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Competitive Pricing

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JEL Classification numbers: D50.
Keywords: Non-convex production sets, competitive pricing rule, competitive pricing equilibrium.

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Abstract

Competitive pricing is a pricing rule that combines two principles that are present in competitive markets. The *profit principle* (an action will be chosen only if it yields maximal payoffs), and the *scarcity principle* (markets make expensive those commodities that restrict production possibilities). It is shown that, under standard assumptions, these principles imply profit maximization at given prices. But also that they can be applied to economies with non-convex production sets (e.g. firms with S-shaped production functions). The chief properties of this pricing rule, as well as the existence and efficiency of the associated equilibria, are analyzed.

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1 Introduction

It is well known that when markets are complete and agents maximize quasi-concave objective functions over convex choice sets, taking prices as given, competitive equilibrium exists and yields efficient outcomes. Here we approach competitive behaviour from a slightly different point of view, by identifying some characteristic features of competitive markets that are independent of price-taking behaviour and can be applied to economies with non-convex production sets. Non-convexities in production naturally arise when there are increasing returns to scale, fixed costs, S-shaped production functions, or when production involves the use of fixed capital (that is, input commodities that are functionally indivisible).[1]

The key concept for the analysis is that of *competitive pricing*. By this we intend to summarize two principles that are present in competitive markets. The first one is the *profit principle*, that says that an action will be chosen only if it yields maximal payoffs. The second one is the *scarcity principle*, that says that agents are ready to pay for the available commodities their maximum worth (i.e. the short-side of the market pays its reservation value). This approach to competitive markets is applicable to situations in which production sets are not convex. Moreover, the application of these two principles turns out to be equivalent to profit maximization at given prices, when production sets are convex.

To discuss these ideas the commodity space is divided into two disjoint subsets: *capital goods* and *standard commodities*. Capital goods are commodities already produced, hence part of the initial endowments, that are inputs to production. These commodities include elements of fixed capital, indivisible goods or other type of inputs that may generate non-convexities in production. Standard commodities consist of durable consumption goods already produced, as well as those goods and services that are obtained from production activities, including both new consumption goods and new input commodities. Within this framework, we say that a firm behaves according to *competitive pricing* whenever it finds acceptable a prices-production combination such that:

(i) At these prices, there is no other production plan yielding higher profits and using fewer capital goods; namely, firms behave as constrained profit maximizers at given prices.

(ii) There is no price vector satisfying (i) with higher prices for capital

[1] Yet a competitive scenario is hardly compatible with pure increasing returns to scale. Therefore, our aim is not to "solve" the problem of market equilibirum with increasing returns to scale, but rather to approach competition in a way that is not totally dependent on the convexity assumption.

goods. In other words, the prices of capital goods are maximal within those satisfying constrained profit maximization.

Part (i) reflects the profit principle and can be regarded as a sort of participation constraint. If this property were not satisfied, firms would be willing (and able, given their restrictions) to change their input-output configurations. Part (ii) expresses the scarcity principle: markets make expensive those commodities that restrict production possibilities. Therefore, firms must be ready to pay maximal prices for the capital goods they use, as long as this is compatible with constrained profit maximization. A situation in which consumers maximize their preferences under their wealth restrictions, firms behave according to the competitive pricing rule, and all markets clear will be called a *competitive pricing equilibrium.*

This approach to competition enables to deal with short-run situations in which firms can be thought of as 'being small' but not necessarily convex. More specifically, we can think of a two-period economy in which firms may use some elements of fixed capital that generate non-convexities in production (land, buildings or heavy machinery, say). The amounts of those commodities are given a priori and restrict production possibilities. The allocation of fixed capital is decided in period one, taken into account current and future prices (these are investment decisions in a complete markets setting). Firms compete in the capital goods market offering the highest prices that are compatible with their incentives. Production and consumption take place in period 2. Now firms choose those production plans that maximize profits, subject to the investment decisions made in period 1, and consumers maximize utility at given prices.

The description of the economy is the subject of section 2. Section 3 introduces the competitive pricing rule and discusses its main properties. It is shown that this pricing rule is an upper hemicontinuous correspondence with non-empty, closed and convex values, and that it coincides with profit maximization when production sets are convex. The existence of equilibrium is taken up in section 4, where we also discuss its (in)efficiency properties. We show that a competitive pricing equilibrium exists under reasonably general conditions, that equilibrium allocations will not be efficient in general, and also that efficiency can be obtained by a public intervention. The most technical part is dealt with in an Appendix.

Let us conclude this section with some references to the literature. In a general equilibrium context, the behaviour of non-convex firms is usually described in terms of pricing rules. A pricing rule is a mapping from the firm's set of efficient production plans to the price space whose graph describes the prices-production pairs which a firm finds *acceptable* (a generalization of the inverse supply mapping). This is the methodological approach followed here to present the notion of competitive pricing. In this way we have benefited from the results already available on the existence of equilibrium [the reader is referred to Bonnisseau & Cornet (1988), Brown (1991) or Villar (2000) for details].

The idea of using quantity restrictions to analyze market equilibria in economies with non-convex production sets is by no means new. The contributions by Scarf (1986) and Dehez & Drèze (1988 a,b) and Villar (1996, ch. 9 and 10) are close antecedents to this work [see also Dierker, Guesnerie & Neuefeiend (1985), Böhm, V.(1986), and Dierker & Neuefeind (1988)].

2 The model

We consider here an economy with ℓ commodities, n firms and m consumers. Commodities are divided into two separate groups: capital goods and standard commodities. The number of commodities, firms and consumers is finite and given *a priori*.

2.1 Commodities and firms

There are ℓ commodities in this economy that can be classified either as **capital goods** (input commodities already available as part of the initial endowments) or **standard commodities** (stocks of pure consumption goods as well as goods and services that become available through the production process). Consequently, the set $\mathcal{L} = \{1, 2, \ldots, \ell\}$ of commodity indices can be partitioned into two disjoint subsets, $\mathcal{L}^K = \{1, 2, ..., k\}$ and its complement, $\mathcal{L}^S = \{k+1, k+2, ..., \ell\}$. Goods in $\mathcal{L}^K$ are capital goods and goods in $\mathcal{L}^S$ are standard commodities. Capital goods are inputs to production that, together with the technology, determine the effective production possibilities.

There is a given number n of firms. For $j = 1, 2, \ldots, n$, $Y_j \subset \mathbb{R}^\ell$ denotes the jth firm's production set, whereas $\mathbf{y}_j \in Y_j$ describes a production plan. Within this framework it is convenient to write production plans in the form $\mathbf{y}_j = (\mathbf{a}_j, \mathbf{b}_j)$, with $\mathbf{a}_j \in -\mathbb{R}^k_+$ and $\mathbf{b}_j \in \mathbb{R}^{\ell-k}$. That is, $\mathbf{a}_j$ is a point in the subspace of capital goods, and $\mathbf{b}_j$ a point in the subspace of standard commodities. Note that no sign restriction is established on the vector of standard commodities that can therefore include both input and output commodities.

Let $\mathbb{F}_j$ stand for the jth firm's set of *weakly efficient production plans.* That is, the set of points $\mathbf{y}_j \in Y_j$ such that if $\mathbf{y}' \in \mathbb{R}^\ell$ is such that $\mathbf{y}' >> \mathbf{y}_j$, then $\mathbf{y}'$ cannot be in Y_j. We shall denote by $\mathbb{F}$ the Cartesian product of the n sets of weakly efficient production plans; namely, $\mathbb{F} \equiv \Pi_{j=1}^{n} \mathbb{F}_j$. A point in $\mathbb{F}$ will be denoted by $\widetilde{\mathbf{y}} = (\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n)$. A price vector is a point $\mathbf{p} \in \mathbb{R}_+^{\ell}$. For each pair $(\mathbf{p}, \mathbf{y}_j) \in \mathbb{R}_+^{\ell} \times \mathbb{F}_j$, the scalar product $\mathbf{p}\mathbf{y}_j$ gives us the associated profits.

The following definitions will help the ensuing discussion:

Definition 1 *A commodity h is* ***relevant*** *for the jth firm if: (i) There exists some $\mathbf{y}_j \in \mathbb{F}_j$ such that $y_{jh} > 0$; or (ii) $y_{jh} \leq 0$ for all $\mathbf{y}_j \in \mathbb{F}_j$, and there exists $\mathbf{y}_j' \in \mathbb{F}_j$such that, for all $\lambda > 0$, $\mathbf{y}_j' + \lambda \mathbf{e}^h \notin Y_j$, where $\mathbf{e}^h$ is the hth canonical vector.*

A commodity is relevant for the jth firm if there exists some production plan for which it is either an effective output or an effective input. A commodity is an effective output when there is some production plan involving positive production of this commodity. A commodity is an effective input when there is some production plan for which it is not possible to reduce the use of such a commodity without affecting production possibilities. Relevant commodities may refer to both capital goods and standard commodities. A commodity which is not relevant for the jth firm is called *irrelevant*.

Definition 2 *A production set $Y_j \subset \mathbb{R}^{\ell}$ is called* ***"convex"*** *(with inverted commas) if it is a convex set in the subspace of its relevant commodities. Similarly, the projection of Y_j on the space $\mathbb{R}^{\ell-k}$ of standard commodities is* ***"convex"****, if it is a convex set on the subspace of standard commodities that are relevant for the jth firm.*

According to this definition we call "convex" those production sets whose non-convexities are inessential, in the sense that they derive from the presence of irrelevant commodities. And the same applies to the projection of Y_j on the space of standard commodities. By extension, a firm is **"non-convex"**, when it is non-convex on the subspace of relevant commodities.

Consider now the following axiom that applies to every $j = 1, 2, ..., n$:

Axiom 1 *(i) Y_j is a closed subset of $\mathbb{R}^{\ell}$, with $Y_j \cap \mathbb{R}_+^{\ell} = \{\mathbf{0}\}$ and $Y_j - \mathbb{R}_+^{\ell} \subset Y_j$.*

(ii) For all $(\mathbf{a}_j, \mathbf{b}_j)$, $(\mathbf{a}_j', \mathbf{b}_j') \in \mathbb{F}_j$, $[\mathbf{a}_j \geq \mathbf{a}_j' \;\&\; \mathbf{b}_j > \mathbf{b}_j'] \Longrightarrow \mathbf{y}_j \leq \mathbf{0}$.

(iii) For every $\mathbf{a}_j \in -\mathbb{R}_+^{k}$ the set

$$B_j(\mathbf{a}_j) \equiv \{\mathbf{b}_j \in \mathbb{R}^{\ell-k} \;\; / \;\; (\mathbf{a}_j', \mathbf{b}_j) \in Y_j \text{ for some } \mathbf{a}_j' \geq \mathbf{a}_j\}$$

is "convex".

Part (i) of axiom 1 is standard in that it assumes closedness, necessity of using up some input to obtain some output, comprehensiveness, and possibility of inaction. It follows from this axiom that the set of weakly efficient production plans $\mathbb{F}_j$ consists exactly of those points in the boundary of Y_j. Part (ii) is a weak monotonicity requirement, that discards the presence of *vertical segments* in that part of the boundary of Y_j associated with positive production. It states that it is not possible to increase production while keeping constant the use of input commodities. Clearly, when Y_j is convex part (ii) is implied by part (i). This property is closely related to that of *bounded marginal returns* (i.e. the marginal rates of transformation are bounded in all points involving positive production). Part (iii) substitutes the assumption of convex production sets by a weaker requirement. It says that, for any given vector of capital goods $\mathbf{a}_j$, the projection on $\mathbb{R}^{\ell-k}$ of those production plans using fewer capital goods than those in $\mathbf{a}_j$ is a convex set, when restricted to the subspace of relevant commodities. This allows us to include within these goods elements of fixed capital that may give rise to non-convexities. Observe that this assumption is compatible with the presence of firms with constant, decreasing or increasing returns to scale, set-up costs or S-shaped production functions. Clearly, when Y_j is convex, part (iii) is automatically satisfied. And, conversely, if a firm does not require using capital goods to develop production activities, then part (iii) implies that it must be convex in the subspace of relevant commodities. Moreover, in the one-input—one-output case, this property is implied by part (i).

The following examples illustrate standard families of production sets satisfying axiom 1 (note that examples 2, 3 and 4 allow for the presence of non-convexities in production):

Example 1 (Convex production sets) *Standard "convex" production sets satisfy these axioms. This is interesting because it ensures that these axioms encompass the conventional way of modelling firms in general competitive analysis.*

Example 2 (Pure fixed-cost) *Those "non-convex" production sets in which non-convexities are only due to the presence of fixed cost. That is to say, once a firm starts producing a positive amount of some good, the technology exhibits constant or decreasing returns to scale [Cf. Moriguchi (1996)].*

Example 3 (Single-production firms) *The production set associated to a monotone production function which produces a single good as a net-output, using only capital goods as input commodities.*

Example 4 (Distributive production sets) *A relevant case of production sets with increasing returns to scale and joint production that satisfies these axioms is that of distributive production sets, introduced by Scarf (1986). He defines a* ***distributive*** *production set* Y_j *as follows: for any collection of points* $(\mathbf{y}^t, \lambda^t)$, $t = 1, 2, \ldots, s$, *with* $\mathbf{y}^t = (\mathbf{a}^t, \mathbf{b}^t) \in Y_j$, $\lambda^t \in \mathbb{R}_+$, *the following condition holds:*

$$\sum_{h=1}^{s} \lambda^h \mathbf{a}^h \leq \mathbf{a}^t, t = 1, 2, \ldots, s \Rightarrow \sum_{h=1}^{s} \lambda^h \mathbf{y}^h \in Y_j$$

In words: A production set is distributive when any nonnegative weighted sum of feasible production plans is feasible, if it does not use fewer capital goods than any of the original plans.

The behaviour of firms will be described by means of **pricing rules**. A pricing rule for the jth firm is a mapping $\Phi_j : \mathbb{F}_j \rightarrow \mathbb{R}^\ell_+$, in which $\Phi_j(\mathbf{y}_j)$ tells us the set of prices that this firm finds *acceptable* in order to produce $\mathbf{y}_j$. We discuss the competitive pricing rule in the next section.

2.2 Consumers

There are m competitive consumers in the economy. The ith consumer is characterized by a tuple $[X_i, u_i, \omega_i, (\theta_{ij})_{j=1}^n]$, where X_i, u_i, ω_i stand for the ith consumer's consumption set, utility function, and initial endowments, respectively, and θ_{ij} denotes the ith consumer's share in the profits of the jth firm (with $0 \leq \theta_{ij} \leq 1$ for all i, j, and $\sum_{i=1}^m \theta_{ij} = 1$ for all j).

Let $(\mathbf{p}, \widetilde{\mathbf{y}}) \in \mathbb{R}^\ell_+ \times \mathbb{F}$ be given. Then, the ith consumer's **demand** is obtained as a solution to the program:

$$\left. \begin{array}{ll} & Max\ u_i(\mathbf{x}_i) \\ \text{s.t.} & \mathbf{x}_i \in X_i \\ & \mathbf{p}\mathbf{x}_i \leq \mathbf{p}\omega_i + \sum_{j=1}^n \theta_{ij}\mathbf{p}\mathbf{y}_j \end{array} \right\}$$

The behaviour of the ith consumer can be summarized by a demand correspondence $\xi_i : \mathbb{R}^\ell_+ \times \mathbb{F} \rightarrow X_i$, where $\xi_i(\mathbf{p}, \widetilde{\mathbf{y}})$ is the set of solutions to the program above.

The next axiom makes it explicit the modelization of consumers:

Axiom 2 *For all* $i = 1, 2, \ldots, m$*:*

(a) X_i *is a closed and convex subset of* $\mathbb{R}^\ell$*, bounded from below.*

(b) $u_i : X_i \rightarrow \mathbb{R}$ *is a continuous and quasi-concave function, which satisfies local non-satiation.*

(c) $\exists\ \mathbf{x}_i \in X_i \ / \ \omega_i >> \mathbf{x}_i$.

Axiom 2 is standard and needs no comment.

2.3 Economies

We call **economy** a specification of the m consumers and the n firms. An economy can thus be described by:

$$E = \{(X_i, u_i, \omega_i, (\theta_{ij})_{j=1}^n)_{i=1}^m; (Y_j,\ \Phi_j)_{j=1}^n\}$$

Let $\omega = \sum_{i=1}^m \omega_i$ stand for the vector of initial resources available for the economy. We can write $\omega = (\omega^K, \omega^S)$, to make it explicit the distinction between capital goods and standard commodities. The set of attainable allocations of an economy is given by:

$$A(\omega) \equiv \{[(\mathbf{x}_i), (\mathbf{y}_j)] \in \Pi_{i=1}^m X_i \times \Pi_{j=1}^n Y_j \ \ / \quad \textstyle\sum_{i=1}^m \mathbf{x}_i - \omega \leq \sum_{j=1}^n \mathbf{y}_j\}$$

The projection of $A(\omega)$ on the spaces containing X_i, Y_j gives us the ith consumer's set of attainable consumption plans and the jth firm's set of attainable production plans, respectively.

We now introduce the following:

Axiom 3 *For every $\omega' \geq \omega$, the set $A(\omega')$ is bounded.*

Axiom 3 states that the set of attainable allocations is bounded. That is to say, it is not possible to obtain unlimited amounts of production from a finite amount of endowments.

3 Competitive pricing

3.1 Constrained profit maximization

We associate the profit principle to the notion of constrained profit maximization: firms maximize profits subject to a capital goods constraint. To define this pricing rule, one has to take into account that the profit principle imposes no restriction on prices when $\mathbf{y}_j = \mathbf{0}$. Hence, to avoid those trivial equilibria in which all firms are inactive, we establish the following:

Definition 3 *The* ***constrained profit maximization pricing rule*** *for the jth firm, is a mapping $\phi_j : \mathbb{F}_j \to \mathbb{R}_+^\ell - \{\mathbf{0}\}$ given by:*
(i) For $\mathbf{y}_j \neq \mathbf{0}$,
$\phi_j(\mathbf{y}_j) \equiv \{\mathbf{q} \in \mathbb{R}_+^\ell - \{\mathbf{0}\} \ / \ \mathbf{q}\mathbf{y}_j \geq \mathbf{q}\mathbf{y}'_j,\ \forall\ \mathbf{y}'_j \in Y_j \ \text{ with } \mathbf{a}'_j \geq \mathbf{a}_j\}$
(ii) $\phi_j(\mathbf{0})$ is the closed convex hull of the set of points $\mathbf{q} \in \mathbb{R}_+^\ell - \{\mathbf{0}\}$ for which there exists a sequence $\{\mathbf{q}^\nu, \mathbf{y}_j^\nu\} \subset \mathbb{R}_+^\ell - \{\mathbf{0}\} \times [\mathbb{F}_j - \{\mathbf{0}\}]$, such that $\{\mathbf{q}^\nu, \mathbf{y}_j^\nu\} \to (\mathbf{q}, \mathbf{0})$, with $\mathbf{q}^\nu \in \phi_j(\mathbf{y}_j^\nu)$.

Therefore, ϕ_j pictures the jth firm as selecting, for each given efficient production plan $\mathbf{y}_j$, prices such that it is not possible to obtain higher profits within the set of production plans which make use of equal or fewer capital goods. Parts (ii) and (iii) of axiom 1 imply that the set $B_j(\mathbf{a}_j)$ is convex and comprehensive, with $\mathbf{b}_j \in \partial B_j(\mathbf{a}_j)$, for all $\mathbf{y}_j \in \mathbb{F}_j$. Both properties are required for this pricing rule be well defined.

Observe that, under axiom 1, ϕ_j is a loss-free pricing rule. That is to say, $\mathbf{q}\mathbf{y}_j \geq 0$ for all $\mathbf{q} \in \phi_j(\mathbf{y}_j)$, all $\mathbf{y}_j \in \mathbb{F}_j$. Also note that $\phi_j(\mathbf{y}_j)$ is contained in the set of marginal prices at $\mathbf{y}_j$, with respect to the truncated production set $Y_j(\mathbf{a}_j) = \{\mathbf{y}'_j \in Y_j \ / \ \mathbf{a}'_j \geq \mathbf{a}_j\}$. When the input restriction is binding, the set of marginal prices to $Y_j(\mathbf{a}_j)$ at $\mathbf{y}_j$ is larger than the normal cone to Y_j at $\mathbf{y}_j$. Therefore, when Y_j is convex, input-constrained profit maximization is a super-correspondence of the profit maximization pricing rule.

The following result summarizes the key properties of this pricing rule, under the assumptions established:

Theorem 1 *[Villar (2000, Th. 8.1)] Under axiom 1 the constrained profit maximization pricing rule satisfies the following properties, for all* $\mathbf{y}_j \in \mathbb{F}_j$*:*

(i) ϕ_j *is a closed correspondence whose values are non-degenerate closed convex cones.*

(ii) $\mathbf{q}\mathbf{y}_j \geq 0$, *for all* $\mathbf{q} \in \phi_j(\mathbf{y}_j)$.

The profit principle may be regarded as a minimal requirement for market economies. Yet, it is not very tight as this principle alone imposes few restrictions on the allocation of capital goods (e.g. zero prices for capital goods cannot be excluded). Therefore, constrained profit maximization may encompass too many market situations; in particular, it does not imply competitive equilibrium when production sets are convex. Put informally, the correspondence ϕ_j is *too large.*

3.2 The competitive pricing rule

We now consider a selection of the constrained profit maximization pricing rule, called competitive pricing, that incorporates another characteristic element of competitive markets: the scarcity principle. The underlying idea is that markets make expensive those commodities that constrain production possibilities. More specifically, competitive pricing requires the firms to pay for the capital goods they actually use the maximum price compatible with constrained profit maximization.

Let us formalize this idea. For a given price vector $\mathbf{q} \in \mathbb{R}_+^\ell$, let us write $\mathbf{q}(j)= (\mathbf{q}_0(j), \mathbf{q}_1(j))$, where $\mathbf{q}_0(j)$ is the vector of prices for the jth firm relevant capital goods, and $\mathbf{q}_1(j)$ its complement. Let now $\widehat{\phi}_j : \mathbb{F}_j \to \mathbb{R}_+^\ell$ be defined as follows:

$$\widehat{\phi}_j(\mathbf{y}_j) \equiv \{\mathbf{q} \in \phi_j(\mathbf{y}_j) \ / \nexists \ \mathbf{q}' \in \phi_j(\mathbf{y}_j) \text{ with } \mathbf{q}'_1(j) = \mathbf{q}_1(j) \text{ and } \mathbf{q}'_0(j) > \mathbf{q}_0(j)\}$$

That is, for each $\mathbf{y}_j \in \mathbb{F}_j$, the jth firm admits price vectors in ϕ_j that include maximal prices for the capital goods required.

It is easy to see $\widehat{\phi}_j(\mathbf{y}_j)$ is closed and non-empty, for all $\mathbf{y}_j \in \mathbb{F}_j$. Yet, $\widehat{\phi}_j$ may not have a closed graph, because it will not be closed at kinks (irrespective of the convexity of Y_j). That shows that this restriction does *not* correspond to (unconstrained) profit maximization when production sets are convex. Put informally, the correspondence $\widehat{\phi}_j$ is *too small.*

It is then natural to define competitive pricing as follows:

Definition 4 *The* ***competitive pricing rule*** *for the jth firm, is a mapping $\phi_j^* : \mathbb{F}_j \to \mathbb{R}_+^\ell$ such that, for each given $\mathbf{y}_j \in \mathbb{F}_j$, $\phi_j^*(\mathbf{y}_j)$ is the convex hull of the set*

$$\{\mathbf{q} \in \mathbb{R}_+^\ell \ / \ \exists \, (\mathbf{q}^\nu, \mathbf{y}_j^\nu) \to (\mathbf{q}, \mathbf{y}_j) \ \ \textit{with } \mathbf{q}^\nu \in \widehat{\phi}_j(\mathbf{y}_j^\nu), \ \ \mathbf{y}_j^\nu \in \mathbb{F}_j\}$$

The competitive pricing rule is made of those prices that satisfy both constrained profit maximization and maximum prices for produced commodities, and the convex hull of those prices that are limits of sequences with these features.

The main properties of this pricing rule are collected in the next theorems:

Theorem 2 *Suppose that axiom 1 holds. Then:*

(i) ϕ_j^* is a closed correspondence whose values are non-degenerate closed convex cones.

(ii) ϕ_j^* is a subcorrespondence of ϕ_j.

Part (i) of this theorem establishes that competitive pricing is "well behaved" (namely, ϕ_j^* is a pricing rule that satisfies a set of conditions that permits one to prove the existence of equilibrium using the results already available). Part (ii) ensures that the participation constraint is not violated, even though we are taking a price correspondence larger than $\widehat{\phi}_j$.

The next theorem refers to the relationship between competitive pricing and profit maximization. It is a generalization of the idea that when production sets are "convex", competitive pricing implies (unconstrained) profit maximization. Formally:

Theorem 3 *Suppose that axiom 1 holds and that the set* $\mathbb{N}(\mathbf{y}_j)$ *of prices that maximize profits unconditionally at* $\mathbf{y}_j \in \mathbb{F}_j$ *is non-empty. Suppose furthermore that there exists* $\varepsilon > 0$ *such that* $\delta(\mathbf{y}_j, \varepsilon) \cap Y_j$ *is a convex set (where* $\delta(\mathbf{y}_j, \varepsilon)$ *denotes a closed ball of center* $\mathbf{y}_j$ *and radius* ε*). Then* $\phi_j^*(\mathbf{y}_j) = \mathbb{N}(\mathbf{y}_j)$.

The way in which this property is formulated aims at encompassing the cases of nonconvex firms with convex parts. Bearing in mind that $\mathbb{N}(\mathbf{y}_j) \neq \emptyset$ for all $\mathbf{y}_j \in \mathbb{F}_j$ when Y_j is "convex", the following result is immediate:

Corollary 1 *Suppose that axiom 1 holds and that* Y_j *is "convex". Then,* ϕ_j^* *implies unconstrained profit maximization.*

4 Equilibrium

4.1 The existence of competitive pricing equilibrium

The next definition makes it precise the equilibrium notion:

Definition 5 *A* ***competitive pricing equilibrium*** *is a price vector* $\mathbf{p}^* \in \mathbb{R}_+^\ell - \{\mathbf{0}\}$*, and an allocation* $[(\mathbf{x}_i^*), \widetilde{\mathbf{y}}^*]$*, such that:*
(α) *For each* $i = 1, 2, \ldots, m,\ \mathbf{x}_i^* \in \xi_i(\mathbf{p}^*, \widetilde{\mathbf{y}}^*)$.
(β) $\mathbf{p}^* \in \bigcap_{j=1}^n \phi_j^*(\mathbf{y}_j^*)$.
(γ) $\sum_{i=1}^m \mathbf{x}_i^* - \sum_{j=1}^n \mathbf{y}_j^* = \omega$.

That is, a competitive pricing equilibrium is a situation in which: (a) Consumers maximize their preferences subject to their budget constraints; (b) The jth firm agrees to produce $\mathbf{y}_j^*$ at prices $\mathbf{p}^*$, for all j (that is, $(\mathbf{p}^*, \widetilde{\mathbf{y}}^*)$ is a *production equilibrium* relative to the competitive pricing rule); and (c) All markets clear. Hence, an equilibrium consists of a price vector and a feasible allocation in which all agents are maximizing their payoff functions within their feasible sets. These feasible sets correspond to budget sets for the case of consumers, and production sets subject to an input constraint, for the case of firms.

Observe that the equality between "supply" and demand is relevant, because it implies that there are no idle capital goods in equilibrium. Hence, the equilibrium allocation of capital goods cannot be arbitrary. The prices of capital goods cannot be arbitrary either: they generate a cost structure and an income distribution that accommodates the equilibrium between "supply" and demand. Moreover, these are maximal prices compatible with constrained profit maximization, which typically implies the exclusion of zero input prices.

The main result of this section is the following:

Theorem 4 *Let E be a market economy that satisfies axioms 1, 2 and 3. Then, there is a competitive pricing equilibrium* $(\mathbf{p}^*, [(\mathbf{x}_j^*), \widetilde{\mathbf{y}}^*])$.

Proof.

It is well established that, under axioms 1 to 3, an equilibrium exists when the behaviour of individual firms can be described by closed pricing rule correspondences, whose values are non-degenerate closed convex cones and are "loss-free" [e.g. Bonnisseau & Cornet (1988, th. 2')]. Theorem 2 shows that the competitive pricing rule satisfies these properties under axiom 1. Hence the result follows. ■

Theorem 4 tells us that there exists a competitive pricing equilibrium under fairly general assumptions. Given the equilibrium allocation of capital goods, the firms maximize profits at given prices within their attainable sets. In equilibrium no firm finds it profitable to operate with fewer inputs.

When production sets are convex a competitive pricing equilibrium is nothing else than a standard competitive equilibrium. This follows immediately from Theorems 3 and 4. Formally:

Corollary 2 *Let E be a market economy satisfying axioms 1 and 2. Suppose furthermore Y_j is "convex" for all $j = 1, 2, ..., n$. Then a competitive pricing equilibrium is a standard competitive equilibrium.*

This corollary states that profit maximization at given prices can be regarded as the outcome of the profit principle and the scarcity principle, when production sets are "convex". This is so because, under convexity, relaxing the input constraint does not make available more profitable options at given prices: local maximization implies global maximization. Therefore, the standard behaviour of competitive firms (profit maximization at given prices), may be regarded as *a reduced model* of a market economy in which firms pay their reservation value for capital goods, due to the competition for those scarce factors that limit production possibilities.

Scarf (1986, Th. 1) shows that the constrained profit maximization pricing rule includes prices for which $\mathbf{q}\mathbf{y}_j = 0$, when the production set is distributive (see the definition in example 4 above). As competitive pricing is a loss-free pricing rule, the next result follows:

Corollary 3 *Let* $(\mathbf{p}^*, [(\mathbf{x}_j^*), \widetilde{\mathbf{y}}^*])$ *be a competitive pricing equilibrium. If Y_j is a distributive production set, then* $\mathbf{p}^*\mathbf{y}_j^* = 0$.

4.2 Efficiency

The efficiency analysis can be summarized in the following claims:

(i) Every competitive pricing equilibrium is efficient conditional on the allocation of capital goods.

(ii) Full efficiency fails because the price system does not ensure a right allocation of capital goods due to the presence of non-convexities.

(iii) When $\mathbf{p}^* \in \cap_{j=1}^n \mathbb{N}(\mathbf{y}_j^*)$ a competitive pricing equilibrium is fully efficient. This happens in particular when Y_j is a convex set for all j.

(iv) Full efficiency can be attained by means of a public intervention that guides the allocation of capital goods and adjsuts the income distribution by taxing the profits that consumers get.

Let us prove those claims.

Claim (i) is rather immediate. Taking the allocation of capital goods as given, the resulting conditional economy is a conventional competitive one (in the subspace of standard commodities). So the first welfare theorem ensures this outcome.

We prove claim (ii) by means of the following example. Consider an economy with two firms and two commodities, corn and labour. Firm 1 is a convex firm which produces efficiently λ units of corn by means of λ units of labour. Firm 2 produces λ units of corn for every $\lambda/2$ units of labour (i.e. it exhibits a better marginal rate of transformation), but requires incurring a fixed cost of -0.5. There are two identical consumers. Each consumer is endowed with a unit of labour and only cares about the consumption of corn (flat indifference curves). At prices $\mathbf{p} = (1, 1)$ consumer i sells one unit of labour and buys one unit of output to firm j, for $j = 1, 2$. This is a competitive pricing equilibrium in which labour is a capital good and two units of output are obtained. It is clear that this equilibrium allocation is not Pareto optimal, because it would be better to operate firm 2 exclusively, as two units of labour applied to that firm produce three rather than two units of output.[2]

Claim (iii) is obvious because in that case total profits are maximized and the result follows again from the standard argument that proves the first welfare theorem.

[2] Indeed, non-convexities in production imply that the allocation of capital goods involves a public good feature that defines a *public environment.* See Hammond & Villar (1998) for an analysis of this aspect.

Finally, let us prove (iv). Define a tax system on consumers' profits $(\tau_i)_{i=1}^m$, as follows:

$$\tau_i(\mathbf{p},\widetilde{\mathbf{y}}) = \sum_{j=1}^{n} \theta_{ij}\mathbf{p}\mathbf{y}_j - \alpha_i \sum_{j=1}^{n} \mathbf{p}\mathbf{y}_j$$

where α_i describes the ith consumer's share in total profits intended by the planner, with $\sum_{i=1}^m \alpha_i = 1$. Then,

Theorem 5 *Let E be a market economy that satisfies axioms 1, 2 and 3 and let $(\tau_i)_{i=1}^m$ be a tax system. Then, there exists a competitive pricing equilibrium with taxes that yields a Pareto optimal allocation.*

Proof.

First recall that an equilibrium allocation is conditionally efficient (that is, is efficient among those allocations that take the equilibrium allotment of capital goods as given). Now observe that the aggregate production set, $Y = \sum_{j=1}^n Y_j$ inherits all properties of the individual sets (i.e. the aggregate production set satisfies axiom 1). So, define a new economy E' that is identical to E except in that now there is a single production set Y whose profits are distributed according the given shares $(\alpha_i)_{i=1}^m$. As profits are non-negative, axioms 1, 2 and 3 ensure that an equilibrium for E' exists (Theorem 4). It is trivial to see that this equilibrium is Pareto efficient and corresponds to a competitive pricing equilibrium with taxes on profits, for the original economy. ■

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5 APPENDIX: Proofs of Theorems 2 and 3

Preliminaries

Let $Y \subset \mathbb{R}^\ell$, $\mathbf{y}^*$ a point in Y, and let $\Phi : Y \to \mathbb{R}^\ell$ be a set valued mapping. The **limsup** of Φ at $\mathbf{y}^*$ is given by:

$$\begin{aligned} LS\Phi(\mathbf{y}^*) &\equiv \{\mathbf{p} = \lim \mathbf{p}^\nu \ / \ \exists\{\mathbf{y}^\nu\} \subset Y, \{\mathbf{y}^\nu\} \to \mathbf{y}^* \text{ and} \\ \mathbf{p}^\nu &\in \Phi(\mathbf{y}^\nu) \ \forall \nu\} \end{aligned}$$

In words: By *limsup* of Φ at $\mathbf{y}^*$ we denote the set of all points which are limits of sequences of points $\mathbf{p}^\nu \in \Phi(\mathbf{y}^\nu)$, when $\mathbf{y}^\nu \to \mathbf{y}^*$. Observe that when Φ is a closed correspondence $LS\Phi(\mathbf{y}^*) = \Phi(\mathbf{y}^*)$, for each $\mathbf{y}^* \in Y$. When this is not so, the *limsup* may be thought of as an operator that "closes the graph" of Φ. Observe that this does not mean that it coincides with $\Phi(\mathbf{y}^*)$ when Φ is closed-valued. It follows from the definition that if $\Phi(\mathbf{y})$ is a cone, then $LS\Phi(\mathbf{y})$ is also a cone.

The following result is known [a proof can be found in Villar (1996, Prop. 5.1)].

Proposition 1 *Let $Y \subset \mathbb{R}^\ell$ be a closed set, and $\Phi : Y \to \mathbb{R}^\ell_+$ a correspondence such that $\Phi(\mathbf{y})$ is a cone, for each $\mathbf{y} \in Y$. Define then a new correspondence $\Gamma : Y \to \mathbb{R}^\ell_+$ that associates to each $\mathbf{y}^* \in Y$ the convex hull of $LS(\mathbf{y}^*)$, that is, $\Gamma(\mathbf{y}^*) = CoLS(\mathbf{y}^*)$. Then Γ has a closed graph.*

Let Y be a closed and convex subset of $\mathbb{R}^\ell$, and $\mathbf{y} \in Y$. The **normal cone** of Y at $\mathbf{y}$, $\mathbb{N}_Y(\mathbf{y})$, is given by:

$$\mathbb{N}_Y(\mathbf{y}) \equiv \{\mathbf{p} \in \mathbb{R}^\ell \mid \mathbf{p}(\mathbf{y}' - \mathbf{y}) \leq 0, \forall \mathbf{y}' \in Y\}$$

Thus when production sets are convex, the cone of normals at a given point $\mathbf{y}$ gives us the set of prices for which this production plan is a profit maximizer at given prices.

The following lemma gives us the relationship between the normal cone to a point $\mathbf{y}$ in the intersection of two convex sets, and the normal cones of each individual set at $\mathbf{y}$:

Lemma 1 (Rockafellar (1970, Corol. 23.8.1)) *Let A, B be convex sets in $\mathbb{R}^\ell$ whose relative interiors have a point in common, and let $\mathbf{y} \in A \cap B$. Then, $\mathbb{N}_{A\cap B}(\mathbf{y}) = \mathbb{N}_A(\mathbf{y}) + \mathbb{N}_B(\mathbf{y})$.*

The following proposition is obtained:

Proposition 2 *Suppose that axiom 1 holds and Y_j is convex. Then, $\widehat{\phi}(\mathbf{y}) \subset \mathbb{N}_Y(\mathbf{y})$.*

Proof.

Let $Y_j(\mathbf{a}'_j) = \{(\mathbf{a}_j, \mathbf{b}_j) \in Y_j \ / \ \mathbf{a}_j \geq \mathbf{a}'_j\}$, a convex set when Y_j is convex. Now observe that:

(i) For all $\mathbf{y}'_j = (\mathbf{a}'_j, \mathbf{b}'_j)$, $\phi_j(\mathbf{y}') = \mathbb{N}_{Y_j(\mathbf{a}'_j)}(\mathbf{y}'_j) \bigcap \mathbb{R}^\ell_+$.

(ii) $Y_j(\mathbf{a}'_j) = Y_j \bigcap C(\mathbf{a}'_j)$, where $C(\mathbf{a}'_j) = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^\ell \ / \ \mathbf{a} \geq \mathbf{a}'_j\}$.

By Lemma 1 we have $\mathbb{N}_{Y_j(\mathbf{a}'_j)}(\mathbf{y}') = \mathbb{N}_{Y_j}(\mathbf{y}') + \mathbb{N}_{C(\mathbf{a}'_j)}(\mathbf{y}')$. It is easy to see that $\mathbb{N}_{C(\mathbf{a}'_j)}(\mathbf{y}') = \{(\mathbf{q}_K, \mathbf{q}_S) \in -\mathbb{R}^\ell_+ \ / \ \mathbf{q}_S = \mathbf{0}\}$. Consequently we can write:

$$\phi_j(\mathbf{y}'_j) = \left[\mathbb{N}_{Y_j}(\mathbf{y}') - \{(\mathbf{q}_K, \mathbf{q}_S) \in \mathbb{R}^\ell_+ \ / \ \mathbf{q}_S = \mathbf{0}\}\right] \bigcap \mathbb{R}^\ell_+$$

From this expression it follows immediately that $\widehat{\phi}(\mathbf{y}) \subset \mathbb{N}_Y(\mathbf{y})$. In particular, when $\mathbb{N}_{Y_j}(\mathbf{y}')$ is a half-line $\widehat{\phi}_j(\mathbf{y}'_j) = \mathbb{N}_Y(\mathbf{y})$. ■

The next result, that appears in Dehez & Drèze (1988a, Lemma 7) as communicated by B. Cornet, says that a closed convex and comprehensive production set is almost everywhere smooth. Formally:

Proposition 3 *Let T be a non-empty, closed, convex and comprehensive subset of $\mathbb{R}^\ell$, with $T \neq \mathbb{R}^\ell$. Then there exists a dense subset $D \subset \partial T$ such that the normal cone is a half-line at every point in D and can be defined, for all $\mathbf{y}^* \in \partial T$ as*

$$\mathbb{N}(\mathbf{y}^*) = co\{\mathbf{p} \in \mathbb{R}^\ell_+ \ / \ \exists(\mathbf{p}^\nu, \mathbf{y}^\nu) \to (\mathbf{p}, \mathbf{y}^*), \ \mathbf{y}^\nu \in D, \, \mathbf{p}^\nu \in \mathbb{N}(\mathbf{y}^\nu) \ \forall \ \nu\}$$

Proof of Theorem 2

(i) Axiom 1 implies that $\phi^*_j(\mathbf{y}_j)$ is a closed convex cone, for every $\mathbf{y}_j \in \mathbb{F}_j$. We have already shown that, under axioms 1 to 3, the constrained profit maximization pricing rule is non-empty valued. From this it follows immediately that $\phi^*_j(\mathbf{y}_j)$ is a non-degenerate closed convex cone, and Proposition 1 ensures that ϕ^*_j is closed.

(ii) As $\widehat{\phi}_j(\mathbf{y}_j) \subset \phi_j(\mathbf{y}_j)$, the only case to be considered is that in which $\phi^*_j(\mathbf{y}_j) \neq \widehat{\phi}_j(\mathbf{y}_j)$. Let $\mathbf{p}$ be a price vector that is in $\phi^*_j(\mathbf{y}_j)$ but is not in $\widehat{\phi}_j(\mathbf{y}_j)$. By definition, $\mathbf{p} = \sum_{i=1}^{\ell+1} \lambda_i \mathbf{q}_i$, where each $\mathbf{q}_i \in \mathbb{R}^\ell_+$ is a limit point of a sequence $(\mathbf{q}^\nu, \mathbf{y}^\nu_j)$ with $\mathbf{q}^\nu \in \widehat{\phi}_j(\mathbf{y}^\nu_j)$, $\mathbf{y}^\nu_j \in \mathbb{F}_j$, $\mathbf{y}^\nu_j \to \mathbf{y}_j$. As ϕ_j is closed

(Theorem 1), each of these $\mathbf{q}_i$ belongs to ϕ_j, that is a closed convex cone. Therefore, $\mathbf{p} \in \phi_j(\mathbf{y}_j)$ as well. Hence, ϕ_j^* is a subcorrespondence of ϕ_j.

Proof of Theorem 3

Suppose first that $\mathbb{N}_{Y_j}(\mathbf{y}_j)$ is a half-line. From Proposition 2 we know that $\widehat{\phi}_j(\mathbf{y}) = \mathbb{N}_{Y_j}(\mathbf{y}_j)$. Moreover, $\phi_j^*(\mathbf{y}_j) = \widehat{\phi}_j(\mathbf{y}_j)$, because in this case $\widehat{\phi}_j$ is closed at $\mathbf{y}_j$. Hence, $\phi_j^*(\mathbf{y}_j) = \mathbb{N}_{Y_j}(\mathbf{y}_j)$.

Let now $T_j(\mathbf{y}_j, \varepsilon) = ch\{B(\mathbf{y}_j, \varepsilon) \cap Y_j\}$, where $ch\{.\}$ denotes the comprehensive hull, and $B(\mathbf{y}_j, \varepsilon)$ is a closed ball of center $\mathbf{y}_j$ and radius $\varepsilon > 0$. By assumption $B(\mathbf{y}_j, \varepsilon) \cap Y_j$ is convex. Thus we can apply Proposition 3 that ensures the existence of a dense subset $D_j \subset T_j(\mathbf{y}_j, \varepsilon)$ such that

$$\mathbb{N}(\mathbf{y}) = co\{\mathbf{p} \in \mathbb{R}_+^\ell \ / \ \exists (\mathbf{p}^\nu, \mathbf{y}_j^\nu) \to (\mathbf{p}, \mathbf{y}_j),\ \mathbf{y}_j^\nu \in D_j,\ \mathbf{p}^\nu \in \widehat{\phi}_j(\mathbf{y}_j^\nu)\ \forall\ \nu\}$$

The desired equality follows when the sequences are taken in $T_j(\mathbf{y}_j, \varepsilon) \setminus D_j$ as well.