Sticky Prices vs. Limited Participation: What Do We Learn From the Data?*

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Abstract

The method of maximum likelihood is used to estimate a Dynamic Stochastic General Equilibrium business cycle model that combines elements of existing sticky-price and limited-participation specifications. Sticky prices are incorporated, following Rotemberg (1982), by assuming that monopolistically competitive firms face a quadratic cost of nominal price adjustment. Limited participation is incorporated, following Cooley and Quadrini (1999), by assuming that households face a quadratic cost of portfolio adjustment.The results support the hypothesis that the degree of the portfolio adjustment is very small in the data, but significant. In addition, the data suggest that the response of the interest rate to deviations of output from the steady state in the interest rate rule should be very close to zero. This is argued by Christiano and Gust (1999) as well. Furthermore, as in Ireland (1999, 2000), the model can not reject the hypothesis of parameter stability for the policy parameters. On the other hand, the model rejects the hypothesis for the rest of the parameters.

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1. Introduction

This paper focuses on the specification and stability of the closed economy dynamic, stochastic, general, equilibrium model that combines sticky prices and limited participation. The model is estimated with maximum likelihood estimation, in order to provide an insight on the degree of the cost of price stickiness and the portfolio adjustment as well as the nature of the interest rate rule, without having any prior assumptions on their level. In addition, the estimation exercise will help us attack the issue of stability as well, in an attempt to show that the structural parameters indeed have remain stable despite the widely believed change in the monetary policy regime that occurred in 1980s.

This methodology, that is proposed in Ireland (1999), combines the dynamic, stochastic, general, equilibrium theory with the flexibility of vector autoregressive time-series models. The purpose is to obtain a hybrid that shares the desirable features of both approaches in macroeconomics: firstly the fact that VARs are designed to be taken directly to the data, are easy to estimate, and can be used for statistical hypothesis tests and forecast analysis, and secondly the fact that DSGE models are based on economic theory. Therefore, the DSGE model is augmented so that its residuals, the movements and co-movements in the data that the theory cannot explain, are described by a VAR allowing us to estimate it and perform hypothesis tests and stability analysis. At the end, the time-series behavior of the endogenous variables is related to the structural parameters that describe private agents' tastes and technologies.

Preliminary results support the hypothesis that the degree of the portfolio adjustment is very small in the data, but significant. In addition, the data suggest that the response of the interest rate to deviations of output from the steady state in the interest rate rule should be very close to zero. This is argued by Christiano and Gust (1999) as well. Turning to the issue of stability, the tests reject the hypothesis of parameter stability of the structural parameters. In addition, the model is not able to account for the instability in the parameters in the policy rule.

The remainder of this paper is organized as follows. Section 2, below, sets up the model. Section 3 describes the data, estimates and tests. Section 4 summarizes and concludes.

2. The Model

2.1. Overview

The model combines elements of existing sticky-price and limited-participation specifications. Sticky prices are incorporated, following Rotemberg (1982), by assuming that monopolistically competitive firms face a quadratic cost of nominal price adjustment. Limited participation is incorporated, following Cooley and Quadrini (1999), by assuming that households face a quadratic cost of portfolio adjustment.

In the model, time periods are indexed by $t = 0, 1, 2, \ldots$. There are five types of agents: a representative household, a representative finished goods-producing firm, a representative bank, a continuum of intermediate goods-producing firms indexed by $i \in [0,1]$, and a monetary authority. Each intermediate goods-producing firm produces a distinct, perishable intermediate good. Hence, the intermediate goods can also be indexed by $i \in [0,1]$, where good i is produced by firm i. Nevertheless, the model contains enough symmetry to allow the analysis to focus on a representative intermediate goods-producing firm, which produces the generic good i. The activities of each agent are described in the subsections below.

2.2. The Representative Household

The representative household enters period t with M_{t-1} units of money and K_t units of capital. Immediately following the realization of the period-t shocks, the household must decide how to divide its funds into an amount D_t to be deposited in the representative bank and an amount $M_{t-1} - D_t$ to be used to facilitate goods purchases. When choosing D_t, the household faces a quadratic portfolio adjustment cost, measured in terms of time and given by

$$\tau_t = \frac{\phi_d}{2}\left(\frac{D_t}{\mu D_{t-1}} - 1\right)^2, \tag{1}$$

where $\phi_d \geq 0$ governs the magnitude of the adjustment cost and where, as noted below, $\mu \geq 1$ denotes the gross steady-state rate of money growth.

During period t, the household supplies $h_t(i)$ units of labor at the nominal wage W_t and $K_t(i)$ units of capital at the nominal rental rate Q_t to each intermediate goods-producing firm $i \in [0,1]$. The household's choices must satisfy

$$h_t = \int_0^1 h_t(i)di,$$

where h_t denotes total hours worked, and

$$K_t = \int_0^1 K_t(i)di$$

for all $t = 0, 1, 2, ...$

During period t, the household purchases output from the representative finished goods-producing firm at the nominal price P_t. It divides its purchases up into an amount C_t to be consumed and an amount I_t to be invested. Since it is assumed that the household receives its wages before making its goods purchases, it faces the cash-in-advance constraint

$$\frac{M_{t-1} - D_t + W_t h_t}{P_t} \geq v_t(C_t + I_t) \tag{2}$$

for all $t = 0, 1, 2, ...$ In (2.2), v_t is a random term that measures the amount of money the household must carry to facilitate its purchases of goods; it is assumed to follow the autoregressive process

$$\ln(v_t) = (1 - \rho_v)\ln(v) + \rho_v \ln(v_{t-1}) + \varepsilon_{vt}, \tag{3}$$

where $v > 0$, $1 > \rho_v > 0$, and the serially uncorrelated innovation ε_{vt} is normally distributed with mean zero and standard deviation σ_v.

By investing I_t units of the finished good during each period t, the household increases the capital stock over time according to

$$K_{t+1} = (1 - \delta)K_t + e_t I_t - \frac{\phi_k}{2}\left(\frac{K_{t+1}}{gK_t} - 1\right)^2 K_t, \tag{4}$$

where the depreciation rate satisfies $1 > \delta > 0$, where the parameter $\phi_k \geq 0$ governs the magnitude of capital adjustment costs, and where, as noted below, g measures the gross steady-state growth rate of the capital stock. The variable e_t is Greenwood, Hercowitz, and Huffman's (1988) shock to the marginal efficiency of investment; it follows the autoregressive process

$$\ln(e_t) = \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{5}$$

where $1 > \rho_e > 0$ and the serially uncorrelated innovation ε_{et} is normally distributed with mean zero and standard deviation σ_e.

At the end of period t, the household receives its rental payments $Q_t K_t$ along with principal plus interest $r_t^d D_t$ from the bank; hence, r_t^d measures the gross interest rate on deposits. The household also receives nominal profits B_t from the representative bank and $F_t(i)$ from each intermediate goods-producing firm $i \in [0,1]$, for a total of $B_t + F_t$ in nominal profits, where

$$F_t = \int_0^1 F_t(i)di.$$

The household then carries M_t units of money into period $t+1$; it faces the budget constraint

$$\frac{M_{t-1} + (r_t^d - 1)D_t + W_t h_t + Q_t K_t + B_t + F_t}{P_t} \geq C_t + I_t + \frac{M_t}{P_t} \tag{6}$$

during each period $t = 0, 1, 2, ...$

Thus, the household chooses C_t, h_t, τ_t, D_t, M_t, I_t, and K_{t+1} for all $t = 0, 1, 2, ...$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [a_t \ln(C_t) - \gamma(h_t + \tau_t)], \tag{7}$$

subject to the constraints imposed by (2.1), (2.2), (2.4), and (2.6) for all $t = 0, 1, 2, ...$ In the utility function, $1 > \beta > 0$, $\gamma > 0$, and the preference shock a_t follows the autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{8}$$

where $1 > \rho_a > 0$ and the serially uncorrelated innovation ε_{at} is normally distributed with mean zero and standard deviation σ_a.

Substitute (2.1) into the utility function and (2.4) into the budget and cash-in-advance constraints. Let Λ_{1t} denote the Lagrange multiplier on the budget constraint (2.6) and let Λ_{2t} denote the Lagrange multiplier on the cash-in-advance constraint (2.2). Then the household's first-order conditions include (2.1), (2.2), (2.4), and (2.6) with equality, along with

$$a_t = (\Lambda_{1t} + v_t \Lambda_{2t})C_t, \tag{9}$$

$$\gamma = (\Lambda_{1t} + \Lambda_{2t})(W_t/P_t), \tag{10}$$

$$\frac{\Lambda_{1t}}{P_t} = \beta E_t \left(\frac{\Lambda_{1t+1} + \Lambda_{2t+1}}{P_{t+1}} \right), \tag{11}$$

$$\begin{aligned} &\gamma\phi_d \left(\frac{D_t}{\mu D_{t-1}} - 1 \right) \frac{D_t}{\mu D_{t-1}} \\ = \; &\frac{[\Lambda_{1t}(r_t^d - 1) - \Lambda_{2t}]D_t}{P_t} + \beta\gamma\phi_d E_t \left[\left(\frac{D_{t+1}}{\mu D_t} - 1 \right) \frac{D_{t+1}}{\mu D_t} \right], \end{aligned} \tag{12}$$

and

$$\begin{aligned}
&(\Lambda_{1t} + v_t\Lambda_{2t})(1/e_t)\left[1 + \frac{\phi_k}{g}\left(\frac{K_{t+1}}{gK_t} - 1\right)\right] \qquad (13)\\
= \; &\beta E_t\left[\Lambda_{1t+1}(Q_{t+1}/P_{t+1}) + (1-\delta)(1/e_{t+1})(\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1})\right]\\
&+\beta\phi_k E_t\left[(\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1})(1/e_{t+1})\left(\frac{K_{t+2}}{gK_{t+1}} - 1\right)\left(\frac{K_{t+2}}{gK_{t+1}}\right)\right]\\
&-\beta(\phi_k/2)E_t\left[(\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1})(1/e_{t+1})\left(\frac{K_{t+2}}{gK_{t+1}} - 1\right)^2\right]
\end{aligned}$$

for all $t = 0, 1, 2, ...$

2.3. The Representative Finished Goods-Producing Firm

During period t, the representative finished goods-producing firm uses $Y_t(i)$ units of each intermediate good $i \in [0, 1]$ to produce Y_t units of the finished good according to the constant returns to scale technology described by

$$\left[\int_0^1 Y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)} \geq Y_t, \qquad (14)$$

with $\theta > 1$. Intermediate good i sells at the nominal price $P_t(i)$, while the finished good sells at the nominal price P_t; given these prices, the finished goods-producing firm chooses Y_t and $Y_t(i)$ for all $i \in [0, 1]$ to maximize its profits,

$$P_tY_t - \int_0^1 P_t(i)Y_t(i)di, \qquad (15)$$

for each $t = 0, 1, 2, ...$

The first-order conditions for this problem can be written as

$$Y_t(i) = [P_t(i)/P_t]^{-\theta}Y_t \qquad (16)$$

for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$ Competition in the market for the finished good requires that the representative firm earn zero profits in equilibrium. This zero-profit condition determines P_t as

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di\right]^{1/(1-\theta)} \qquad (17)$$

for all $t = 0, 1, 2, ...$

2.4. The Representative Bank

At the beginning of period t, the representative bank accepts deposits D_t from the representative household. At the beginning of period t, the bank also receives a lump-sum nominal transfer X_t from the monetary authority. Thus, the bank can lend $L_t(i)$ to each intermediate goods-producing firm $i \in [0,1]$, subject to the constraint

$$D_t + X_t \geq L_t, \tag{18}$$

where

$$L_t = \int_0^1 L_t(i)di.$$

At the end of period t, the bank collects $r_t L_t(i)$ in principal and interest from each intermediate goods-producing firm $i \in [0,1]$; hence, r_t denotes the gross nominal interest rate on loans. Since the bank owes $r_t^d D_t$ to its depositors, its profits are given by

$$B_t = r_t L_t + D_t + X_t - L_t - r_t^d D_t. \tag{19}$$

Competition among banks for loans and deposits guarantees that

$$r_t = r_t^d \tag{20}$$

for all $t = 0, 1, 2, ...$ So long as the net nominal interest rate $r_t - 1$ is positive, the bank will lend out all of its funds and (2.18) will hold with equality.

2.5. The Representative Intermediate Goods-Producing Firm

The representative intermediate goods-producing firm hires $h_t(i)$ units of labor and $K_t(i)$ units of capital from the representative household during period t in order to produce $Y_t(i)$ units of intermediate good i according to the constant returns to scale technology described by

$$K_t(i)^\alpha [g^t z_t h_t(i)]^{1-\alpha} \geq Y_t(i), \tag{21}$$

where $1 > \alpha > 0$ and where $g \geq 1$ denotes the gross rate of labor-augmenting technological progress. The aggregate technology shock z_t follows the autoregressive process

$$\ln(z_t) = (1 - \rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{22}$$

where $z > 0$, $1 > \rho_z > 0$, and the serially uncorrelated innovation ε_{zt} is normally distributed with mean zero and standard deviation σ_z.

The firm rents capital on credit, but must pay its wage bill with funds $L_t(i)$ borrowed from the representative bank. It therefore faces the finance constraint

$$L_t(i) \geq W_t h_t(i) \tag{23}$$

for all $t = 0, 1, 2, ...$ Since these funds are borrowed at the gross rate r_t, the bank must repay principal plus interest $r_t L_t(i)$ at the end of the period.

Since intermediate goods substitute imperfectly for one another as inputs to producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market; during each period t, it sets a nominal price $P_t(i)$ subject to the requirement that it satisfy the representative finished goods-producing firm's demand, taking P_t and Y_t as given.

In addition, each intermediate goods-producing firm faces a quadratic cost of adjusting its nominal price, measured in terms of the finished good and given by

$$\frac{\phi_p}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 Y_t, \tag{24}$$

where $\phi_p \geq 0$ governs the magnitude of the adjustment cost and where, as noted below, $\pi \geq 1$ denotes the gross steady-state rate of inflation.

These costs of price adjustment make the firm's problem dynamic; it chooses $h_t(i)$, $K_t(i)$, $Y_t(i)$, $L_t(i)$, and $P_t(i)$ for all $t = 0, 1, 2, ...$ to maximize its total market value, equal to

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_{1t}[F_t(i)/P_t], \tag{25}$$

where $\beta^t \Lambda_{1t}/P_t$ represents the marginal utility to the representative household provided by an additional dollar of profits during period t and where

$$\begin{aligned} F_t(i) &= P_t(i)Y_t(i) + [L_t(i) - W_t h_t(i)] - Q_t K_t(i) \\ &\quad -r_t L_t(i) - \frac{\phi_p}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 P_t Y_t, \end{aligned}$$

subject to the constraints imposed by its production possibilities, by the finance constraint (2.23), and by the demand curve

$$Y_t(i) = [P_t(i)/P_t]^{-\theta} Y_t$$

for all $t = 0, 1, 2, ...$

When the net nominal interest rate $r_t - 1$ is positive, the finance constraint (2.23) will hold with equality. In this case, the firm's problem simplifies to one of choosing $h_t(i)$, $K_t(i)$, and $P_t(i)$ to maximize its total market value, where

$$\frac{F_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta} Y_t - \frac{Q_t K_t(i) + r_t W_t h_t(i)}{P_t} - \frac{\phi_p}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 Y_t, \quad (26)$$

subject to the single constraint

$$K_t(i)^\alpha [g^t z_t h_t(i)]^{1-\alpha} \geq [P_t(i)/P_t]^{-\theta} Y_t \quad (27)$$

for all $t = 0, 1, 2, ...$ The first-order conditions for this problem are (2.27) with equality,

$$\Lambda_{1t} r_t (W_t/P_t) h_t(i) = (1-\alpha)\Xi_t K_t(i)^\alpha [g^t z_t h_t(i)]^{1-\alpha}, \quad (28)$$

$$\Lambda_{1t}(Q_t/P_t) K_t(i) = \alpha \Xi_t K_t(i)^\alpha [g^t z_t h_t(i)]^{1-\alpha}, \quad (29)$$

and

$$\begin{aligned} 0 \;=\; & (1-\theta)\Lambda_{1t}\left[\frac{P_t(i)}{P_t}\right]^{-\theta}\left(\frac{Y_t}{P_t}\right) + \theta \Xi_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta-1}\left(\frac{Y_t}{P_t}\right) \\ & -\phi_p \Lambda_{1t}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]\left[\frac{Y_t}{\pi P_{t-1}(i)}\right] \\ & +\beta \phi_p E_t \left\{\Lambda_{1t+1}\left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right]\left[\frac{P_{t+1}(i) Y_{t+1}}{\pi P_t(i)^2}\right]\right\} \end{aligned} \quad (30)$$

for all $t = 0, 1, 2, ...$, where Ξ_t is the Lagrange multiplier on (2.27).

2.6. The Monetary Authority

The monetary authority conducts monetary policy by adjusting the short-term nominal interest rate r_t in response to deviations of detrended output $y_t = Y_t/g^t$, inflation $\pi_t = P_t/P_{t-1}$, and money growth $\mu_t = M_t/M_{t-1}$ from their steady-state values y, π, and μ according to the policy rule

$$\ln(r_t/r) = \rho_y \ln(y_t/y) + \rho_\pi \ln(\pi_t/\pi) + \rho_\mu \ln(\mu_t/\mu) + \varepsilon_{rt}, \quad (31)$$

where r is the steady-state value of r. In (2.31), the parameters ρ_y, ρ_π, and ρ_μ should all be positive. The serially uncorrelated innovation ε_{rt} is normally distributed with mean zero and standard deviation σ_r.

2.7. Symmetric Equilibrium

In a symmetric equilibrium, all intermediate goods-producing firms make identical decisions, so that $h_t(i) = h_t$, $K_t(i) = K_t$, $F_t(i) = F_t$, $Y_t(i) = Y_t$, $P_t(i) = P_t$, and $L_t(i) = L_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$ In addition, the market-clearing condition $M_t = M_{t-1}+X_t$ must hold for all $t = 0, 1, 2, ...$ These equilibrium conditions, together with the first-order conditions for the representative agents' problems, the laws of motion for the aggregate shocks, and the policy rule, form a system of difference equations describing the model's equilibrium. In the absence of shocks, the economy converges to a steady state. The system is log-linearized around its steady state, and the methods of Blanchard and Kahn (1980) can be is applied to obtain a solution of the form

$$\mathrm{f}_t = U s_t \tag{32}$$

and

$$s_t = \Pi s_{t-1} + W \varepsilon_t \tag{33}$$

for all $t = 0, 1, 2, ...$

In (2.32) and (2.33), f_t is the vector of the model's flow variables which includes output $y_t = Y_t/g^t$, inflation π_t, money growth μ_t, consumption $c_t = C_t/g^t$, investments $i_t = I_t/g^t$, the real factor prices $w_t = (W_t/P_t)/g^t$, and $q_t = Q_t/P_t$, the interest rate r_t, the nominal transfers x_t, banks profits $b_t = B_t/M_t$, bank loans $l_t = L_t/M_t$, hours worked h_t, real profits $f_t = (F_t/P_t)/g^t$, the bank deposits $d_t = D_t/M_t$, the multipliers $\lambda_{1t} = g^t\Lambda_{1t}$, $\lambda_{2t} = g^t\Lambda_{2t}$, and $\xi_t = g^t\Xi_t$. s_t is the vector of the model's endogenous state variables and the five shocks in the model. The model's endogenous state variables are the lagged values of the bank deposits $d_{t-1} = D_{t-1}/M_{t-1}$, the lagged values of real balances $m_{t-1} = (M_{t-1}/P_{t-1})/g^{t-1}$, and the current values of the capital stock k_t. The five shocks in the model are the money demand shock v_t, the shock to the marginal efficiency of investment e_t, the preference shock a_t, the technology shock z_t and the policy shock ε_{rt}. The vector ε_t includes the four innovations ε_{vt}, ε_{at}, ε_{zt}, and ε_{rt} and is assumed to be normally distributed with zero mean and covariance matrix

$$V = E\varepsilon_t\varepsilon_t' = \begin{bmatrix} \sigma_v^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_a^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}$$

The parameters that describe private agents' tastes, technologies and the policy rule determine the elements of the matrices Π, W, and U. The model's solution as described by (2.32) and (2.33) takes the form of a state-space econometric model, driven by the five innovations in ε_t. Therefore, maximum likelihood estimations of the parameters in Π, W, and U can be obtained as described in Hamilton (1994) using the Kalman filter together with the data on five variables: consumption C_t, investment I_t, money M_t, prices P_t, and interest rates r_t.

3. Data, Estimates and Tests

3.1. Data

In the data consumption is measured by personal consumption expenditure, investments are measured by private gross domestic investment, real balances are measured by dividing the M2 money stock by the GDP deflator, inflation is measured by changes in the GDP deflator and the interest rate is measured by the yield on three-month Treasure bills. All series, except for the interest rate, are seasonally adjusted; the series for consumption, investments and real balances are expressed in per-capita terms by dividing by the civilian, noninstitutional population, age 16 and over.

The data are quarterly and run from 1959:1 through 2001:1. The data are divided into two subsamples, the first covering the period 1959:1 through 1979:2, and the second covering the period from 1979:3 through 2001:1. The breakpoint of the sample corresponds to the widely believed change in monetary policy that occurred in 1979:2, when Paul Volker was appointed Chairman of the Board of Governors of the Federal Reserve System.

Distinct upward trends appear in the series for consumption, investments and real balances, because of growth. Ireland (1997) accounts for these trends in the data by including a deterministic trend in the production function that captures the effect of labor-augmenting technological progress. Thus the model implies that C_t, I_t, and m_t grow at the same rate g along a balanced growth path.

The data don't contain enough information to estimate all of the model's parameters. Therefore some must be fixed prior to estimation. Thus, the weight γ on leisure is set equal to 1.5, implying that the household spends about one third of its time working. The depreciation rate δ is set equal to 0.025. Lastly, θ is set equal to 6, implying an average markup of price over marginal cost equal to 20 percent.

3.2. Estimates

Table 1 displays maximum likelihood estimates of the model's remaining 20 parameters, together with their standard errors, that are computed by taking the square roots of the diagonal elements of the inverted matrix of second derivatives of the maximized log likelihood function.

The results support the hypothesis that the degree of the portfolio adjustment is statistically significant in the data. In addition, the data suggest that the response of the interest rate to deviations of output from the steady state in the interest rate rule should be very close to zero. This is also argued by Christiano and Gust (1999). Models that incorporate limited participation, should incorporate interest rate rules with the interest rate reacting very little to deviations of output from the steady state in order to have non-explosive results.

Table 2 displays the maximum likelihood estimates for the two subsamples, pre- and post-1980s. It is observed that again the portfolio adjustment cost is significant in both periods, and that the degree of the interest rate response to output deviations from the steady state is very small in both subsamples, especially pre-1980s.

Tables 3 and 4 display the results of the forecast error variance decompositions, in an attempt to find what fraction of the observed consumption and investment variation comes from the five shocks that are incorporated in the model. The estimated model is used to decompose the k-step-ahead forecast error variances in consumption and investments into five orthogonal components: one attributable to each shock, the money demand, the investment, the preference, the technology and the policy shocks. We observe that for $k = \infty$ investment shocks account for nearly 99 percent of the unconditional variance in detrended output and investment. For one- to twenty-step-ahead forecast error variances though it is indicated that both preference and technology shocks are those that account for the variation in consumption. Concerning the variation in investments, the shocks that account for its variation for one- to thirty-step-ahead forecast error variance are investment and technology. These results indicate that in addition to technology shocks that are important for the variation of the components of output, consumption is specifically influenced by preference shocks and investment from investment shocks.

3.3. Tests

An advantage of the real business cycle models is that they are structural, meaning that they are able to link the behavior of real variables in the economy with private agent's tastes, technologies. These structural parameters, in order to be consistent with the Lucas critique, have to remain invariant to changes in the monetary policy regime.

As discussed above, table 2 displays the maximum likelihood estimates for the two subsamples, pre- and post-1980s. Therefore, the hybrid model can be used to test for parameter stability across the two sub-samples. Andrews and Fair (1988) describe procedures that can be used to test for the stability of the model's estimated parameters across the two subsamples. Let the vector Θ_q^1 contain q parameters estimated with pre-1979 data, let Θ_q^2 contain the same q parameters estimated with post-1979, and let H_q^1 and H_q^2 denote the covariances matrices of Θ_q^1 and Θ_q^2. Then the Wald statistic can be written more simply as

$$W = (\Theta_q^1 - \Theta_q^2)'(H_q^1 + H_q^2)^{-1}(\Theta_q^1 - \Theta_q^2). \tag{34}$$

According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with q degrees of freedom under the null hypothesis of stability, where q is the number of parameters being tested for stability.

The Wald statistics in table 5 indicate that the model is not able to accept the null hypothesis that the structural parameters remain stable across the two subsamples. Evidently, there has been a major change in the data between pre- and post- 1980s, and the hybrid model cannot capture its source. On the other hand, the tests indicate that the monetary policy regime has remained stable, since the model can not reject the hypothesis of parameter stability for the policy parameters. This is something puzzling that needs further investigation.

4. Conclusions

This paper, focuses on the specification and stability of the estimated model that incorporates sticky prices and limiter participation in the financial markets. The model is estimated with maximum likelihood following the methodology in Ireland (1999).

Preliminary results indicate that limited participation is statistically significant, although small in the data. In addition, they suggest that the degree of the interest rate response to output deviations from the steady state in the monetary policy rule should be very small in order to have stability and non-explosive results, something that is argued by Christiano and Gust (1999) as well.

Concerning stability between pre- and post-1980s, where there is believed that a major change in the monetary policy has been occurred, the tests are not able to capture the stability of the structural parameters. Therefore the estimated model is not consistent with the Lucas critique. In addition, the model is not able to account for the instability in the parameters in the policy rule.

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Table 1. Full Sample Maximum Likelihood Estimates and Standard Errors

Parameter	Full Sample Estimate	Standard Error
β	0.9997	0.0005
α	0.1685	0.0064
ϕ_d	0.0275	0.0035
ϕ_k	4.8556	0.0812
ϕ_p	25.5239	0.0000
μ	1.0002	0.0128
ρ_y	0.0003	0.0015
ρ_π	0.7832	0.0081
ρ_μ	0.4101	0.0082
v	1.0115	0.0091
z	1098.9	0.4940
ρ_v	0.9999	0.0002
ρ_e	1.0000	0.0000
ρ_a	0.9673	0.0118
ρ_z	0.9063	0.0191
σ_v	0.0110	0.0006
σ_e	0.0267	0.0017
σ_a	0.0099	0.0009
σ_z	0.0130	0.0008
σ_r	0.0066	0.0004

Table 2. Subsample Maximum Likelihood Estimates and Standard Errors

Parameter	Pre-1979 Estimate	Standard Error	Post-1979 Estimate	Standard Error
β	0.9977	0.0005	0.9909	0.0008
α	0.2201	0.0065	0.2166	0.0173
ϕ_d	0.0040	0.0017	0.0290	0.0086
ϕ_k	7.9017	1.3240	4.9625	0.0127
ϕ_p	323.8098	3.8239	21.6284	0.0060
μ	1.0613	0.0165	1.0049	0.0113
ρ_y	0.0000012	0.0000	0.00046	0.0009
ρ_π	0.6348	0.0514	0.8321	0.0165
ρ_μ	0.3709	0.0505	0.4002	0.0076
v	3.5779	0.5479	0.8210	0.0135
z	4043.4	0.9274	50425	0.0343
ρ_v	0.9991	0.0011	0.9999	0.0001
ρ_e	0.7391	0.0878	1.0000	0.0000
ρ_a	0.9656	0.0267	0.9173	0.0335
ρ_z	0.9904	0.0041	0.9300	0.0209
σ_v	0.0105	0.0008	0.0114	0.0010
σ_e	0.0118	0.0024	0.0210	0.0016
σ_a	0.0064	0.0005	0.0075	0.0013
σ_z	0.0184	0.0035	0.0110	0.0012
σ_r	0.0045	0.0004	0.0041	0.0003

Table 3. Forecast Error Variance Decompositions for Consumption

Full Sample

Quarters Ahead	Money Demand	Investment	Preference	Technology	Policy
1	13.9776	8.8818	31.4929	26.6605	18.9872
4	4.4291	3.8451	37.7840	47.0659	6.8759
8	2.4074	2.1296	40.8737	50.8343	3.7550
12	1.8202	1.8423	43.1226	50.3740	2.8409
20	1.3802	3.6297	45.6078	47.2275	2.1548
40	0.9926	16.3689	43.1939	37.8957	1.5488
∞	0.0009	99.921	0.0423	0.0343	0.0014

Pre-1979

Quarters Ahead	Money Demand	Investment	Preference	Technology	Policy
1	35.5868	5.1292	4.3401	37.3261	17.6178
4	13.5804	1.3140	3.4016	74.0560	7.6479
8	6.2522	1.0756	2.5695	86.9132	2.9201
12	4.1187	1.0756	2.0867	91.1145	1.6045
20	2.7590	0.6777	1.5115	94.2649	0.7868
40	2.1141	0.3332	0.8670	96.3514	0.3343
∞	11.9625	0.1058	0.2910	87.5387	0.1020

Post-1979

Quarters Ahead	Money Demand	Investment	Preference	Technology	Policy
1	19.2401	9.2971	28.3054	32.8098	10.3475
4	6.0060	3.8894	28.6690	57.8700	3.5656
8	3.4320	2.2472	25.7685	66.5350	2.0174
12	2.6628	2.5662	23.2514	69.9720	1.5476
20	2.0491	7.2419	19.2831	70.2611	1.1648
40	1.3918	30.0134	12.7701	55.0696	0.7551
∞	0.0054	99.9668	0.0050	0.0225	0.0003

Table 4. Forecast Error Variance Decompositions for Investment

Full Sample

Quarters Ahead	Money Demand	Investment	Preference	Technology	Policy
1	24.5314	24.8265	4.8024	12.2071	33.6327
4	8.7318	32.2586	1.5503	44.0357	13.4235
8	5.2209	36.7312	1.3356	48.6830	8.0293
12	4.3109	41.0208	1.3557	46.6791	6.6335
20	3.7002	47.2553	1.3878	41.9587	5.6981
40	3.2550	53.3663	1.3312	37.0317	5.0158
∞	0.0415	99.4021	0.0171	0.4754	0.0639

Pre-1979

Quarters Ahead	Money Demand	Investment	Preference	Technology	Policy
1	40.5289	35.2055	4.1320	1.5548	18.5788
4	21.8284	22.5599	1.5888	43.0170	11.0058
8	10.8738	10.9819	0.6117	72.9083	4.6243
12	7.2419	6.6440	0.3665	83.0233	2.7243
20	4.8920	3.8189	0.2240	89.5011	1.5640
40	3.7724	2.2577	0.1398	92.9079	0.9221
∞	13.8510	1.2548	0.0784	84.3039	49.9613

Post-1979

Quarters Ahead	Money Demand	Investment	Preference	Technology	Policy
1	31.2121	32.9433	6.6388	12.1430	17.0627
4	10.2936	41.2419	2.2353	40.1360	6.0931
8	6.0866	45.3085	1.3234	43.6712	3.6104
12	4.9029	49.0295	1.0659	42.0857	2.9160
20	4.0752	54.6030	0.8864	38.0041	2.4313
40	3.5058	60.7106	0.7635	32.9252	2.0950
∞	0.0397	99.7291	0.0049	0.2129	0.0134

Table 5. Tests of Parameter Stability

20 Estimated Parameters	W = 3189555350***
5 Policy Parameters: μ, ρ_y, ρ_π, ρ_μ, σ_r	W = 10.0318
Portfolio Adjustment Cost: ϕ_d	W = 8.11331***
Price Adjustment Cost: ϕ_p	W = 6244.97476***
2 Adjustment Parameters: ϕ_d, ϕ_p	W = 6349.9029***
3 Money Demand Parameters: v, ρ_v, σ_v	W = 27.6580***
2 Investment Parameters: ρ_e, σ_e	W = 15.6494***
3 Preference Parameters: β, ρ_a, σ_a	W = 52.2749***
3 Technology Parameters: z, ρ_z, σ_z	W = 2581737967***

Note: *** denotes significance at the 1% level.

Technical Appendix

5. Characterizing the Equilibrium

5.1. Symmetric Equilibrium

In a symmetric equilibrium, $h_t(i) = h_t$, $K_t(i) = K_t$, $F_t(i) = F_t$, $Y_t(i) = Y_t$, $P_t(i) = P_t$, and $L_t(i) = L_t$ for all $i \in [0,1]$ and $t = 0,1,2,....$ In addition, the market-clearing condition

$$M_t = M_{t-1} + X_t$$

or

$$\mu_t = 1 + X_t/M_{t-1} \tag{24}$$

must hold for all $t = 0,1,2,....$ It is useful to note that these equilibrium conditions, together with (13)-(15), (17), and (18), can be used to rewrite the household's budget constraint (6) as the aggregate resource constraint

$$Y_t = C_t + I_t + \frac{\phi_p}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 Y_t, \tag{6}$$

which must also hold for all $t = 0,1,2,....$

Collecting and simplifying (1)-(24) yields

$$\tau_t = \frac{\phi_d}{2}\left(\frac{D_t}{\mu D_{t-1}} - 1\right)^2, \tag{1}$$

$$\frac{M_t}{P_t} = v_t(C_t + I_t), \tag{2}$$

$$\ln(v_t) = (1-\rho_v)\ln(v) + \rho_v \ln(v_{t-1}) + \varepsilon_{vt}, \tag{3}$$

$$K_{t+1} = (1-\delta)K_t + e_t I_t - \frac{\phi_k}{2}\left(\frac{K_{t+1}}{gK_t} - 1\right)^2 K_t, \tag{4}$$

$$\ln(e_t) = \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{5}$$

$$Y_t = C_t + I_t + \frac{\phi_p}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 Y_t, \tag{6}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{7}$$

$$a_t = (\Lambda_{1t} + v_t\Lambda_{2t})C_t, \tag{8}$$

$$\gamma = (\Lambda_{1t} + \Lambda_{2t})(W_t/P_t), \tag{9}$$

$$\frac{\Lambda_{1t}}{P_t} = \beta E_t \left(\frac{\Lambda_{1t+1} + \Lambda_{2t+1}}{P_{t+1}} \right), \tag{10}$$

$$\begin{aligned} &\gamma\phi_d \left(\frac{D_t}{\mu D_{t-1}} - 1 \right) \frac{D_t}{\mu D_{t-1}} \\ = \quad &\frac{[\Lambda_{1t}(r_t^d - 1) - \Lambda_{2t}]D_t}{P_t} + \beta\gamma\phi_d E_t \left[\left(\frac{D_{t+1}}{\mu D_t} - 1 \right) \frac{D_{t+1}}{\mu D_t} \right], \end{aligned} \tag{11}$$

$$\begin{aligned} &(\Lambda_{1t} + v_t\Lambda_{2t})(1/e_t) \left[1 + \frac{\phi_k}{g} \left(\frac{K_{t+1}}{gK_t} - 1 \right) \right] \\ = \quad &\beta E_t \left[\Lambda_{1t+1}(Q_{t+1}/P_{t+1}) + (1-\delta)(1/e_{t+1})(\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1}) \right] \\ &+\beta\phi_k E_t \left[(\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1})(1/e_{t+1}) \left(\frac{K_{t+2}}{gK_{t+1}} - 1 \right) \left(\frac{K_{t+2}}{gK_{t+1}} \right) \right] \\ &-\beta(\phi_k/2) E_t \left[(\Lambda_{1t+1} + v_{t+1}\Lambda_{2t+1})(1/e_{t+1}) \left(\frac{K_{t+2}}{gK_{t+1}} - 1 \right)^2 \right] \end{aligned} \tag{12}$$

$$D_t + X_t = L_t, \tag{13}$$

$$B_t = r_t X_t, \tag{14}$$

$$r_t = r_t^d, \tag{15}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{16}$$

$$L_t = W_t h_t, \tag{17}$$

$$\frac{F_t}{P_t} = Y_t - \frac{Q_t K_t + r_t W_t h_t}{P_t} - \frac{\phi_p}{2} \left(\frac{\pi_t}{\pi} - 1 \right)^2 Y_t, \tag{18}$$

$$Y_t = K_t^\alpha (g^t z_t h_t)^{1-\alpha}, \tag{19}$$

$$\Lambda_{1t} r_t (W_t/P_t) h_t = (1-\alpha)\Xi_t Y_t, \tag{20}$$

$$\Lambda_{1t}(Q_t/P_t)K_t = \alpha \Xi_t Y_t, \tag{21}$$

$$\begin{aligned} 0 \quad = \quad &(1-\theta)\Lambda_{1t} + \theta\Xi_t - \phi_p \Lambda_{1t} \left(\frac{\pi_t}{\pi} - 1 \right) \left(\frac{\pi_t}{\pi} \right) \\ &+\beta\phi_p E_t \left[\Lambda_{1t+1} \left(\frac{\pi_{t+1}}{\pi} - 1 \right) \left(\frac{\pi_{t+1}}{\pi} \right) \left(\frac{Y_{t+1}}{Y_t} \right) \right], \end{aligned} \tag{22}$$

$$\ln(r_t/r) = \rho_y \ln(y_t/y) + \rho_\pi \ln(\pi_t/\pi) + \rho_\mu \ln(\mu_t/\mu) + \varepsilon_{rt}, \tag{23}$$

and

$$\mu_t = 1 + X_t/M_{t-1}. \tag{24}$$

Together with the definitions $y_t = Y_t/g^t$, $\pi_t = P_t/P_{t-1}$, and $\mu_t = M_t/M_{t-1}$, these 24 equations determine the behavior of the 24 variables τ_t, D_t, M_t, P_t, v_t, C_t, I_t, K_t, e_t, Y_t, a_t, Λ_{1t}, Λ_{2t}, W_t, r_t, Q_t, X_t, B_t, r_t^d, z_t, L_t, h_t, F_t, and Ξ_t.

5.2. Transformed System

As a first step in solving the model, define the transformed variables $d_t = D_t/M_t$, $m_t = (M_t/P_t)/g^t$, $\mu_t = M_t/M_{t-1}$, $c_t = C_t/g^t$, $i_t = I_t/g^t$, $k_t = K_t/g^t$, $y_t = Y_t/g^t$, $\lambda_{1t} = g^t\Lambda_{1t}$, $\lambda_{2t} = g^t\Lambda_{2t}$, $w_t = (W_t/P_t)/g^t$, $q_t = Q_t/P_t$, $x_t = X_t/M_{t-1}$, $b_t = B_t/M_t$, $l_t = L_t/M_t$, $f_t = (F_t/P_t)/g^t$, and $\xi_t = g^t\Xi_t$. Use (15) to eliminate r_t^d from the system, and rewrite (1)-(14) and (16)–(24) as

$$\tau_t = \frac{\phi_d}{2}\left(\frac{\mu_t d_t}{\mu d_{t-1}} - 1\right)^2, \tag{1}$$

$$m_t = v_t(c_t + i_t), \tag{2}$$

$$\ln(v_t) = (1-\rho_v)\ln(v) + \rho_v \ln(v_{t-1}) + \varepsilon_{vt}, \tag{3}$$

$$gk_{t+1} = (1-\delta)k_t + e_t i_t - \frac{\phi_k}{2}\left(\frac{k_{t+1}}{k_t} - 1\right)^2 k_t, \tag{4}$$

$$\ln(e_t) = \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{5}$$

$$y_t = c_t + i_t + \frac{\phi_p}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{6}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{7}$$

$$a_t = (\lambda_{1t} + v_t\lambda_{2t})c_t, \tag{8}$$

$$\gamma = (\lambda_{1t} + \lambda_{2t})w_t, \tag{9}$$

$$g\lambda_{1t} = \beta E_t\left(\frac{\lambda_{1t+1} + \lambda_{2t+1}}{\pi_{t+1}}\right), \tag{10}$$

$$\begin{aligned} &\gamma\phi_d\left(\frac{\mu_t d_t}{\mu d_{t-1}} - 1\right)\frac{\mu_t d_t}{\mu d_{t-1}} \\ = \; &[\lambda_{1t}(r_t - 1) - \lambda_{2t}]d_t m_t + \beta\gamma\phi_d E_t\left[\left(\frac{\mu_{t+1}d_{t+1}}{\mu d_t} - 1\right)\frac{\mu_{t+1}d_{t+1}}{\mu d_t}\right], \end{aligned} \tag{11}$$

$$\begin{aligned} &g(\lambda_{1t} + v_t\lambda_{2t})(1/e_t)\left[1 + \frac{\phi_k}{g}\left(\frac{k_{t+1}}{k_t} - 1\right)\right] \\ = \; &\beta E_t\left[\lambda_{1t+1}q_{t+1} + (1-\delta)(1/e_{t+1})(\lambda_{1t+1} + v_{t+1}\lambda_{2t+1})\right] \\ &+\beta\phi_k E_t\left[(\lambda_{1t+1} + v_{t+1}\lambda_{2t+1})(1/e_{t+1})\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right] \\ &-\beta(\phi_k/2)E_t\left[(\lambda_{1t+1} + v_{t+1}\lambda_{2t+1})(1/e_{t+1})\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)^2\right] \end{aligned} \tag{12}$$

$$d_t + x_t/\mu_t = l_t, \tag{13}$$

$$b_t\mu_t = r_t x_t, \tag{14}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{16}$$

$$m_t l_t = w_t h_t, \tag{17}$$

$$f_t = y_t - q_t k_t - r_t w_t h_t - \frac{\phi_p}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{18}$$

$$y_t = k_t^\alpha (z_t h_t)^{1-\alpha}, \tag{19}$$

$$\lambda_{1t} r_t w_t h_t = (1-\alpha)\xi_t y_t, \tag{20}$$

$$\lambda_{1t} q_t k_t = \alpha \xi_t y_t, \tag{21}$$

$$\begin{aligned} 0 \;=\; & (1-\theta)\lambda_{1t} + \theta\xi_t - \phi_p \lambda_{1t}\left(\frac{\pi_t}{\pi} - 1\right)\left(\frac{\pi_t}{\pi}\right) \\ & +\beta\phi_p E_t\left[\lambda_{1t+1}\left(\frac{\pi_{t+1}}{\pi} - 1\right)\left(\frac{\pi_{t+1}}{\pi}\right)\left(\frac{y_{t+1}}{y_t}\right)\right], \end{aligned} \tag{22}$$

$$\ln(r_t/r) = \rho_y \ln(y_t/y) + \rho_\pi \ln(\pi_t/\pi) + \rho_\mu \ln(\mu_t/\mu) + \varepsilon_{rt}, \tag{23}$$

and

$$\mu_t = 1 + x_t. \tag{24}$$

Note also that the definitions of π_t, m_t, and μ_t imply

$$g m_t \pi_t = \mu_t m_{t-1}. \tag{25}$$

These 24 equations determine the behavior of the 24 stationary variables y_t, π_t, τ_t, d_t, m_t, μ_t, v_t, c_t, i_t, k_t, e_t, a_t, λ_{1t}, λ_{2t}, w_t, r_t, q_t, x_t, b_t, z_t, l_t, h_t, f_t, and ξ_t.

5.3. Steady State

In the absence of shocks, the economy converges to a steady state, in which each of the stationary variables is constant. Let μ be chosen by policy. Equations (3), (5), (7), and (16) determine v, $e = 1$, $a = 1$, and z. Equations (1), (24), and (25) determine

$$\tau = 0,$$

$$\pi = \mu/g,$$

and

$$x = \mu - 1.$$

Equations (10)-(12), (14), and (22) determine

$$r = \pi(g/\beta),$$

$$b = rx/\mu,$$

$$\lambda_2 = (r-1)\lambda_1,$$

$$q = (g/\beta - 1 + \delta)[1 + v(r-1)],$$

and

$$\xi = \left(\frac{\theta - 1}{\theta}\right)\lambda_1.$$

Equations (8) and (9) determine

$$c = \frac{1}{\lambda_1 + v\lambda_2}$$

and

$$w = \frac{\gamma}{\lambda_1 + \lambda_2}.$$

Equations (2), (4), (6), (18), (20), and (21) determine

$$k = \frac{c}{(\lambda_1 q/\alpha\xi) - g + 1 - \delta},$$

$$i = (g - 1 + \delta)k,$$

$$m = v(c + i),$$

$$y = \frac{\lambda_1 qk}{\alpha\xi},$$

$$h = \frac{(1-\alpha)\xi y}{\lambda_1 rw},$$

and

$$f = y - qk - rwh.$$

Equations (13) and (17) determine

$$l = wh/m$$

and

$$d = l - x/\mu.$$

Finally, (19) determines

$$\lambda_1 = \left[\frac{\gamma}{(1-\alpha)z}\right]\left(\frac{\theta}{\theta-1}\right)^{1/(1-\alpha)}\left(\frac{q}{\alpha}\right)^{\alpha/(1-\alpha)}.$$

5.4. Linearized System

Equations (1)-(14) and (16)–(25) can be log-linearized to describe the behavior of the stationary variables as the fluctuate about their steady-state values in response to shocks. Let $\hat{y}_t = \ln(y_t/y_t)$, $\hat{\pi}_t = \ln(\pi_t/\pi)$, $\hat{\tau}_t = \ln(\tau_t/\tau)$, $\hat{d}_t = \ln(d_t/d)$, $\hat{m}_t = \ln(m_t/m)$, $\hat{\mu}_t = \ln(\mu_t/\mu)$, $\hat{v}_t = \ln(v_t/v)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{\imath}_t = \ln(i_t/i)$, $\hat{k}_t = \ln(k_t/k)$, $\hat{e}_t = \ln(e_t/e)$, $\hat{a}_t = \ln(a_t/a)$, $\hat{\lambda}_{1t} = \ln(\lambda_{1t}/\lambda_1)$, $\hat{\lambda}_{2t} = \ln(\lambda_{2t}/\lambda_2)$, $\hat{w}_t = \ln(w_t/w)$, $\hat{r}_t = \ln(r_t/r)$, $\hat{q}_t = \ln(q_t/q)$, $\hat{x}_t = \ln(x_t/x)$, $\hat{b}_t = \ln(b_t/b)$, $\hat{z}_t = \ln(z_t/z)$, $\hat{l}_t = \ln(l_t/l)$, $\hat{h}_t = \ln(h_t/h)$, $\hat{f}_t = \ln(f_t/f)$, and $\hat{\xi}_t = \ln(\xi_t/\xi)$. Then a log-linear approximation of (1) implies that $\hat{\tau}_t = 0$, while log-linear approximations to (2)-(14) and (16)–(25) yield

$$m\hat{m}_t = m\hat{v}_t + vc\hat{c}_t + vi\hat{\imath}_t, \tag{2}$$

$$\hat{v}_t = \rho_v \hat{v}_{t-1} + \varepsilon_{vt}, \tag{3}$$

$$gk\hat{k}_{t+1} = (1-\delta)k\hat{k}_t + i\hat{e}_t + i\hat{\imath}_t, \tag{4}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \tag{5}$$

$$y\hat{y}_t = c\hat{c}_t + i\hat{\imath}_t, \tag{6}$$

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{7}$$

$$\hat{a}_t = \lambda_1 c\hat{\lambda}_{1t} + v\lambda_2 c\hat{v}_t + v\lambda_2 c\hat{\lambda}_{2t} + \hat{c}_t, \tag{8}$$

$$0 = \lambda_1 w\hat{\lambda}_{1t} + \lambda_2 w\hat{\lambda}_{2t} + \gamma\hat{w}_t, \tag{9}$$

$$r\hat{\lambda}_{1t} = E_t\hat{\lambda}_{1t+1} + (r-1)E_t\hat{\lambda}_{2t+1} - rE_t\hat{\pi}_{t+1}, \tag{10}$$

$$\begin{aligned} & \gamma\phi_d\hat{\mu}_t + \gamma\phi_d(1+\beta)\hat{d}_t - \gamma\phi_d\hat{d}_{t-1} \\ = \; & \lambda_1(r-1)dm\hat{\lambda}_{1t} + \lambda_1 rdm\hat{r}_t - \lambda_2 dm\hat{\lambda}_{2t} \\ & +\beta\gamma\phi_d E_t\hat{\mu}_{t+1} + \beta\gamma\phi_d E_t\hat{d}_{t+1}, \end{aligned} \tag{11}$$

$$\begin{aligned} & g\lambda_1\hat{\lambda}_{1t} + gv\lambda_2\hat{v}_t + gv\lambda_2\hat{\lambda}_{2t} - g(\lambda_1 + v\lambda_2)\hat{e}_t - \phi_k(\lambda_1 + v\lambda_2)\hat{k}_t \\ = \; & \beta\lambda_1(q+1-\delta)E_t\hat{\lambda}_{1t+1} + \beta\lambda_1 qE_t\hat{q}_{t+1} + \beta(1-\delta)v\lambda_2 E_t\hat{v}_{t+1} \\ & +\beta(1-\delta)v\lambda_2 E_t\hat{\lambda}_{2t+1} - \beta(1-\delta)(\lambda_1 + v\lambda_2)E_t\hat{e}_{t+1} \\ & -\phi_k(1+\beta)(\lambda_1 + v\lambda_2)\hat{k}_{t+1} + \beta\phi_k(\lambda_1 + v\lambda_2)E_t\hat{k}_{t+2} \end{aligned} \tag{12}$$

$$d\hat{d}_t + (x/\mu)\hat{x}_t - (x/\mu)\hat{\mu}_t = l\hat{l}_t, \tag{13}$$

$$\hat{b}_t + \hat{\mu}_t = \hat{r}_t + \hat{x}_t, \tag{14}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \tag{16}$$

$$\hat{m}_t + \hat{l}_t = \hat{w}_t + \hat{h}_t, \tag{17}$$

$$f\hat{f}_t = y\hat{y}_t - qk\hat{q}_t - qk\hat{k}_t - rwh\hat{r}_t - rwh\hat{w}_t - rwh\hat{h}_t, \tag{18}$$

$$\hat{y}_t = \alpha\hat{k}_t + (1-\alpha)\hat{z}_t + (1-\alpha)\hat{h}_t, \tag{19}$$

$$\hat{\lambda}_{1t} + \hat{r}_t + \hat{w}_t + \hat{h}_t = \hat{\xi}_t + \hat{y}_t, \tag{20}$$

$$\hat{\lambda}_{1t} + \hat{q}_t + \hat{k}_t = \hat{\xi}_t + \hat{y}_t, \tag{21}$$

$$\phi_p\hat{\pi}_t = (\theta-1)\hat{\xi}_t - (\theta-1)\hat{\lambda}_{1t} + \beta\phi_p E_t\hat{\pi}_{t+1} \tag{22}$$

$$\hat{r}_t = \rho_y\hat{y}_t + \rho_\pi\hat{\pi}_t + \rho_\mu\hat{\mu}_t + \varepsilon_{rt}, \tag{23}$$

$$\mu\hat{\mu}_t = x\hat{x}_t, \tag{24}$$

and

$$\hat{m}_t + \hat{\pi}_t = \hat{\mu}_t + \hat{m}_{t-1}. \tag{25}$$

These 23 equations determine the behavior of the 23 variables $\hat{y}_t$, $\hat{\pi}_t$, $\hat{d}_t$, $\hat{m}_t$, $\hat{\mu}_t$, $\hat{v}_t$, $\hat{c}_t$, $\hat{\imath}_t$, $\hat{k}_t$, $\hat{e}_t$, $\hat{a}_t$, $\hat{\lambda}_{1t}$, $\hat{\lambda}_{2t}$, $\hat{w}_t$, $\hat{r}_t$, $\hat{q}_t$, $\hat{x}_t$, $\hat{b}_t$, $\hat{z}_t$, $\hat{l}_t$, $\hat{h}_t$, $\hat{f}_t$, and $\hat{\xi}_t$. In preparing to solve the model, it is convenient to use (25) to rewrite (2) and (17) as

$$m\hat{\mu}_t + m\hat{m}_{t-1} = m\hat{\pi}_t + m\hat{v}_t + vc\hat{c}_t + vi\hat{\imath}_t \tag{2}$$

and

$$\hat{\mu}_t + \hat{m}_{t-1} + \hat{l}_t = \hat{\pi}_t + \hat{w}_t + \hat{h}_t. \tag{17}$$

It is also convenient to use (3)-(5) to rewrite (12) as

$$\begin{aligned} & g\lambda_1\hat{\lambda}_{1t} + v\lambda_2[g - \beta(1-\delta)\rho_v]\hat{v}_t + gv\lambda_2\hat{\lambda}_{2t} - \phi_k(\lambda_1 + v\lambda_2)\hat{k}_t \\ & -\{\beta(\lambda_1 + v\lambda_2)[\phi_k(i/k)(1/g) - (1-\delta)]\rho_e + g(\lambda_1 + v\lambda_2)\}\hat{e}_t \\ = \; & \beta\lambda_1(q + 1 - \delta)E_t\hat{\lambda}_{1t+1} + \beta\lambda_1 q E_t\hat{q}_{t+1} \\ & +\beta(1-\delta)v\lambda_2 E_t\hat{\lambda}_{2t+1} + \beta\phi_k(\lambda_1 + v\lambda_2)(i/k)(1/g)E_t\hat{\imath}_{t+1} \\ & +\phi_k(\lambda_1 + v\lambda_2)\{\beta[(1-\delta)/g] - (1+\beta)\}\hat{k}_{t+1} \end{aligned} \tag{12}$$

5.5. The Linear System in Matrix Form

Let

$$f_t^0 = \left[\begin{array}{ccccccccccccc} \hat{y}_t & \hat{\pi}_t & \hat{\mu}_t & \hat{c}_t & \hat{\imath}_t & \hat{w}_t & \hat{r}_t & \hat{q}_t & \hat{x}_t & \hat{b}_t & \hat{l}_t & \hat{h}_t & \hat{f}_t \end{array} \right]',$$

$$s_t^0 = \left[\begin{array}{ccccccc} \hat{d}_{t-1} & \hat{m}_{t-1} & \hat{k}_t & \hat{d}_t & \hat{\lambda}_{1t} & \hat{\lambda}_{2t} & \hat{\xi}_t \end{array} \right]',$$

and

$$z_t^0 = \left[\begin{array}{ccccc} \hat{v}_t & \hat{e}_t & \hat{a}_t & \hat{z}_t & \varepsilon_{rt} \end{array} \right]'.$$

Then (2), (6), (8), (9), (13), (14), (17)-(21), (23), and (24) can be written as

$$Af_t^0 = Bs_t^0 + Cz_t^0, \tag{26}$$

where A is 13×13, B is 13×7, and C is 13×5.

Equation (2) implies $a_{12} = m$, $a_{13} = -m$, $a_{14} = vc$, $a_{15} = vi$, $b_{12} = m$, $c_{11} = -m$.

Equation (6) implies $a_{21} = y$, $a_{24} = -c$, $a_{25} = -i$.

Equation (8) implies $a_{34} = 1$, $b_{35} = -\lambda_1 c$, $b_{36} = -v\lambda_2 c$, $c_{31} = -v\lambda_2 c$, $c_{33} = 1$.

Equation (9) implies $a_{46} = \gamma$, $b_{45} = -\lambda_1 w$, $b_{46} = -\lambda_2 w$.

Equation (13) implies $a_{53} = x/\mu$, $a_{59} = -x/\mu$, $a_{511} = l$, $b_{54} = d$.

Equation (14) implies $a_{63} = 1$, $a_{67} = -1$, $a_{69} = -1$, $a_{610} = 1$.

Equation (17) implies $a_{72} = 1$, $a_{73} = -1$, $a_{76} = 1$, $a_{711} = -1$, $a_{712} = 1$, $b_{72} = 1$.

Equation (18) implies $a_{81} = y$, $a_{86} = -rwh$, $a_{87} = -rwh$, $a_{88} = -qk$, $a_{812} = -rwh$, $a_{813} = -f$, $b_{83} = qk$.

Equation (19) implies $a_{91} = 1$, $a_{912} = \alpha - 1$, $b_{93} = \alpha$, $c_{94} = 1 - \alpha$.

Equation (20) implies $a_{101} = 1$, $a_{106} = -1$, $a_{107} = -1$, $a_{1012} = -1$, $b_{105} = 1$, $b_{107} = -1$.

Equation (21) implies $a_{111} = 1$, $a_{118} = -1$, $b_{113} = 1$, $b_{115} = 1$, $b_{117} = -1$.

Equation (23) implies $a_{121} = \rho_y$, $a_{122} = \rho_\pi$, $a_{123} = \rho_\mu$, $a_{127} = -1$, $c_{125} = -1$.

Equation (24) implies $a_{133} = \mu$, $a_{139} = -x$.

Equations (4), (10)-(12), (22), and (25) can be written as

$$DE_t s_{t+1}^0 + FE_t f_{t+1}^0 = Gs_t^0 + Hf_t^0 + Jz_t^0, \tag{27}$$

where D and G are 7×7, F and H are 7×13, and J is 7×5.

Equation (4) implies $d_{13} = gk$, $g_{13} = (1 - \delta)k$, $h_{15} = i$, $j_{12} = i$.

Equation (10) implies $d_{25} = 1$, $d_{26} = r - 1$, $f_{22} = -r$, $g_{25} = r$.

Equation (11) implies $d_{34} = \beta\gamma\phi_d$, $f_{33} = \beta\gamma\phi_d$, $g_{31} = -\gamma\phi_d$, $g_{34} = \gamma\phi_d(1 + \beta)$, $g_{35} = -\lambda_1(r - 1)dm$, $g_{36} = \lambda_2 dm$, $h_{33} = \gamma\phi_d$, $h_{37} = -\lambda_1 rdm$.

Equation (12) implies $d_{43} = \phi_k(\lambda_1 + v\lambda_2)\{\beta[(1-\delta)/g] - (1+\beta)\}$, $d_{45} = \beta\lambda_1(q+1-\delta)$, $d_{46} = \beta(1-\delta)v\lambda_2$, $f_{45} = \beta\phi_k(\lambda_1 + v\lambda_2)(i/k)(1/g)$, $f_{48} = \beta\lambda_1 q$, $g_{43} = -\phi_k(\lambda_1 + v\lambda_2)$, $g_{45} = g\lambda_1$, $g_{46} = gv\lambda_2$, $j_{41} = v\lambda_2[g - \beta(1-\delta)\rho_v]$, $j_{42} = -\{\beta(\lambda_1 + v\lambda_2)[\phi_k(i/k)(1/g) - (1-\delta)]\rho_e + g(\lambda_1 + v\lambda_2)\}$.

Equation (22) implies $f_{52} = \beta\phi_p$, $g_{55} = \theta - 1$, $g_{57} = 1 - \theta$, $h_{52} = \phi_p$.

Equation (25) implies $d_{62} = 1$, $g_{62} = 1$, $h_{62} = -1$, $h_{63} = 1$.

The presence of d_t in s^0_{t+1} and s^0_t implies $d_{71} = 1$, $g_{74} = 1$.

Finally, (3), (5), (7), and (16) can be written as

$$z^0_{t+1} = Pz^0_t + \varepsilon_{t+1}, \tag{28}$$

where

$$P = \begin{bmatrix} \rho_v & 0 & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 & 0 \\ 0 & 0 & \rho_a & 0 & 0 \\ 0 & 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{vt} & \varepsilon_{et} & \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}'.$$

6. Solving the Model

Rewrite (26) as

$$f^0_t = A^{-1}Bs^0_t + A^{-1}Cz^0_t,$$

and substitute it into (27) to obtain

$$E_t s^0_{t+1} = Ks^0_t + Lz^0_t, \tag{29}$$

where

$$K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$$

and

$$L = (D + FA^{-1}B)^{-1}(J + HA^{-1}C - FA^{-1}CP).$$

If the 7×7 matrix K has three eigenvalues inside the unit circle and four eigenvalues outside the unit circle, then the system has a unique solution. If K has more than four eigenvalues outside the unit circle, then the system has no solution. If K has less than four eigenvalues outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980).

Assuming from now on that there are exactly four eigenvalues outside the unit circle, write K as

$$K = M^{-1}NM,$$

where

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

The diagonal elements of N are the eigenvalues of K, with those in the 3×3 matrix N_1 inside the unit circle and those in the 4×4 matrix N_2 outside the unit circle. The columns of M^{-1} are the eigenvectors of K; M_{11} is 3×3, M_{12} is 3×4, M_{21} is 4×3, and M_{22} is 4×4. In addition, let

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where L_1 is 3×5 and L_2 is 4×5.

Now (29) can be rewritten as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} E_t s_{t+1}^0 = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s_t^0 + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} z_t^0$$

or

$$E_t s_{1t+1}^1 = N_1 s_{1t}^1 + Q_1 z_t^0 \tag{30}$$

and

$$E_t s_{2t+1}^1 = N_2 s_{2t}^1 + Q_2 z_t^0, \tag{31}$$

where

$$s_{1t} = M_{11} \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + M_{12} \begin{bmatrix} \hat{d}_t \\ \hat{\lambda}_{1t} \\ \hat{\lambda}_{2t} \\ \hat{\xi}_t \end{bmatrix}, \tag{32}$$

$$s_{2t} = M_{21} \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + M_{22} \begin{bmatrix} \hat{d}_t \\ \hat{\lambda}_{1t} \\ \hat{\lambda}_{2t} \\ \hat{\xi}_t \end{bmatrix}, \tag{33}$$

$$Q_1 = M_{11}L_1 + M_{12}L_2,$$

and

$$Q_2 = M_{21}L_1 + M_{22}L_2.$$

Since the eigenvalues in N_2 lie outside the unit circle, (31) can be solved forward to obtain

$$s_{2t} = -N_2^{-1}Rz_t^0,$$

where the 4×5 matrix R is given by

$$\begin{aligned} vec(R) &= vec\sum_{j=0}^{\infty} N_2^{-j}Q_2P^j = \sum_{j=0}^{\infty} vec(N_2^{-j}Q_2P^j) \\ &= \sum_{j=0}^{\infty}[P^j \otimes (N_2^{-1})^j]vec(Q_2) = \sum_{j=0}^{\infty}(P \otimes N_2^{-1})^j vec(Q_2) \\ &= [I_{(20\times20)} - P \otimes N_2^{-1}]^{-1}vec(Q_2). \end{aligned}$$

Use this result, along with (33), to solve for

$$\begin{bmatrix} \hat{d}_t \\ \hat{\lambda}_{1t} \\ \hat{\lambda}_{2t} \\ \hat{\xi}_t \end{bmatrix} = S_1 \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + S_2 z_t^0, \tag{34}$$

where

$$S_1 = -M_{22}^{-1}M_{21}$$

and

$$S_2 = -M_{22}^{-1}N_2^{-1}R.$$

Equation (32) now provides a solution for s_{1t}^1:

$$s_{1t} = (M_{11} + M_{12}S_1)\begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + M_{12}S_2z_t^0.$$

Substitute this result into (30) to obtain

$$\begin{bmatrix} \hat{d}_t \\ \hat{m}_t \\ \hat{k}_{t+1} \end{bmatrix} = S_3 \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + S_4 z_t^0, \tag{35}$$

where

$$S_3 = (M_{11} + M_{12}S_1)^{-1}N_1(M_{11} + M_{12}S_1)$$

and

$$S_4 = (M_{11} + M_{12}S_1)^{-1}(Q_1 + N_1M_{12}S_2 - M_{12}S_2P).$$

Finally, return to

$$\begin{aligned} f_t^0 &= A^{-1}Bs_t^0 + A^{-1}Cu_t \\ &= A^{-1}B \begin{bmatrix} I_{(3\times3)} \\ S_1 \end{bmatrix} \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + A^{-1}B \begin{bmatrix} 0_{(3\times5)} \\ S_2 \end{bmatrix} z_t^0 + A^{-1}Cz_t^0, \end{aligned}$$

which can be written more simply as

$$f_t^0 = S_5 \begin{bmatrix} \hat{d}_{t-1} \\ \hat{m}_{t-1} \\ \hat{k}_t \end{bmatrix} + S_6 z_t^0, \tag{36}$$

where

$$S_5 = A^{-1}B \begin{bmatrix} I_{(3\times3)} \\ S_1 \end{bmatrix}$$

and

$$S_6 = A^{-1}B \begin{bmatrix} 0_{(3\times5)} \\ S_2 \end{bmatrix} + A^{-1}C.$$

Equations (28) and (34)-(36) provide the model's solution:

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1} \tag{37}$$

and

$$f_t = Us_t, \tag{38}$$

where

$$s_t = \begin{bmatrix} \hat{d}_{t-1} & \hat{m}_{t-1} & \hat{k}_t & \hat{v}_t & \hat{e}_t & \hat{a}_t & \hat{z}_t & \hat{\varepsilon}_{rt} \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{y}_t & \hat{\pi}_t & \hat{\mu}_t & \hat{c}_t & \hat{\imath}_t & \hat{w}_t & \hat{r}_t & \hat{q}_t & \hat{x}_t & \hat{b}_t & \hat{l}_t & \hat{h}_t & \hat{f}_t & \hat{d}_t & \hat{\lambda}_{1t} & \hat{\lambda}_{2t} & \hat{\xi}_t \end{bmatrix}',$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{vt} & \varepsilon_{et} & \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}',$$

$$\Pi = \begin{bmatrix} S_3 & S_4 \\ 0_{(5\times3)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(3\times5)} \\ I_{(5\times5)} \end{bmatrix},$$

and

$$U = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix}.$$

7. Estimating the Model

Suppose that data are available on consumption C_t, investment I_t, money M_t, prices P_t, and interest rates r_t. These data can be used to construct a series $\{d_t\}_{t=1}^T$, where

$$d_t = \begin{bmatrix} \hat{c}_t \\ \hat{\imath}_t \\ \hat{m}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \ln(C_t) - t\ln(g) - \ln(c) \\ \ln(I_t) - t\ln(g) - \ln(i) \\ \ln(M_t) - \ln(P_t) - t\ln(g) - \ln(m) \\ \ln(P_t) - \ln(P_{t-1}) - \ln(\pi) \\ \ln(r_t) - \ln(r) \end{bmatrix}.$$

Equations (37) and (38) then given rise to an empirical model of the form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{39}$$

and

$$d_t = Cs_t, \tag{40}$$

where $A = \Pi$, $B = W$, C is formed from the rows of Π and U as

$$C = \begin{bmatrix} U_4 \\ U_5 \\ \Pi_2 \\ U_2 \\ U_7 \end{bmatrix},$$

and the vector of serially uncorrelated innovations

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{vt+1} & \varepsilon_{et+1} & \varepsilon_{at+1} & \varepsilon_{zt+1} & \varepsilon_{rt+1} \end{bmatrix}'$$

is assumed to be normally distributed with zero mean and diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}' = \begin{bmatrix} \sigma_v^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_a^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}.$$

The model defined by (39) and (40) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (39) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(8\times1)} \tag{41}$$

and

$$vec(\Sigma_{1|0}) = vec(Es_1s_1') = [I_{(64\times64)} - A\otimes A]^{-1}vec(BVB'). \tag{42}$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (40) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_tu_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned}\hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1}u_t\\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.\end{aligned}$$

Hence, from (39),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Using this last result, along with (39) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t|d_{t-1}, d_{t-2}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_t u_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t|d_{t-1}, d_{t-2}, ..., d_1),$$

$$Eu_t u_t' = C\Sigma_t C' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_t C'(C\Sigma_t C')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'(C\Sigma_t C')^{-1}C\Sigma_t A',$$

and the initial conditions $\hat{s}_1$ and Σ_1 are provided by (41) and (42).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln L = -\frac{5T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^{T}u_t'\Omega_t^{-1}u_t.$$

The model's 24 parameters are

$\phi_d \geq 0$	$\phi_k \geq 0$	$1 > \rho_a \geq 0$	$\sigma_z > 0$
$\mu \geq 1$	$g \geq 1$	$\sigma_a > 0$	$\phi_p \geq 0$
$v > 0$	$1 > \rho_e \geq 0$	$\theta > 1$	$\rho_y > 0$
$1 > \rho_v \geq 0$	$\sigma_e > 0$	$1 > \alpha > 0$	$\rho_\pi > 0$
$\sigma_v > 0$	$1 > \beta > 0$	$z > 0$	$\rho_\mu > 0$
$1 > \delta > 0$	$\gamma > 0$	$1 > \rho_z \geq 0$	$\sigma_r > 0$

8. Evaluating the Model

8.1. Testing for Parameter Stability

The procedures described by Andrews and Fair (1988) can be used to test for the stability of the model's estimated parameters. Let Θ^1 and Θ^2 denote the estimated parameters from two disjoint subsamples, and let H^1 and H^2 denote the associated covariance matrices, so that asymptotically,

$$\Theta^1 \sim N(\Theta^{10}, H^1)$$

and

$$\Theta^2 \sim N(\Theta^{20}, H^2).$$

One way of testing for the stability of all of the estimated parameters is with the likelihood ratio statistic

$$LR = 2[\ln L(\Theta^1) + \ln L(\Theta^2) - \ln L(\Theta)],$$

where $\ln L(\Theta^1)$, $\ln L(\Theta^2)$, and $\ln L(\Theta)$ are the maximized log likelihood functions for the first subsample, the second subsample, and the third entire sample. According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with q degrees of freedom under the null hypothesis of stability, where q is the number of estimated parameters.

Alternatively, the stability of some or all of the parameters can be tested with the Wald statistic

$$W = g(\Theta^1, \Theta^2)'(G\hat{H}G')^{-1}g(\Theta^1, \Theta^2),$$

when the stability restrictions are written as

$$g(\Theta^1, \Theta^2) = 0$$

and where

$$G = \frac{\vartheta g(\Theta^1, \Theta^2)}{\vartheta(\Theta^1, \Theta^2)'}$$

and

$$\hat{H} = \begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix}.$$

If Θ_q^1 and Θ_q^2 denote the subsets of Θ^1 and Θ^2 of interest, and if H_q^1 and H_q^2 denote the covariances matrices of Θ_q^1 and Θ_q^2, then this Wald statistic can be written more simply as

$$W = (\Theta_q^1 - \Theta_q^2)'(H_q^1 + H_q^2)^{-1}(\Theta_q^1 - \Theta_q^2).$$

According to Andrews and Fair, this statistic will be asymptotically dstributed as a chi-square random variable with q degrees of freedom under the null hypothesis of stability, where q is the number of parameters being tested for stability.

8.2. Variance Decompositions

Begin by considering (39), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t$$

or

$$(1 - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B\varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\ &= BVB' + ABVB'A' + A^2BVB'A^{2'} + ... + A^{k-1}BVB'A^{k-1\prime}.\end{aligned}$$

In addition, (39) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s$$

is given by

$$vec(\Sigma^s) = [I_{(64x64)} - A \otimes A]^{-1} vec(BVB').$$

Next, consider (38) and (39), which imply that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U\Sigma_k^s U',$$

$$\Sigma^f = \lim_{k\to\infty} \Sigma_k^f = U\Sigma^s U',$$

$$\Sigma_k^d = E(d_{t+k} - E_t d_{t+k})(d_{t+k} - E_t d_{t+k})' = C\Sigma_k^s C',$$

and

$$\Sigma^d = \lim_{k\to\infty} \Sigma_k^d = C\Sigma^s C'.$$

Let Θ denote the vector of estimated parameters, and let H denote the covariance matrix of these estimated parameters, so that asymptotically,

$$\Theta \sim N(\Theta^0, H).$$

Note that the elements of Σ_k^s, Σ^s, Σ_k^f, Σ^f, Σ_k^d, and Σ^d can all be expressed as nonlinear functions of Θ :

$$\Sigma = g(\Theta),$$

so that asymptotic standard errors for these elements can be found by calculating

$$\bigtriangledown g H \bigtriangledown g'.$$

In practice, the gradient $\bigtriangledown g$ can be evaluated numerically, as suggested by Runkle (1987).