# ADVERTISING VERSUS SALES IN DEMAND CREATION

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# Advertising Versus Sales In Demand Creation

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**Abstract**

Using an analytical model, we investigate the dynamics of a firm with market power whose advertisements and sales contribute to its customers' stock of goodwill. An advertising campaign precedes the firm's sales when customers are not familiar with its product, (e.g., movies), whereas sales of a new brand of a familiar product may start without advertising (e.g. Crocs shoes). For constant demand elasticity, both advertising and sales take place from the start. Two different types of solutions then emerge: one for low demand elasticity and one for high demand elasticity. These solutions are analyzed by phase diagrams. We also perform a numerical sensitivity analysis.

*Keywords: Dynemic Advertisement, Diffusion, Adoption, Goodwill, Learning by Buying, Phase Diagram.*

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# 1 Introduction

During the 1984 Super Bowl XVIII, the Apple Macintosh personal computer was introduced to the world in a TV commercial that cost 1.5 million dollars. Advertisement campaigns had also preceded commercial marketing of Window's 95. Some products, however, are launched without a single advertisement but yet rapidly become well known with fast increasing sales, e.g., Crocs shoes. Why did Apple launch an advertising campaign before sales began, whereas Crocs started sales without any advertisement? Under what circumstances are such strategies optimal? This paper addresses these questions as well as other similar ones.

To tackle these issues, we develop a dynamic model of a firm with market power that is engaged in production, sales and advertising. At each point in time the firm faces a downward sloping demand curve, which shifts up and outward with an increase in the stock of customers' goodwill for the product. Both advertising and sales are controlled by the firm and contribute to the rate of growth of the stock of customer goodwill. This stock of goodwill has a finite upper limit and its rate of growth declines with proximity of the goodwill stock to this upper limit. There is a constant depreciation rate of the goodwill stock so that sales and/or advertising are needed to prevent the goodwill from decaying. Such a setup characterizes a sole producer of a good (monopolist) and the sole producer of a brand of a differentiated product that engage in monopolistic competition. Such producers possess market power. We first solve the model in general, and afterwards in greater detail in subsequent models with the same general specification but with more specific functional forms.

In the literature, models view advertisement as a tool that is controlled by a firm and enhances consumers' demand for the firm's differentiated product and thus increases the firm's sales. Dorfman and Steiner's (1954) positive model is one of the first formal static models of an advertising monopoly. Most of the earlier advertising static models belong to the 'persuasive view' that assumes advertising changes the utility function and creates brand loyalty (e.g., Comanor and Wilson, 1974). Another approach, known as the 'informative approach', assumes that advertisement facilitates purchases by carrying information to consumers. Ozga (1960) and Stigler (1961) laid the foundations to this methodology and Dukes (2004) elaborated on reasons for its variety of levels. A third approach assumes advertising directly enters consumers' preferences in a manner complementary to the consumption of the advertised product (e.g., Stigler and Becker, 1977, Hochman and Luski, 1988). A comprehensive survey of all this literature can be found in Bagwell (2005) and references therein.

Static models, however, cannot capture the effects that a firm's past performances have on its current performance, for which dynamic models have been created. One of the earlier dynamic models is the 1957 Vidale and Wolfe model which maximizes the accumulated discounted net sales while changes in current sales are caused by advertising and depreciation. The concept of a stock of goodwill influencing sales was introduced by Nerlove and Arrow (1962). According to their approach advertising accumulates into goodwill the same way that investment accumulates into capital stock. For details on the literature emanating from these two papers see, for example Feichtinger, Hartl and Sethi (1994), and Feinberg (2001). Another branch of the dynamic marketing literature concentrates on the process of diffusion of sales information by 'word of mouth' and 'repeated sales' and its adoption.[1] In most of these dynamic models, however, the production process and demand considerations that constitute the main issues addressed by static models are ignored and only a single sales variable (that is actually the net revenue) exists.[2]

[1] A comprehensive literature review, which covers these concepts among other, can be found in Sethi (1977); Dodson and Muller (1978); Bass (1969); Feinberg (2001) and Feichtinger, Hartl and Sethi (1994).

[2] One of the exceptions is a paper by Hochman and Hochman (1975), which extends Nerlove and Arrow's model to include advertising and goodwill stock as well as investment and capital stock.

In this paper we include the elements of both static and dynamic models. Thus, our firm is facing a downward sloping demand curve that is shifted by a change in the stock of goodwill and the firm's cost is a function of the firm's output and advertising at each point in time. In addition, current sales as well as advertising contribute to the stock of goodwill. We term the process of sales affecting goodwill "learning by buying". Consequently, our model, unlike the existing literature, includes two processes affecting consumers' goodwill: advertising and learning by buying.[3]

Our results demonstrate the importance of including the demand and production processes into our dynamic model. It turns out that two different strategies for new firms exist in our solution: One strategy is for firms facing low demand elasticity and the other is for firms facing high demand elasticity. The strategies differ in the investment rate in advertising relative to sales during the initial phase of operations of the firm.

In the next section we sum up the results of the paper. The model is described in Section 2 and we derive the general solution in Section 3. The optimal path to steady state for more specific functional forms is characterized in Section 4. Numerical solutions and sensitivity analysis are simulated in Section 5.

## 1.1 The Model's Results

In myopic equilibrium the quantity of a good produced equates the marginal cost of production to the marginal revenue. In contrast, our results show that when sales take place the quantity of the good sold exceeds the myopic quantity. The "loss" in current revenue due to a reduction in myopic mark-up is balanced by the added future revenues obtained from the rise in goodwill attributed to the increased sales. Another conclusion

[3] In Kalish (1985) and Horsky (1990), as in our model, prices are determined endogenously unlike other works that either have no prices or in which prices are kept constant. However, unlike our paper, both Kalish and Horsky have no advertising or goodwill and instead diffusion and adoption of durable goods are captured by accumulated sales affecting current sales.

of our model is that when both advertising and sales take place simultaneously, there is equality between the marginal net cost of goodwill generated by either current sales or by current advertisement. We also find that the quantity produced along the optimal path increases when either goodwill or advertising increase while all other factors stay constant. The price along the optimal path, however, increases only when advertising decreases and is not affected by changes in goodwill.

By two examples satisfying our general specifications we learn that when customers have little or no knowledge about the nature of the product, the firm may launch an advertising campaign prior to sales, a strategy typical to the movies and publishing industries, among others. On the other hand, when a firm decides to produce a new brand of a familiar product, circumstances exist in which the firm should commence sales and delay advertising to a later stage or even terminate advertising entirely, e.g., Crocs shoes.

We then assume, in order to obtain an interior solution, full convexity of the model's functions. We also assume more specific functional forms, which also include constant demand elasticity. Then, two distinct alternative strategies emerge, one for high demand elasticity and one for low. When the demand elasticity the firm faces is low, the firm starts its operations with a high level of advertising and gradually reduces it to the steady state level. During that time the stock of goodwill and the price increase continuously to their steady state levels while sales can increase, decrease or fluctuate.

In a solution typical for high demand elasticity, a firm begins with low levels of sales and advertising that gradually increase. This process may continue until steady state levels are attained or, alternatively, at some point in time when the advertising level reaches a maximum from which it descends to its steady state level. During the decline of advertising, sales may increase or decrease. In both cases, from inception up to the time it reaches steady state, the stock of goodwill monotonously increases. The price along the optimal path is affected only by advertising, increasing when advertising declines and vice versa.

The economic intuition of the difference between the solutions of high and low demand elasticity is as follows: A firm initially seeks a quick build-up of its stock of goodwill in order to quickly raise the demand for its product. When the demand elasticity is low, a massive increase in the quantity sold will cause a considerable drop in the product's price. Hence, in the low elasticity case, the firm prefers to build-up goodwill by using advertising rather than excess sales. In the case of high demand elasticity, however, a massive increase in the quantity sold reduces the price only marginally, thus learning-by-buying becomes a relatively cheap way to build-up goodwill compared to advertising.

When the model with specific functional forms is solved numerically for several sets of parameter values, the results of these simulations support the analytical solution. In addition, by running alternative values of some of the parameters the simulations provide a partial sensitivity analysis of the model.

## 2 The Model

Consider a firm that produces and markets a unique good that no other firm is producing. Such a good may be either a differentiated product in an industry in which other firms produce close substitutes of the good with different brand names, or there are no close substitutes in the market and the firm is a monopoly. In both cases, the firm has market power and can generate customer goodwill (GW), $A(t)$, to increase the quantity of its product sold at a given price. Thus, $X(t)$,the quantity of the firm's product demanded at time $t$, is a decreasing function of $P(t)$, the price at time $t$, and an increasing function of the stock of GW prevailing at time $t$, $A(t)$. Accordingly, $X(t) = X(P(t), A(t))$, where

$\frac{\partial X}{\partial P} < 0$ and $\frac{\partial X}{\partial A} > 0$. The elasticity of the current demand is defined as $\epsilon_{X,P} = -\frac{\partial X}{\partial P}\frac{P}{X}$ and by assumption $1 < \epsilon_{X,P} < \infty$.[4]

For convenience, and without loss of generality, we choose $X(t)$ and $A(t)$ as the independent variables in the current demand function facing the firm so that the price becomes a function of the two of them, i.e., $P(t) = P(X(t), A(t))$, $\frac{\partial P}{\partial X} < 0$, $\frac{\partial^2 P}{\partial X^2} \geq 0$ and $\frac{\partial P}{\partial A} > 0$. The GW, $A(t)$, is a stock variable generated by the firms advertisement $I$ and the quantity of its product sold, $X$, according to the following equation of motion:

$$\dot{A} \equiv \frac{dA}{dt} = (g(X(t)) + h(I(t)))(1 - A(t)) - \psi A(t). \tag{1}$$

The term $g(X)$ in the above equation of motion (1) is the current sales multiplier in the generation of GW, where $\frac{\partial g}{\partial X} > 0$. The contribution of current sales to the stock of GW is either due to the effect it has on the willingness of the customer to buy in the near future more of the product (i.e., repeated sales) or by the buyer informing others about the product he just bought and consumed, thus creating goodwill in others as well (i.e., word of mouth) and this increases the willingness to buy. The variable $I(t)$ denotes advertisement, an investment in GW independent of the firm's sales. $h(I)$ denotes the advertisement multiplier in the generation of GW, where $\frac{\partial h}{\partial I} > 0$. The parameter $\psi$ is the constant rate of GW decay as a result of forgetfulness and entrance of newer products. The index $(1 - A)$ measures the remaining potential of the firm's GW and in the equation of motion it implies decreasing returns to investment in the stock of GW. Note that (1) implies that the maximum amount of GW is normalized to 1, and $0 \leq A(t) < 1$, since, as $A$ approaches 1, $\dot{A}$ turns negative and $A$ declines. Thus, $A(t)$ is the proportion of maximum goodwill utilized by the firm to enhance the demand for its product. To sum up the arguments: the GW, $A$, is affected by advertisement, $I$, and by

[4] In an industry with many firms, each of which producies a unique good that is a close substitute of each of the other goods produced by the firms in the industry, the demand elasticity that each firm faces is a finite number much larger than one, i.e., $1 \ll \epsilon_{X,P} < \infty$ .

If there is a single monopoly in an industry without close substitutes, its demand elasticity is a number higher than one but not much higher, i.e., $1 < \epsilon_{X,P} \ll \infty$.

learning-by-buying (word of mouth and repeated sales) represented by the firm's output $X$. In addition, $A$ is bounded between zero and one and displays decreasing returns to investment.[5]

The firm bears the costs of producing output $X$, $C_1(X)$ as well as the cost of advertisement $I$, $C_2(I)$. Both cost functions have positive first derivatives and non-negative second derivatives. Accordingly, the current profit function, $\pi(t)$, is

$$\pi(t) = P(t) X(t) - C_1(X(t)) - C_2(I(t)).$$

The discount rate in the economy is a given positive parameter $r < 1$. All variables are functions of time $t$, although in what follows $t$ is omitted unless it is essential for understanding the equations. Accordingly, the variables of our model are $A$, a state variable with $\lambda$ as its co-state, and the control variables $X$ and $I$. Note that both $X$ and $I$ can only be non-negative. In this paper we are interested in the optimal strategy of a firm around its inception, i.e., we are essentially interested in the solution beginning with $A(0) = 0$. The firm maximizes the following inter-temporal profit function at time $t = 0$ at which time the firm is founded,

$$\begin{aligned} \mathcal{L}(0) &= \max \int_0^\infty e^{-rt} \left[P(X, A) X - C_1(X) - C_2(I)\right] dt \\ & \quad s.t. \;\; \dot{A} = (g(X) + h(I))(1 - A) - \psi A; \\ & \quad A(0) = 0, \;\; -I \leq 0 \;\; and \quad -X \leq 0. \end{aligned} \tag{2}$$

We adopt the plausible assumption of $r \leq \psi$, i.e., $r$, now standing for the alternative cost of aggregate consumption in terms of enhanced investment in the stock of GW, is smaller than the decay rate of GW, $\psi$.

[5] Note that for $g(X) \equiv 0$ and $h(I) \equiv I$, we obtain an equation of motion similar to that of Gould as well as to that of Vidale and Wolfe. Except for the term $(1 - A)$ this equation is also similar to the equation of motion of Nerlove and Arrow.

In what follows we specify necessary conditions for an optimal solution for the firm and try to typify some solutions that satisfy these conditions.

# 3 The Model's Solution

We first derive the overall necessary conditions for any of these three types of the model's solutions: a corner solution, an internal solution or a null solution. To this end, let $\mathcal{H}(t)$ designate the Hamiltonian formulated from $\mathcal{L}(0)$ below:

$$\begin{aligned}&\mathcal{H}(t) = P(X(t), A(t))\, X(t) - C_1(X(t)) - C_2(I(t)) + \\ &+\lambda\left[(g(X(t)) + h(I(t)))(1 - A(t)) - \psi A(t)\right] + \delta(t)\, I(t) + \eta(t)\, X(t),\end{aligned}$$

where $\delta(t)$ and $\eta(t)$ are the Kunn-Tucker multipliers associated with the two inequalities. Three conditions then follow:

1) $\frac{\partial\mathcal{H}}{\partial X} = P_X(X, A)\, X + P(X, A) - C_1'(X) + \lambda g'(X)(1 - A) + \eta = 0$, where $\eta \geq 0$, and $\eta X = 0$.

2) $\frac{\partial\mathcal{H}}{\partial I} = -C_2'(I) + \lambda h'(I)(1 - A) + \delta = 0$, where $\delta \geq 0$, and $\delta I = 0$.

3) $\dot{\lambda} = r\lambda - \frac{\partial\mathcal{H}}{\partial A} = \lambda(r + \psi + g(X) + h(I)) - P_A(X, A)\, X$,

where an apostrophe in a superscript of a single-variable function designates the first derivative of the function, a multi-variable function with a variable in its subscript stands for the derivative of the function with respect to the variable and a dot above a function denotes derivation with respect to time.

The necessary conditions specified in section 3.1 follow from the three conditions above. Then, the necessary conditions are used to obtain two examples of corner solutions and to characterize in detail a model with specific functional forms as described in Section 4. It should be noted that in our model there is always a maximum that satisfies the necessary conditions, i.e., there is always either an internal solution or a corner one. The null solution $X(t) = I(t) = A(t) = 0$ is always a local optimum and if costs are higher than benefits it is also the global optimum. In what follows, we characterize solutions with positive-valued variables.

### 3.1 Necessary Conditions

The differentiation of the Hamiltonian with respect to advertising, $I$ (condition 2 above), yields the (in)equality of the shadow value of the marginal product of advertisement $I$, $\lambda h'(I)(1-A)$ to the marginal current cost $C_2'(I)$, i.e.,

$$\lambda h'(I)(1-A) \leq C_2'(I) \Longrightarrow \lambda \leq \frac{C_2'(I)}{h'(I)(1-A)}, \tag{3}$$

where $0 \leq A < 1$ and equality holds in (3) when $I > 0$. When advertising is positive, equality holds between $\frac{C_2'(I)}{h'(I)(1-A)}$, the imputed cost of a unit of $\dot{A}$ generated by advertising, and $\lambda$, the shadow price of a unit of GW.

The differentiation of the Hamiltonian with respect to the output $X$ (see condition 1 above) yields the (in)equality between the sum of the marginal revenue of $X$ plus the value of the marginal contribution of $X$ to $\dot{A}$, and the current marginal cost of $X$, $MC(X)$, namely

$$MR_X(X,A) + \lambda g'(X)(1-A) \leq C_1'(X) \Longrightarrow \lambda \leq \frac{C_1'(X) - MR_X(X,A)}{g'(X)(1-A)} \tag{4}$$

where $0 \leq A < 1$, and for $X > 0$ equality holds. $MR_X(X,A)$ is the marginal revenue of $X$ and $MR_X(X,A) \equiv \frac{\partial(XP(X,A))}{\partial X} = P(X,A) + XP_X(X,A)$. The marginal contribution of $X$ to $A$ is $\lambda g'(X)(1-A)$, and $MC(X) = C_1'(X)$.

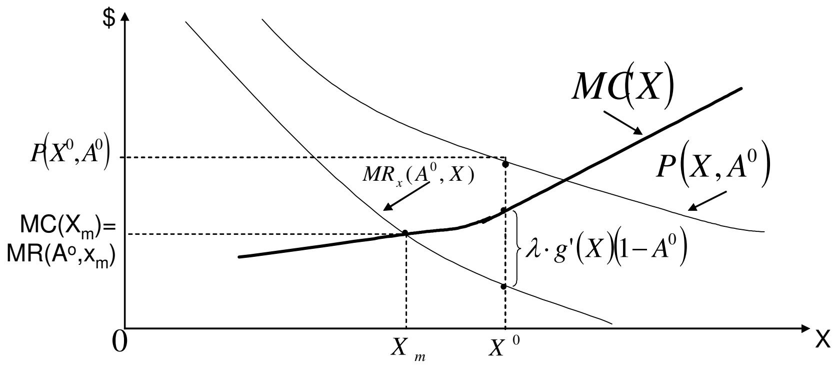


Figure 1: Investment in Goodwill of Learning by Buying

The case of positive output $X$ and equality in (4) is depicted in Figure 1. While a monopolistic myopic firm in a static model equates $MR_X$ to $MC(X)$, i.e., produces the output $X_m$ shown in Figure 1, in our dynamic model the firm produces an output $X^o$, where $X^o > X_m$. The additional marginal cost above the marginal revenue at the optimal output $X^o$ is equal to the shadow value of the marginal contribution of sales to the creation of future GW.

The rate of change of the shadow price, $\dot{\lambda}$, is obtained by differentiating the Hamiltonian with respect to the stock of GW, $A$, (see 3 above) and is given in (5) below, for $\lambda > 0$.

$$\begin{aligned} \dot{\lambda} &= (r + \psi + g(X) + h(I))\,\lambda - MR_A(X, A) \Longrightarrow \\ \frac{\dot{\lambda} + MR_A(X, A)}{\lambda} &= (r + \psi + g(X) + h(I)) \;\; for\ \lambda > 0, \end{aligned} \tag{5}$$

where $MR_A(X, A) \left(= \frac{\partial(XP(X,A))}{\partial A}\right) = XP_A(X, A)$, is the marginal revenue of $A$. In the second line of (5) the two terms on the right-hand side of the equation are the rate of growth of the GW price, $\dot{\lambda}/\lambda$ plus the current marginal revenue of GW, $MR_A$, normalized by its price $\lambda$. On the right-hand side of (5), the first two terms, $r + \psi$, are the maintenance costs per unit of GW and the last two terms measure the contribution to $\dot{A}$ of learning by buying, $g(X)$ and advertisement, $h(I)$.

The necessary conditions $(3)$, $(4)$ and (5), the equation of motion (1), the initial condition and the non-negativity restrictions specified in (2), all together determine the solution of the maximization problem specified in (2).

## 3.2 The Model's Internal and Corner Solutions

From equations (3) and (4) and after eliminating $\lambda$ we obtain

$$L(X, A) \equiv \frac{C_1'(X) - MR_X(X, A)}{g'(X)} \gtreqless \frac{C_2'(I)}{h'(I)} \equiv R(I) \tag{6}$$

Since the variable $A$ fulfills $0 \leq A < 1$, the term $(1 - A)$ is positive and it therefore, was eliminated from (6). The above equation states that the optimum can consist either of an internal solution in which both $X$ and $I$ are positive or of a corner solution in which either $X$ or $I$ vanish.

### 3.2.1 The Internal Solution

When in the optimum both $I$ and $X$ are positive and $A$ fulfills $0 \leq A < 1$, then equality holds in $(6)$. This is a case of an internal solution. Additional conditions on the functions of the model, on top of the conditions assumed in section 2 are needed to ensure the existence of an internal solution. Such additional conditions are henceforth assumed: 1) additional conditions on the demand function $P(X, A)$, i.e., $\frac{\partial P}{\partial X} < 0, \frac{\partial^2 p}{\partial X^2} > 0$ and $\frac{\partial P}{\partial A} > 0$, 2) conditions on the advertisement multiplier $h(I)$, i.e., $\frac{\partial h}{\partial I} > 0$, $\frac{\partial^2 h}{\partial I^2} \leq 0$ and 3) conditions on the sales multiplier $g(X)$, i.e., $\frac{\partial g}{\partial X} > 0$, $\frac{\partial^2 g}{\partial X^2} \leq 0$.

In an internal solution, the firm has two current policy instruments. The first is advertisement $I$ that increases the future stock of GW, $A$, and thus shifts upward the future price for a given quantity demanded with its future revenues. The second is the quantity sold of the product, $X$, with its double impact of current income generation through sales and the increase of future stock of GW, $A$, and with it future revenues.

Accordingly, in the internal solution equalities hold in (3) and (4) from which we obtain,

$$\frac{C_1'(X) - MR_X(X,A)}{g'(X)} = \frac{C_2'(I)}{h'(I)}. \tag{7}$$

The equality in (7) is between the marginal net cost of GW generated either by current sales or by current advertisement. By totally differentiating (7) we obtain,

$$\begin{aligned}&\left[\frac{C_1''(X) - \frac{\partial MR_X(X,A)}{\partial X}}{g'(X)} - \frac{g''(X)\left(C_1'(X) - MR_X(X,A)\right)}{\left(g'(X)\right)^2}\right] dX + \\ &+ \left[-\frac{1}{g'(X)}\frac{\partial MR_X(X,A)}{\partial A}\right] dA + \left[-\frac{h'(I)\,C_2''(I) - C_2'(I)\,h''(I)}{\left(h'(I)\right)^2}\right] dI = 0.\end{aligned} \tag{8}$$

From (8) we obtain the following two derivatives (Eqs. (9) and (10)):

$$\begin{aligned}&0 < \frac{\partial X}{\partial I}|_{dA=0} = \\ &\left(\frac{(g'(X))}{(h'(I))}\right)^2 \frac{\left[h'(I)\,C_2''(I) - h''(I)\right]}{\left[g'(X)\left(C_1''(X) - \frac{\partial MR_X}{\partial X}\right) - g''(X)\left(C_1'(X) - MR_X\right)\right]},\end{aligned} \tag{9}$$

where $MR_X$ is a function of $X$ and $A$ only. The inequality in (9) is proved as follows: Since the first-order derivatives $g'(X)$, $h'(I)$, $C_1'(X)$ and $C_2'(I)$ are positive, the second order derivatives of the cost functions $C_1$ and $C_2$ are also positive and the second order derivatives of $h$ and $g$ are non-positive, it follows that the numerator of the equation above is positive. Since $C_1'(X) - MR_X(X,A)$ must be positive due to the non-negativity of $\lambda$ in (4) and because $\frac{\partial MR_X(X,A)}{\partial X}$ is negative due to the negative slope of the demand curve, the denominator of the equation above is also positive. Positive numerator and denominator imply that the whole expression is positive.

In a similar way we obtain (10) below,

$$\begin{aligned}&0 < \frac{\partial X}{\partial A}|_{dI=0} \cdot \\ &= \frac{(g'(X))\left[\frac{\partial MR_X(X,A)}{\partial A}\right]}{g'(X)\left(C_1''(X) - \frac{\partial MR_X(X,A)}{\partial X}\right) - g''(X)\left(C_1'(X) - MR_X(X,A)\right)}.\end{aligned} \tag{10}$$

The inequality in (10) is proved similarly to the proof of the inequality in (9).

In (3) we obtained $\lambda$ as a function of $I$ and $A$ alone. By differentiating the expression with respect to time we now obtain,

$$\dot{\lambda} = \frac{h' C_2" - C_2' h''}{(h')^2 (1-A)} \dot{I} + \frac{C_2'}{h' (1-A)^2} \dot{A}. \tag{11}$$

Let $E_{C_2'} = \frac{C_2"}{C_2'} I$ be the elasticity with respect to advertising, $I$, of $C_2'$ − the marginal cost of advertising. In the same way $E_{h'} = \frac{h''}{h'} I$ is the elasticity of $h'$ − the marginal advertising multiplier of GW with respect to advertising. Then by substituting $\lambda$ from (3) and the elasticities $E_{C_2'}$ and $E_{h'}$ defined above into (11) and rearranging terms, we obtain

$$\frac{\dot{\lambda}}{\lambda} = \left(E_{C_2'} - E_{h'}\right) \frac{\dot{I}}{I} + \frac{\dot{A}}{1-A}. \tag{12}$$

Equation (12) implies that the rate of change, $\frac{\dot{\lambda}}{\lambda}$ , is decomposed into two components: current and inter-temporal. The current component measures the net cost of the rate of change of advertising, $\frac{\dot{I}}{I}$, and the inter-temporal component measures the growth of GW, $\dot{A}$, relative to the remaining growth potential of GW, $(1-A)$.

By substituting (11) into (5) together with (3) and (1) we obtain the expression for the change over time in advertising,[6]

$$\dot{I} = \frac{1}{\left(\frac{C_2"}{C_2'} - \frac{h''}{h'}\right)} \left\{ r + \frac{\psi}{(1-A)} - \frac{h'}{C_2'} (1-A) MR_A \right\}, \tag{13}$$

for $0 \leq A < 1$ and $0 < I$. Later on we will characterize the internal solution by using the equations derived here.

[6]When $I = 0$, either there is an inequality in (3) and then $\dot{I} = I = 0$ or there is an equality in (3), i.e., $\lambda = \frac{C_2'(0)}{h'(0)(1-A)}$ and then $\dot{\lambda}$ in (5) must be positive for both $\lambda$ and $I$ to grow. Only then can we obtain (11). From $\dot{\lambda} > 0$ we obtain $(r + \psi + g(X(0,A)) + h(0)) > \frac{h'(0)}{C_2'(0)} (1-A) MR_A(X,A)$ and substituting that into (13) yields $\dot{I} < \frac{1}{\left(\frac{C_2"}{C_2'} - \frac{h''}{h'}\right)} \left\{ \frac{A\psi}{(1-A)} - (g(X(0,A)) + h(0)) \right\}$. For sufficiently small $A$ the term in the RHS is negative, but $\dot{I}$ cannot be negative when $I = 0$. Hence, in an internal solution, $I$ cannot be zero for small $A$, i.e., $I$ is discontinuous at $I = 0$. Therefore, $I = 0 \Longrightarrow \dot{I} = 0$. Note that it can also be shown that $\dot{I}(A = 1, I = 0) = 0$.

### 3.2.2 Corner Solution I: $I\,(t = 0) = 0$

The relation between the (in)equalities in the necessary conditions and corner solution I is as follows: If in the optimum $I\,(t) = 0$ and $X\,(t) > 0$, then $0 \leq A\,(t) < 1$, equality holds in (4) and (weak) inequality holds in $(3)$. In this case a weak inequality holds in (6) as well and the left-hand side of the equation, $L\,(X\,(t)\,, A\,(t))$ and the right-hand side of $(6)$, $R\,(I\,(t))$ fulfill $L\,(X\,(t)\,, A\,(t)) \leq R\,(I\,(t))$.

If at time $t$ the left-hand side of (6), $L\,(X, A\,(t))$, is for all non-negative pairs $(I, X)$ less than $R\,(I)$, the equation's right-hand side, then the optimal solution at time $t$ is a corner solution with $I\,(t) = 0$ and $X\,(t) > 0$, i.e., at time $t$ the firm produces a positive quantity of good $X$ and does not advertise.

To demonstrate such a solution consider the following example:

First, let $h\,(I) = I^2 + 2I$ and $C_2\,(I) = I^2 + I$. Then, for $I > 0$, the right-hand side of (6), $R\,(I) = \frac{1+1/2I}{1+1/I}$. $R\,(I)$ is monotonic increasing with $I$ in the domain $0 < I < \infty$, and obtains its values in the interval $1/2 < R\,(I) < 1$.

Second, we designate $C_1\,(X) = 2X$, $g(X) = X$ and

$$P\,(X) = \begin{cases} 1.75 + 10A - AX, & for\ 0 \leq X < \frac{1.75+10A}{A} \\ 0, & otherwise. \end{cases} \tag{14}$$

Then, the left-hand side of (6) is $L\,(X, A) = \frac{1}{4} - A\,(10 - 2X)$.

When $A = 0$, the price function is $P\,(X) = 1.75$, i.e., the firm is facing then a demand curve of infinite elasticity and is therefore a price-taker. The left-hand side of (6), $L\,(X, 0) = 0.25 < R\,(I)$,the right-hand side of (6) for all $I$. Therefore at $A = 0$ the solution is a corner solution with $X\,(t = 0) > 0$ and $I\,(t = 0) = 0$. Due to learning-by-buying, the initial sale creates a positive GW, $A$. Once $A$ is positive, there are sufficiently large $X's$ so that $L\,(X, A)$ is larger than $R\,(I = 0)$ =0.5. Thus, when $A$ is positive there may be an internal solution where both $X$ and $I$ are positive. Initially, however, $I(0) = 0$ and $X(0) > 0$, which is a corner solution.

Such a pattern can be traced in markets of differentiated products with many brands and high demand elasticity for each brand. A start-up firm in such a market will first have to distinguish its product through learning by buying and commence with advertising at a later stage. Examples of such firms include start-up restaurants, small stores, confection manufacturers and other small producers of a differentiated product (Crocs shoes).

### 3.2.3 Corner Solution II: $X(t = 0) = 0$

If in the optimum at time $t$ there is a corner solution in which $I(t) > 0$, $X(t) = 0$ and $0 \leq A(t) < 1$, then $L(0, A(t))$, the left-hand side of equation $(6)$, is larger or equal to the equation's right hand side, $R(I(t))$. Equality must hold in (3) and (weak) inequality hold in $(4)$.

If in the optimum at time $t$ the left-hand side of (6), $L(X, A(t))$ where $0 \leq A(t) < 1$, is larger than the equation's right-hand side, $R(I)$ for all non-negative pairs $(I, X)$, then the optimal solution is a corner solution with $I(t) > 0$ and $X(t) = 0$. Namely, at time $t$ the firm advertises but does not produce.

To demonstrate such a corner solution, consider the example in corner solution I but with the inverse demand function replaced by

$$P(X) = \begin{cases} A(100 - X), & for\ 0 \leq X \leq 100 \\ 0, & otherwise. \end{cases} .$$

In this case, at time $t = 0$ since $A(0) = 0$ there is no demand for the firm's product and $R(I) < L(X, 0)$ for all $X$ and $I$. Therefore, at time $t = 0$ the firm advertises but does not sell. It may begin to sell as well only when the GW exceeds the level of $A = 0.01$, which is the level at which the demand price may justify sales. Indeed, when $A \geq 0.01$ there are positive pairs $(X, I)$ for which equality prevails in $(6)$. Such a pattern is typical to the movies industry, sport events, publishing of celebrities' books, among others. Note that in such cases the beginning of sales is preceded by a promotion campaign.

In the next section we introduce specific functional forms into our model in order to gain more insight into the behavior of the firm along the optimal path in an internal solution.

# 4 The Model with Specific Functional Forms

In this section we assume specific functional forms that satisfy the conditions for an internal solution. These functional forms are sufficiently general to typify the industry, yet they are sufficiently specific to allow full characterization of the model's solution.

## 4.1 The Functional Forms

First, we assume that the inverted demand function, $P(X, A)$, is the constant elasticity function $P = P_0 (1 + sA) X^{-\gamma}$, where $P_0, \gamma$, and $s$ are given parameters, $P_0 > 0$, $0 < \gamma < 1$ and $1 \leq s < 10$. The parameter $s$ measures the maximum capacity of goodwill. Note that the demand elasticity, $\epsilon_{X,p} = \frac{1}{\gamma}$, fulfills $1 < \frac{1}{\gamma} < \infty$.[7]

Next we assume that $g(X) = \beta X$ and $h(I) = \alpha I$ where $\alpha$ and $\beta$ are positive given parameters. The cost functions are $C_1(X) = bX$ and $C_2(I) = \frac{c}{2} I^2$ where $c$ is a given positive parameter. Thus, the revenue function is $R(X, A) = P_0 (1 + sA) X^{1-\gamma}$, with

[7] Such a demand function is, for example, the demand of a community of $N$ households, all of which have the same quasi-linear utility function $u(x_i, A, z_i) = K_o (1 + sA) x_i^{(1-\gamma)} + z_i$, where $x_i$ is the quantity of the firm's product consumed by household $i$ and $z_i$ stands for all other goods consumed by the household. Then, the inverted demand function, $P_0 (1 + sA) X^{-\gamma}$, is the inverted aggregate demand function of the community of $N$, where $P_o = K_o (1 - \gamma) N^{\gamma}$ and $X = \sum x_i$.

marginal revenue functions, $MR_X = P_0 (1-\gamma)(1+sA) X^{-\gamma} = (1-\gamma) P(X,A)$ and $MR_A = P_0 s X^{1-\gamma}$. Substituting the above in (2) we obtain the Lagrangian $\mathcal{L}^s(0)$ in (15), for the specific functional forms model.

$$\begin{aligned}
\mathcal{L}^s(0) &= \max \int_0^\infty e^{-rt}\left[P_0(1+sA)X^{1-\gamma} - bX - \frac{c}{2}I^2\right]dt \\
& \quad s.t.\ \ \dot{A} = (\beta X + \alpha I)(1-A) - \psi A; \\
& \quad A(0) = 0, \ \ -I \le 0 \ \ and \quad -X \le 0. \qquad (15)
\end{aligned}$$

The solutions to the maximization of $\mathcal{L}^s(0)$ in (15) is internal and has no corner solutions (see Appendix A).

## 4.2 The Solution of the Model with the Specific Functional Forms

We first compute the current quantity and price of the product by substituting the specific functional forms specified in section 4.1 into (7). We obtain,

$$X = \left[\frac{P_0(1-\gamma)(1+sA)}{\beta\left(\frac{b}{\beta} - \frac{c}{\alpha}I\right)}\right]^{\frac{1}{\gamma}}. \qquad (16)$$

Since $I$ cannot be negative, the lower bound of the quantity sold is $X_{\min}(A) = \left[\frac{P_0(1-\gamma)(1+sA)}{b}\right]^{\frac{1}{\gamma}}$ and since $X$ must be finite there is an upper bound of advertising $I_{\max} = \frac{b\alpha}{\beta c}$. It should be noted that $I_{\max}$ is independent of $\gamma$, and has the same value for all $\gamma$. Therefore,

$$X = \left[\frac{P_0(1-\gamma)(1+sA)}{\frac{\beta c}{\alpha}(I_{\max} - I)}\right]^{\frac{1}{\gamma}}. \qquad (17)$$

From (17) we obtain,

$$\begin{aligned}
\frac{\partial X}{\partial I} &= \frac{X}{\gamma(I_{\max} - I)} > 0 \\
\frac{\partial X}{\partial A} &= \frac{sX}{\gamma(1+sA)} > 0 \qquad (18)
\end{aligned}$$

Note that by substituting $X$ into $P = P_0 (1 + sA) X^{-\gamma}$ we obtain the price along the optimal path as a function of $I$ alone as follows:

$$P = \frac{\beta\left(\frac{b}{\beta} - \frac{c}{\alpha}I\right)}{(1-\gamma)}, \tag{19}$$

where $\frac{\partial P}{\partial A} = 0$ and $\frac{\partial P}{\partial I} < 0$. Hence, a change in the stock of GW does not affect the equilibrium price, whereas an increase of advertisement along the optimal path reduces the price.

### 4.2.1 The Singular Curves

The characterization of the model's solution is done by the technique of phase diagrams: in the $(A, I)$ plane we plot the singular curves $\dot{I} = 0$ and $\dot{A} = 0$ and their intersection, the steady state point. We will then determine the direction of movement over time on the $(A, I)$ plane and use this information to find the optimal path to the steady state. At first we calculate the equations of the singular curves.

By substituting from section 4.1 the specific functional forms into (13) and equating the result to zero, we obtain the singular curve $\dot{I} = 0$.

$$\dot{I} \equiv I\left(r + \frac{\psi}{(1-A)}\right) - \frac{\alpha s}{c}(1-A) P_0 (X)^{1-\gamma} = 0, \tag{20}$$

where $0 < I < I_{\max}$. By substituting the specific functional forms into (1) and equating the result to zero we obtain the singular curve $\dot{A} = 0$,

$$\dot{A} \equiv (\beta X + \alpha I)(1-A) - \psi A = 0, \tag{21}$$

from which we obtain the following expression for $A$ as a function of $I$ and $X$ along the curve $\dot{A} = 0$,

$$A|_{\dot{A}=0} = \frac{\beta X + \alpha I}{\beta X + \alpha I + \psi}. \tag{22}$$

Consequently, $0 \leq A|_{\dot{A}=0} < 1$.

## 4.3 Phase Diagrams Analysis

In this section we characterize and analyze the solution of the functional forms model introduced in the previous section by using the phase diagrams technique. In the previous section we calculate the singular curves and in Appendix B we calculate the layouts and slopes of the two singular curves $\dot{I} = 0$ and $\dot{A} = 0$ in the $[A, I]$ plane. The two curves are displayed in figures 2 and 3. The directions of growth of $A$ and $I$ in various locations on the $[A, I]$ plane are depicted by arrows in figures 2 and 3 and are calculated in Appendix B.

There are two typical solutions to our model with specific functional forms, which are described by cases I and case II below. In Case I are included models whose demand elasticities are relatively low, i.e., $1 < \epsilon_{Xp} \leq \left(\frac{\psi}{s(r+\psi)} + \left(\frac{1}{s} + 1\right)\right)$, where $\epsilon_{Xp}\left(= -\frac{\partial X(P,A)}{\partial P}\frac{P}{X}\right)$ is the demand elasticity and in the model with specific functional forms $\epsilon_{Xp} = 1/\gamma$. Case II includes models with higher demand elasticities, i.e., $\left(\frac{\psi}{s(r+\psi)} + \left(\frac{1}{s} + 1\right)\right) < \epsilon_{Xp} < \infty$. Below we present the two cases.

### 4.3.1 Case I: $\frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)} \leq \gamma < 1$.

Case I is depicted in Figure 2. The domain of $\gamma$ in case I can be divided into two consecutive intervals, $\frac{s}{1+s} \leq \gamma < 1$, and $\frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)} \leq \gamma < \frac{s}{1+s}$. The price elasticities of the inverted demand function $P(X, A)$ are: $\mu_{P,X}\left(= -\frac{\partial P}{\partial X}\frac{X}{P} = 1/\epsilon_{X,P}\right) = \gamma$ and $\mu_{P,A} = \frac{\partial P}{\partial A}\frac{A}{P} = \frac{sA}{1+sA}$. Because $\frac{s}{1+s} \geq \frac{sA}{1+sA} = \mu_{p,A}$, for all $A$ (since $0 \leq A \leq 1$), in the first interval $\gamma = \mu_{p,X} > \frac{s}{1+s} > \mu_{p,A}$. Thus, in the first interval of case I, the price elasticity with respect to quantity is larger than the price elasticity with respect to GW. The second interval in Case I extends the validity of the results depicted in Figure 2 to slightly lower $\gamma's$. Note that for firms with $\psi$ much larger than $r$, the optimal path depicted in Figure 2 still prevails, even if $\gamma$ is considerably smaller than $\frac{s}{1+s}$.

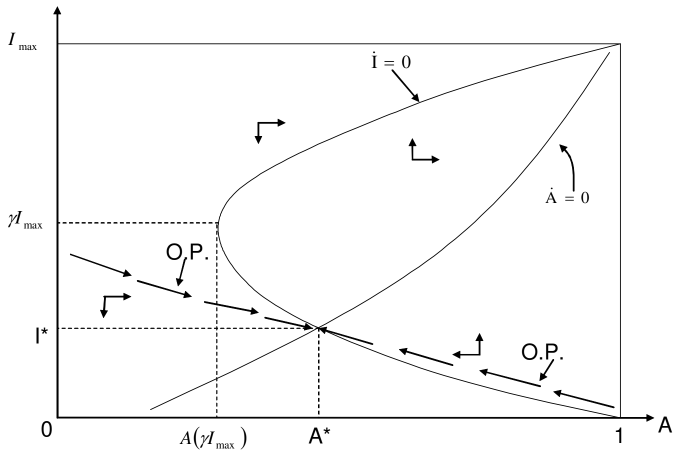


Figure 2: The Optimal Path in Case I.

The singular curve $\dot{I} = 0$ in Figure 2 has two branches that connect at point $(\gamma I_{\max}, A(\gamma I_{\max}))$ where its slope is infinite. The slope of the lower branch of the curve is negative while the upper branch has a positive slope. Both branches of $\dot{I} = 0$ terminate at $A = 1$, the upper branch at $I = I_{\max}$ and the lower at $I = 0$. The singular curve $\dot{A} = 0$ in Figure 2 starts from the $A$-axis at $A = \underline{A} > 0$ and ends at $(I_{\max}, 1)$ at the upper right hand corner and its slope is always positive.

The intersection of the two singular curves, $\dot{I} = 0$ and $\dot{A} = 0$ yields the steady state point $(I^*, A^*)$. The optimal path of a new firm depicted in Figure 2 is characterized by initial high advertising expenditure and sizable sales. As the stock of GW increases due to advertisement and learning by buying, advertising declines. The quantity sold may, however, increase or decrease while the GW increases. The reason for this ambiguity is that while $A$ increases along the optimal path and contributes to an increase of $X$ (see (18)), $I$ declines along the path and therefore has the opposite effect on $X$ thus leaving the end result of $X$ ambiguous. The price $P(X, A)$ however increases along the optimal path, as indicated by $(19)$. When the firm starts with an initial stock of GW $A_0$, $A_0 > A^*$ , advertisement increases while GW decreases and the price decreases. The quantity sold, $X$, is ambiguous along the optimal path, as it is for the case of $A_0 < A^*$. The rationale behind the ambiguity of the change in sales along the optimal path is that two opposite effects are influencing the sale process. The first causes an increase of $X$ by an upward shift of the demand curve, generated by an increase of GW. The second effect is due to a movement on the demand curve which cause a decline of sales and a price raise.

Low demand elasticity products are often essential goods produced by a few firms in markets that are difficult to enter.[8]

### 4.3.2 Case II: $0 < \gamma < \frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)}$.

The solution of this case is depicted in Figures 3a and 3b, where $\gamma$ is relatively low.

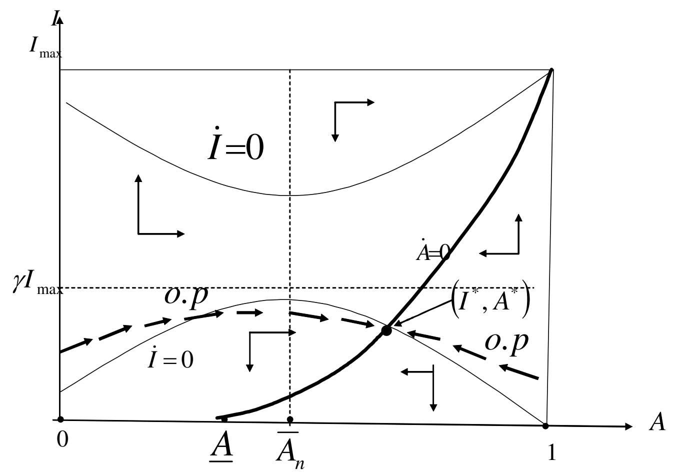


Figure 3a: The Optimal Path of Case II- Steady State to the Right of $\overline{A}_n$.

In Figures 3a and 3b the two branches of the singular curve $\dot{I} = 0$ are separated. The lower branch is bell-shaped with a peak at $\overline{A}_n$ and the upper branch is U-shaped with its bottom also at $\overline{A}_n$ . As in case I, the singular curve $\dot{A} = 0$ intersects the lower branch

[8] An example to a low demand elasticity product is Lipitor, which is a drug taken by patients for cholesterol reduction. Lipitor is an essential drug protected by a patent. Another example to Case I product is aeroplanes. Note that potential buyers to which these firms' advertisement aims are not the general public but a specific part of it. In the case of a new drug the targeted population are medical doctors and in the case of new aeroplanes the target population are aviators and aviation corporations.

of $\dot{I} = 0$ at the steady state point $(I^*, A^*)$ . In Figure 3a the steady state is to the left of $\overline{A}_n$ and in Figure 3b it is to the right.

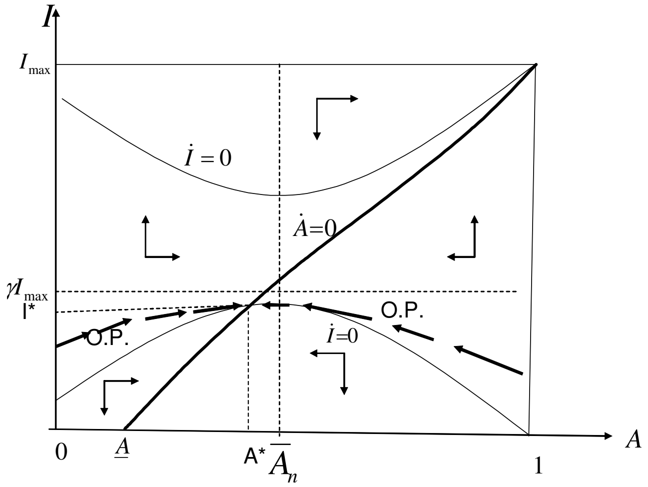


Figure 3b: The Optimal Path of Case II- Steady State to the Left of $\overline{A}_n$.

In both figures, the optimal path of a firm with initial zero GW starts from low levels of advertising and sales that increase as the GW increases. In Figure 3b, where the steady state is to the left of $\overline{A}_n$, advertising and sales increase monotonously towards steady state. In Figure 3a, on the other hand, the steady state is to the right of $\overline{A}_n$; advertising, $I$, overshoots its steady state level $I^*$ and approaches it from above. Nevertheless, cases I and II do not differ significantly. Note that the optimal path on the right-hand side of the steady state point in 3a (3b) is the mirror image of the optimal path on the left-hand side of the steady state point in 3b (3a).

High demand elasticity that prevails in case II is typical of differentiated products, where all brands of a differentiated product are close substitutes of each other and each brand is produced by a different firm. Consequently, each firm is facing a demand curve of high elasticity and has limited monopoly power.[9]

[9]Examples of markets of perishable goods with high demand elasticity are markets of prepared food items, confections, clothing and flights of commercial airlines. Flats in condominiums, cars and inexpensive watches are examples of high demand elasticity of durable goods.

This situation raises the question of why the optimal paths in Case I and Case II differ? When answering this question we should first note that a massive build-up of GW through learning-by-buying requires sales beyond the myopic equilibrium level of sales. In Case I, since the demand elasticity is higher than one but close to it, the shape of the demand curve is such that a sizeable increase in sales entails a considerable price reduction. Therefore, a build-up of GW in Case I by learning-by-buying will drive the price level far below the price of maximum current profits, thus rendering learning-by-buying far more expensive than advertising. Accordingly, in our model, the initial build-up of a GW stock in Case I is done by advertising and as sales and GW stock grow, advertising gradually declines. In Case II, in which demand elasticity is high, the reverse is true and a massive expansion of sales causes only a slight reduction of price, rendering the learning-by-buying process the preferred engine for building up initial GW.

# 5 Simulations of the Model

In the present section, specific numerical values are chosen for the parameters of the model with specific functional forms discussed in section 4 and numerical solutions are calculated by simulations. The simulation design is based on the set of necessary conditions for the optimal solution of the specific functional forms model.

In the following analysis we focus on two values of $\gamma$, the demand price elasticity, namely $\gamma = 0.7$ of Case I and $\gamma = 0.3$ of Case II. For each of these $\gamma$'s we assign a Base Model with the same set of parameters other than $\gamma$, as follows:

**Table 1: The Base Model**

| | | | |
|---|---|---|---|
| $\alpha = 0.0007$ | $\beta = 0.0007$ | $\psi = 0.2$ | $s = 1.5$ |
| $r = 0.1$ | $b = 2.0$ | $c = 0.0005$ | $T = 50$ |

where $T$ is the number of years simulated. Figures 4 and 5 describe the optimal path of the Base Models for $\gamma$=0.7 and $\gamma$=0.3, respectively, where variables $I, A, X$ and $P$ are each depicted as a function of time. Figure 4 corresponds to Case I as depicted by the phase diagram in Figure 2, and Figure 5 corresponds to Case II as depicted by the phase diagram in Figure 3a. Note that while the endogenous variables in the phase diagrams are implicit functions of time, in the simulation results time appears as an independent variable.

Another feature that distinguishes the simulation solution from the analytic is that the former portrays a numerical solution to a numerical problem, whereas the analytic solution covers the entire range of the problem parameters.

The following table presents the initial, final, maximum and minimum values of the resulting variables of the two base cases.

**Table 2: Results of The Base Models**

| $\backslash\ \gamma's$ variables$\backslash$ | $\gamma = 0.7$ initial | final | $\gamma = 0.3$ initial | final |
|---|---|---|---|---|
| $I$ | 29.72 | 27.89 | 174.42 | 174.09 [@] |
| $A$ | 0 | 0.09 | 0 | 0.43 |
| $X$ | 0.67 | 0.80 | 7.49 | 39.03 |
| $P$ | 6.62 | 6.62 | 2.73 | 2.73 |
| $\pi^*$ [§] | 3.52 | | 21.03 | |

[@] The maximum value of *I(t)* is *I(t=3.5)=182.85*, see also Figure 5.

Thus, Table 2 presents an example of Case II as depicted in Figure 3a.

[§] In annual current values at the steady state.

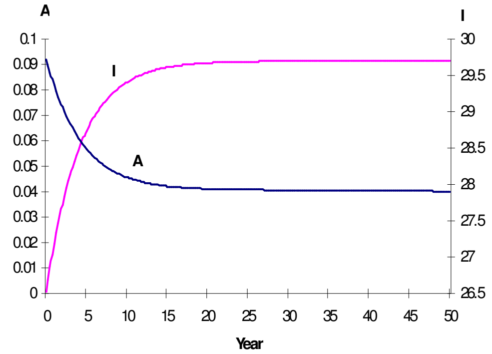


Figure 4a: Optimal path of *I* and *A* Along the Temporal Axis for $\gamma$ =0.7.

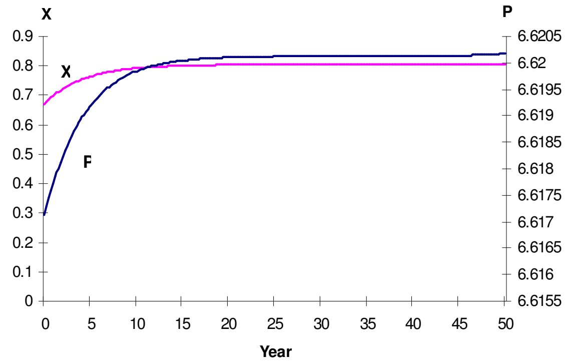


Figure 4b: Optimal path of *X* and *P* Along the Temporal Axis for $\gamma$ =0.7.

Figures 4 and 5 and Table 2 teach us about the optimal behavioral patterns over time of a firm with market power in two distinct cases: one, of a firm with a low demand elasticity of 1.43 ($\gamma$=0.7) and the second of a firm facing a higher demand elasticity of 3.33 ($\gamma$=0.3). In the first case, the firm faces low marginal revenues (of $X$ and $A$) relative to the marginal costs (of $X$ and $I$). Consequently, the firm rapidly approaches steady state levels typified by low levels of sales, advertising and GW. Thus, case I is characterized by practically a myopic behavior of the firm (see the first column in Table 2.). In the second case of higher demand elasticity, the firm increases its sales over time, creating through the learning-by-buying effect an increasing stock of GW. The resulting demand creation (a shift of the demand curve to the right) is strengthened by relatively high levels of advertising expenditure that further increase the stock of GW. Consequently, the convergence to steady state is relatively long. Thus, the second case is characterized by the firm's dynamic behavior (see the second column in Table 2).

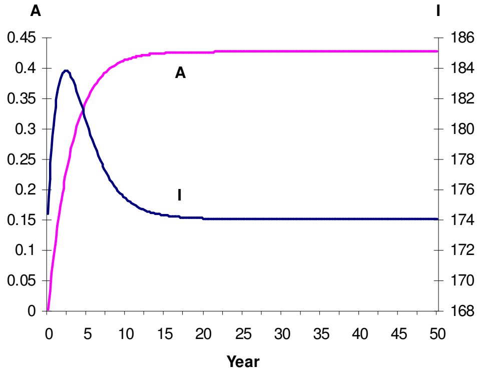


Figure 5a: The Optimal Path of *I* and *A* *Along The Temporal Axis for* $\gamma$ *=0.3*

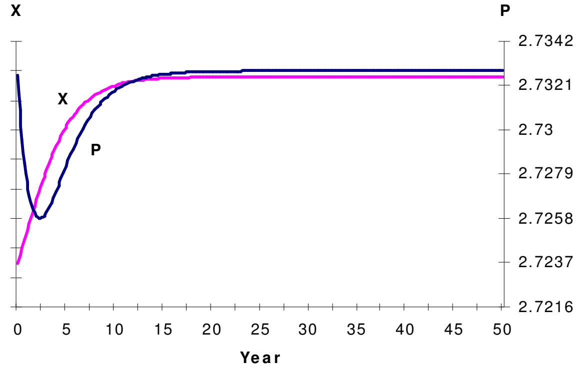


Figure 5b:The Optimal Path of *X* and *P* Along the Temporal Axis for $\gamma$=0.3

Table 3 below presents results of a sensitivity analysis we performed on the base case of $\gamma = 0.3$. In each run, represented by a row in Table 3, we changed one of the base case parameters while keeping the rest of the parameters unchanged . We do not present results of the sensitivity analysis based on the case of $\gamma = 0.7$ because these results are similar to those of the case of $\gamma = 0.3$ and since the dynamics in the case of $\gamma = 0.7$ are less significant. The Base Model with all its variations is characterized by the phase diagram depicted in Figure 3a, except for the variation (raw) of $b = 2.5$ (see Table 3) that is typified by the phase diagram in Figure 3b. In all these cases, the firm adopts a dynamic policy with aggressive sales and advertising that result in increasing GW and with it demand creation. Indeed, the higher the initial advertising (column 2) and sales (column 4), which, in turn are associated with lower initial current profits (which are sometimes even negative, see column 6), the higher are the discounted profits, $\int_0^T e^{-rt}\pi dt$ (column 6). In general, as listed in Table 3, an increase in a cost parameter (such as $\psi$ and $b$) reduces total output, profits, advertising and GW stock, while an increase in beneficial parameters (such as $s, \alpha$ and $\beta$) has the opposite effect.

**Table 3: Sensitivity Analysis of the Base Case for $\gamma = 0.3$**

Initial and Steady State Values

| \variables parameters\ | | $A$ 1 | $I$ 2 | $\hat{I}$ @ 3 | $X$ 4 | $P$ 5 | $\pi$ ★ 6 | $\int_0^T e^{-rt}\pi dt$ |
|---|---|---|---|---|---|---|---|---|
| Base Case | initial | 0 | 174.4 | 183.8 | 7.49 | 2.7 | -2.12 | 138.5 |
| | final | 0.427 | 174.1 | | 39.05 | | 21.03 | |
| $s = 1$ | initial | 0 | 94.48 | 95.33 | 6.89 | 2.8 | 3.29 | 81.2 |
| | final | 0.275 | 92.83 | | 15.70 | | 10.26 | |
| $s = 2$ | initial | 0 | 304.9 | 324.7 | 8.41 | 2.7 | -17.9 | 807.6 |
| | final | 0.569 | 273.7 | | 103.0 | | 49.39 | |
| $\psi = .25$ | initial | 0 | 143.5 | 154.7 | 7.29 | 2.7 | 0.35 | 112.4 |
| | final | 0.337 | 153.3 | | 28.8 | | 15.66 | |
| $b = 2.5$ | initial | 0 | 94.33 | 104.5 | 3.27 | 3.5 | 1.08 | 56.8 |
| | final | 0.287 | 104.5 | | 10.9 | | 8.12 | |
| $\alpha$=.001 $\beta$=.001 | initial | 0 | 249.2 | 252.8 | 8.00 | 2.7 | -10.1 | 225.0 |
| | final | 0.422 | 202.4 | | 59.8 | | 32.33 | |

@The optimal path of $I$ in the case of $\gamma = 0.3$ corresponds to Figure 3a, and has a maximum designated by $\hat{I}$. This is true for all cases other than $b$ =2.5, which is described in Figure 3b.

★ $\pi$ is the value of the current profit and $\int_0^T e^{-rt}\pi dt$ is the sum of the discounted profits.

Another feature worth noting in the simulation runs is the lack of price variations, not only within each run where variations are less than one percent of the price, but also between the different runs. Except for the case of $b = 2.5$, in which the price is 3.5, the price in all the other cases is about 2.7. It stands to reason that the marginal cost of $X$, $b$, is the essential factor in the price determination. It is interesting to note that most of the literature cited in this paper *assumes* the price is constant over time, whereas in our case, this is an *endogenous* result.

Below are additional features about the sensitivity analysis:

- *Changes in $s$, the effectiveness coefficient of $A$:* Initially, the quantity of $X$ is determined mostly by the elasticity of demand that is a static consideration. Therefore, the spread of the initial quantities sold for the different levels of $s$ is relatively small. Final sales, however, are determined mainly by the firm's policy along the optimal path and therefore the spread of final sales is greater. The impact of the dynamic behavioral patterns of the firm is further demonstrated by the different advertisement expenditures for the various levels of $s$ and the corresponding GW stocks.

- *Changes in $\psi$, the coefficient of GW's decay*: An increase in the decay coefficient hinders the growth of stock $A$ and reduces its dynamic impact. Since in this case of $\gamma = 0.3$ the dynamic impact is considerable, changes in $\psi$ are significant.

- *An increase in $b$, the marginal cost of production of $X$, to 2.5:* The build-up of $X$, $I$, and $A$ is gradual and steady, maintaining a high and constant price level. Of all runs depicted in Table 3, this is the only one typified by Figure 3b whereas all the other runs are typified by Figure 3a.

- *An increase to .001 of $\alpha$ and $\beta$,* the *coefficients of advertisement and sales in the equation of motion*, respectively*: The* result is similar to the result from an increase in $s$. The result is an increase in $I$, $X$ and $A$ along the optimal path.

# 6 Appendices

## 6.1 Appendix A: Nonexistence of Corner Solutions in the Model with Specific Functional Forms

In this appendix, we argue that the two conditions in the general case for Corner Solutions 1 and 2 in section 3.2, are not fulfilled in the solution to our specific functional forms model.

To show this we substitute the parameters of the specific functional forms model for $A = 0$ into (6) to obtain (23),

$$L(X) = \frac{b - P_0 (1-\gamma)(X)^{-\gamma}}{\beta} \gtreqless \frac{cI}{\alpha} = R(I). \tag{23}$$

In Corner Solution 2 initially advertising $I(0) > 0$ and $X(0) = 0$. For this outcome to occur $L(0)$ in (23) must be greater than $R(I)$. However, since in (23) $L(X = 0) = -\infty$, there is no non-negative $I$, for which $R(I) < -\infty$. This means that in our specific example, advertising without production is never optimal.

Initially, i.e., when $A(0) = 0$, Corner Solution 1 has only production and no advertising; hence, the initial advertising must be $I(0) = 0$. Substituting this initial investment into (23) yields

$$L(X(0)) = \frac{b - P_o (1-\gamma) X(0)^{-\gamma}}{\beta} = \lambda(0) \leq R(I(0)) = 0.$$

Since $L(X = 0) = -\infty$ and $L(X = \infty) = \frac{b}{\beta} > 0$, a positive constant $\underline{x}$ must exist such that $L(\underline{x}) = 0$. Since $\lambda(t = 0)$ is non-negative, the triplet $X(t = 0) = X_{\min}$ , $I(t = 0) = 0$ and $\lambda(t = 0) = 0$, is the only possible corner solution.[10] However, with positive parameters in(23), a whole range of $X > X_{\min}$ and positive $I$ and $\lambda$ exist, which

[10]Note that $\underline{x}$ must be equal to $x_{\min}(0)$, and this solution is technically not a corner solution since equality holds in (23).

constitutes feasible interior solutions. The optimal solution is, therefore, an interior solution, possibly with positive $X$ and $I$.[11]

## 6.2 Appendix B: The Phase Diagrams

### 6.2.1 Calculating the Singular Curve $\dot{I} = 0$.

To determine the slope of $\dot{I} \mid_{I>0} = 0$, we differentiate totally the equation in (20) to obtain $TD$ below,

$$TD\left[\dot{I} \mid_{I>0} = 0\right] \equiv dI\left[\left(r + \frac{\psi}{(1-A)}\right) - \frac{\alpha s}{c}(1-\gamma)(1-A)P_0 X^{-\gamma}\frac{\partial X}{\partial I}\right] +$$
$$+dA\left[\frac{I\psi}{(1-A)^2} + \frac{\alpha s}{c}P_0 X^{1-\gamma} - \frac{\alpha s}{c}(1-\gamma)(1-A)P_0 X^{-\gamma}\frac{\partial X}{\partial A}\right] = 0.$$

By substituting the first term that multiplies $dI$ above, which is also the first term in $(20)$, with the second term in (20) and then substituting for $X$ and its partial differentials with respect to $I$ and $A$ from (18) and (16) while keeping in mind that $I_{\max} = \frac{b\alpha}{\beta c}$, we obtain,

$$TD\left[\dot{I} \mid_{I>0} = 0\right] \equiv dI\left[\frac{\alpha s}{c}(1-A)P_0(X)^{1-\gamma}\left(\frac{\gamma I_{\max} - I}{\gamma I(I_{\max} - I)}\right)\right]$$
$$+dA\left[\frac{I\psi}{(1-A)^2} + \frac{\alpha s}{c}P_0 X^{1-\gamma}\left(1 - \frac{(1-\gamma)}{\gamma}\frac{s(1-A)}{(1+sA)}\right)\right] = 0$$

The above equation implies,

[11] However, if $b$, for example, is sufficiently small, $\underline{x}$ is sufficiently large and $\underline{x}$, $I = 0$ and $\lambda = 0$, can constitute the optimal solution. Note that in this case, $I = 0$ and $\lambda = 0$ for all $t$, and the solution is entirely myopic. Yet, this solution is not a typical corner solution since for all $t$ equality holds in $(23)$.

$$\frac{dI}{dA}\Big|_{\substack{\dot{I}=0\\ I>0}} = \left\{ -\frac{\frac{I\psi}{(1-A)^2} + \frac{\alpha s}{c} P_0 X^{(1-\gamma)} \left(1 - \frac{(1-\gamma)}{\gamma}\frac{s(1-A)}{(1+sA)}\right)}{\frac{\alpha s}{c}(1-A)\, P_0 X^{(1-\gamma)} \left(\frac{\gamma I_{\max} - I}{\gamma I (I_{\max} - I)}\right)} \right\} \tag{24}$$

$$= -\frac{\frac{I\psi}{(1-A)^2} + \frac{\alpha s}{c} P_0 X^{(1-\gamma)} \frac{\gamma + s(A+\gamma-1)}{\gamma(1+sA)}}{\frac{\alpha s}{c}(1-A)\, P_0 X^{(1-\gamma)} \left(\frac{\gamma I_{\max} - I}{\gamma I (I_{\max} - I)}\right)}$$

Consider the denominator in (24), which is a multiplication of two terms of which the first is always positive. The second term in the multiplication is negative when $\gamma I_{\max} < I < I_{\max}$ and positive when $I$ fulfills $0 < I < \gamma I_{\max}$. At $I = \gamma I_{\max}$ the last term of the multiplication in the denominator vanishes and with it the whole denominator.

Since $X(I=0) > 0$, it follows that $A(I=0)\,|_{\dot{I}=0}$ must be equal to 1 for equation (20) to be fulfilled. To see this note that when $I = 0$ the first term in (20) vanishes; hence, the second term must disappear as well and this holds if and only if $\lim_{I\to 0} A|_{\left\{\substack{\dot{I}=0\\ I>0}\right.} \to 1$. In addition, note that since $X(\ I_{\max}) = \infty$, it must be that $\lim_{I\to I_{\max}} A|_{\left\{\substack{\dot{I}=0\\ I>0}\right.} \to 1$ for $\dot{I}$ in (20) to vanish. Otherwise, if $A|_{\left\{\substack{\dot{I}=0\\ I>0}\right.}$ is less than one, it is clear that in equation (20) $\dot{I}$ does not disappear. Therefore, the only way that equation (20) holds is for $\lim_{I\to I_{\max}} A|_{\left\{\substack{\dot{I}=0\\ I>0}\right.} \to 1$.

Suppose the numerator in (24) is positive for all $I$, $0 \leq I \leq I_{\max}$ and $A(I)$, $0 \leq A(I) \leq 1$ (e.g., when $\gamma \geq \frac{s}{1+s}$) and let $A(\gamma I_{\max})\,|_{\dot{I}=0}$ be the value of $A$ in $\dot{I} = 0$ when $I = \gamma I_{\max}$. Then, for each $A$ such that $A(\gamma I_{\max})\,|_{\dot{I}=0} < A \leq 1$ there are two $I's$ which satisfy (20). For $A's$ such that $A < A(\gamma I_{\max})\,|_{\dot{I}=0}$, there are no $I's$ that belong to $\dot{I} = 0$ and the curve does not exist there. In other words, for $A$, $A(\gamma I_{\max})\,|_{\dot{I}=0} < A \leq 1$, the curve $\dot{I} = 0$ has two branches: in the upper branch $I$ and $A$ are both increasing, and in the lower branch, when $I$ is decreasing, $A$ is increasing. These two branches of $\dot{I} = 0$ are depicted in Figures 2a and 2b (in 2a $A(\gamma I_{\max})\,|_{\dot{I}=0} > 0$ and in 2b $A(\gamma I_{\max})\,|_{\dot{I}=0} < 0$).[12]

[12] To find out when the curve $\dot{I} = 0$ intersects the $I$ axis we do the following.We substitute $A = 0$ into (20) to obtain $I = \frac{\alpha s}{c(r+\psi)} \left[\frac{P_0(1-\gamma)}{\left(b - \beta\frac{c}{\alpha} I\right)}\right]^{\frac{1-\gamma}{\gamma}}$. Next, we change the variable $I$ into a variable $\theta$ so that

We now extend the investigation of the curve $\dot{I} = 0$ to include cases in which the numerator of $\frac{dI}{dA}|_{\dot{I}=0}$ in (24) is negative. First we find out when the numerator of (20) is positive or negative. The first term in the numerator is always positive and the second term is non-negative if $\frac{\gamma}{1-\gamma} \geq s\frac{1-A}{1+sA}$. The right-hand side of the inequality above is a function of $A$ and $s$, and is bounded between zero and $s$. The left-hand side is determined by $\gamma$ alone and for $\gamma \geq s/(1+s) \Longrightarrow \frac{\gamma}{1-\gamma} \geq s \geq s\frac{1-A}{1+sA}$. In this case both terms in the numerator are positive and so is the numerator. In Figure 2a the curve $\dot{I} = 0$ is depicted as discussed above.

From here on we adopt the plausible assumption $r \leq \psi$. The numerator of $\frac{dI}{dA}|_{\dot{I}=0}$ in (24) is given by $Num = \frac{I\psi}{(1-A)^2} + \frac{\alpha s}{c}P_0X^{(1-\gamma)}\frac{\gamma+s(A+\gamma-1)}{\gamma(1+sA)}$. We now replace in $Num$ the term $\frac{\alpha s}{c}P_0(X)^{1-\gamma}$ with the term $\frac{I}{(1-A)^2}(r(1-A)+\psi)$ by using (20) to obtain,

$$Num = \frac{I}{\gamma(1+sA)(1-A)^2}\left\{(\psi\gamma + \psi\gamma sA) + (r(1-A)+\psi)(\gamma + sA + s\gamma - s)\right\}.$$

The sign of $Num$ is characterized by the following cases in which we make the plausible assumption $r \leq \psi$.

**Case $\boldsymbol{I_1}$: $\gamma \geq \frac{s}{1+s}$.** In this case all the terms in the numerator are positive for all feasible $A$ as we already showed above (see Figure 2). Note also that in this case $|\eta_{p,X}| = \gamma \geq \eta_{p,A} = \frac{sA}{1+sA} \leq \frac{s}{1+s}$, for all $0 < A < 1$, where $|\eta_{p,X}|$ is the price elasticity

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$I = \theta I_{\max} = \theta\frac{b\alpha}{\beta c}$. The resulting equation in $\theta$ is: $\frac{s}{(r+\psi)}\left[\frac{P_0(1-\gamma)}{b(1-\theta)}\right]^{\frac{1-\gamma}{\gamma}} = \frac{b}{\beta}\theta$. For $\gamma = 1/2$ the equation can be solved, i.e., $m - \theta + \theta^2 = 0$ where the parameter $m$ is,

$$m = \frac{P_0\beta s}{2b^2(r+\psi)}.$$

There are two real solutions to $\theta$ if $0 < m < 0.25$: $0 \leq \theta_{1,2} = \frac{1\pm\sqrt{1-4m}}{2} \leq 1$, which means that the curve $\dot{I} = 0$ intersects the $I$ axis twice when $m < 0.25$. If $m > 0.25$ there is no real solution to $\theta$ which means that $\dot{I} = 0$ does not intersect the $I$ axis and the line $\dot{I} = 0$ is not defined over small $A$ and never reaches the $I$ axis. When $\gamma \neq 1/2$ the situation is essentially the same. In other words we may have two or more intersection points of $\dot{I} = 0$ with $A = 0$, one intersection point or none at all.

with respect to the quantity demanded, and $\eta_{p,A}$ is the price elasticity with respect to the GW.

**Case $I_2$:** $\frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)} \leq \gamma < \frac{s}{1+s}$ **.** Like case $I_1$, $Num$ in this case is positive for $0 < A \leq 1$. To see that, first note that $Num$ is positive for $A = 1$ for all $\gamma$, because then $sA$ and $(-s)$ cancel each other out and the remaining terms are positive. Furthermore, continuity implies that for each $\gamma$ there are feasible $A's$ sufficiently close to 1 for which $Num$ is positive. Next, we find out if and when $Num$ changes sign when $\gamma < \frac{s}{1+s}$. Since the numerator is continuous in $A$, to change signs it must vanish at some feasible $A$, say $0 < \bar{A}_n(\gamma) < 1$, so that for $A > \bar{A}_n(\gamma)$ the numerator is positive and for $A < \bar{A}_n(\gamma)$ the numerator is negative. We solve $\bar{A}_n(\gamma)$ from the second order equation in $A$ below, in which the numerator of $Num$ is equated to zero,

$$-A^2(rs) + A(\psi s(\gamma+1) + \ r(s(2-\gamma)-\gamma)) + [\psi\gamma + (r+\psi)(\gamma(1+s)-s)] = 0.$$

The above is an equation in $A^2$ and therefore may have zero, one or two solutions of $A$ in the relevant range $(0,1)$. If there is no solution in the relevant segment $(0,1)$, $Num$ does not change sign. If there is a single solution in the relevant range the sign changes once and if there are two solutions the sign changes twice. The solution of the above equation is given below in (25),

$$\bar{A}_n(\gamma) = rd \pm re = \underbrace{\frac{[\psi s(\gamma+1) + \ r(s(2-\gamma)-\gamma)]}{2rs}}_{rd} \pm \tag{25}$$

$$\underbrace{\frac{\sqrt{(\psi s(\gamma+1) + \ r(s(2-\gamma)-\gamma))^2 + 4(rs)[\psi\gamma + (\gamma(1+s)-s)]}}{2rs}}_{re}$$

Dividing the first term (in the square brackets) in the numerator of (25) by the denominator, $(2rs)$, yields, $rd = \frac{1}{2}\left\{\frac{\psi}{r}(\gamma+1) + \left[2 \ -\gamma\left(1+\frac{1}{s}\right)\right]\right\} > 1.5 - \frac{\gamma}{2} > 1$. The above inequalities follow for $\psi/r \geq 1$, $s \geq 1$ and $0 < \gamma < 1$. The fact that $rd$ is larger than 1 implies that the solution with the $+$ sign before $re$ in (25) is always larger than 1 and therefore it is not in the relevant range $(0,1)$.

Next, note that because $\left[\frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)} < \gamma < \frac{s}{1+s}\right]$, the second term in the root in $re$ is positive. Therefore, the root of the sum of the two terms is larger than the root of the first term alone. Hence, in (25) $re > rd$ and the difference between them is negative and therefore $\bar{A}_n(\gamma) = re - rd < 0$ is not in the relevant range. In conclusion, as in case $I_1$, $Num$ does not change its sign. Case I in the text is Case $\mathrm{I}_1$+Case $\mathrm{I}_2$.

**Case *II:*** $0 < \gamma < \frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)}$. In this case, the second term in the root in (25) is negative and its absolute value is smaller than the first term. Accordingly, contrary to case $I_2$, the whole root is smaller than the root of only the first term. Thus, in $(25)$, for $\gamma$ smaller than $\frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)}$, $re$ is less then $rd$; hence $\bar{A}_n(\gamma) = rd - re > 0$. Note that for $\gamma$ smaller than but close to $\frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)}$, $re$ is smaller but close to $rd$. Therefore, $\bar{A}_n(\gamma)$ is positive and close to zero. This means that for small $\bar{A}_n(\gamma)$, the following inequality holds $A(\gamma I_{\max}) > \bar{A}_n(\gamma) > 0$. On the other hand, for $\gamma << \frac{s}{\left(\frac{\psi}{(r+\psi)}+(1+s)\right)}$ there is no $A(\gamma I_{\max})$ in $\dot{I} = 0$.

We now return to the determination of the slope of $\dot{I} = 0$. In cases $I_1$ and $I_2$ the numerator of $\frac{\partial I}{\partial A}/_{\dot{I}=0}$, $Num$, is positive; hence $\dot{I} = 0$ is as depicted in Figure 2. In case $II$, the numerator, $Num$, changes sign at a single point $\bar{A}_n$. When this happens, a change in the sign of the slope of $\dot{I} = 0$ may occur at $\bar{A}_n$ as depicted in Figures 3.[13]

Next we characterize the curve $\dot{A} = 0$.

### 6.2.2 Calculating the Singular Curve $\dot{A} = 0$

By differentiating (21) we obtain,

$$\frac{dI}{dA}|_{\dot{A}=0} = -\frac{\beta\frac{\partial X}{\partial A}(1-A) - (\beta X + \alpha I + \psi)}{\left[\beta\frac{\partial X}{\partial I} + \alpha\right](1-A)} > 0 \tag{26}$$

[13]If $\bar{A}_n < A(\gamma I_{\max})$ there is no change of sign and the curve $\dot{I} = 0$ is as depicted in Figure 2. The curve $\dot{I} = 0$ changes signs only if there is no $A(\gamma I_{\max})$.

From (1) it can be verified that $\dot{A}$ vanishes at $A = 0$ for $I$ negative, i.e., the intercept of the curve $\dot{A} = 0$ is negative. On the other hand, when $A$ is close to, but less than 1, $\dot{A}$ vanishes for $0 < I < I_{\max}$.. To prove this, consider (21) from which we obtain

$$(\beta X + \alpha I) = \psi \frac{A}{(1-A)}. \tag{27}$$

The right-hand side of (27) can obtain any value in the interval $(0, \infty)$ and is a monotonic increasing function of $A$. The expression $\beta X_{\min}(A_0)$ is a positive number equal to the value of $X$ when $I = 0$. Thus, $I$ solved from (27) for an $A$ that fulfills $\beta X_{\min}(A = 1) < \psi \frac{A}{(1-A)}$, is positive. The above analysis implies that the curve $\dot{A} = 0$ must intersect the $A$-axis at a point $\underline{A}$, $0 < \underline{A} < 1$.

When $A$ approaches 1 from below, the expression, $(\beta X + \alpha I)$ in $\dot{A} = 0$ must approach infinity to offset the effect of the decreasing $(1 - A)$. Note that for the expression $(\beta X + \alpha I)$ to approach infinity $I$ must approach $I_{\max}$, since then $X$ approaches infinity. Thus, the curve $\dot{A} = 0$ ends in $(I = I_{\max}, A = 1)$.

Consider now the slope of $\dot{A} = 0$ in the interval $(\underline{A}, 1)$. First, replace $\frac{\partial X}{\partial A}$ in the numerator in (26) with the term $\frac{sX}{\gamma(1+sA)}$ from (18) and then substitute $(\beta X + \alpha I)$ in (26) with $\frac{\psi A}{(1-A)}$ from (21). After some manipulations the numerator of $\frac{dI}{dA}|_{\dot{A}=0}$ becomes,

$\beta s \frac{X}{\gamma(1+sA)}(1 - A) - \frac{\psi A}{(1-A)} - \psi.$

The first term in the expression above is positive while the next two terms are negative. The first term is dominant when $A$ is close to zero, the second term becomes dominant when $A$ is close to 1. Since the denominator of $\frac{dI}{dA}|_{\dot{A}=0}$ is always positive, when the numerator is negative the slope $\frac{dI}{dA}|_{\dot{A}=0}$ is positive.

For small $A'$s, the values of $I$ in the curve $\dot{A} = 0$ is negative. Therefore, for the curve $\dot{A} = 0$ to intersect the $A$-axis its slope must become positive already for negative $I's$. Once the slope becomes positive for a given $A$ it remains positive for all larger $A's$.

Thus in Figures 2 and 3a the curve $\dot{A} = 0$ intersects $\dot{I} = 0$ at the lower branch with a negative slope. In Figure 3b, the two curves intersect where both have a positive slope.

### 6.2.3 Determining the Direction of Growth

Next we proceed with the evaluation of the directions of growth in time of $A$ and $I$. We first differentiate $\dot{A}$ in (15) with respect to $I$ to obtain $\frac{\partial \dot{A}}{\partial I}|_{dA=0}$ below,

$$\frac{\partial \dot{A}}{\partial I}|_{dA=0} = \left(\beta \frac{\partial X}{\partial I} + \alpha\right)(1-A) > 0. \tag{28}$$

This implies that in Figures 2 and 3, above the curve $\dot{A} = 0$, $\dot{A}$ is positive and below the curve $\dot{A}$ is negative, as indicated by the horizontal arrows.

Next we substitute the values of the specific example into (13) and differentiate the resulting $\dot{I}$ with respect to $A$, to obtain,

$$\frac{\partial \dot{I}}{\partial A}|_{dI=0} = \frac{I\psi}{(1-A)^2} + \frac{\alpha s}{c} P_0 X^{(1-\gamma)} - \frac{\alpha s}{c}(1-\gamma)(1-A) P_0 X^{-\gamma} \frac{\partial X}{\partial A},\ for\ I > 0.$$

Then, by substituting $\frac{\partial X}{\partial A}$ from (18) into $\frac{\partial \dot{I}}{\partial A}|_{dI=0}$ above, we obtain,

$$\frac{\partial \dot{I}}{\partial A}|_{dI=0} = \frac{I\psi}{(1-A)^2} + \frac{\alpha s}{c} P_0 X^{(1-\gamma)} \left(1 - \frac{(1-\gamma)(1-A)}{\gamma(1+sA)}\right) \overset{\substack{\text{If } 1-\varepsilon < A < 1 \\ for\ \varepsilon\ sufficiently\ small}}{>} 0. \tag{29}$$

Equation (29) implies that for values of $A$ sufficiently close to but still smaller than 1, $\dot{I}$ is positive to the right of the curve $\dot{I} = 0$ and negative to the left. Note that the sign of $\dot{I}$ reverses itself, if and only if it crosses the singular curve $\dot{I} = 0$. These two characteristics are reflected in the direction of the vertical arrows as depicted in Figures 2 and 3.

The intersection $(I^*, A^*)$ of the two curves, $\dot{I} = 0$ in (20) and $\dot{A} = 0$ in (21), yields the steady states depicted in Figures 2 and 3. The optimal path that leads to the steady state is determined by the directions of the arrows.

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