$, in recessions the $NSC$ will bind on a larger fraction of new jobs than in booms. For positive values of $b/\phi$ the expected surplus share of firms is therefore smaller than $(1-\beta)$ even though the wage distribution is not degenerate.

## 4.2 Acyclical job destruction

The case of acyclical job destruction is illustrated in figure 2. Since $S(R_1, 2) < (\mu/(r + \lambda + \mu + \delta)) * (b/\phi)$, firms cannot maintain fragile jobs because the increase in the flow wage needed to enduce effort would make the profits of the firm negative. Hence, $R_1 = R_2 = R$ and given that $S(x, 1)$ and $S(x, 2)$ have the same slope we can write

$$S(x, 2) = S(x, 1) + j(R, 2) \tag{52}$$

The size of the jump $j(R, 2)$ can be determined using the surplus equations associated with the case of acyclical job destruction:

$$(r + \lambda + \mu + \delta) S(x, 2) = x + \zeta_2 - rU(2) + \lambda \int_R^{\overline{x}} S(z, 2) \, dF(z) + \mu S(x, 1) \tag{53}$$

$$(r + \lambda + \mu + \delta) S(x, 1) = x + \zeta_1 - rU(1) + \lambda \int_R^{\overline{x}} S(z, 1) \, dF(z) + \mu S(x, 2) \tag{54}$$

Subtracting (54) from (53), and using (52) to replace $S(x, 2) - S(x, 1)$ by $j(R, 2)$, we get:

$$(r + \lambda F(R) + \delta + 2\mu) j(R, 2) = (\zeta_2 - \zeta_1) - r(U(2) - U(1)) \tag{55}$$

The above equation defines the jump as the appropriately discounted difference between the operating surplus $x + \zeta_X - r(U(X))$ in both periods. This difference is increasing in the size of the aggregate shocks, $\zeta_2 - \zeta_1$, and decreasing in the frequency of aggregate shocks, $\mu$.

On the basis of the above solutions we can write the surplus functions as:

$$S(x, 1) = \frac{x - R}{r + \lambda + \mu + \delta} + b/\phi \tag{56}$$

$$S(x, 2) = S(x, 1) + j(R, 2) \tag{57}$$

where $j(R, 2)$ satisfies (55).

Together with job creation conditions (49) and the job destruction condition for state 1 (condition 54 evaluated at $R$ with $S(R, 1) = b/\phi$ ) these equations deliver a solution for $\theta_1$, $\theta_2$ and $R$.

# 5 Numerical results

In this section we present some (preliminary) numerical results for the case of three aggregate states. The baseline parameters are shown in Table 1. The model period corresponds to a quarter, the idiosyncratic shocks are uniformly distributed and the relative bargaining strength of workers, $\beta$ is chosen to satisfy Hosios' (1990) condition. The decentralized equilibrium with $b/\phi = 0$ is therefore constrained efficient.

| Parameter | Values |
|---|---|
| $\zeta_X$ | $[-0.053, 0, 0.053]$ |
| $x$ | $U \sim [0, 1]$ |
| $r$ | 0.01 |
| $b$ | 0.9 |
| $m(u, v)$ | $u^{0.5}v^{0.5}$ |
| $\beta$ | 0.5 |
| $c$ | 0.125 |
| $\mu$ | 0.067 |
| $\lambda$ | 0.081 |
| $\delta$ | 0 |

In Table 2 we present results for the case in which new jobs start with the maximum value of the idiosyncratic shock $\overline{x} = 1$. In the first column we present the business cycle facts for the post-war period in the U.S. reported in Shimer (2005). Comparing the data to the efficient decentralized equilibrium (Column 2), we see that the baseline model performs well in many respects. In particular, our baseline model generates virtually the same coefficient of variation for the unemployment rate as observed in the U.S.. The level of the average unemployment rate in the baseline model is 5.8% which is also close to the U.S. average. The feature that is responsible for this high degree of cyclical volatility in $u$ is our choice of $b$. The value of leisure is equivalent to 90% of the initial match value in the intermediate state. As a result, the match surplus is small and tiny variations in productivity generate large variations in the surplus value and entry.

| | | *Model* | | | |
|---|---|---|---|---|---|
| *Coefficient of variation* | *Data* | *Efficient* | *Low* | *Medium* | *High* |
| $u$ | 0.188 | 0.1950 | 0.1620 | 0.1577 | 0.2283 |
| $v$ | 0.183 | 0.0933 | 0.0952 | 0.1233 | 0.2670 |
| $y(x, X)$ | 0.0306 | 0.0282 | 0.0282 | 0.0286 | 0.0290 |
| $w(x, X)$ | 0.013 | 0.0265 | 0.0206 | 0.0180 | 0.0211 |
| *job creation* $(jc)$ | 0.117 | 0.0979 | 0.0636 | 0.0414 | 0.0625 |
| *job destruction* $(jd)$ | 0.197 | 0.2390 | 0.1639 | 0.1161 | 0.0575 |
| *Correlations* | | | | | |
| $corr(u, y)$ | -0.367 | -0.877 | -0.9015 | -0.8995 | -0.9146 |
| $corr(v, y)$ | 0.362 | 0.6032 | 0.8581 | 0.9515 | 0.9717 |
| $corr(u, v)$ | -0.896 | -0.2274 | -0.6135 | -0.8075 | -0.8212 |
| $corr(jc, jd)$ | -0.65 | 0.3349 | 0.2327 | 0.1984 | -0.0697 |

The next three columns report the results for different values of $b/\phi$: Low (0.05), Medium (0.10) and High (0.15). The main variable of interest is the coefficient of variation for job destruction (Row 6). Inspection of Table 2 shows that this variable decreases monotonically from a value of 0.2390 in the efficient outcome to 0.0575 in the case where $b/\phi = 0.15$. These results confirm our claim that moral hazard problems tend to smooth the cyclical fluctuations in job destruction. Moving from Low to High, the reservation productivities in all three states increase, but the increase is higher in the good aggregate state than in the bad aggregate state.

On the contrary, the coefficient of variation of job creation follows a non-monotonic pattern. Initially, the introduction of moral hazard reduces the values of jobs due to inefficient separations. Moreover, this effect is stronger in good states than in bad states because in relatively good states firms are forced to terminate some fragile jobs that do not survive after a negative productivity shock. However, beyond $b/\phi = 0.10$ we find the opposite result. When we move from Medium to High, the coefficient of variation of job creation increases from 0.0414 to 0.0625. The explanation is simple. In the last column the $NSC$ is binding on all jobs in a recession ($\zeta = -0.053$). In other words, in a recession workers get a larger share of the surplus than in the remaining two states. As a result, the surplus share of jobs becomes pro-cyclical and this gives rise to an increase in the cyclical volatility of job creation.

The above results for the variation in the pattern of job creation and job destruction also explain the changes in the evolution of the coefficient of variation of unemployment. As in the case of job creation, this variable follows a U-shape pattern in $b/\phi$.

Another feature of the model that deserves attention is the Beveridge curve relation. In our benchmark model the negative correlation between $u$ and $v$ is much smaller than in the data. But when we introduce a lowerbound on the match surplus of 0.10 or 0.15 we observe that our model generates a very realistic Beveridge curve relation. This feature is noteworthy because models with endogenous job destruction typically face problems to generate this feature. The failure of the standard matching model to produce a strong Beveridge curve relation is due to the so-called "echo-problem". Since firms shed many workers in recessions, this is typically a good period to search for a new worker. An increase in job destruction is therefore followed by a strong increase in vacancy creation, leading to a positive rather than a negative correlation between $u$ and $v$. In our model this is not the case because the inefficient separations reduce the gains from job creation.[8]

# References

Costain, James, and Michael Reiter (2005), "Business Cycles, Unemployment Insurance, and the Calibration of Matching Models." Univ. Pompeu Fabra WP# 872.

Den Haan, Wouter, and Gary Ramey and Joel Watson (1999), "Contract-Theoretic Approaches to Wages and Displacement". NBER Working Paper No. 6972.

[8]We are currently completing the numerical simulations. We repeat the above numerical exercise for the case of stochastic initial match value; we consider different values for b; also we consider an example in which we offset the increase in overall job destruction and unemployment via a reduction in the exogenous separation rate which is set to zero for the moment.

Hall, Robert

Hall, Robert (2005), "Employment Fluctuations with Equilibrium Wage Stickiness" *American Economic Review* 95(1), pp. 50-65.

Kiley

Kimball, Miles (1994), "Labor-Market Dynamics when Unemployment is a Worker Discipline Device." *American Economic Review* 84 (4), pp. 1045-59.

Jansen, Marcel (2001), "Equilibrium Unemployment with Matching Frictions and Worker Moral Hazard", mimeo.

Mortensen, Dale, and Christopher Pissarides (1994), "Job Creation and Job Destruction in the Theory of Unemployment." *Review of Economic Studies* 61 (3), pp. 397-415.

Mortensen, Dale, and Christopher Pissarides (1999), "New Developments in Models of Search in the Labor Market", Chapter 39 in Orly Ashenfelter and David Card (eds.): Handbook of Labor Economics, Vol. 3b, North-Holland Publishers, Amsterdam.

Ramey, Gary, and Joel Watson (1997), "Contractual Fragility, Job Destruction, and Business Cycles." *Quarterly Journal of Economics* 112 (3), pp. 873-911.

Rustichini, Aldo (1998), "Dynamic Programming Solution of Incentive Constrained Problems." *Journal of Economic Theory* 78, pp. 329-54.

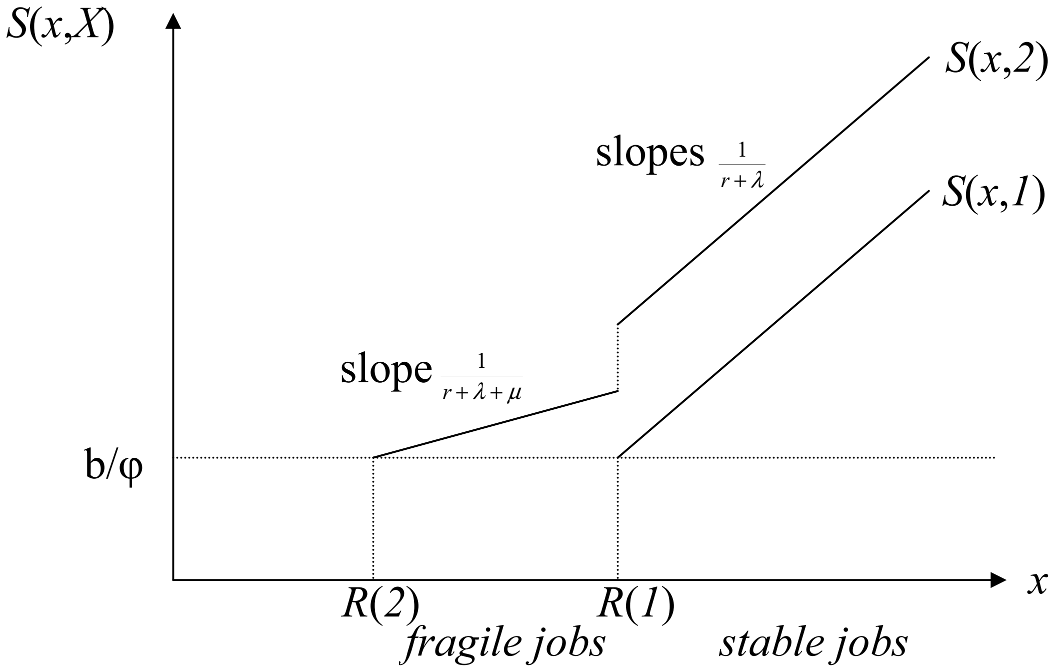
Shapiro and Stiglitz (1984), "Involuntary Unemployment as a Worker Discipline Device." *American Economic Review* 74 (3), pp. 433-44.

Shimer, Robert (2004), "The Consequences of Rigid Wages in Search Models." *Journal of the European Economic Association (Papers and Proceedings* 2, pp. 469-479.

Shimer, Robert (2005) "Cyclical Behavior of Equilibrium Unemployment." *American Economic Review* 95(1), pp. 25-49.

Shimer and Wright (2005) "Competitive Search Equilibrium with Asymmetric Information". mimeo.

# Countercyclical job destruction



# Acyclical job destruction