On the performance of the Shapley Shubik and Banzhaf power indices for the allocations of mandates

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Summary: A classical problem in the power index literature is to design a voting mechanism such as the distribution of power of the different players is equal (or closer) to a pre established target. This tradition is especially popular when considering two tiers voting mechanisms: each player votes in his own jurisdiction to designate a delegate for the upper tier; and the question is to assign a certain number of mandates for each delegate according the population of the jurisdiction he or she represents. Unfortunately, there exist several measures of power, which in turn imply different distributions of the mandates for the same pre established target. The purposes of this paper are twofold: first, we calculate the probability that the two most important power indices, the Banzhaf index and the Shapley-Shubik index, lead to the same voting rule when the target is the same. Secondly, we determine which index on average comes closer to the pre established target.

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1 Introduction

The concept of power index is probably the most famous application of game theory to political sciences. The objective of a power index is to evaluate the a priori influence of a given player for a given voting rule by computing the number of times he or she is decisive. In a 'yes' or 'no' binary decision, a player is said to be decisive each time he or she can reverse the decision by changing his vote. Most of the literature on power indices is then devoted to the evaluation of the influence of the different players in different institutions, *e.g.* shareholders in a firm, political parties in a parliament, countries in the United Nations, US states in the Electoral College, etc.

The power indices can also be used from a normative perspective rather than from a descriptive one. If the objective is to allocate the power to the players according to a pre established target, what are the political institutions that come closer to this objective? This institution design problem is known as the 'inverse problem' in the power index literature. It is an issue as old as the field: Penrore [10] already remarked in 1952 that, according to his measure of power[1], the best way to equalize the influence of the citizens of different states in a two tiers voting system should be to allocate to each state a number of mandates proportionally to the square root of its population.

In this paper, we study the inverse problem from a more general perspective. Consider the whole class of weighted quota games, where each player is endowed with a certain number of mandates, or weights, and where a decision is approved if and only if the number of players that vote 'yes' altogether strictly gathers more than q votes. Then knowing that the power must be as close as possible to some pre established normative target (typically, the distribution of population of the different members of a political union) which is the best voting rule? The problem is tricky and time consuming (see Leech [8]) and an algorithm to solve the inverse problem is proposed in a companion paper (see Barthélémy and Martin [2]). But apart from computational issues, another question with the inverse problem approach is the choice of power index. The literature on power indices has suggested several ways to measure power, the most famous indices being the ones proposed by Shapley and Shubik [11] and Banzhaf [1]. Thus, depending on the choice of the power index, the results of

[1]Penrose's index is also now known as the absolute Banzhaf index. For more on the history of the power indices and the fact that they have been rediscovered several times in different fields, see the paper by Felsenthal and Machover [5].

the inverse program may lead to different optimal weighted quota games. The primary objective of this paper is to examine to which extend the choice of different power indices may change the solution of the optimization problem. As a by product, we will also observe which index, on average, performs better in terms of minimizing the total distance to a given target.

The rest of the paper is organized as follows. In Section 2, we present the definitions and the basic concepts. Section 3 is devoted to a detailed analysis of the three player case. The results for the general case are presented in Section 4. Section 5 concludes the paper.

2 Definitions

2.1 Weighted quota games

Let $N = \{1, 2, ..., n\}$ be the set of players (elsewhere voters, states, cities, etc), of cardinality n. Let $G = [q; w_1, ..., w_n]$ be a weighted quota game where q is the quota and w_i is the weight attached to the player i, with $\sum_{i=1}^{n} w_i = \bar{w}$ and $q < \bar{w}$. We assume that $w_1 \geqslant w_2 \geqslant ... \geqslant w_n$. We say that a coalition $S \subseteq N$ (that is a group of players) is winning if and only if $\sum_{i \in S} w_i > q$.

For example, consider the following weighted quota game $G = [9; 8, 4, 4, 1]$. The coalition $S = \{1, 2\}$ is winning (we write $S \in W$ with W the set of all winning coalitions) since $w_1 + w_2 > q$. If $S \in W$, we attribute a value 1 to S, denoted $v(S) = 1$ and if $S \notin W$, we have $v(S) = 0$. We only consider proper voting games, that is voting games such that if $S \in W$ then $N \backslash S \notin W$. In particular, this implies $q \geq \frac{1}{2}\bar{w}$. A weighted quota game is a majority weighted game if $q = \bar{w}/2$.

2.2 Power indices

Several power indices are proposed in the literature and all of them admit the importance of a particular player, the decisive player. A player is decisive in a coalition $S \in W$ if this coalition becomes a loosing coalition when the player leaves it. In the previous example, player 1 is decisive since $w_2 < q$, that is $S \backslash \{1\} \notin W$. In this paper we only consider the two most important power indices, the Shapley-Shubik power index and the Banzhaf power index (for a complete description of the power indices, see Straffin [12], Felsenthal and Machover [5] or Laruelle [7]).

The Shapley-Shubik index [11] takes into account the following reasoning: consider a player to construct a coalition and analyze if this coalition is winning. If it is not the case, add a second player and check again whether this coalition is winning and so on. When a player, in joining a coalition, makes the coalition winning, we call him a pivotal player. Now, consider all the possible permutations of the player ($n!$). The Shapley-Shubik index is the number of times where a player is pivotal divided by the total number of permutations of the voters. The Shapley-Shubik index of the player i for a game G reads:

$$\phi_i(G) = \frac{number \;\; of \; orders \; with \; i \; pivotal}{n!} \tag{1}$$

We derive the following formula

$$\phi_i(G) = \sum_{S \subseteq N,\, i \in S} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \backslash \{i\})] \tag{2}$$

with s the number of players in S. Since $v(S) = 0$ or $v(S) = 1$, $[v(S) - v(S\backslash\{i\})]$ is non-null if and only if the player i is pivotal in S. $\phi(G) = (\phi_1(G), \phi_2(G), ..., \phi_n(G))$ is the power vector associated with the Shapley-Shubik index for the game G.

The calculus of the Banzhaf index [1] is simpler. We just have to identify the number of winning coalitions where the player i is decisive among the $2^n - 1$ coalitions he belongs to, irrespective of the order of arrivals in the coalition, and divide it by the total number of decisive players. The Banzhaf index of the player i for game G is then

$$\beta_i(G) = \frac{number \; of \; times \; for \; which \; player \; i \; is \; decisive}{total \; number \; of \; decisive \; players} \tag{3}$$

We can write

$$\beta_i(G) = \frac{\sum_{S \subseteq N} [v(S) - v(S \backslash \{i\})]}{\sum_{j \in N} \sum_{S \subseteq N} [v(S) - v(S \backslash \{j\})]} \tag{4}$$

$\beta(G) = (\beta_1(G), \beta_2(G), ..., \beta_n(G))$ is the power vector associated with the Banzhaf index for the game G.

2.3 Meeting the target

In some voting problems, there might exist a pre existing norm concerning a distribution of the power that we may wish to attain. For example, if the players are the countries of a federal union, one may wish the power of a player to be proportional to its population. In a charity trust, the different participants may wish their influence to be proportional to the amount of their donation. In a firm, some minority shareholders may wish their interests to be protected from the decisions of a major shareholder. Thus, we assume that there exists a pre established target vector on the ideal repartition of power. We denote by $p = (p_1, \ldots, p_n) \in \mathcal{P}^n$ the target, with $\mathcal{P}^n = \{p \in \mathbb{R}^n : p_i \geq 0 \textit{ and } \sum_{i=1}^n p_i = 1\}$. Knowing this distribution p, we then wish the distribution of power to be as close as possible to p. The problem is then to determine, given the "population" vector p representing the ideal influence of the players, the distribution of the weights $(w_1, \ldots w_n)$ and the quota q such that the sum of the differences between the target and the power is minimal. This approach comes from the studies of Leech [8], Pajala [9] or Bisson, Bonnet and Lepelley [3], among others. In this paper, we only consider the variance[2].

Let $d(x, y)$ be the variance

$$d(x, y) = \sum_{i=1}^{n} (x_i - y_i)^2 \,,\; x \in \mathbb{R}^n \textit{ and } y \in \mathbb{R}^n \tag{5}$$

We assume that there exists a link between the target and the weight of player i: we impose $p_1 \geqslant p_2 \geqslant ... \geqslant p_n$ (a more important target implies a more important weight by hypothesis). The distance between the Shapley-Shubik index and the target is written d^{SS}. Similarly, the distance between the Banzhaf index and the target is denoted by d^B.

Obviously, the distribution of the weights allocated to the players and so the distribution of power depends on the choice of the power index. In particular, it can be argued that no power index is ultimately better than another one from a normative point of view, so there is no specific reason to privilege one of the index while adopting an inverse problem perspective. It means that the choice of the power index to determine the distribution of the weight is of crucial importance. Our main purpose is to examine whether the situations where the optimal weighted quota games are different while using different power indices are frequent. In other words, can we search for the best weighted quota game irrespective of the choice of power measure when the objective is to equalize influence with a pre determined target vector p? As a by product, we will also discover which power index, in average, minimizes the distance to a target and best performs for the inverse program. More precisely, we study three points:

[2] All the results obtained with the variance were also obtained with the other important measure of distance in the literature, the sum of differences in absolute value. Since the results were almost the same, irrespective the choice for a distance, we only present here the results by considering the variance.

- First, we compute the probability that both power indices give an identical vector of power for a target chosen randomly from an uniform distribution in the unit simplex of dimension n. For n players, this probability is denoted $P_n(B = SS)$.

- Let $G_B(p)$ (resp. $G_{SS}(p)$) be the weighted quota game obtained as a solution of the inverse problem for a target vector p for the Banzhaf index (resp. the Shapley-Shubik index). The probability that the inverse problem gives the same institutional solution is thus denoted by $P_n(G_B = G_{SS})$ for n players.

- Thirdly, we compute the probability that the Shapley-Shubik index of power for $G_{SS}(p)$ implies a minimal distance inferior to the one given by the Banzhaf index for $G_B(p)$ (denoted $P_n(B > SS)$) and vice-versa (denoted $P_n(B < SS)$).

Before turning to the general case, we will now study in detail the 3 player game in order to familiarise the reader with the concepts.

3 The 3 player case

In this section, we give some analytical results for the particular case of 3 players. We compute the probability that the minimal distance between the power and the target is the same with the two power indices, that is the probability $P_3(B = SS)$. For this, we have to determine first all the possible vectors of power. Next, we derive $P_3(G_B = G_{SS})$ and check which power index is "closer" to the target by deriving $P_3(B > SS)$ and $P_3(SS > B)$.

These probabilities are obtained for any quota and any sum of weights. But we can add some constraints to the inverse program, for example, by fixing the quota to focus on majority games or by fixing an a priori $\bar{w}$. Some particular cases are studied in the third subsection.

3.1 The number of possible vectors of power

To our knowledge, there does not exist a general formula to determine all the possible power vectors for a given n and a given power index. However, for n small, it is possible to enumerate all the possible cases. The details of the calculus for $n = 3$ are given in the Appendix and we can summarize these results in table 1. The first column gives the conditions on the weight vector w. Column 2 and 3 indicate the corresponding Banzhaf and

Shapley-Shubik power vectors. The last column displays one game G for each class, where the weights and the quota are integers; notice that all the possible games for $n = 3$ G_1, G_2, G_3 and G_4 can be described as majority games.

Table 1: The four different weighted quota games for $n = 3$, with examples.

Conditions on w	Banzhaf	Shapley-Shubik	Example
$w_1 > q$	$\beta^1 = (1, 0, 0)$	$\phi^1 = (1, 0, 0)$	$G_1 = (1; 2, 0, 0)$
$w_1 + w_3 \leqslant q$ *and* $w_1 + w_2 > q$	$\beta^2 = (\frac{1}{2}, \frac{1}{2}, 0)$	$\phi^2 = (\frac{1}{2}, \frac{1}{2}, 0)$	$G_2 = (1; 1, 1, 0)$
$w_1 \leqslant q$ *and* $w_2 + w_3 > q$	$\beta^3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\phi^3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$G_3 = (3; 2, 2, 2)$
$w_1 \leqslant q$ *and* $w_2 + w_3 \leqslant q$ *and* $w_1 + w_3 > q$	$\beta^4 = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$	$\phi^4 = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	$G_4 = (4; 4, 2, 2)$

3.2 A graphic computation of $P_3(B = SS)$, $P_3(G_B = G_{SS})$ and $P_3(B > SS)$

For the three players case, we can illustrate the inverse problem with simple graphics[3]. Since we assume that $p_1 + p_2 + p_3 = 1$, we can represent all the possible targets in a simplex. Furthermore, we assume that $p_1 \geqslant p_2 \geqslant p_3$: Thus, the dotted area in Figure 1 represents all the admissible targets. We assume that any point in this area is equally likely to be an admissible target vector. Note that the constraint $p_1 \geqslant p_2 \geqslant p_3$ implies $p_2 \geqslant \frac{1}{2} - \frac{p_1}{2}$.

The surface of the admissible area A is equal to

$$A = \int_{1/3}^{1/2} p_1 dp_1 + \int_{1/2}^{1} (1 - p_1) dp_1 - \int_{1/3}^{1} (\frac{1}{2} - \frac{1}{2} p_1) dp_1 = \frac{1}{12} \simeq 0.0833 \qquad (6)$$

If we calculate the distances d^B and d^{SS} between a vector of power and a target p, we obtain easily:

- $d^B = d^{SS} = \sum_{i=1}^n p_i^2 - \frac{1}{3}$ for the vector $(1/3, 1/3, 1/3)$,
- $d^B = \sum_{i=1}^n p_i^2 + \frac{1}{25} - \frac{4p_1}{5}$ for the vector $(3/5, 1/5, 1/5)$,
- $d^{SS} = \sum_{i=1}^n p_i^2 + \frac{1}{6} - p_1$ for the vector $(2/3, 1/6, 1/6)$,
- $d^B = d^{SS} = \sum_{i=1}^n p_i^2 + \frac{1}{2} - p_1 - p_2$ for the vector $(1/2, 1/2, 0)$,
- $d^B = d^{SS} = \sum_{i=1}^n p_i^2 + 1 - 2p_1$ for the vector $(1, 0, 0)$.

[3] A similar graphic interpretation of the power indices has been presented by Jones [6] for the analysis of the paradoxes of power indices.

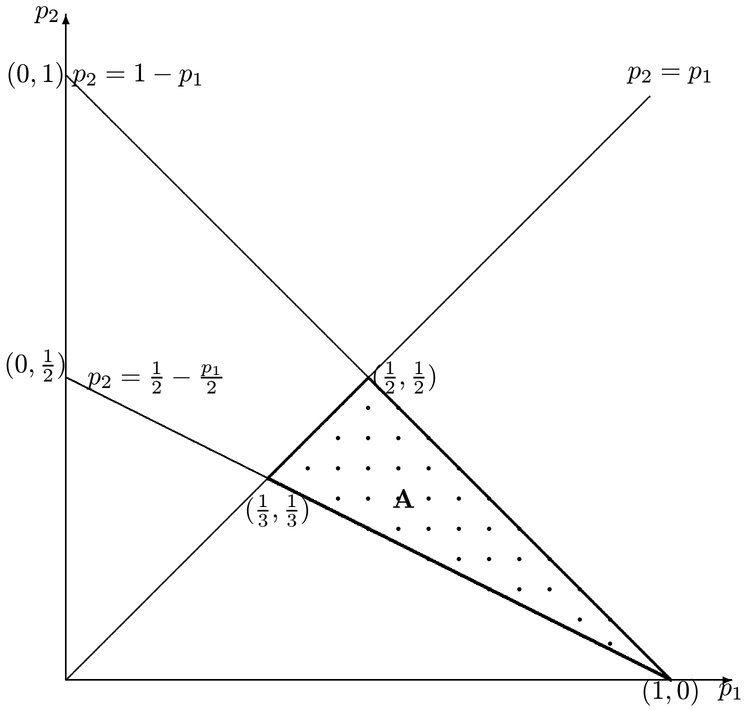

Figure 1: The possible target vectors

We can compare the distances and we obtain that, for the Banzhaf index:

-The vector of power $\beta^1 = (1,0,0)$ and G_1 minimize d^B if $p_1 > 5/6$.

-The vector of power $\beta^2 = (1/2, 1/2, 0)$ and G_2 minimize d^B if $p_2 > 1/3$ and $p_2 > 5/6 - p_1$.

-The vector of power $\beta^3 = (1/3, 1/3, 1/3)$ and G_3 minimize d^B if $p_2 < 1/2$ and $p_2 < 5/6 - p_1$.

-The vector of power $\beta^4 = (2/3, 1/6, 1/6)$ and G_4 minimize d^B otherwise, that is if $7/15 < p_1 < 4/5$ and $p_1 + 5p_2 < 23/10$.

The different zones corresponding to the optimal games are depicted in Figure 2.

The same reasoning for Shapley Shubik enable us to define the following domains:

-The vector of power $\phi^1 = (1,0,0)$ and G_1 minimize d^{SS} if $p_1 > 4/5$.

-The vector of power $\phi^2 = (1/2, 1/2, 0)$ and G_2 minimize d^{SS} if $p_2 > 23/50 - p_1/5$ and $p_2 > 5/6 - p_1$.

-The vector of power $\phi^3 = (1/3, 1/3, 1/3)$ and G_3 minimize d^{SS} if $p_2 < 7/15$ and $p_2 < 5/6 - p_1$.

-The vector of power $\phi^4 = (2/3, 1/6, 1/6)$ and G_4 minimize d^{SS} otherwise, that is if

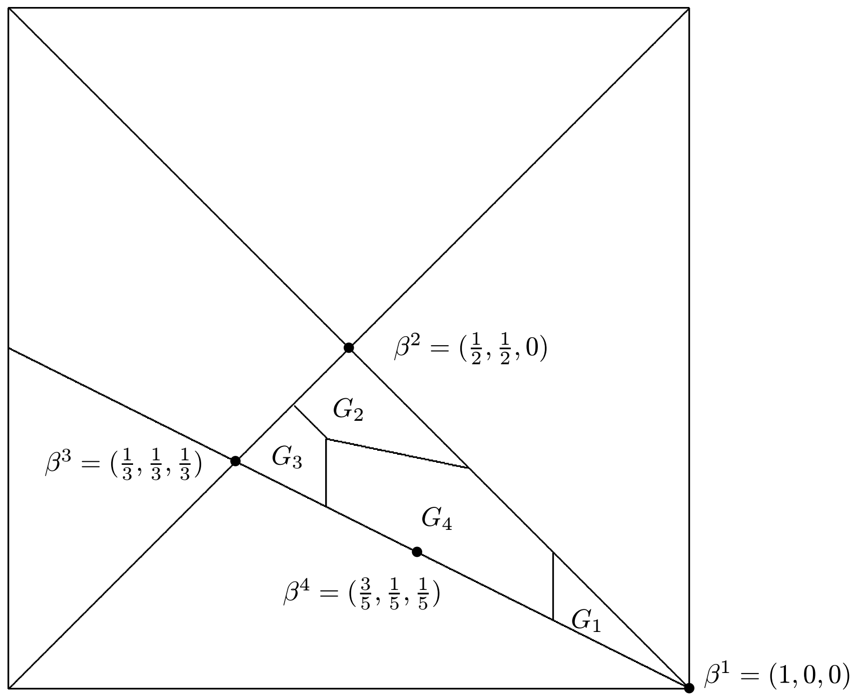

Figure 2: The different closest games for the Banzhaf index for $n = 3$

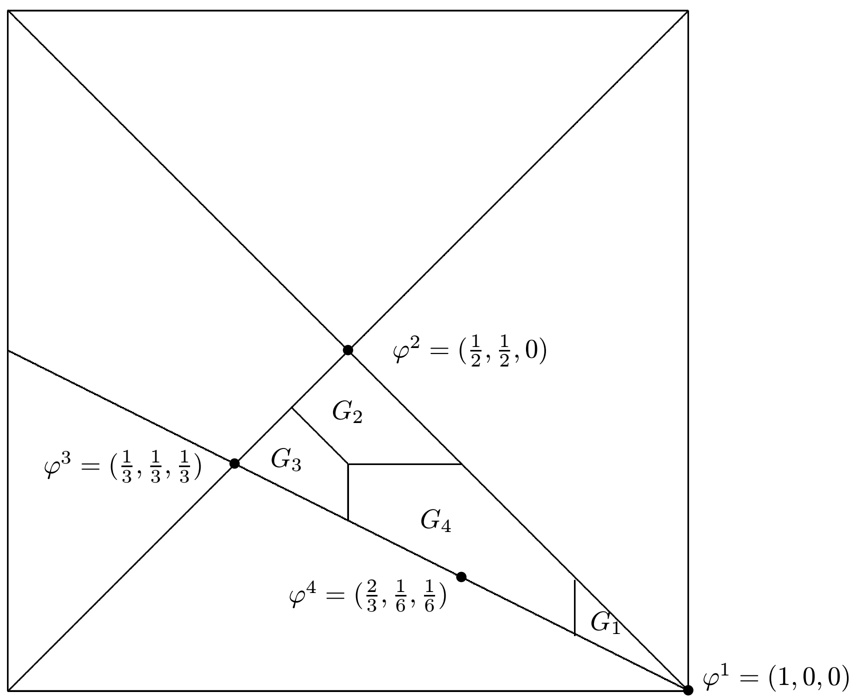

Figure 3: The different closest games for the Shapley-Shubik index for $n = 3$

$7/15 < p_1 < 4/5$ and $p_1 + 5p_2 < 23/10$.

The corresponding zones are depicted in Figure 3. Now, by comparing Figure 2 and 3, we can immediately identify in Figure 4 the regions where the minimizing distance process does not lead to the same vector of power. The dotted area D in Figure 4 is equal to

$$D = \int_{7/15}^{19/30} (\frac{23}{50} - \frac{p_1}{5})dp_1 + \int_{19/30}^{2/3} \frac{1}{3}dp_1 + \int_{2/3}^{5/6} (1-p_1)dp_1 - \int_{7/15}^{5/6} (\frac{1}{2} - \frac{1}{2}p_1)dp_1$$

$$D = \frac{169}{3600} \tag{7}$$

Thus the probability that the Banzhaf index and the Shapley-Shubik index give the same vector of power and the same minimal distance to a target is

$$P_3(B = SS, d) = 1 - \frac{D}{1/12} = \frac{131}{300} \simeq 0.4366 \tag{8}$$

Similarly, the dotted area D' in Figure 5 represents the target vectors which lead to different games as solution of the inverse problem. We derive :

$$\begin{aligned} P_3(G_B = G_{SS}) &= \frac{1}{12}\left(\int_{\frac{7}{15}}^{\frac{19}{30}} (\frac{23}{50} - \frac{1}{5}p_1)dp_1 - \int_{\frac{19}{30}}^{\frac{27}{40}} (\frac{23}{50} - \frac{1}{5}p_1)dp_1 - \int_{\frac{4}{5}}^{\frac{5}{6}} (\frac{1}{2} - \frac{1}{2}p_1)dp_1 \right. \\ &\quad - \int_{\frac{7}{15}}^{\frac{1}{2}} (\frac{1}{2} - \frac{1}{2}p_1)dp_1 - \int_{\frac{1}{2}}^{\frac{19}{30}} \frac{1}{3}dp_1 + \int_{\frac{19}{30}}^{\frac{2}{3}} \frac{1}{3}dp_1 \\ &\quad \left. + \int_{\frac{2}{3}}^{\frac{27}{40}} (1-p_1)dp_1 + \int_{\frac{4}{5}}^{\frac{5}{6}} (1-p_1)dp_1 \right) \\ &= \frac{61}{6} \approx 0.1016 \end{aligned} \tag{9}$$

3.3 Particular cases: majority weighted games and other constraints

When designing a voting rule, several constraints may complete our model: we may focus on the majority rule only, or set $\bar{w}$ to be an odd integer, for example. We deal here with the most important voting game in the literature, that is the majority voting game where $q = \frac{\bar{w}}{2}$. All the games presented in the table 1 are majority games. However, we show in the appendix that when $\bar{w}$ is odd, only the games G_1 and G_3 can appear. Hence, the possible vectors are the same ones with the Banzhaf and the Shapley-Shubik indices, and trivially, $P_3(B = SS) = P_3(G_B = G_{SS}) = 1$.

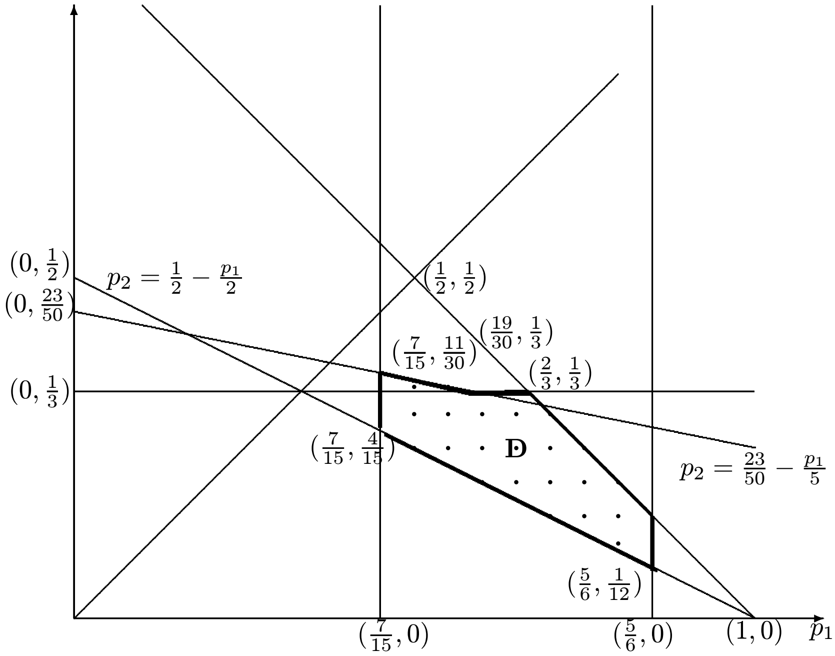

Figure 4: The area corresponding to $P_3(B = SS)$

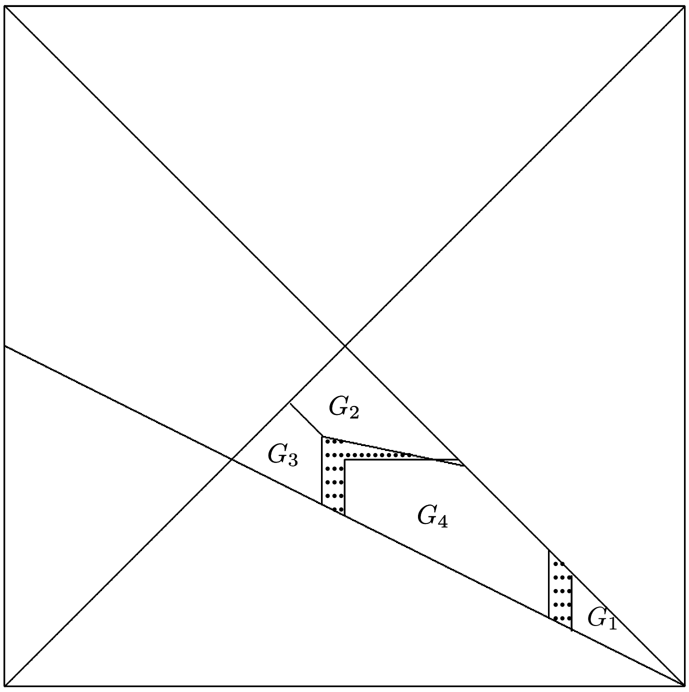

Figure 5: Different Shapley-Shubik and Banzhaf inverse games for $n = 3$

If $\bar{w}$ is even and greater than 4, the four games G_i can appear, thus the results with the majority game are equivalent to the ones obtain in section 3.2. However, if $\bar{w} = 4$ with integer weights, the situation is different since the game G_3 and its $(1/3, 1/3, 1/3)$ power repartition is no longer possible. The detail of the calculus of $P_3(B = SS)$ with a graphic representation are omitted, but with a reasoning similar to the one presented in the previous subsection, we derive $P_3(B = SS) = \frac{49}{150} \approx 0.3266$.

In conclusion, in this simple three player game, we can see that by slightly changing the class of voting rules we wish to obtain, the results of the inverse program can be radically different. This drastic changes will also be illustrated in the next section, depending whether we impose some conditions on the set of possible games.

3.4 Is Shapley-Shubik better than Banzhaf?

In the previous subsection, we have seen that there exists an area D in Figure 4 for which the minimal distance to a vector of power is different and depends on the chosen index of power. We now determine the area which is such that the Banzhaf index "does better", that is minimizes the minimal distance to a vector of power. In other words, for a target p, we search whether p is closer to a β or a ϕ vector. We calculate the probability, denoted $P_3(B < SS)$, that the Banzhaf index gives a minimal distance smaller than the one given by the Shapley-Shubik index.

By the previous section, we derived $d^B = \sum_{i=1}^n p_i^2 + \frac{1}{25} - \frac{4p_1}{5}$ for the vector $(3/5, 1/5, 1/5)$ and $d^{SS} = \sum_{i=1}^n p_i^2 + \frac{1}{6} - p_1$ for the vector $(2/3, 1/6, 1/6)$. Thus, Banzhaf minimizes the minimal distance if $\sum_{i=1}^n p_i^2 + \frac{1}{25} - \frac{4p_1}{5} < \sum_{i=1}^n p_i^2 + \frac{1}{6} - p_1$. We obtain $p_1 < 19/30$. In Figure 4, we can split the dotted area D in two parts: for $p_1 < 19/30$, the choice of the Banzhaf index implies a smaller distance than the choice of the Shapley-Shubik index. For $p_1 > 19/30$, the contrary holds. Therefore, we have

$$\begin{aligned} P_3(B < SS) &= 12\left(\int_{19/30}^{2/3} \frac{1}{3} dp_1 + \int_{2/3}^{5/6} (1 - p_1) dp_1 - \int_{19/30}^{5/6} (\frac{1}{2} - \frac{1}{2} p_1) dp_1\right) \\ &= \frac{1}{4} = 25\% \end{aligned} \quad (10)$$

Since $P_3(B < SS) + P_3(B = SS) + P_3(B > SS) = 1$, we have

$$P_3(B > SS) = \frac{47}{150} \simeq 0.3133 \quad (11)$$

Thus, when the objective is to find a distribution of power as close as possible to a pre established target, the Shapley-Shubik index tends to perform slightly better than the Banzhaf index. When there is no constraint on the choice of the possible games, this result will be confirmed for a higher number of players by the following section.

4 The general case, $n \geqslant 3$

When the number of players increases, the analytical approach becomes very tedious or simply impossible when we wish to evaluate the different probabilities. For example, there are at least 14 710 vectors of power with 7 players when we consider the Banzhaf index! A graphic illustration is no longer possible. Thus, the only solution is to rely upon computer simulations:

- Firstly, using a computer program presented in a companion paper [2] we enumerate (and store) the different possible vectors of power obtained with the Banzhaf index and the Shapley-Shubik index. This is done for $n = 3$ to $n = 8$. This method may not catch all the possible vectors, and we will just get a lower bound on the number of possible vectors. However, we are quite confident that we come extremely close to the exact value.

- Secondly, using this information, we search for the closest games $G_{SS}(p)$ and $G_B(p)$ for vectors p drawn from the uniform distribution in the unit simplex $\mathcal{P}^n$ (with the constraint that $p_1 \geq p_2 \geq \ldots \geq p_n$), and then estimate $P_n(G_B = G_{SS})$.

- Thirdly, we estimate the probability that the same Banzhaf and Shapley-Shubik vector of power minimizes the distance with the target, that is we estimate $P_n(B = SS)$.

- Fourthly, we estimate the probability that Shapley-Shubik "does better" than Banzhaf, that is $P_n(SS < B)$. This corresponds to the case where the minimal distance to a Shapley-Shubik vector of power is smaller than the one get with the Banzhaf index.

4.1 The number of vectors of power

The first column ($q \geqslant \bar{w}/2$) of table 2, present a lower bound for the total number of vectors of power (for the details of these simulations, see Barthélémy and Martin [2]). Note that these vectors are obtained for all the possible values of the quota q and for all the possible values of sum of weight $\bar{w}$.

Constraints may be added in order to get the number of vectors of power in more particular cases. In the two other columns of table 2, the number of vectors of power given for two relative quotas, the majority game (denoted $q = \bar{w}/2$) and the 2/3 majority game($q = 2\bar{w}/3$). In these last cases, it is obvious that the number of vectors of power is less important[4].

Table 3 gives the number of vectors of power in the case where there is no dummy player (a dummy player is a player without power). In fact, we may impose, while minimizing the distance to a vector of power, that each player has at least a positive power, which means that we should exclude some power vectors and some games as possible outputs.

4.2 Computation of $P_n(G_B = G_{SS})$ for $n \geq 3$

The first column of table 4 displays the probabilities $P_n(G_B = G_{SS})$ for $n = 3, 4, 5, 6$. The figures in this table tell us whether the choice of a power index has an impact on the choice of the weighted game that better fits to a pre established target. The answer is clear: the probability that the optimal weighted quota games are the same declines steadily as n increases. The choice of a "best voting mechanism" cannot be done irrespective of the power index that we choose. Notice that we have computed $P_n(G_B = G_{SS})$ with no restriction of the game, allowing in particular for games with dummy players and any value for the quota. By focussing on majority games only (Column 2, Table 4), the higher values that we observe neither prevent us from a steady decline for $P_n(G_B = G_{SS})$ as n increases. Quite surprisingly, the first column of Table 5 shows that considering games without dummy players tends to slightly increase the probability of agreement when all quota games are possible.

[4]Notice that when q is fixed, we have always obtained an equal number of vectors of power for the two indices. The difference in the number of vectors for the Shapley-Shubik index and the Banzhaf index is due to the fact that the same vectors can be obtained for different fixed values of q and $\bar{w}$.

	All q		$q = \bar{w}/2$		$q = 2\bar{w}/3$	
n	Banzhaf	Shapley Shubik	Banzhaf	Shapley Shubik	Banzhaf	Shapley Shubik
3	4	4	4	4	4	4
4	12	11	9	9	9	9
5	57	53	27	27	27	27
6	555	536	138	138	133	133
7	14 710	14 178	1 663	1 663	1 440	1 440
8			63 583	63 583	44 934	44 934

Table 2: *An estimation of the number of vectors of power for the Banzhaf and Shapley-Shubik indices of power.*

	All q		$q = \bar{w}/2$		$q = 2\bar{w}/3$	
n	Banzhaf	Shapley Shubik	Banzhaf	Shapley Shubik	Banzhaf	Shapley Shubik
3	2	2	2	2	2	2
4	8	7	5	5	5	5
5	45	42	18	18	18	18
6	498	483	111	111	106	106
7	14 155	13 642	1 509	1 509	1 298	1 298

Table 3: *An estimation of the number of vectors of power with no dummy player for the Banzhaf and Shapley-Shubik indices of power.*

Table 4: $P_n(G_B = G_{SS})$ *for* $n = 3, 4, 5, 6, 7$.

n	*all* q	$q = \bar{w}/2$	$q = 2\bar{w}/3$	$q = 3\bar{w}/4$
3	89.92	89.92	89.92	89.92
4	65.68	86.23	62.68	59.87
5	44.52	73.98	39.92	21.27
6	34.60	62.63	30.82	06.93
7		45.08	14.69	04.46

4.3 Probability of equal minimal distance for Banzhaf and Shapley Shubik indices without or with constraints

To get the result displayed on tables 4 and 5, we have generated the different vectors of power for the Banzhaf and Shapley Shubik power indices for a given number of players ($n = 3, \dots, 8$). Hence, for a given target vector $p = \{p_1, \dots, p_n\}$, the *optimal* vector of power can be found, *optimal* in the sense that it minimizes a the distance between the index of power and the target vector (see the previous section, for the case where $n = 3$). Let's denote, $d^{B^*}(p)$ and $d^{SS^*}(p)$ the minimal distances in the two studied cases for a target vector p.

Generating P target vectors by simulation[5], we can also evaluate the proportion of cases where $d^{B^*}(p) = d^{SS^*}(p)$. Table 6 presents the estimated probabilities for the three different values of q studied in the previous section. For the case where $q > \bar{w}/2$, 0.4396 is an estimation of the theoretical value 0.4366 calculated in the previous section.[6] As one may have guessed, the probability of having the same optimal repartition of the power among the players with two different indices quickly crashes for all the cases (different relative quotas, presence or absence of dummy players, see table 7).

[5] In order to avoid noise due to sampling variations, the same P target vectors have been used for all the cases. The same seed has been used to generate uniform (pseudo) random numbers for all the simulation. P is set to 10 000 for the estimations.

[6] The 99% confident interval $[0.4238; 0.4494]$ contains the true probability $Prob(B = SS)$ which is equal to 0.4366.

4.4 Is Shapley-Shubik better than Banzhaf?

The previous subsection focused on the probability of having the same vector of power for Banzhaf and Shapley-Shubik when minimizing the distance between the indices of power and a target. This result may be extended and we may estimate:

- the probability that the minimal distance obtained with the Banzhaf index is less than the one obtained with the Shapley-Shubik index; the estimation is the proportion of cases where $d^{B^*}(p) < d^{SS^*}(p)$.
- the probability that the minimal distance obtained with the Banzhaf index is greater than the one obtained with the Shapley-Shubik index; the estimation is the proportion of cases where $d^{B^*}(p) > d^{SS^*}(p)$.

These two estimated probabilities as well as the probability of equality between those two indices are presented in tables 8 (with dummies) and 9 (without dummies) for the case where $q > \bar{w}/2$ with all the vectors of power. The numbers in bold correspond to the probabilities presented in the table 6. The slight "advantage" of the Shapley-Shubik compared to the Banzhaf that we observed for $n = 3$ still prevails and seems to increase when we impose a positive power for all the players.

The fact that the possible Shapley-Shubik vectors perform better when we wish to come as close as possible to a pre established target is still observed with majority games (see tables 10 and 11 but disappears for the 2/3 and 3/4 quotas (tables 12, 13, 14, 15).

Table 5: $P_n(G_B = G_{SS})$ *for games without dummy player.*

n	all q	$q = \bar{w}/2$	$q = 2\bar{w}/3$	$q = 3\bar{w}/4$
3	91.50	91.50	91.50	91.50
4	70.15	95.47	70.06	65.78
5	48.94	75.13	44.15	27.61
6	35.46	62.46	25.79	07.05
7		42.17	10.48	03.38

Table 6: *Probability of having the same minimal distance for Banzhaf and Shapley-Shubik indices of power.*

n	all q	$q = \bar{w}/2$	$q = 2\bar{w}/3$	$q = 3\bar{w}/4$
3	43.96	43.96	43.96	43.96
4	32.09	61.95	25.41	12.67
5	08.74	27.51	18.27	03.21
6	02.18	13.60	06.59	01.51
7		05.95	02.01	00.65

Table 7: *Probability of having the same minimal distance for Banzhaf and Shapley-Shubik indices of power without dummy player.*

n	all q	$q = \bar{w}/2$	$q = 2\bar{w}/3$	$q = 3\bar{w}/4$
3	16.38	16.38	16.38	16.38
4	29.83	74.02	18.97	02.24
5	05.14	09.84	10.94	00.06
6	01.68	05.02	00.00	00.61
7		02.16	00.00	00.03

Table 8: *Probability of having a Banzhaf minimal distance under, equal or over the Shapley-Shubik's one, all q*

n	$P_n(B < SS)$	$P_n(B = SS)$	$P_n(B > SS)$
3	24.75	**43.96**	31.29
4	32.33	**32.09**	35.58
5	42.07	**08.74**	49.19
6	46.36	**02.18**	51.46

Table 9: *Probability of having a Banzhaf minimal distance under, equal or over the Shapley-Shubik's one, all q, without dummy player*

n	$P_n(B < SS)$	$P_n(B = SS)$	$P_n(B > SS)$
3	43.44	**16.38**	40.18
4	30.59	**29.83**	39.58
5	41.89	**05.14**	52.97
6	44.77	**01.68**	53.55

Table 10: *Probability of having a Banzhaf minimal distance under, equal or over the Shapley-Shubik's one, $q = \bar{w}/2$.*

n	$P_n(B < SS)$	$P_n(B = SS)$	$P_n(B > SS)$
3	24.75	**43.96**	31.29
4	19.87	**61.95**	18.18
5	29.97	**27.51**	42.52
6	34.89	**13.60**	51.51
7	38.13	**05.95**	55.92

Table 11: *Probability of having a Banzhaf minimal distance under, equal or over the Shapley-Shubik's one, $q = \bar{w}/2$, without dummy player.*

n	$P_n(B < SS)$	$P_n(B = SS)$	$P_n(B > SS)$
3	43.44	**16.38**	40.18
4	17.31	**74.02**	08.67
5	35.51	**09.84**	54.65
6	35.56	**05.02**	59.42
7	38.24	**02.16**	59.60

Table 12: *Probability of having a Banzhaf minimal distance under, equal or over the Shapley-Shubik's one,* $q = 2\bar{w}/3$.

n	$P_n(B < SS)$	$P_n(B = SS)$	$P_n(B > SS)$
3	24.75	**43.96**	31.29
4	41.29	**25.41**	33.30
5	50.99	**18.27**	30.74
6	61.58	**06.59**	31.83
7	69.12	**02.01**	28.87

Table 13: *Probability of having a Banzhaf minimal distance under, equal or over the Shapley-Shubik's one,* $q = 2\bar{w}/3$, *without dummy player.*

n	$P_n(B < SS)$	$P_n(B = SS)$	$P_n(B > SS)$
3	43.44	**16.38**	40.18
4	45.06	**18.97**	35.97
5	55.10	**10.94**	33.96
6	66.99	**00.00**	33.01
7	70.94	**00.00**	29.06

Table 14: *Probability of having a Banzhaf minimal distance under, equal or over the Shapley-Shubik's one,* $q = 3\bar{w}/4$.

n	$P_n(B < SS)$	$P_n(B = SS)$	$P_n(B > SS)$
3	24.75	**43.96**	31.29
4	46.79	**12.67**	40.54
5	60.66	**03.21**	36.13
6	68.77	**01.51**	29.72
7	74.98	**00.65**	24.37

Table 15: *Probability of having a Banzhaf minimal distance under, equal or over the Shapley-Shubik's one,* $q = 3\bar{w}/4$*, without dummy player.*

n	$P_n(B < SS)$	$P_n(B = SS)$	$P_n(B > SS)$
3	43.44	**16.38**	40.18
4	52.65	**02.24**	45.11
5	62.11	**00.06**	37.83
6	69.48	**00.61**	29.91
7	75.71	**00.03**	24.26

4.5 Graphical representations

The two minimal distances relative to the two indices of power can be represented in a two dimension space: the Shapley-Shubik minimal distance as a function of the Banzhaf minimal distance. To each target vector p corresponds a point $(d^{B^*}(p), d^{SS^*}(p))$in this space. Then, as we have generated $P = 1\ 000\ 000$ target vectors, we may have a scatter plot of a 1 000 000 points. In the case of perfect adequation between these two minimal distances, the scatter plots would be linear (it would be the first bisecting line of the two dimensional space). In such a graph, hundred percents of the generated population vectors would lead to the same minimal distance for the two indices of power. Then, the first bisecting line will be the reference.

The percentage of points plotted on the first bisecting line corresponds to the estimated probability of having the same minimal distance for the indices of power, this estimated probability being listed in tables 6 and 8. The estimated probability that the minimal distance obtained with the Banzhaf index is less than the one obtained with the Shapley-Shubik index is graphically the percentage of points plotted over the bisecting line (with Shapley-Shubik represented on the y axes). In the same way, the percentage of points plotted below the first bisecting line corresponds to the estimated probability that the minimal distance obtained with the Banzhaf index is higher than the one obtained with the Shapley-Shubik index.

For instance, for the majority game, the following results appear from the graphical representations (figures 6 and 7):

- the relative weight of the bisecting line is decreasing as n increases.

- the distribution of points above and over this line is not symmetric.

- the higher values of distance are computed with the Banzhaf index.

Moreover, the minimal distance cumulative distributions for a given number of player can be estimated (using the P simulations). Figure 8 illustrates in the case of 3 players, the fact that Banzhaf minimal distances may be higher than the Shapley-Shubik's (this is linked to the asymmetry around the bisecting line shown in the scatter plots).

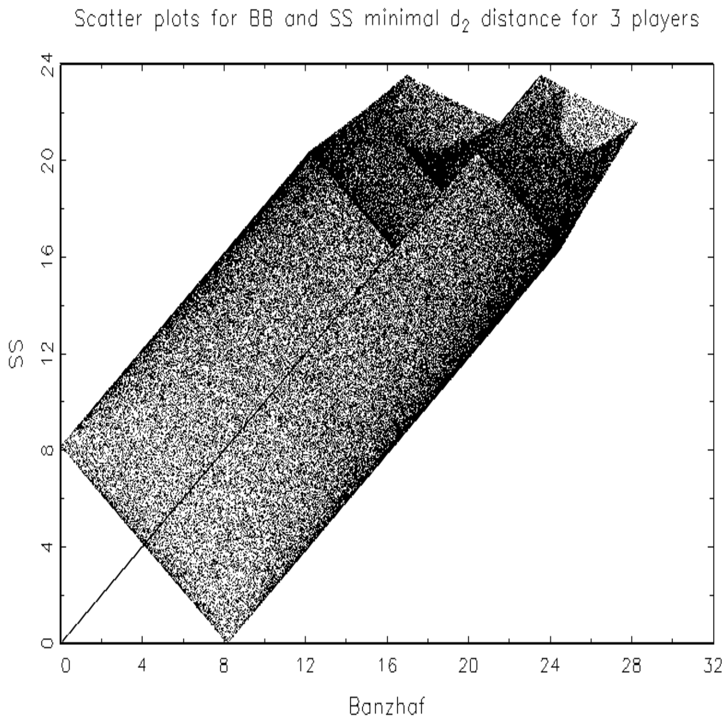

Figure 6: Scatter plots with $n = 3$

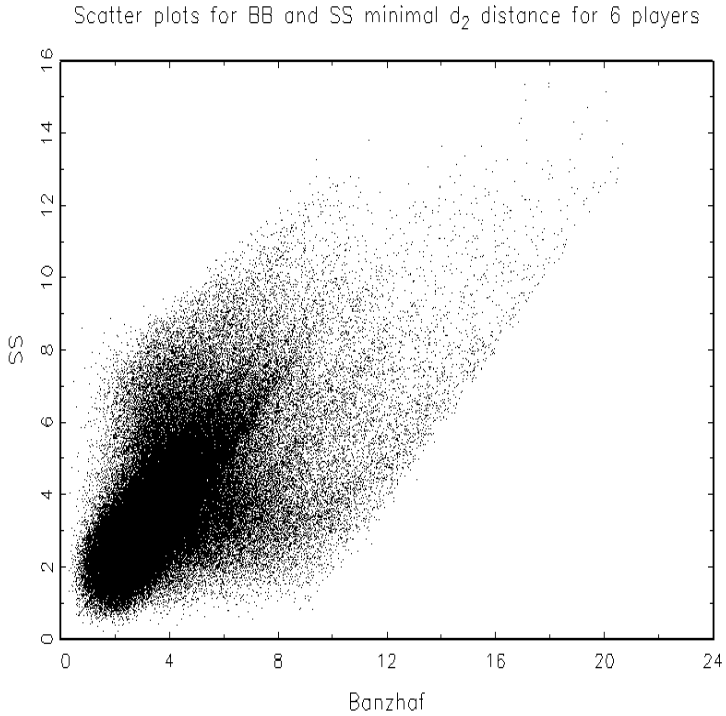

Figure 7: Scatter plots with $n = 6$

5 Conclusion

This paper was partly designed as an application of the method proposed by Barthélémy and Martin [2] for the enumeration of all the possible weighted quota games. When designing a voting rule, we may wish the voters not to have the same influence in the decision process. By choosing adequately the weights and the quota in a weighted quota game, we may try to come as close as possible to the desired repartition of influence among the players. To realize this objective, we effectively need to know what are all the possible weighted quota games at our disposal. Thus, the main objective of this paper was to prove that this exercise could not be done irrespectively of the choice of the power index. For 6 players, the Shapley Shubik index and the Banzhaf index already disagree on the optimal game to choose for about two third of the cases. Unfortunately, the large number of possible games for small values of n (at least 14710 for $n = 7$) did not allow us to estimate the probability for a larger number of players, and it may be difficult to obtain results for significatively higher values, even if we restrict ourselves to majority games with no dummy players (see Table

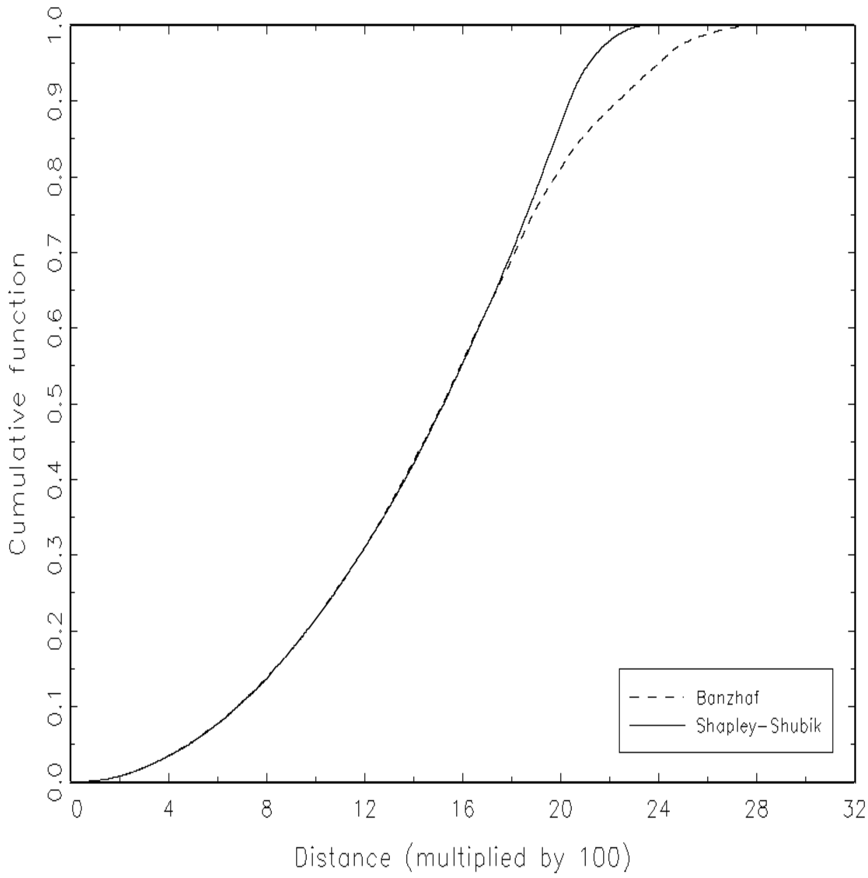

Figure 8: Banzhaf and Shapley-Shubik minimal distances repartition function in the 3 player case

7).

We have also observed that the Shapley-Shubik index seems to performs slightly better than the Banzhaf index when the objective is to minimize the distance to a target. However, the results we have for $n = 3$ to $n = 6$ depend on the fact that all the possible weighted quota games are available at the same time. They cannot be extrapolated for fixed relative quotas. On this ground, we can compare our results to the ones obtained by Chang, Chua and Machover [4] which tested the Penrose's law for high values of n. Penrose's law asserts that, under certain conditions, the ration between the Banzhaf power of any two voters converges to the ratio between their weights as n increases. Using the sum of differences in absolute values, Chang, Chua and Machover proved that the conjecture is true for the Banzhaf index, going from $n = 15$ to $n = 55$, and q close to $\bar{w}/2$. They also performed the same exercise for the Shapley-Shubik index, showing that then Penrose's conjecture is true for almost all the values of q. From their tables, we can also derive that, on average, the proportionality seems to be slightly better for the Banzhaf index than for the Shapley Shubik index. This is in fact corroborated by the results we obtained when we set the relative quota to be fixed at the level 2/3 or 3/4 for $n = 3$ to $n = 7$.

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6 Appendix

Let $\{q, w_1, w_2, w_3\}$ be a voting game. Firstly, we determine all the different vectors of power with the two power indices. Let us begin with the Banzhaf index. All the coalitions are $\{1,2,3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1\}$, $\{2\}$ and $\{3\}$. Since $w_1 \geqslant w_2 \geqslant w_3$ and the voting game is proper, $v(\{2\}) = 0$ and $v(\{3\}) = 0$. Assume $v(\{1\}) = 1$. It means that the player 1 belongs to all the winning coalitions, it has all the power (it is always a decisive player) and the vector of power is $(1,0,0)$. Assume now that $v(\{1\}) = 0$ and $v(\{2,3\}) = 1$. Thus we have $v(\{1,2,3\}) = 1$, $v(\{1,3\}) = 1$ and $v(\{1,2\}) = 1$. In $\{1,2,3\}$, there is no decisive player while in $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$, every player is decisive. Therefore the vector of power is $(1/3, 1/3, 1/3)$. Assume now that $v(\{2,3\}) = 0$ and $v(\{1,3\}) = 1$. Thus we have $v(\{1,2,3\}) = 1$ and $v(\{1,2\}) = 1$. In $\{1,2,3\}$, only the player 1 is decisive while in $\{1,2\}$, and $\{1,3\}$ every player is decisive. Therefore the vector of power is $(3/5, 1/5, 1/5)$. Assume now that $v(\{1,3\}) = 0$ and $v(\{1,2\}) = 1$. Thus we have $v(\{1,2,3\}) = 1$. In $\{1,2,3\}$, only the player 3 is not decisive while in $\{1,2\}$ every player is decisive. Therefore the vector of power is $(1/2, 1/2, 0)$. Assume now that $v(\{1,2\}) = 0$, we have $v(\{1,2,3\}) = 1$. In $\{1,2,3\}$, every player is decisive and the vector of power is $(1/3, 1/3, 1/3)$. Finally, there are 4 vectors of power, $(1/3, 1/3, 1/3)$, $(3/5, 1/5, 1/5)$, $(1/2, 1/2, 0)$ and $(1,0,0)$.

For the Shapley-Shubik index with 3 players, there are 6 possible orders. If $v(\{1\}) = 1$, then the player 1 is the only pivotal, even if it arrives last in the coalition, thus the vector of power is $(1,0,0)$. Assume now that $v(\{1\}) = 0$ and $v(\{2,3\}) = 1$. Thus we have $v(\{1,2,3\}) = 1$, $v(\{1,3\}) = 1$ and $v(\{1,2\}) = 1$. For each order, the player which is in second position is pivotal, therefore the vector of power is $(1/3, 1/3, 1/3)$. Assume now that $v(\{2,3\}) = 0$ and $v(\{1,3\}) = 1$. Thus we have $v(\{1,2,3\}) = 1$ and $v(\{1,2\}) = 1$. When the player 1 is not first in the order, it is always pivotal and when the player 1 is first in the orders, the pivotal is the player which arrives second in the order. Therefore the vector of power is $(2/3, 1/6, 1/6)$. Assume now that $v(\{1,3\}) = 0$ and $v(\{1,2\}) = 1$. Thus we have $v(\{1,2,3\}) = 1$. When the player 1 is first in the order, the player 2 is pivotal and when the player 2 is first in the order, the player 1 is pivotal. In the orders 312 and 321, the player which arrives last is pivotal. Therefore the vector of power is $(1/2, 1/2, 0)$. Assume now that $v(\{1,2\}) = 0$, we have $v(\{1,2,3\}) = 1$. Thus it is always the player who arrives last in the orders the pivotal and the vector of power is $(1/3, 1/3, 1/3)$. Finally, there are 4 vectors of power, $(1/3, 1/3, 1/3)$, $(2/3, 1/6, 1/6)$, $(1/2, 1/2, 0)$ and $(1,0,0)$.

In the reasoning above, the quota is not fixed since our purpose is to determine all the possible vectors of power. We show now that the result can be different if we consider the majority games. The result is different in function of the parity of $\bar{w}$.

***** $\bar{w}$ ***is odd.*** It means that $v(S) = 0 \iff v(N\backslash S) = 1$. Assume that $v(1) = 1$, the vector of power is thus $(1, 0, 0)$, with the two power indices. Actually, the player 1 is the only player who is decisive since $v(2, 3) = 0$. Assume now that $v(1) = 0$, thus $v(2, 3) = 1$, and $v(1, 3) = v(1, 2) = 1$. All the players are decisive the same number of times with the two power indices : the vector of power is $(1/3, 1/3, 1/3)$. Therefore we have two possible solutions and these are the same for the Banzhaf index and the Shapley-Shubik index.

$*\bar{w}$ ***is even.*** We show that, for any $\bar{w} > 4$, the four vectors of power are possible. In this case, the probability is the same as above since the conditions are not modified.

For $(1, 0, 0)$, it is obvious if we suppose that $w_1 = \bar{w}$.

For $(1/2, 1/2, 0)$, we must have $w_1 + w_2 > q$, $w_1 + w_3 \leqslant q$ and $w_1 \leqslant q$. Let $w_1 = w_2 = \frac{\bar{w}}{2}$, this implies $w_3 = 0$, the conditions are verified.

For $(3/5, 1/5, 1/5)$ for Banzhaf and $(2/3, 1/6, 1/6)$ for Shapley-Shubik, we must have $w_1 + w_3 > q$, $w_2 + w_3 \leqslant q$ and $w_1 \leqslant q$. Assume that $w_1 < \frac{\bar{w}}{2}$. This implies that $w_2 + w_3 > \frac{\bar{w}}{2}$ or $w_2 + w_3 > q$ which is not possible. Thus $w_1 = \frac{\bar{w}}{2}$ and $w_2 + w_3 = q$. Assume $w_3 = 1$ and $w_2 = \bar{w} - w_1 - w_3$ and the conditions are trivially verified.

For $(1/3, 1/3, 1/3)$, we must have $w_1 \leqslant q$ and $w_2 + w_3 > q$ or $w_1 + w_2 \leqslant q$. Let $w_1 = w_2 = q - 1$ and $w_3 = 2$, which is possible since $\bar{w}$ is even and the conditions $w_1 \leqslant q$ and $w_2 + w_3 > q$ are verified. Notice that if $\bar{w} = 4$, all the vectors of power are not possible. In this case, w_1 is necessary equal to 2, the condition $w_2 + w_3 > q$ or $w_1 + w_2 \leqslant q$ are not possible.