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ALLOCATION PROBLEMS WITH INDIVISIBILITIES WHEN PREFERENCES ARE SINGLE-PEAKED*

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Abstract

We consider allocation problems with indivisible goods when agents' preferences are single-peaked. In this paper we identify the family of efficient, fair and non-manipulable solutions. We refer to such a family as *M-temporary satisfaction methods*. Besides, we provide arguments to defend these methods as extensions to the indivisible case of the so-called *uniform rule*.

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1 Introduction

We face up in this paper the problem of allocating an amount of indivisible units of an homogeneous good among a group of agents whose preferences are single-peaked. Consider the following situation. At a health center there are some shifts to cover by doctors. Each of these doctors has a *single-peaked* preference over worked hours. This means that she has a most preferred amount of hours to work. And, if she has to work more than this preferred amount, the less the better. Analogously, if she has to work less than this preferred amount the more the better. Similar problems arise in the allocation of crew members to flights or teaching hours to faculty members, among other examples.

The above situation is a particular instance of a general set of problems called *allocation problems with indivisibilities when preferences are single-peaked*. These problems come described by three elements. First, a set of *agents*. Second, an amount of indivisible units to distribute, called *task*. And third, a profile of *single-peaked preferences* over the number of units to consume. A *rule or solution* is a mechanism to distribute the task among the agents according to their preferences.

In the axiomatic method, solutions are justified in terms of the properties they fulfil, and, in general, suitable combinations of different desirable properties are used to differentiate among rules. In this paper we identify the class of rules that, applied to the allotment of indivisible units, are efficient, fair, non-manipulable, and robust. We refer to such a family of rules as *M-temporary satisfaction methods*. These methods are defined by using *monotonic standards of comparison* defined over the cartesian product potential agents-integer numbers.[1] *M-temporary satisfaction methods* start by giving all agents their preferred consumptions, and then move away from this provisional allocation, unit by unit by using the standard. Here, the numbers paired with the agents are interpreted either as agents' peaks or the opposite peaks, depending upon the type of problem at hand (either an excess demand or excess supply problem).

By far, the best-known rule in the continuous case, when the task is perfectly divisible, is the *uniform rule*, introduced in Sprumont (1991). It proposes to treat all agents as equally as possible, subject to efficiency. Characterizations of this rule also appear in Ching (1994), Sönmez (1994), Thomson (1994a,b), and Dagan (1996), among others. The uniform rule is efficient, equitable, and non-manipulable. It is worth noting that *M-temporary satisfaction methods* are close to the idea inspiring the *uniform rule*. This statement is supported by two facts. First, *M-temporary satisfaction methods* can be characterized by combinations of properties similar to those supporting the characterizations of the *uniform rule*. And second, for any problem, the allocation prescribed by the *uniform rule* is the ex-ante expectation of the agents under the application of *M-temporary satisfaction methods*, if all plausible monotonic standards are equally likely.

The rest of the paper is structured as follows: In Section 2 we set up the model. In Section 3 we analyze the properties our families of rules may fulfil. In Section 4 we introduce standards of

[1] This subclass of standards of comparison has been formulated by Herrero and Martínez (2006) in the context of claims problems with indivisibilities. The general class of standards was introduced by Young (1994).

comparison and use them to construct an allotment procedure (*temporary satisfaction methods*); we present our characterization result as well. In Section 5 we establish the connections between the *M-temporary satisfaction methods* and the *uniform rule*. Finally, in Section 6 we conclude with some final remarks. Examples providing the tightness of the characterizations are relegated to Appendix A, while proofs are in Appendix B.

2 Statement of the model

A preference relation, R, defined over $\mathbb{Z}_+$ is **single-peaked** if there exists an integer number $p(R) \in \mathbb{Z}_+$ (called the **peak** of R) such that, for each $a, b \in \mathbb{Z}_+$,

$$aPb \Leftrightarrow [(b < a < p(R)) \text{ or } (p(R) < a < b)],$$

where P is the strict preference relation induced by R. Let $\mathbb{S}$ denote the class of all single-peaked preferences defined over $\mathbb{Z}_+$. Let $\mathbb{N}$ be the set of all potential agents and $\mathcal{N}$ be the family of all finite non-empty subsets of $\mathbb{N}$.

An allocation problem with single-peaked preferences, or simply a **problem**, is a triple $\boldsymbol{e = (N, T, R)}$ in which a fixed number of units T (called **task**) has to be distributed among a group of **agents**, $N \in \mathcal{N}$, whose **preferences** over consumption are **single-peaked**, $R = (R_i)_{i \in N} \in \mathbb{S}^N$. Let $\mathbb{A}^N$ denote the class of problems with fixed-agent set N, and $\mathbb{A}$ the class of all problems, that is,

$$\mathbb{A}^N = \left\{e = (N, T, R) \in \{N\} \times \mathbb{Z}_+ \times \mathbb{S}^N\right\}$$

and

$$\mathbb{A} = \bigcup_{N \in \mathcal{N}} \mathbb{A}^N.$$

For each problem, we face the question of finding a division of the task among the agents. An **allocation** for $e \in \mathbb{A}$ is a list of integer numbers, $\boldsymbol{x} \in \mathbb{Z}_+^N$, satisfying the condition of being a complete distribution of the task, i.e., $\sum_{i \in N} x_i = T$. Let $\boldsymbol{X(e)}$ be the set of all allocations for $e \in \mathbb{A}$. A **rule** is a function, $\boldsymbol{F} : \mathbb{A} \longrightarrow \mathbb{Z}_+^N$, that selects, for each problem $e \in \mathbb{A}$, a unique allocation $F(e) \in X(e)$.

3 Properties

Our goal in this paper is to obtain *efficient*, *fair*, *non-manipulable*, and *robust* rules to solve allocation problems with indivisibilities when preferences are single-peaked.

One basic requirement for a rule to satisfy is *efficiency*. An allocation is *efficient* if there is no other allocation in which all the agents are better off. *Efficiency* requires the rule to select efficient allocations.

Efficiency: For each $e \in \mathbb{A}$, there is no allocation $x \in X(e)$ such that, for each $i \in N$, $x_i R_i F_i(e)$, and for some $j \in N$, $x_j P_j F_j(e)$.

As Sprumont (1991) points out, the principle of efficiency is equivalent to asking for each agent to consume, no more than her peak when the task is too little, and no less than her peak when the task is too much. Equivalently, if F is efficient, $F_i(e) \leq p(R_i)$ for each $i \in N$ if $\sum_{i \in N} p(R_i) \geq T$, and $F_i(e) \geq p(R_i)$ for each $i \in N$ if $\sum_{i \in N} p(R_i) \leq T$.

In any rationing framework, a minimal fairness condition is always desirable. The *equal treatment of equals* principle says that agents with identical preferences should be indifferent among their respective allocations. Paired with the requirement of efficiency, it simply means that agents with identical preferences should be allotted the same amount. Unfortunately, no rule can fulfill equal treatment of equals in the context of problems with indivisibilities. To illustrate such a fact it is enough to consider a two-agent problem with identical preferences and only one unit to assign. Young (1994), and Herrero and Martínez (2006) formulate a milder version of this condition: *balancedness*. It postulates that equal agents should be treated, if not equal, at least as equal as possible. Balancedness requires the awards of equal agents to differ, at most, by one unit (representing this unit the size of the indivisibility).

Balancedness: For each $e \in \mathbb{A}$ and each $\{i, j\} \subseteq N$, if $R_i = R_j$ then $|F_i(e) - F_j(e)| \leq 1$.

In this class of problems is quite common that agents report their preferences directly to a central planer. This opens the door to some possible manipulation from agents by lying about their preferences. Next property prevents this type of manipulation to occur. *Strategy-proofness* requires that truthtelling should be a (weakly) dominant strategy for all agents, or, in other words, that no agent should benefit from misrepresenting her preference.[2]

Strategy-proofness: For each $e = (N, T, (R_i, R_{-i})) \in \mathbb{A}$ and each $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}$, $F_i(e) R_i F_i(e')$.

Next property refers to an stability condition with respect to changes in population. Suppose that, after solving the problem $e = (N, T, R) \in \mathbb{A}$, a proper subset of agents, $S \subset N$, decides to reallocate the total amount they have received, that is, they face a new allocation problem: $(S, \sum_{i \in S} a_i, R_S)$, where $R_S = (R_i)_{i \in S}$ and a is the allocation corresponding to apply the rule to the problem e. *Consistency* requires no agent to take advantage from this change in the population, i.e., each agent $i \in S$ receives the same amount of units in problem $(S, \sum_{i \in S} a_i, R_S)$ as she did in problem e. In other words, the new reallocation is only a restriction to the subset S of the initial one.[3]

Consistency: For each $e \in \mathbb{A}$, each $S \subset N$, and each $i \in S$, $F_i(e) = F_i(S, \sum_{j \in S} F_j(e), R_S)$.

[2]The notation R_{-i} refers to the preference profile R where agent i has been deleted, i.e., $R_{-i} = R_{N \setminus \{i\}}$.

[3]The reader is referred to Thomson (2004) for a widely exposition of consistency and its converse.

4 A family of rules. Temporary satisfaction methods

We face now the question of finding rules satisfying the properties formulated in Section 3.

A standard of comparison is a linear order (complete, antisymmetric and transitive binary relation) over the cartesian product potencial agent-integer number, $\mathbb{N} \times \mathbb{Z}$, such that, for each agent, larger integer numbers have priority over smaller integer numbers.[4]

Standard of comparison $\sigma : \mathbb{N} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ such that, for each $i \in \mathbb{N}$, and each $a \in \mathbb{Z}$, $\sigma(i, a+1) < \sigma(i, a)$. Let Σ denote the class of all standards of comparison.[5]

By using standards of comparison, we can construct rules to solve allocation problems. We may accommodate all units of either excess demand or excess supply one by one, after giving (temporarily) all agents their peaks. We call those rules *temporary satisfaction methods.*

Let $\{(i, a_i)\}_{i \in M}$ be a collection of pairs agent-number. Let $\sigma \in \Sigma$ be a standard of comparison. The **pair with the highest priority** in $\{(i, a_i)\}_{i \in M}$, according to σ, is the pair (i, a_i) such that $\sigma(i, a_i) < \sigma(j, a_j)$ for all $j \in M \setminus \{i\}$.

Temporary satisfaction method associated to $\boldsymbol{\sigma}$, $\boldsymbol{TS^{\sigma}}$: Let $e \in \mathbb{A}$. Start by giving to each agent her peak. Now we distinguish two cases.

(1) If the task is too little, i.e., $\sum_{i \in N} p(R_i) \geq T$. In this case we have to remove some units from the temporary allocation. Associate to each agent her peak, that is, $a_i = p(R_i)$. Identify the pair with the highest priority according to σ. Subtract one unit from this agent (allocation), and reduce her number by one unit. Identify again the pair with the highest priority according to σ, and proceed in the same way until reaching the task.

(2) If the task is too large, i.e., $\sum_{i \in N} p(R_i) \leq T$. In this case we have to allocate some extra units, $T' = T - \sum_{i \in N} p(R_i)$. We shall proceed in the following way. Associate to each agent the opposite of her peak, that is, $a_i = -p(R_i)$. Identify the pair with the highest priority according to σ. Then assign one unit of the remaining task, T', to this agent, and reduce her number by one unit. Identify again the pair with the highest priority according to σ, and proceed in the same way until the task T' runs out.

Next example illustrates how temporary satisfaction methods work.

Example 4.1. Assume that the standard of comparison σ is such that, restricted to agents in $N = \{1, 2, 3\}$, it happens that $\sigma(2, x) < \sigma(1, y) < \sigma(3, z)$, for all $x, y, z \in \mathbb{Z}$. Now, consider the allocation problem where $N = \{1, 2, 3\}$, $T = 6$, and $R = (R_1, R_2, R_3)$ such that $R_2 = R_3$ and $p(R) = (1, 5, 5)$. Note that, in this case, $\sum_{i \in N} p(R_i) > T$.

We start by fully satisfying all agents, that is, by giving to each agent her peak. This implies allocating 11 units, but we only have 6 units available. Thus we need to remove 5 units. To do

[4]The notion of standard of comparison was formulated by Young (1994).

[5]If $\sigma(i, a) < \sigma(j, b)$ we will understand that the pair (i, a) has priority over the pair (j, b).

that we procede as follows. Since we are in case (1) ($\sum_{i\in N} p(R_i) = 11 \geq 6 = T$), we identify each agent with her peak, that is, we consider the pairs $(1,1)$, $(2,5)$, and $(3,5)$. According to σ, the pair with the highest priority is $(2,5)$. Then we subtract one unit from agent 2, and we now consider the new pairs $(1,1)$, $(2,4)$, and $(3,5)$. According to σ, the pair with the highest priority is $(2,4)$. Then we subtract one unit from agent 2, we consider the new pairs $(1,1)$, $(2,3)$, and $(3,5)$. The table shows the rest of the procedure until the 5 units have been removed. The first column shows the kth unit of the task. We start from 11 units and we remove one by one up to reach 6 units. The second column shows the allocation up to that unit, $x^{(k)}$. The third column shows the updated vector of numbers, $a^{(k)}$.

T	$x^{(k)}$	$a^{(k)}$
11	(1,5,5)	(1,5,5)
10	(1,4,5)	(1,4,5)
9	(1,3,5)	(1,3,5)
8	(1,2,5)	(1,1,5)
7	(1,1,5)	(1,1,5)
6	(1,0,5)	(1,0,5)

Imagine now that the task were $T = 14$. In this case, $T = 14 > 11 = \sum_{i\in N} p(R_i)$. Again, we start by giving to each agent her peak. This implies allocating 11 units, but there are $T' = 3(= T - (p(R_1) + p(R_2) + p(R_3)))$ remaining units to allot. Since we are in case (2), we identify each agent with her opposite peak, that is, we consider the pairs $(1,-1)$, $(2,-5)$, and $(3,-5)$. According to σ, the pair with the highest priority is $(2,-5)$. Then we give one unit to agent 2, and we now consider the new pairs $(1,-1)$, $(2,-6)$, and $(3,-5)$. According to σ, the pair with the highest priority is $(2,-6)$. Then we give one unit to agent 2 and we consider the new pair $(1,-1)$, $(2,-7)$, and $(3,-5)$. The table shows the rest of the procedure until the $T' = 3$ units have been assigned.

T'	$x^{(k)}$	$a^{(k)}$
0	(1,5,5)	(-1,-5,-5)
1	(1,6,5)	(-1,-6,-5)
2	(1,7,5)	(-1,-7,-5)
3	(1,8,5)	(-1,-8,-5)

Previous example illustrates how the temporary satisfaction methods work. Additionally, they show that these methods may violate balancedness. In Example 4.1 both the second and the third agent have identical preferences. Nevertheless the amount they receive are far away. To recover balancedness we have to concentrate in a particular subclass of standard of comparison.

Monotonic standard of comparison: For each $\{i,j\} \subseteq \mathbb{N}$, and each $x,y \in \mathbb{Z}$, if $x > y$, then $\sigma(i,x) < \sigma(j,y)$. Let Σ^M denote the subfamily of all monotonic standards of comparison.

In other words, monotonic standards of comparison always give priority to agents with larger integer numbers. Next example illustrates the allocation temporary satisfaction methods result in when a monotonic standard of comparison is imposed.

Example 4.2. Let $N = \{1,2,3\}$, and assume that the standard of comparison is monotonic. Furthermore, $\sigma(1,x) < \sigma(2,x) < \sigma(3,x)$ if x is odd, and $\sigma(1,x) < \sigma(3,x) < \sigma(2,x)$ if x is even. Now, as in Example 4.1, consider the allocation problem where $N = \{1,2,3\}$, $T = 6$, and $R = (R_1, R_2, R_3)$ such that $R_2 = R_3$ and $p(R) = (1,5,5)$. Applying the aforementioned procedure, we obtain that $TS^\sigma(e) = (1,2,3)$. If the number of units to allot were $T = 14$, then $TS^\sigma(e) = (4,5,5)$.

We call **M-temporary satisfaction methods** to the temporary satisfaction methods associated to monotonic standards of comparison. As Example 4.2 suggests, M-temporary satisfaction methods satisfy balancedness. Moreover, next result sets that only temporary satisfaction methods induced by a monotonic standard of comparison satisfy such a property.

Theorem 4.1. *Let $\sigma \in \Sigma$ be an standard of comparison. Then, a temporary satisfaction methods, TS^σ, satisfies balancedness if and only if σ is monotonic.*

As we mentioned above, our goal is to identify efficient, fair (balanced), non-manipulable (strategy proof), and robust (consistent) rules. In our main result, we obtain that there is only one family of rules compatible with these four requirements: M-temporary satisfaction methods. The proof, preceded by some lemmas, is relegated to Appendix B.

Theorem 4.2. *A rule F satisfies efficiency, balancedness, strategy proofness, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = TS^\sigma$.*

5 Relations between the discrete and the continuum

In the previous section we obtained a characterization for the family of M-temporary satisfaction methods. The properties used in such a result (Theorem 4.2) are very much related to those used by Ching (1994) to characterize the uniform rule.[6] We may interpret this fact as a suggestion of a relationship between our family of methods and the uniform rule. Any M-temporary satisfaction method can be interpreted as a discrete version of the uniform rule in the following sense. The allocation prescribed by the uniform rule is the ex-ante expectations of the agents under the application of M-temporary satisfaction methods, if all monotonic standard of comparison are equally likely.

[6] Under the assumption that the task were completely divisible, one of the most widely studied rule is the so-called uniform rule. The idea underlying this solution is equality distribution of the task.

Uniform rule, $\boldsymbol{u}$: For each $e \in \mathbb{A}$, selects the unique vector $u(e) \in \mathbb{R}^N$ such that: If $\sum_{i\in N} p(R_i) \geq T$, then $u(e) = \min\{p(R_i), \lambda\}$ for some $\lambda \in \mathbb{R}$ such that $\sum_{i\in N} \min\{p(R_i), \lambda\} = T$. And, if $\sum_{i\in N} p(R_i) \leq T$, then $u(e) = \max\{p(R_i), \lambda\}$ for some $\lambda \in \mathbb{R}$ such that $\sum_{i\in N} \max\{p(R_i), \lambda\} = T$.

Proposition 5.1. *Let $e \in \mathbb{A}$. Let Σ_e^M denote the subset of Σ^M of the different standards involved in problem e.*[7] *Then*

$$\frac{1}{|\Sigma_e^M|} \sum_{\sigma \in \Sigma_e^M} TS^\sigma(e) = u(e)$$

6 Final Remarks

In this work we have considered allocation problems with indivisible goods when the agents' preferences are single-peaked, that is, problems in which the task, the allocations and the preferences are only defined over the set of integer numbers. Our goal has been obtaining solutions fulfilling efficiency, fairness and non-manipulability conditions. To do that, we have defined the *temporary satisfaction methods*. These methods are *efficient*, *strategy-proof*, and *consistent*. Nevertheless, unlike the continuos model, in which we may impose *anonymity*, when indivisible units are allotted anonymous rules cannot be obtained. As Example 4.1 shows, in the model with indivisibilities, the simply requirement of *efficiency*, *strategy-proofness*, and *consistency* may result in extremely unfair allocations. In order to recover a minimal condition of equity we imposed *balancedness*. We find that the *M-temporary satisfaction methods* (*temporary satisfaction methods* associated to a *monotonic standard of comparison*) are the unique efficient, fair (balanced), and non-manipulable (strategy-proof and consistent) rules.

We have also noticed that, our family of solutions can be characterized by combinations of properties similar to those supporting the characterizations of the *uniform rule*. Besides, the allocation selected by the uniform rule can be interpreted as the expected allocation of the *M-temporary satisfaction methods*, if all plausible monotonic standards were equally likely.

[7] In Σ^M we consider all monotonic standards over $\mathbb{N} \times \mathbb{Z}$. Notice that, for a given e, not all of them rank the pairs (i, a_i) involved in that particular problem in different ways. Σ_e^M denotes precisely the subset of those different standards.

Appendix A. On the tightness of characterization result

We present now a collection of examples to illustrate the independence of properties used in Theorem 4.2.

Example 6.1. Let $\succ: \mathbb{N} \longrightarrow \mathbb{Z}_{++}$ be an order defined over the set of potential agents such that agent labeled i has priority over agent labeled $i+1$, i.e., $i \succ i+1$. The rule $G^{\succ}$ works as follows. Let $e \in \mathbb{A}$. Give to each agent the integer part of the equal split allocation,[8] that is $\lfloor \frac{T}{n} \rfloor$ for each $i \in N$. If no unit remains we have finished. If some units, $T' = T - n \cdot \lfloor \frac{T}{n} \rfloor$, remain, then allot each one of them to each one of the T' agents with the highest priority according to $\succ$.

Example 6.2. Consider the standard σ_1 such that $\sigma_1(i,x) < \sigma_1(i+1,y)$. That is, σ_1 is an standard in which the smaller the agent's label the higher the priority. Let us consider now the temporary satisfaction method associated to this standard, TS^{σ_1}.

Example 6.3. Alternative to the temporary satisfaction methods, in which we start by fully satisfy all the agents, we may think in allocating the task unit by unit starting from giving nothing to each agent. Let $\sigma \in \Sigma$. Then

Up method associated to $\boldsymbol{\sigma}$, $\boldsymbol{U^{\sigma}}$: Let $e \in \mathbb{A}$. Start by associating to each agent his peak, and then identifying the agent with the strongest number (peak) according to σ. Then give one unit of the task, T, to this agent. Reduce his number (peak) by one unit. Now identify the agent with the new strongest number for σ, and proceed in the same way. Repeat this process until the task runs out.

If $\sigma \in \sigma^M$, the resulting method is called **M-up method**.

Example 6.4. This rule, F, can be defined as follows. Let $\sigma_1, \sigma_2 \in \Sigma^M$ be two different monotonic standards such that $\sigma_1(i,x) < \sigma_1(i+1,x)$ and $\sigma_2(i+1,x) < \sigma_2(i,x)$. Then, we define the solution $F^{(\sigma_1,\sigma_2)}$ as

$$F^{(\sigma_1,\sigma_2)}(e) = \begin{cases} TS^{\sigma_1}(e) & \text{if } |N| = 2 \\ TS^{\sigma_2}(e) & \text{otherwise} \end{cases}$$

Next table shows the independence of properties in Theorem 4.2 are independent.

Property	$G^{\succ}$	TS^{σ_1}	$U^{\sigma}(\sigma \in \Sigma^M)$	$F^{(\sigma_1,\sigma_2)}$
Efficiency	N	Y	Y	Y
Balancedness	Y	N	Y	Y
Strategy-proofness	Y	Y	N	Y
Consistency	Y	Y	Y	N

Table 1: Independence of properties.

[8] We denote by $\lfloor a \rfloor$ the smallest integer number non greater than a.

Appendix B. Proofs of the results

Lemma 6.1 (Elevator Lemma, (Thomson (2004)))**.** *If a rule F is consistent and coincides with a conversely consistent rule F' in the two agent case, then it coincides with F' in general.*[9]

Lemma 6.2. *Efficiency and strategy-proofness together imply peaks only.*[10]

Proof. Let F be a rule fulfilling *strategy-proofness*. Let $e = (N, T, (R_i, R_{-i})) \in \mathbb{A}$ and $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}$ such that $p(R'_i) = p(R_i)$. Let us show that $x_i = F(e) = F(e') = x'_i$ when $T \leq \sum_{j \in N} p(R_j)$. Let us suppose that this is not true, and $x_i \neq x'_i$. We can assume without loss of generality that $x_i < x'_i$. If this is the case, *efficiency* implies that $x_i < x'_i \leq p(R_i) = p(R'_i)$. Then, $x'_i P_i x_i$, which means that $F_i(N, T, (R'_i, R_{-i})) P_i F_i(N, T, (R_i, R_{-i}))$. This implies a contradiction with *strategy-proofness*. Therefore $x_i = x'_i$.

The case when $T \geq \sum_{j \in N} p(R_j)$ is analogous. □

Lemma 6.3. *One-sided resource monotonicity together with consistency imply converse consistency.*[11]

Proof. Let $e \in \mathbb{A}$. By *consistency* the set $c.con(e; F) \neq \phi$. Let $x, y \in c.con(e; F)$ with $x \neq y$. We distinguish two cases.

Case 1. If $\sum_{i \in N} p(R_i) \geq T$. Since, $x \neq y$, there exists $k \in N$ such that $x_k > y_k$. Consider each two-agent set $S = \{k, j\}$ with $j \in N$ and $j \neq k$. Since $x, y \in c.con(e; F)$, $x_S = F(S, x_j + x_k, R_S)$ and $y_S = F(S, y_j + y_k, R_S)$. By *efficiency* and *one-sided resource monotonicity*, $x_j \geq y_j$. This fact, join with $x_k > y_k$, and $\sum_{i \in N} x_i = T = \sum_{i \in N} y_i$ yields a contradiction.

Case 2. If $\sum_{i \in N} p(R_i) \leq T$. There exists $k \in N$ such that $x_k < y_k$. Consider each two-agent set $S = \{k, j\}$ with $j \in N$ and $j \neq k$. Since $x, y \in c.con(T, R; F)$, $x_S = F(S, x_j + x_k, R_S)$ and $y_S = F(S, y_j + y_k, R_S)$. By *efficiency* and *one-sided resource monotonicity*, $x_j \leq y_j$. This fact, join with $x_k < y_k$, and $\sum_{i \in N} x_i = T = \sum_{i \in N} y_i$ yields a contradiction.

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[9] Let us consider an allocation for a problem with the following feature: For each two-agents subset, the rule chooses the restriction of that allocation for the associated reduced problem to this agent subset. *Converse consistency* requires the allocation to be the one selected by the rule for the original problem. This property was formulated by Chun (1999) in the context of claims problems.

Let $c.con(e; F) \equiv \{x \in \mathbb{Z}^N_+ : \sum_{i \in N} x_i = T$ and for all $S \subset N$ such that $|S| = 2, x_S = F(S, \sum_{i \in S} x_i, R_S)\}$

Converse consistency: For each $e \in \mathbb{A}$, $c.con(e; F) \neq \phi$, and if $x \in c.con(e; F)$, then $x = F(e)$.

[10] *Peaks only* says that an agent's allocation depends only on his preferred consumption.

Peaks only: For each $e = (N, T, (R_i, R_{-i})) \in \mathbb{A}$ and each $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}$ such that $p(R'_i) = p(R_i)$, then $F_i(e) = F_i(e')$.

[11] *One-sided resource monotonicity*, considers the case in which the change in the task does not alter the type of rationing associated to the initial problem, i.e, if initially we have to ration labor, it is still labor to be rationed after the task increasing, or else, if in the initial problem we have to ration leisure, then again, we have too much labor to allocate even after the decreasing of the task. In either case, the property states that no agent should suffer.

One-sided resource monotonicity: For each $e, e' \in \mathbb{A}$ such that $e = (N, T, R)$ and $e' = (N, T', R)$. If (a) $\sum_{j \in N} p(R_j) \geq T' > T$, or (b)$\sum_{j \in N} p(R_j) \leq T' < T$. Then for each $i \in N$, $F_i(e') R_i F_i(e)$.

□

Lemma 6.4. *Let F be a rule satisfying efficiency, balancedness, and strategy-proofnes. Let $e, e' \in \mathbb{A}$ be two problems involving two agents, $\{i, j\}$, such that $e = (\{i, j\}, T, (R, R))$; $e' = (\{i, j\}, T, (R', R'))$ such that either both $2p(R)$, $2p(R')$ are strictly larger or both strictly smaller than T. Then, $F(e) = F(e')$.*

Proof. Consider first the case where $2p(R) > T$, $2p(R') > T$. Let R'' be such that $p(R'') = \frac{T+1}{2}$, and let $e'' = (\{i, j\}, T, (R'', R''))$. We shall prove that $F(e) = F(e'') = F(e')$.

If T is even, *balancedness* implies the result. Let $T = 2\lambda + 1$, for some $\lambda \in \mathbb{Z}$, and suppose, w.l.o.g., that $F(e'') = (\lambda, \lambda + 1)$, whereas $F(e) = (\lambda + 1, \lambda)$. This is the only possibility of discrepancy because of *efficiency* and *balancedness*. Since $p(R) \geq p(R'') = \lambda + 1$, agent j is happier in problem e'' than he is in problem e, and it is the other way around for agent i. Additionally, *strategy proofness* implies that

$$F_i(\{i, j\}, T, (R, R'')) \leq \lambda; \qquad F_j(\{i, j\}, T, (R, R'')) \leq \lambda$$

The first inequality follows from agent i's inability to get a better result when misrepresenting his preferences in problem e'', while the second inequality follows from agent j's inability to benefit from misrepresenting his preferences in problem e. But, if this is the case,

$$F_i(\{i, j\}, T, (R, R")) + F_j(\{i, j\}, T, (R, R")) \leq 2\lambda < T$$

which is a contradiction with F being a rule.

The case where $2p(R) < T$, $2p(R') < T$ is analogous. □

Proof of Theorem 4.1.

It is straightforward to check that any M-temporary satisfaction method satisfies balancedness. Conversely, we will show that if TS^σ is *balanced* then $\sigma \in \Sigma^M$. Let us suppose that this is not true and there exists $\sigma \in \Sigma \setminus \Sigma^M$ such that TS^σ is *balanced*. Since $\sigma \notin \Sigma^M$, there exist $\{i, j\} \in N$ and $x, y \in \mathbb{Z}$ such that $x > y$ and $\sigma(j, y) < \sigma(i, x)$. By definition of standard of comparison, we have that $\sigma(j, x) < \sigma(j, y) < \sigma(i, x)$. Consider now the problem $e = (\{i, j\}, 2x - 2, (R_i, R_j))$ where $R_i = R_j$ and $p(R_i) = p(R_j) = x$; then $TS^\sigma(e) = (x, x - 2)$ violating *balancedness*. We reach in the way a contradiction and, therefore, $\sigma \in \Sigma^M$.

Proof of Theorem 4.2

It is easy to check that each TS^σ satisfies the four properties. Conversely, let F be a rule satisfying all the properties. We divide the rest of the proof into two steps.

Step 1. *Definition of the monotonic standard of comparison.* Let us define the order $\sigma \in \Sigma^M$ as follows

$$a > b \Rightarrow \sigma(i, a) < \sigma(j, b)$$
$$a = b \Rightarrow [\sigma(i, a) < \sigma(j, b) \Leftrightarrow F_i(\{i, j\}, 2a - 1, (R_i, R_j)) = a - 1],$$

where R_i and R_j are two single-peaked preference relations such that $p(R_i) = a = b = p(R_j)$ (by Lemma 6.2 it is enough to consider the peaks). It is straightforward to see that such a σ is complete and antisymmetric. Let us show that σ is transitive. Suppose that there exist $\{i, j, k\} \subseteq \mathbb{N}$ such that $\sigma(i, x) < \sigma(j, y)$, $\sigma(j, y) < \sigma(k, z)$, but $\sigma(i, x) > \sigma(k, z)$. By construction and *peaks only* (implied by *efficiency* and *strategy proofness* according to Lemma 6.2), this can only happen when $x = y = z$. By the definition of σ, in such a case, $F_i(\{i, j\}, 2x - 1, (R_i, R_j)) = x - 1$, $F_j(\{j, k\}, 2x - 1, (R_j, R_k)) = x - 1$, and $F_k(\{k, i\}, 2x - 1, (R_k, R_i)) = x - 1$, where $p(R_i) = p(R_j) = p(R_k) = x = y = z$. Consider the problem $(\{i, j, k\}, 3x - 2, (R_i, R_j, R_k))$. There are only three possible allocations: $(x - 1, x - 1, x)$, $(x - 1, x, x - 1)$, and $(x, x - 1, x - 1)$. Suppose that $F(\{i, j, k\}, 3x - 2, (R_i, R_j, R_k)) = (x - 1, x - 1, x)$, by *consistency*, $F_k(\{i, k\}, 2x - 1, (R_i, R_k)) = x$, achieving in this way a contradiction with $F_k(\{i, k\}, 2x - 1, (R_i, R_j)) = x - 1$. An analogous argument is applied if $F(\{i, j, k\}, 3x-2, (R_i, R_j, R_k)) = (x-1, x, x-1)$, or if $F(\{i, j, k\}, 3x-2, (R_i, R_j, R_k)) = (x, x - 1, x - 1)$. Therefore $\sigma(i, x) < \sigma(k, z)$, and then σ is transitive.

Step 2. *Let us prove now that $F = TS^\sigma$*. It is straightforward that TS^σ is *efficiency*, *one-sided resource monotonic*, and *consistent*, then, by Lemma 6.3, TS^σ is *conversely consistent*. Therefore, in application of Lemma 6.1, it is enough to show that $F = TS^\sigma$ in the two-agent case. Then, let us consider the problem $e = (S, T, R) \in \mathbb{A}$ where $S = \{i, j\}$. Without loss of generality we can assume that $p(R_i) \leq p(R_j)$. We analyze the case in which $p(R_i) + p(R_j) \geq T$. The other case is completely analogous. We distinguish the following cases:

Case 1. If $R_i = R_j$ and T is even. By *balancedness*, $F(e) = \left(\frac{T}{2}, \frac{T}{2}\right) = TS^\sigma(e)$.

Case 2. If $R_i = R_j$ and T is odd. If $T = 2p(R_i) - 1$, by the definition of the standard of comparison, $F(e) = TS^\sigma(e)$. If $T < 2p(R_i) - 1$, by Lemma 6.4, $F(e) = F(\{i, j\}, T, (R'_i, R'_j))$, where $R'_i = R'_j$ and $p(R'_i) = p(R'_j) = \frac{T+1}{2}$. And then, $F(e) = F(\{i, j\}, T, (R'_i, R'_j)) = TS^\sigma(\{i, j\}, T, (R'_i, R'_j)) = T^\sigma(e)$.

Case 3. If $F_i(e) \leq F_j(e) \leq p(R_i) \leq p(R_j)$. By *efficiency* and *strategy proofness*, $F_i(e) = F_i(S, T, (R_j, R_j)) = TS^\sigma_i(S, T, (R_j, R_j)) = TS^\sigma_i(e)$.

Case 4. If $F_j(e) \leq F_i(e) \leq p(R_i) \leq p(R_j)$. By *efficiency* and *strategy proofness*, $F_j(e) = F_j(S, T, (R_i, R_i)) = TS^\sigma_j(S, T, (R_i, R_i)) = TS^\sigma_j(e)$.

Case 5. If $F_i(e) \leq p(R_i) < F_j(e) \leq p(R_j)$. By *efficiency* and *strategy proofness*, $F_i(e) = F_i(S, T, (R_j, R_j)) = TS^\sigma_i(S, T, (R_j, R_j)) = TS^\sigma_i(e)$. If $F_i(e) = TS^\sigma_i(e) = p(R_i)$, then $F_j(e) = T - F_i(e) = T - TS^\sigma_i(e) = TS^\sigma_j(e)$. If $F_i(e) \leq p(R_i) - 1$, then $F_j(e) \leq p(R_i)$, which is a contradiction.

Then, F coincides with TS^σ in the two agents case, and therefore they do so in general.

Proof of Proposition 5.1.

On one hand, it is known that the uniform rule satisfies *converse consistency*. On the other hand, it is easy to check that the M-temporary satisfaction methods are *consistent*. Then the average given by the left hand side in the formula is also consistent (see Thomson (2004)). By using the Lemma 6.1 it is enough to consider the two-agent case. But it is straightforward that in this case both the uniform rule and the average coincide.

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