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Value-at-Risk Calculations with Time Varying Copulae

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Value-at-Risk (VaR) of a portfolio is determined by the multivariate distribution of the risk factors increments. This distribution can be modelled through copulae, where the copulae parameters are not necessarily constant over time. For an exchange rate portfolio, copulae with time varying parameters are estimated and the VaR simulated accordingly. Backtesting underlines the improved performance of time varying copulae.

Value-at-Risk and Copulae

At time t a linear portfolio composed of d positions $w = (w_1, \dots, w_d)^\top$ on assets with prices $S_t = (S_{1,t}, \dots, S_{d,t})^\top$ and log prices $Z_t = \ln S_t$, has value

(1) $$V_t = \sum_{j=1}^{d} w_j e^{Z_{j,t}}$$

The *profit and loss* (P&L) function is defined as $L_{t+1} = (V_{t+1} - V_t)$. Defining $X_{t+1} = (Z_{t+1} - Z_t)$ as the time increment in the risk factors from period t to $t+1$, the P&L can be expressed as:

(2) $$L_{t+1} = \sum_{j=1}^{d} w_j S_{j,t}(e^{X_{j,t+1}} - 1)$$

The *Value-at-Risk* (VaR) is calculated as the α-quantile from F_L, the distribution of L:

(3) $$VaR = F_L^{-1}(\alpha)$$

The 1-dimensional distribution F_L depends on the d-dimensional distribution F_X. Using *copulae*, the marginal distributions F_{X_j} from each univariate increment can be *separately* modelled from their dependence structure and then *coupled* together to form the multivariate distribution F_X.

In the following, the dependence parameter $\hat{\theta}$ and the joint distribution $\hat{F}_X$ from a sample $\{X_t\}_{t=1}^{T}$ of log returns from exchange rate positions are estimated with copulae. A Monte Carlo simulation based on $\hat{F}_X$ generates different P&L samples. The quantiles at different levels from these simulation samples are then used as estimators for the Value-at-Risk of differem portfolio.

Computing Value-at-Risk with Copulae

A copula is a d-dimensional distribution function $C : [0,1]^d \to [0,1]$ with uniform marginals on the interval $[0,1]$. As in Nelsen (1998), multivariate distributions can be modelled via:

Theorem 1 (Sklar's theorem) *Let F be a d-dimensional distribution function with marginals $F_{X_1} \dots, F_{X_d}$. Then there exists a copula C with*

(4) $$F(x_1, \dots, x_d) = C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\}$$

for every $x_1, \dots, x_d \in \overline{\mathbb{R}}$. If $F_{X_1}, \dots, F_{X_d}$ are continuous, then C is unique. On the other hand, if C is a copula and $F_{X_1}, \dots, F_{X_d}$ are distribution functions, then the function F defined in (4) is a joint distribution function with marginals $F_{X_1}, \dots, F_{X_d}$.

The estimation of the Value-at-Risk, based on an *i.i.d.* sample $\{X_t\}_{t=1}^T$ is implemented in the following procedure:

1. specification of marginal distributions $F_{X_j}(x_j)$
2. specification of copula $C(u_1, \dots, u_d; \theta)$
3. fitting the copula C to obtain $\hat{\theta}$
4. generation of Monte Carlo data $X_{T+1} \sim C(u_1, \dots, u_d; \hat{\theta})$
5. generation of a sample of portfolio losses $L_{T+1}(X_{T+1})$
6. estimation of $\widehat{VaR}_{T+1}$, the empirical quantile at level α from $L_{T+1}(X_{T+1})$.

For copulae belonging to a parametric family $C = \{C_\theta, \theta \in \Theta\}$ and univariate marginals $F_{X_j}(x_j; \delta_j)$, the density of X is given by:

$$f(x_1, \dots, x_d; \delta_1, \dots, \delta_d, \theta) = c\{F_{X_1}(x_1; \delta_1), \dots, F_{X_d}(x_d; \delta_d); \theta\} \prod_{j=1}^{d} f_j(x_j; \delta_j)$$

where

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d}$$

In the IFM (*inference for margins*) method, the log-likelihood function for each of the marginal distributions

$$\ell_j(\delta_j) = \sum_{t=1}^{T} \ln f_i(x_{j,t}; \delta_j), j = 1, \dots, d \tag{5}$$

is maximized to obtain estimates $(\hat{\delta}_1, \dots, \hat{\delta}_d)^\top$. The function

$$\ell(\theta, \hat{\delta}_1, \dots, \hat{\delta}_d) = \sum_{t=1}^{T} [\ln c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \dots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}] \tag{6}$$

is then maximized over θ to get the dependence parameter estimate $\hat{\theta}$. The estimates $\hat{\theta}_{IFM} = (\hat{\delta}_1, \dots, \hat{\delta}_d, \hat{\theta})^\top$ solve

$$(\partial \ell_1 / \partial \delta_1, \dots, \partial \ell_d / \partial \delta_d, \partial \ell / \partial \theta) = 0$$

Backtesting

This procedure is applied to a daily exchange rate portfolio (DEM/USD and GBP/USD from 01.12.1979 to 01.04.1994) with $T = 250$. The univariate risk factor increments (log returns) are assumed to be Gaussian distributed with parameters estimated from the data. The selected copulae belong to the bivariate one-parametric Gumbel family:

$$C(u, v) = \exp(-[(\ln u)^\theta + (\ln v)^\theta]^{\theta^{-1}}), 1 \leq \theta \leq \infty \tag{7}$$

To evaluate the performance of the copula in the VaR calculations, different portfolio compositions are used to generate P&L samples. The quantiles from the samples at four levels $\alpha_1 = 0.05$, $\alpha_2 = 0.01$, $\alpha_3 = 0.005$ and $\alpha_4 = 0.001$ are used as estimators for VaR.

The estimated VaR is compared with the realization of the P&L function, an *exceedance* occuring for each P&L value smaller than the estimated VaR. The ratio of the number of exceedances to the number of observations gives the empirical level $\hat{\alpha}$.

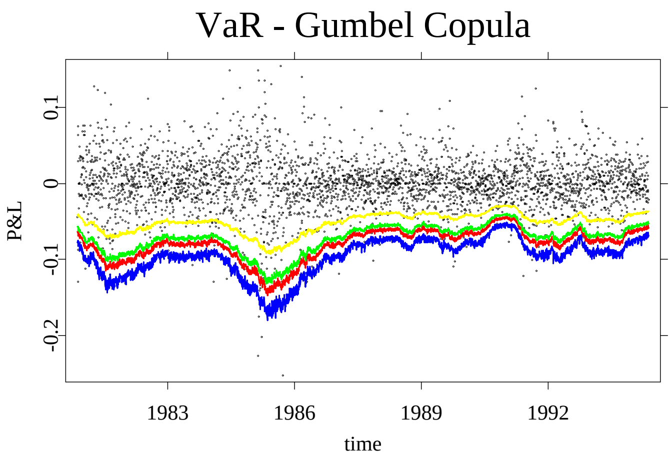

Figure 1: Value-at-Risk at levels $\alpha_1 = 0.05$ (yellow), $\alpha_2 = 0.01$ (green), $\alpha_3 = 0.005$ (red), and $\alpha_4 = 0.001$ (blue), P&L (black), estimated at each time from a Monte Carlo sample of 10.000 P&L values generated with Gumbel copula, $w = (2, 1)^\top$.

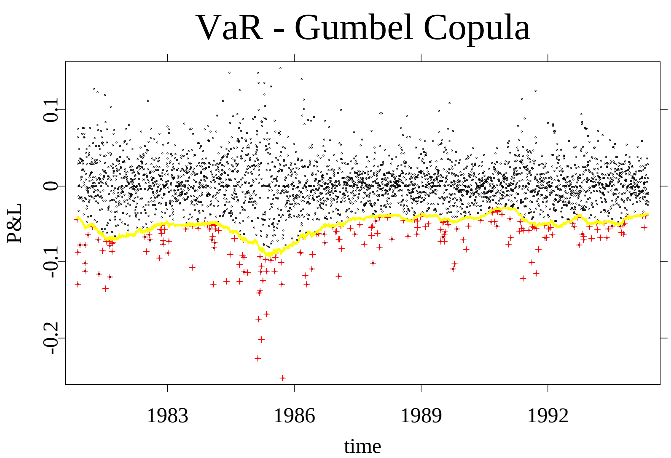

Figure 2: Value-at-Risk at level $\alpha = 0.05$ (yellow), P&L (black) and exceedances (red), $\hat{\alpha} = 0.0573$, $w = (2, 1)^\top$. P&L samples generated with Gumbel copula.

Table 1: Gumbel copula, empirical levels $\hat{\alpha}$ for different portfolios.

	level $\alpha(\times 10^2)$			
	5	1	0.5	0.1
Portfolio $w^\top$	empirical level $\hat{\alpha}(\times 10^2)$			
$(1,1)$	6.05	2.45	1.75	0.83
$(1,2)$	6.34	2.74	1.75	1.00
$(2,1)$	5.73	2.24	1.58	0.69
$(2,3)$	6.22	2.56	1.75	0.92
$(3,2)$	5.99	2.30	1.55	0.74
$(-1,2)$	1.64	0.37	0.20	0.11
$(1,-2)$	2.01	0.51	0.43	0.11
$(-2,1)$	4.44	1.49	0.95	0.40
$(2,-1)$	4.09	1.35	1.09	0.49

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RÉSUMÉ

La Value at Risk (VaR) d'un portefeuille est déterminée par la distribution multivariée des incréments des facteurs de risques. Cette distribution peut être modélisée par des copules dont les paramètres ne sont pas nécessairement constants par rapport au temps. Pour un portefeuille de taux de change, des copules dépendant du temps sont estimés et la VaR est ainsi simulée. Le backtesting confirme l'amélioration apportée par les copules dépendant du temps.

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