BEST-OF-THREE ALL-PAY AUCTIONS

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Abstract

We study a three-stage all-pay auction with two players in which the first player to win two matches wins the best-of-three all-pay auction. The players have values of winning the contest and may have also values of losing, the latter depending on the stage in which the contest is decided. It is shown that without values of losing, if players are heterogenous (they have different values) the best-of-three all-pay auction is less competitive (the difference between the players' probabilities to win is larger) as well as less productive (the players' total expected effort is smaller) than the one-stage all-pay auction. If players are homogenous, however, the productivity and obviously the competitiveness of the best-of-three all-pay auction and the one-stage all-pay auction are identical. These results hold even if players have values of losing that do not depend on the stage in which the contest is decided. However, the best-of-three all-pay auction with different values of losing over the contest's stages may be more productive than the one-stage all-pay auction.

1 Introduction

The National Basketball Association (NBA) playoffs begin with eight teams in each conference who qualify for the playoffs. Each team plays against a rival in a best-of-seven contest, with the first team to win four games advancing into the next round, and the other team being eliminated from the playoffs. In the next round, the successful team plays against another advancing team from the same conference. Thus, all but

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one team in each conference are eliminated from the playoffs. The final playoff round, which is a best-of-seven contest between the winners of both conferences, is known as the NBA Finals. Basketball is only one of numerous sporting contests that are based on the form of the best-of-k contest in which the winner is the one who is first to win $\frac{k+1}{2}$ games (k is an odd integer). Another classical best-of-three contest is tennis in which the first player to win two sets wins the contest. In certain prominent tennis tournaments for men, including the all four Grand Slam tournaments (the Australian Open, French Open, Wimbledon, and US Open) the first player to win three sets wins the best-of-five contest. While best-of-k contests are common in sport, they also appear in political races (see Klumpp and Polborn (2006)). For example, the US presidential primary election is a sophisticated version of the best-of-k contest, as in this contest every game (state-election) actually has a different weight (party delegates) and the first to win the majority of the total party delegates is elected as the nominee of the party.

One of the main issues we address in this paper is why a contest designer would choose to use a best-of-k contest over a one-stage contest. One possible reason is that best-of-k contests, such as the NBA playoffs, have a relative advantage over the one-stage contest in that there is a higher number of matches. In other words, if the contest designer makes a profit per match, then he might prefer the best-of-k contest. On the other hand, in tennis, which is a best-of-three contest, the designer does not make a profit from every set, in which case the advantage of the best-of-three tennis contest over a contest based on a long single set is not so clear.

We study a best-of-three all-pay auction with two players where the players' abilities are common knowledge. Each match among the players is modelled as an all-pay auction. We first show that if players have only values of winning and they are heterogeneous (they have different abilities or different values), then the best-of-three all-pay auction is less productive (the total effort is smaller) and less competitive (the difference between the players' probabilities to win is higher) than the standard one-stage all-pay auction. If players are homogenous, however, the productivity and obviously the competitiveness of the best-of-three all-pay auction and the one-stage all-pay auction are identical. By this comparison it would seem that the contest designer who wishes to maximize the players' total effort would prefer the one-stage all-pay auction over the best-of-three all-pay auction. However, this is not inevitably the case. The reason is that in one-stage contests we usually assume that players have values of winning but not values of losing, since the difference between the values of winning and losing is the crucial parameter and therefore the value of losing is usually normalized to zero. However, in multi-stage contests the role of the value of losing is more complex in that the difference between the values of winning and losing is not sufficient to reflect the effect of these values on the players' behavior. This is because the values of losing indirectly change the players' expected payoffs in the previous stages and, accordingly, the players' probabilities of winning in each match. To illustrate, in tennis tournaments, for example, Wimbledon 2008, the prize for the winner was £750,000, and the prize for the runner-up was £375,000. We can find the same ratio in the US Open 2008 where the prize for the winner was $1,500,000 and the prize for the runner-up was $750,000. These runner-up prizes, which represent the values of losing in our model, are the same no matter at what stage the contest is decided. However, we assume here that the values of losing may depend on the contest stage, the reason being that the player not only has a benefit from the monetary value of the prize, but also may benefit from coming close to winning. As such, it seem reasonable that the value of losing increases in stages regardless of whether these values are positive or negative. Thus, in our model we assume that players have values of losing and that these values may be different during the stages of the contest. In particular, we assume that the value of losing the contest in the third stage is larger or equal to the value of losing in the second stage.

We show that our results about the comparison of productivity between the best-of-three all-pay auction and the one-stage all-pay auction hold even if the players have values of losing, given that the values are the same over the stages in the best-of-three all-pay auction. However, if the values of losing are different over the stages, such that the value of losing in the third stage is sufficiently larger than the value of losing in the second stage, then the total effort in the best-of-three all-pay auction might be larger than in the one-stage all-pay auction. This result holds when the value of losing in the one-stage auction is either larger or smaller than both values of losing in the best-of-three all-pay auction. Hence, given the natural assumption that players have values of losing that depend on the stage in which the contest is decided, we provide a possible explanation for why a contest designer who wishes to maximize the total effort would prefer the best-of-k contest over the one-stage contest. Obviously, as we mentioned, the contest designer may have other reasons to prefer the best-of-k contest, particularly if he makes a profit for each match of the contest.

1.1 Related literature

Similar to the comparison made in this paper between the best-of-three all-pay auction and the one-stage all-pay auction, in the literature on contests we can find comparisons between multi-stage contests and one-stage contests. Gradstein and Konrad (1999) studied a rent-seeking contest à la Tullock (with homogenous players) and found that simultaneous contests are strictly superior if the contest's rules are sufficiently discriminatory (as in an all-pay auction). Groh et al. (2009), on the other hand, showed that in a setting with heterogenous valuations, for the Gradstein-Konrad result to hold it is necessary that the multi-stage contest induce a positive probability that the two strongest players do not reach the final with probability one. Otherwise, if this happens, the total expected effort in the elimination tournament equals the total effort in the all-pay auction where all players compete simultaneously. Konrad and Kovenock (2005) studied a tug-of-war contest in which two players match in a sequence of all-pay auctions and the first to win a sufficiently higher number of games receives the prize. They showed that the total effort in their model is lower than in the one-stage all-pay auction. Clark and Riis (1998) analyzed all-pay auctions with multiple identical prizes and compared simultaneous versus sequential designs from the point of view of a revenue-maximizing designer. Their results indicated that if there is a dominant player (one who has a much higher value than his colleagues) a designer would maximize the expected total bid in the contest by distributing prizes simultaneously, whereas if no player were dominant, the designer would prefer a sequential distribution.

Several studies in the literature deal with the problem of finding the optimal allocation of prizes in contests. For example, in all-pay auctions, Moldovanu and Sela (2001) showed that under incomplete information when cost functions are linear or concave in effort, it is optimal to allocate the entire prize sum to a single first prize.[1] In symmetric all-pay auctions under complete information, Barut and Kovenock (1998) showed that the revenue maximizing prize structure allows any combination of $k-1$ prizes, where k is the number of players. These findings indicate that a prize for the player with the lowest effort should not be allocated, that is, players do not have (monetary) values of losing (they are equal to zero). On the other hand, Moldovanu, Sela and Shi (2008) claimed that even if punishment is costly, punishing the weak players (players with low

[1] Szymanski and Valletti (2005) studied the optimal number of prizes in Tullock's model, and showed that the allocation of two prizes may be more profitable than the allocation of a single prize.

efforts) may be more effective than rewarding the strong players (players with high efforts) in eliciting effort input. In other words, the value of losing may play a stronger role than the value of winning.

Akerlof and Holden (2008) studied rank-order tournaments with players who are homogeneous in ability where the probability of winning a match is a stochastic function of players' efforts. Their results indicated that prizes for players with low effort levels are usually more profitable for the designer who maximizes total effort than prizes for players with high effort levels. In other words, in one-stage contests, other than all-pay auctions, values of losing could be even more significant than values of winning.[2]

In the literature on contests, several papers deal with best-of-k contests. Klumpp and Polborn (2006) used Tullock's model to demonstrate that the winner of the first match is more likely to win the contest, and Malueg and Yates (2006), using a generalization of Tullock's model, showed that best-of three contests are more likely to end in two rounds rather than three. These results do not necessarily hold in our model since these authors assumed that players have homogenous abilities[3] while in our paper the players may have heterogeneous abilities.

The rest of the paper is organized as follows: Section 2 introduces the best-of-three all-pay auction and Section 3 presents the unique equilibrium in the one-stage all-pay auction with a value of losing. In Section 4 we show the unique subgame-perfect equilibrium in the best-of-three all-pay auction with differential values of losing. Section 5 analyzes the players' probabilities to win the contests, while Section 6 analyzes the players' total efforts. Section 8 concludes.

2 The model

In the model, two players (or teams) $i = 1, 2$ compete in a best-of-three all-pay auction. The players compete in sequential matches, and the first who wins two matches wins the contest. We model each match among the two players as an all-pay auction: both players exert efforts, and the one exerting the higher effort wins. Participating in the contest generates a (sunk) cost e_i/c_i for player i, where e_i is the the effort of player i and

[2] Other works on allocation of resources in sequential contests include, among others, Rosen (1986), Warneryd (1998) and Konrad (2004).

[3] It should be mentioned though that Klumpp and Polborn (2006) do provide some numerical results about players with heterogenous abilities.

c_i is his ability. Player i's ability c_i is common knowledge. We assume that $c_1 \geq c_2$. The value of winning the contest is v for both players. The players also have values of losing the contest which are either positive or negative, and we assume that the value of losing depends on the stage that the contest is decided, namely, the value of losing in the third stage is a and the value of losing in the second stage is b where $v > a \geq b$. Note that our model is equivalent to a model where the value of winning the contest for player i is $v_i = vc_i$ and the value of losing for player i is $a_i = ac_i$ in the third stage and $b_i = bc_i$ in the second stage. In this equivalent model both players have the same cost function $c(e_i) = e_i$, and we have $\frac{c_2}{c_1} = \frac{v_2}{v_1} = \frac{a_2}{a_1} = \frac{b_2}{b_1}$. We use this equivalence for the analysis of the players' equilibrium strategies in the best-of-three all-pay auction as well as in the one-stage all-pay auction.

3 The one-stage all-pay auction

We begin with the analysis of the standard one-stage all-pay auction which plays a key role in our analysis of the best-of-three all-pay auction. Consider a one-stage all-pay auction with two players $1, 2$ where the players' values for winning are $v_1 \geq v_2 > 0$ and their values for losing are $a_1 \geq a_2$. According to Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996, 2007), there is always a unique mixed-strategy equilibrium in which the players randomize on the interval $[0, v_2 - a_2]$ according to their effort cumulative distribution functions which are given by

$$\begin{aligned} v_1 F_2(x) + a_1(1 - F_2(x)) - x &= v_1 - v_2 + a_2 \\ v_2 F_1(x) + a_2(1 - F_1(x)) - x &= a_2 \end{aligned}$$

Thus, player 1's effort is uniformly distributed

$$F_1(x) = \frac{x}{v_2 - a_2}$$

while player 2's effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{v_1 - v_2 + a_2 - a_1 + x}{v_1 - a_1}$$

Given these mixed strategies, player 1's winning probability against 2 is

$$p_1^* = 1 - \frac{v_2 - a_2}{2(v_1 - a_1)} \tag{1}$$

Without loss of generality, assume that $v_1 = 1$. Substituting $h = \frac{c_2}{c_1} = \frac{v_2}{v_1} = \frac{a_2}{a_1}$ in (1) yields

$$p_1^* = 1 - \frac{h(1-a_1)}{2(1-a_1)} = 1 - \frac{h}{2}$$

That is, the value of losing a does not have any effect on the players' probabilities to win the contest.

The total expected effort is given by

$$TE^* = \frac{v_2 - a_2}{2}(1 + \frac{v_2 - a_2}{v_1 - a_1}) \tag{2}$$

Now let $v_1 = 1$. Substituting $h = \frac{v_2}{v_1} = \frac{a_2}{a_1}$ in (2) implies

$$TE^*(a) = \frac{h(1-a_1)}{2}(1+h)$$

That is, the total effort decreases in the value of losing a.

Using the analysis of the one-stage all-pay auction we can now turn to analyzing the players' equilibrium strategies in the best-of-three all-pay auction.

4 The best-of-three all-pay auction

In the analysis of the subgame-perfect equilibrium of the best-of-three all-pay auction we assume that,

$$2v_2 - v_1 - 2a_2 > 0 \tag{3}$$

Note that without losing values, the requirement $2v_2 - v_1 > 0$ is necessary, since otherwise, by the analysis below, it would be verified that player 2 will not have any incentive to compete in the first stage. Assumption (3) guarantees that player 2 has an incentive to compete in the first stage and also that given the existing of values of losing, the winner in the first stage has a relative advantage in the second stage. Thus, without this assumption, the best-of-three all-pay auction is less competitive and therefore less interesting since the player with the higher value (ability) wins some matches without a real competition.

In order to analyze the subgame-perfect equilibrium of the best-of-three all-pay auction we begin with the last stage of the contest and go backwards to the previous stages.

4.1 Stage 3

The players compete in the last stage only if each player won one of the previous matches. Therefore, the expected value of player i if he wins the match in stage 3 is v_i and if he loses, it is a_i. Thus, based on the analysis of the one-stage all-pay auction, players 1 and 2 randomize on the interval $[0, v_2 - a_2]$ according to their cumulative distribution functions $F_i^{(3)}, i = 1, 2$ which are given by

$$\begin{aligned} v_1 F_2^{(3)}(x) + a_1(1 - F_2^{(3)}(x)) - x &= v_1 - v_2 + a_2 \\ v_2 F_1^{(3)}(x) + a_2(1 - F_1^{(3)}(x)) - x &= a_2 \end{aligned} \tag{4}$$

4.2 Stage 2

Assume first that player 1 won the first match in stage 1. Then, if player 2 wins in this stage, by (4) his expected payoff in the next stage is a_2, and if player 2 loses in this stage, by (4) his payoff is b_2. Similarly, if player 1 wins in this stage, he wins the contest, and his payoff is v_1, and if player 1 loses in this stage, by (4) his expected payoff in the next stage is $v_1 - v_2 + a_2$. Thus, since our assumption (3) implies that $v_2 - a_2 \geq a_2 - b_2$, we obtain that players 1 and 2 randomize on the interval $[0, a_2 - b_2]$ according to their effort cumulative distribution functions $F_i^{(2)}, i = 1, 2$ which are given by

$$\begin{aligned} v_1 F_2^{(2)}(x) + (v_1 - v_2 + a_2)(1 - F_2^{(2)}(x)) - x &= v_1 - a_2 + b_2 \\ a_2 F_1^{(2)}(x) + b_2(1 - F_1^{(2)}(x)) - x &= b_2 \end{aligned} \tag{5}$$

Assume now that player 2 won the first match in stage 1. Then, if player 1 wins in this stage, by (4) his expected payoff in the next stage is $v_1 - v_2 + a_2$ and if player 1 loses, his payoff is b_1. Similarly, if player 2 wins in this stage, he wins the contest, and then his payoff is v_2, and if he loses, by (4) his expected payoff in the next stage is a_2. Thus, since our assumption (3) implies that $v_2 - a_2 \geq v_1 - v_2 + a_2 - b_1$, we obtain that players 1 and 2 randomize on the interval $[0, v_1 - v_2 + a_2 - b_1]$ according to their effort cumulative distribution functions $F_i^{(2)}, i = 1, 2$ which are given by

$$\begin{aligned} (v_1 - v_2 + a_2) F_2^{(2)}(x) + b_1(1 - F_2^{(2)}(x)) - x &= b_1 \\ v_2 F_1^{(2)}(x) + a_2(1 - F_1^{(2)}(x)) - x &= 2v_2 - v_1 - a_2 + b_1 \end{aligned} \tag{6}$$

4.3 Stage 1

If player 1 wins, by (5) his expected payoff in the next stage is $v_1 - a_2 + b_2$, and if he loses, by (6) his expected payoff in the next stage is b_1. Similarly, if player 2 wins, by (6) his expected payoff in the next stage is $2v_2 - v_1 - a_2 + b_1$, and if he loses by (5) his expected payoff is b_2. Thus, since $v_1 - a_2 + b_2 - b_1 \geq 2v_2 - v_1 - a_2 + b_1 - b_2$ and by our assumption (3) $2v_2 - v_1 - a_2 + b_1 - b_2 > 0$, we obtain that players 1 and 2 randomize on the interval $[0, 2v_2 - v_1 - a_2 + b_1 - b_2]$ according to their effort cumulative distribution functions $F_1^{(1)}, i = 1, 2$, which are given by

$$\begin{aligned} (v_1 - a_2 + b_2)F_2^{(1)}(x) + b_1(1 - F_2^{(1)}(x)) - x &= 2v_1 - 2v_2 + 2b_2 - b_1 \\ (2v_2 - v_1 - a_2 + b_1)F_1^{(1)}(x) + b_2(1 - F_1^{(1)}(x)) - x &= b_2 \end{aligned} \tag{7}$$

5 Probabilities of winning

By the above analysis of the subgame-perfect equilibrium, we obtain that in the best-of-three all-pay auction, player 1 (the player with the higher ability) wins the contest if:

1. He wins the first two matches. This happens with the probability of

$$(1 - \frac{2v_2 - v_1 - a_2 + b_1 - b_2}{2(v_1 - a_2 + b_2 - b_1)})(1 - \frac{a_2 - b_2}{2(v_2 - a_2)})$$

2. He loses the first match and wins matches 2 and 3. This happens with the probability of

$$(\frac{2v_2 - v_1 - a_2 + b_1 - b_2}{2(v_1 - a_2 + b_2 - b_1)})(\frac{v_1 - v_2 + a_2 - b_1}{2(v_2 - a_2)})(1 - \frac{v_2 - a_2}{2(v_1 - a_1)})$$

3. Player 1 wins matches 1 and 3 and loses match 2. This happens with the probability of

$$(1 - \frac{2v_2 - v_1 - a_2 + b_1 - b_2}{2(v_1 - a_2 + b_2 - b_1)})(\frac{a_2 - b_2}{2(v_2 - a_2)})(1 - \frac{v_2 - a_2}{2(v_1 - a_1)})$$

Hence, player 1's probability to win the contest is:

$$\begin{aligned} p_1 = & (1 - \frac{2v_2 - v_1 - a_2 + b_1 - b_2}{2(v_1 - a_2 + b_2 - b_1)})(1 - \frac{a_2 - b_2}{2(v_2 - a_2)}) \\ & +(\frac{2v_2 - v_1 - a_2 + b_1 - b_2}{2(v_1 - a_2 + b_2 - b_1)})(\frac{v_1 - v_2 + a_2 - b_1}{2(v_2 - a_2)})(1 - \frac{v_2 - a_2}{2(v_1 - a_1)}) \\ & +(1 - \frac{2v_2 - v_1 - a_2 + b_1 - b_2}{2(v_1 - a_2 + b_2 - b_1)})(\frac{a_2 - b_2}{2(v_2 - a_2)})(1 - \frac{v_2 - a_2}{2(v_1 - a_1)}) \end{aligned} \tag{8}$$

A comparison of player 1's probability to win the one-stage all-pay auction and his probability to win the best-of-three all-pay auction yields the following intuitive result.

Proposition 1 *If players do not have values of losing, the probability of the player with the higher ability (value) to win the best-of-three all-pay auction is higher than his probability to win the one-stage all-pay auction with the same values of winning.*

Assume now that players have a value of losing a in the one-stage all-pay auction as well as in both stages (2 and 3) of the best-of-three all-pay auction, that is, $a = b$. Then we have,

Proposition 2 *If players have the same value of losing in both stages ($a = b$), then the probability of the player with the higher ability to win increases in the value of losing, and in particular, it is higher than his probability to win the one-stage all-pay auction with the same values of winning and losing.*

It can be shown that even when the players' values of losing in both stages of the best-of-three all-pay auction are not identical ($a > b$), the probability that the strong player wins is still larger than his probability to win the one-stage all-pay auction. In other words, the values of losing do not have a significant effect on the players' probabilities to win in the best-of-three all-pay auction. On the other hand, the values of losing have a significant effect on the players' total effort as we show in the following section.

6 Total effort

By the analysis of the subgame-perfect equilibrium of the best-of-three all-pay auction in Section 4, the total effort for the different stages is as follows:

The total effort in the third stage is

$$(\frac{v_2 - a_2}{2})(1 + \frac{v_2 - a_2}{v_1 - a_1})$$

which is obtained with the probability of

$$(\frac{2v_2 - v_1 - a_2 + b_1 - b_2}{2(v_1 - a_2 + b_2 - b_1)})(\frac{v_1 - v_2 + a_2 - b_1}{2(v_2 - a_2)}) + (1 - \frac{2v_2 - v_1 - a_2 + b_1 - b_2}{2(v_1 - a_2 + b_2 - b_1)})(\frac{a_2 - b_2}{2(v_2 - a_2)})$$

If player 1 wins the first match, the total effort in the second stage is

$$(\frac{a_2-b_2}{2})(1+\frac{a_2-b_2}{v_2-a_2})$$

and this effort is obtained with the probability of

$$1-\frac{2v_2-v_1-a_2+b_1-b_2}{2(v_1-a_2+b_2-b_1)}$$

Otherwise, if player 2 wins the first match, the total effort in the second stage is

$$(\frac{v_1-v_2+a_2-b_1}{2})(1+\frac{v_1-v_2+a_2-b_1}{v_2-a_2})$$

and this effort is obtained with the probability of

$$\frac{2v_2-v_1-a_2+b_1-b_2}{2(v_1-a_2+b_2-b_1)}$$

The total effort in the first stage is

$$(\frac{2v_2-v_1-a_2+b_1-b_2}{2})(1+\frac{2v_2-v_1-a_2+b_1-b_2}{v_1-a_2+b_2-b_1})$$

Hence, the total effort in the best-of-three all-pay auction is given by

$$\begin{aligned} TE &= (\frac{v_2-a_2}{2})(1+\frac{v_2-a_2}{v_1-a_1})(\frac{2v_2-v_1-a_2+b_1-b_2}{2(v_1-a_2+b_2-b_1)})(\frac{v_1-v_2+a_2-b_1}{2(v_2-a_2)}) \\ &+(\frac{v_2-a_2}{2})(1+\frac{v_2-a_2}{v_1-a_1})(1-\frac{2v_2-v_1-a_2+b_1-b_2}{2(v_1-a_2+b_2-b_1)})(\frac{a_2-b_2}{2(v_2-a_2)}) \\ &+(\frac{a_2-b_2}{2})(1+\frac{a_2-b_2}{v_2-a_2})(1-\frac{2v_2-v_1-a_2+b_1-b_2}{2(v_1-a_2+b_2-b_1)}) \\ &+(\frac{v_1-v_2+a_2-b_1}{2})(1+\frac{v_1-v_2+a_2-b_1}{v_2-a_2})(\frac{2v_2-v_1-a_2+b_1-b_2}{2(v_1-a_2+b_2-b_1)}) \\ &+(\frac{2v_2-v_1-a_2+b_1-b_2}{2})(1+\frac{2v_2-v_1-a_2+b_1-b_2}{v_1-a_2+b_2-b_1}) \end{aligned} \quad (9)$$

A comparison of the total effort in the standard one-stage all-pay auction and in the best-of-three all-pay auction yields the following unambiguous result:

Proposition 3 *If players do not have values of losing, the total effort in the best-of-three all-pay auction with heterogenous players is smaller than the total effort in the one-stage all-pay auction with the same values of winning.*

Assume now that players have the value of losing a in the one-stage all-pay auction and the same value in both stages of the best-of-three all-pay auction, that is, $a = b$. Then we have,

Proposition 4 *If players have the same value of losing in both stages (a=b), then the total effort in the best-of-three all-pay auction with heterogenous players is smaller than the total effort in the one-stage all-pay auction with the same values of winning and losing.*

However, if the values of losing are not the same over the stages in the best-of-three all-pay auction, the comparison of the total effort in this contest and the one-stage all-pay auction is more complex. The following result shows the effects of both values of losing on the players' total effort in the best-of-three all-pay auction.

Proposition 5 *Assume that players are homogeneous. Then, the players' total effort in the best-of-three all-pay auction decreases in the value of losing in the second stage and increases in the value of losing in the third stage. If the value of losing is the same in both stages, the total effort decreases in the value of losing and, in particular, it is equal to the total effort in the one-stage all-pay auction with the same values of winning and losing.*

Assume that players are homogeneous. Also assume that the value of losing in the one-stage all-pay auction is either 0 or $a > 0$, and the value of losing in the best of three all-pay auction is 0 in the second stage and a in the third stage. Then the total effort in the one-stage all-pay auction is smaller or equal to the total effort without a value of losing, and by Proposition 5 the total effort in the best-of-three all-pay auction is larger than the total effort without values of losing. Since without values of losing both forms of the contest yield the same total effort, we obtain that

Conclusion 1 *If players are homogeneous, the total effort in the best of three all-pay auction may be larger than the total effort in the one-stage all-pay auction where the values of losing in the best-of-three all-pay auction may be larger or smaller than the value of losing in the one-stage all-pay auction.*

Because of the continuity of the players' total effort function in the asymmetry parameter $h = \frac{c_2}{c_1}$, it is clear that the above result according to which the expected total effort in the best-of-three all-pay auction may be larger than the total effort in the one-stage all-pay auction holds even for heterogenous players.

7 Concluding remarks

We studied the best-of-three all-pay auction where each match is modeled as an all-pay auction. We showed that if players do not have values of losing (the values of losing are equal to zero) and these players are heterogenous, the best-of-three all-pay auction is inferior to the standard one-stage all-pay auction from two points of view: 1) the difference between the players' probabilities of winning is always larger in the best-of-three all-pay auction than in the standard one-stage all-pay auction, and 2) the total effort in the best-of three all-pay auction is always smaller than in the standard one-stage all-pay auction. From these points of view, if players are homogenous, the best-of-three all-pay auction and the one-stage all-pay auction are equivalent. This superiority of the one-stage all-pay auction over the best-of-three all-pay auction holds even if players have values of losing when they are the same over all the stages of the best-of-three all-pay auction. However, if the values of losing are not the same, the best-of-three all-pay auction is not necessarily inferior to the one-stage all-pay auction and the total effort in the best-of-three all-pay auction might be larger than in the one-stage all-pay auction. Our findings explain why best-of-k contests are sometimes preferred. We may conclude that if the contest designer wishes to maximize players' total effort, the choice between best-of-k contests and one-stage contests could depend on the players' values of losing.

8 Appendix

8.1 Proof of Proposition 1

Without loss of generality, assume that $v_1 = 1$. Denote by $h = \frac{c_2}{c_1} = \frac{v_2}{v_1} = \frac{a_2}{a_1} = \frac{b_2}{b_1} > \frac{1}{2}$, $a_1 = a$ and $b_1 = b$.

Then, by (8), player 1's probability to win the best-of-three all-pay auction is given by

$$\begin{aligned} p_1 &= (1 - \frac{2h-1-ah+b(1-h)}{2(1-ah+b(h-1))})(1 - \frac{a-b}{2(1-a)}) \\ &+ (\frac{2h-1-ah+b(1-h)}{2(1-ah+b(h-1))})(\frac{1-h+ah-b}{2(h-a)})(1-\frac{h}{2}) \\ &+ (1 - \frac{2h-1-ah+b(1-h)}{2(1-ah+b(h-1))})(\frac{a-b}{2(1-a)})(1-\frac{h}{2}) \end{aligned} \tag{10}$$

When the values of losing approach zero we obtain,

$$\lim_{a,b\to 0} p_1(a,b) = (1 - \frac{(2h-1)}{2}) + (\frac{(2h-1)}{2})(\frac{(1-h)}{2h})(1-\frac{h}{2})$$

By (1), the probability of player 1 to win the one-stage all-pay auction is

$$p_1^* = 1 - \frac{h}{2}$$

Thus,

$$\begin{aligned} p_1 - p_1^* &= (1 - \frac{(2h-1)}{2}) + (\frac{(2h-1)}{2})(\frac{(1-h)}{2h})(1-\frac{h}{2}) - (1-\frac{h}{2}) \\ &= \frac{1}{2h}(1-h)\left(h-\frac{1}{2}\right)\left(1-\frac{1}{2}h\right) + \frac{1}{2}(1-h) \end{aligned} \tag{11}$$

Note that by our assumption (3) $h > \frac{1}{2}$ and then $p_1 - p_1^* > 0$. *Q.E.D.*

8.2 Proof of Proposition 2

Denote $a_1 = a$ and $b_1 = b$ and suppose that $a = b$. Then, by (8), player 1's probability to win the best-of-three all-pay auction is given by

$$p_1 = -\frac{1}{8a-8h}\left(19h - 10a + 7ah^2 - 2ah^3 + ah - 15h^2 + 2h^3 - 2\right)$$

The derivative of player 1's probability to win is

$$\frac{dp_1(a)}{da} = -\frac{1}{8(a-h)^2}(h-1)^2\left(2h^2 - 5h + 2\right) \tag{12}$$

Note that since $2h^2 - 5h + 2 < 0$ for all $0.5 < h \leq 1$, by (12) we obtain that player 1's probability to win the best-of-three all-pay auction increases in a. This argument together with Proposition 1 and the fact that the value of losing does not affect the probability to win in the one-stage all-pay auction yields the desirable result. *Q.E.D.*

8.3 Proof of Proposition 3

Without loss of generality, assume that $v_1 = 1$. Let $h = \frac{v_2}{v_1} = \frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} > \frac{1}{2}$, $a_1 = a$ and $b_1 = b$. Then, by (9), the total effort in the best-of-three all-pay auction is

$$\begin{aligned} TE &= (\frac{h(1-a)}{2})(1+h)((\frac{2h-1-ah+b(1-h)}{2(1-ah+b(h-1))})(\frac{1-h+ah-b}{2h(1-a)}) + (1 - \frac{2h-1-ah+b(1-h)}{2(1-ah+b(h-1))})(\frac{a-b}{2(1-a)})) \\ &+(\frac{(a-b)h}{2})(1+\frac{a-b}{(1-a)})(1-\frac{2h-1-ah+b(1-h)}{2(1-ah+b(h-1))}) \\ &+(\frac{1-h+ah-b}{2})(1+\frac{1-h+ah-b}{h(1-a)})(\frac{2h-1-ah+b(1-h)}{2(1-ah+b(h-1))}) \\ &+(\frac{2h-1-ah+b(1-h)}{2})(1+\frac{2h-1-ah+b(1-h)}{1-ah+b(h-1)}) \end{aligned} \tag{13}$$

When the values of losing approach zero, the total effort is

$$\begin{aligned} \lim_{a,b\to 0} TE(a,b) &= \frac{h}{2}(1+h)(\frac{2h-1}{2})(\frac{1-h}{2h}) \\ &+(\frac{1-h}{2})(1+\frac{1-h}{h})(\frac{2h-1}{2}) \\ &+(\frac{2h-1}{2})2h \end{aligned}$$

By (2), the total effort in the standard one-stage all-pay auction is

$$TE^* = \frac{h}{2}(1+h)$$

Thus, the difference of the total efforts in both contests is

$$\begin{aligned} TE - TE^* &= \frac{h}{2}(1+h)(\frac{2h-1}{2})(\frac{1-h}{2h}) + (\frac{1-h}{2})(1+\frac{1-h}{h})(\frac{2h-1}{2}) + (\frac{2h-1}{2})2h - \frac{h}{2}(1+h) \\ &= \frac{13}{8}h^2 - \frac{5}{4}h - \frac{1}{4}h^3 + \frac{1}{4h}\left(3h - 2h^2 - 1\right) - \frac{1}{8} = \frac{13h^3 - 14h^2 - 2h^4 + 5h - 2}{8h} \end{aligned}$$

It can be verified that $\Delta(h) = 13h^3 - 14h^2 - 2h^4 + 5h - 2 < 0$ for all $0.5 < h < 1$ and $\Delta(h) = 0$ for $h = 1$. *Q.E.D.*

8.4 Proof of Proposition 4

By (9) and (2), if $a = b$, the difference of the total effort in both contests is given by

$$\Delta = TE(a = b) - TE^{*} = -\frac{1}{8h}\left(13ah^3 - 5h - 14ah^2 - 2a - 2ah^4 + 5ah + 14h^2 - 13h^3 + 2h^4 + 2\right)$$

The first order derivative is

$$\frac{d\Delta}{dh} = \frac{1}{4h^2}\left(a-1\right)\left(3h^4 - 13h^3 + 7h^2 - 1\right)$$

It can be verified that $\left(3h^4 - 13h^3 + 7h^2 - 1\right) < 0$ for all $0.5 < h \leq 1$, and since $a < 1$, we have $\frac{d\Delta}{dh} > 0$ for all $0.5 < h \leq 1$.

Note that $\Delta(h = 1) = 0$. Since $\frac{d\Delta}{dh} > 0$, we obtain that $\Delta(h) < 0$ for all $0.5 < h < 1$ and $\Delta(h) = 0$ for $h = 1$. *Q.E.D.*

8.5 Proof of Proposition 5

Let h approach 1 in (13). Then since $a, b < 1$ we obtain,

$$\frac{d}{da}\lim_{h\to 1} TE(a,b) = \frac{1}{2\left(a-1\right)^2}\left(-a^2 + 2a + b^2 - 2b\right) = \frac{1}{2\left(a-1\right)^2}(a-b)(2-a-b) > 0$$

and,

$$\frac{d}{db}\lim_{h\to 1} TE(a,b) = \frac{1}{1-a}\left(b-1\right) < 0$$

If $a = b$, we obtain that the total effort of homogenous players in the best-of three all-pay auction is the same as in the one-stage all-pay auction and is given by

$$\lim_{h\to 1} TE(b = a) = 1 - a$$

Q.E.D.

References

[1] Akerlof, R., Holden, R.: The Nature of Tournaments. Working paper (2008)

[2] Barut, Y., Kovenock, D.: The Symmetric Multiple Prize All-Pay Auction with Complete Information. European Journal of Political Economy 14, 627-644 (1998)

[3] Baye, M., Kovenock, D., de Vries, C.: The All-Pay Auction with Complete Information. Economic Theory 8, 291-305 (1996)

[4] Baye, M., Kovenock, D., de Vries, C.: Contests with Rank-Order Spillover. Working paper (2007)

[5] Clark, D., Riis, C.: Competition over More than One Prize. American Economic Review 88, 276-289 (1998)

[6] Groh, C., Moldovanu, B., Sela, A., Sunde, U.: Optimal Seedings in Elimination Tournaments. Economic Theory, forthcoming (2009)

[7] Hillman, A., Riley, J.: Politically Contestable Rents and Transfers. Economics and Politics 1, 17-39 (1989)

[8] Gradstein, M., Konrad, K.: Orchestrating Rent Seeking Contests. The Economic Journal 109, 536-545 (1999)

[9] Klumpp, T., Polborn, M.: Primaries and the New Hampshire Effect. Journal of Public Economics, 90, 1073-1114 (2006)

[10] Konrad, K.: Bidding in Hierarchies. European Economic Review 48, 1301-1308 (2004)

[11] Konrad, K., Kovenock, D.: Equilibrium and Efficiency in the Tug-of-War. CEPR Discussion Paper, (2005)

[12] Malueg, D., Yates, A.: Best-of Three Contests Between Equally-Skilled Players. Working paper (2006)

[13] Moldovanu, B., Sela, A.: The Optimal Allocation of Prizes in Contests. American Economic Review 91, 542-558 (2001)

[14] Moldovanu, B., Sela, A.: Contest Architecture. Journal of Economic Theory 126(1), 70-97 (2006)

[15] Moldovanu, B., Sela, A., Shi, X.: Carrots and Sticks: Prizes and Punishments in Contests. Working paper (2008)

[16] Rosen, S.: Prizes and Incentives in Elimination Tournaments. American Economic Review 76, 701-715 (1986)

[17] Szymanski, S., Valletti, T.: First and Second Prizes in Imperfectly Discriminating Contests. European Journal of Political Economy 21, 467-481 (2005)

[18] Warneryd, K.: Distributional Conflict and Jurisdictional Organization. Journal of Public Economics 69, 435-450 (1998)