THE EDGEWORTH, COURNOT AND WALRASIAN CORES OF AN ECONOMY

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October 2003

COWLES FOUNDATION DISCUSSION PAPER NO. 1439

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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August 1, 2003

Abstract

Three variations of the core of a market game representing and exchange economy are considered and compared. The possibility for utilizing the Walrasian core to reflect certain monetary phenomena is noted.

Keywords: Market game, Strategic market game, exchange economy, core, characteristic function.

JEL Classification: C71, D5

Contents

1 Walrasian Cores

The core of a market game and its relationship to an exchange economy is well known in economic theory and provides a valuable link in our understanding of the nature of both competition and collaboration in an exchange economy with many agents. The papers of Shubik [12], Debreu and Scarf [3] and many others have explored the relationship between exchange economies and market games in many basic ways.

Part of the program of Shapley and Shubik [11] in their development of Market Games was to explore many cooperative game theoretic solutions and to understand their relationship to the competitive equilibria an exchange economy with many agents. Thus not only was the core solution considered, so was the value, nucleolus (Schmeidler [8]) and several others.

Notably missing from the list of comparisons of the closed competitive equilibrium model and the market game treatment of exchange was the Cournot–Nash noncooperative equilibrium. A little reflection on why one of the oldest mathematical models in economics should be missing from the list of game theoretic solutions considered, shows that the distinction in modeling between the cooperative approaches to the study of the price system and the noncooperative approach is fundamental. The Market Game approach uses the Coalitional Form of a game with its primitive concept being the Characteristic Function[1] of an n-person Game. The models to which the non-cooperative theory are applied utilize the Strategic or the Extensive Form of an n-person game. A key distinction between an exchange economy modeled in coalitional form and one modeled in strategic form is that in the former there is no explicit mechanism for price formation, whereas in the latter the need to be explicit about the rules of motion within the game calls for the specification of how price is formed. It was the observation that in order to be able to consider the properties of Cournot competition in a closed economy, the economy had to be modeled in strategic form, that led to the construction of Strategic Market Games [14], [11], [9].

1.1 Institutional versus Non-institutional Approaches

In viewing some technical problems in trying to construct an intrinsically symmetric game with n traders trading in n commodities, the old problem of selecting the commodity to be a unit of account made the treatment of symmetric roles for all traders difficult unless one introduced an extra commodity. This extra commodity had a natural interpretation in terms of a money in the system. Furthermore giving each agent a supply of at least one commodity and a supply of the money enabled one to construct an elementary simple price formation device.

At first glance the seemingly innocent action of adding a commodity to provide an extra of degree freedom to facilitate the construction of symmetric strategy sets can be done utilizing a fictional unit of account whose net supply is zero or by using a magical u-money in finite supply. There is a distinct difference between the two. One reflects the pure numeraire properties of a money and the other its properties as a

[1] More precisely we should distinguish between the characteristic function of an n-person game with side-payments and the more complicated characterizing function for no-side-payments.

special commodity. When considered from the viewpoint of cooperative game theory, the first leads naturally to a treatment via no-sidepayment (NSP) game theory and (when the supply of the special monetary good is large enough) the second leads to a treatment via side-payment (SP) games.

The approach of cooperative game theory to the study of economic exchange is at a higher level of abstraction than is general equilibrium theory. This can be seen immediately from the observation that the relationship between a market game and an exchange economy is such that there are many exchange economies that map onto the same market game. The concerns of Shapley and Shubik in their study of the convergence properties of various game theoretic solutions to Market Games were to show the emergence of price rather than to assume its existence.

The mathematization of general equilibrium theory has been carried out in a way that minimizes institutional content. Markets and the mechanisms for forming price are not specified beyond comments on supply and demand or excess demand conditions. Essentially the institutional detail is swept away by the implicit assumption of a "law of one price" in each market.[2]

In contrast with the two above, the formulation of Strategic Market Games was devoted explicitly to fully defining market exchange in the strategic form of a game. If institutions are regarded as the carriers of process, the act of completely defining a playable game in strategic form calls for, at the very minimum, the specification of mechanism which can be interpreted as minimal institutions. This provides for a way to consider the invariant properties of markets and other financial institutions.

1.2 Three Approaches Considered

When one considers Edgeworth's [4] treatment of exchange it is naturally generalized to a no-sidepayment (NSP) game theoretic model. The modeling problems faced in translating the Edgeworth model of trading among subsets of agents provides a direct and simple way to define the characteristic function of a market game. We consider the possibility that any subset S of traders in a game involving N traders can choose to limit exchange to their own group without being tied in any way to the others. In the terminology suggested by Shapley and Shubik such a game can be described as a c-game, i.e., a game in coalition form where the underlying rational for assigning the structure of the feasible set of imputations of wealth achievable by the players adequately reflects the physical process being considered. This statement is, of necessity somewhat vague, but it identifies one of the considerable difficulties in the examination of games in coalitional form. This difficulty was present in the original work of von Neumann and Morgenstern and they attempted to solve it by introducing a dummy $N+1$st player into a N-person nonzero sum game.

Although at first glance the Edgeworth approach to the emergence of the price system might appear to be easily consistent with Walras or Cournot, there are several

[2] Anyone who has looked at an actual market in real time knows that, at best, "one price" is an extremely crude approximation as the price formation depends on a myriad of detailed empirically observable variables such as transportation costs, information and evaluation, packaging and preserving, inventorying, accounting, ...etc.

subtle points among all three. In essence, although all three were writing well before the development of pure game theory methods, they all have an immediate and direct interpretation in terms of game theory. Edgeworth's approach was clearly coalitional and was a natural precursor of the core of a market game [10]. Cournot's approach [2] was clearly non-cooperative and was a natural precursor of the noncooperative equilibrium approach to games in strategic form [5]. Walras' [16] work placed emphasis on the influence of the price system as a device for the efficient decentralization of individual decision making. From the game theoretic point of view, the competitive equilibria of an exchange economy[3] may be regarded as the limit of the core or other solutions as the number of agents in the economy becomes large.

There are several ways in which one can portray the meaning of the number of agents becoming large. We can consider replication, replacing one butcher or baker by k butchers or bakers or we can consider a continuum of agents. When we do the latter, mathematical links between general equilibrium theory and noncooperative game theory methods can be forged. But for many purposes of economic analysis, one can make the simple assumption that in a Walrasian economy the individual small agents are price takers and we do not need to be concerned with group action or coalitions.

1.3 Pecuniary Externalities and Markets as a Weak Public Good

An article by Viner [15] in the 1930s noted that developments in one industry may change costs of inputs to another industry. He termed this phenomenon to be a "pecuniary externality" and not an externality caused by the interaction of physical processes. The externality suggested by Viner can best be described as a network externality caused by the connections in an economy where all agents are interlinked via a market system utilizing some form of money, credit and prices. I suggested [13] that a way to view the concept of a "pecuniary externality" is as a weak externality which may more or less attenuate as the number of agents becomes large.

In the following sections the distinctions among the three approaches noted above are considered at the level of the formation of the characteristic function.

1.4 The Characteristic and the Generalized Characteristic Function of an N-Person Game

A key feature in the theory of cooperative side-payment n-person games is the Characteristic Function. The idea is to capture in a single numerical index the potential worth of each coalition of players. The characteristic function is the final distillation of the descriptive phase of the theory. It may also be regarded as a "presolution" or as a first crude cut at a solution.

With the characteristic function in hand, all questions of tactics, information, and physical transactions are removed. As much as possible it is "institution free." The

[3] The qualification "exchange economy" is utilized here because the introduction of production brings in considerably more difficulties which are not germaine to the discussion here.

characteristic function is primarily a device for dividing difficulties and for eliminating as many distractions as possible in preparation for the confrontation with the indeterminacy of the n-person problem.

Unfortunately, not all games admit a clear separation between strategic and coalitional questions, and for those that do not, the characteristic-function approach must be modified or abandoned.

1.4.1 The Concept of c-game

There are at least two major limitations to the application of the characteristic function. Games where one group can threaten or damage the others are not easily modeled with a characteristic function. Many games do not allow the clean separation that we need between questions of strategic optimization and questions of negotiation. For example, a player may have a number of possible threat strategies against a bargaining opponent. Although we can still define the characteristic function for such variable-threat situations, we cannot analyze them properly without additional information about the actual rules of play. This information is lost when one passes to the characteristic-function form.

A second problem with the characteristic function as defined here is that representing a coalition's worth by a single number implies freely transferable utility. For the nontransferable case a different notion of characteristic function is required.

To simplify discussion, Shapley and Shubik coined the term "c-game" for a game that is adequately represented by its characteristic function. We do not attempt a categorical definition of this term. What is adequate in a given instance may well depend on the solution concept that we intend to employ. To say that a game is a c-game is merely a way of asserting that nothing essential to the ultimate purpose of the model is lost in the process of condensing the extensive or strategic description into a characteristic function.

Although the definition of c-game is not categorical, two important and easily recognized classes of games qualify quite generally as c-games, no matter what solution is adopted. The first are the constant-sum games, in which the total payoff is a fixed quantity, regardless of the strategies chosen. In such a game, the characteristic value of a coalition accurately represents its true worth, because the worst threats against the coalition are precisely the plays that maximize the payoff to the players outside the coalition. The characteristic function pessimistically assumes the worst, and in this case "the worst" is a reasonable assumption.

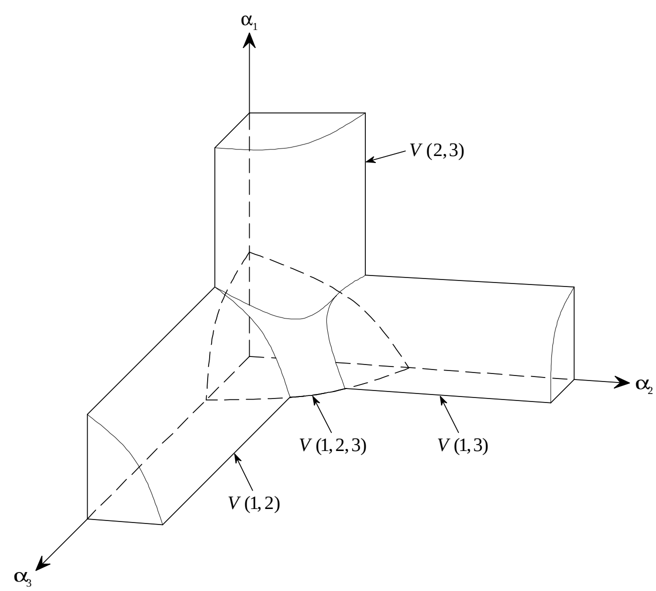
Of more immediate interest to economics is another class of c-games, called games of consent or games of orthogonal coalitions. The idea here is that nothing can happen to change a player's fortune (payoff) unless he himself is a party to the action. Either you cooperate with someone or you ignore him; you cannot actively hurt him. In this case the only threat by outsiders against a coalition is not to belong to it-that is, the boycott. To sum up, in a constant-sum game "you're either with us or against us"; in a game of consent "you're either with us or we don't care what you do." In both cases a coalition might as well assume the worst and equate the outsiders' intentions with their capabilities.

While constant-sum games are relatively uncommon in economic theory, games satisfying the consent condition arise in many place, most notably in models of pure competition without externalities.

1.4.2 The generalized characteristic function

In games without "utility-money" (u-money), a single index number $v(S)$ cannot completely express the worth of the coalition S. For one thing, it makes no sense to lump together the payoffs to the members of S, since they are no longer free to redistribute the total as they please. In fact, the total may be meaningless if the payoffs are in incomparable units. But even a vector of payoffs will not usually be enough to represent the "best" efforts of the coalition; a set of vectors will generally be required-defining a sort of coalition limited "Pareto" or maximal surface to capture the true worth or effectiveness of S.

There are several ways to define this characteristic set, which is traditionally denoted by $V(S)$. Some authors place it in the space of S-vectors, since only the payoffs to members of S are significant. We treat each space of S-vectors as a linear subspace of the space of N-vectors (where N denoted the all-player set); this has the effect of making each element of $V(S)$ a complete payoff vector of the game by arbitrarily assigning zero to the players outside S. Zero has no intrinsic meaning in this formulation (Billera and Bixby [1]). The most popular device, however, is to make $V(S)$ a full-fledged subset of N-space, defining it as though the players outside S could get all possible payoffs. This makes $V(S)$ a "cylinder," that is, the Cartesian product of a subset of the S-vectors with a linear space of dimension $|N - S|$.(Scarf [7]). This expedient simplifies several basic formulas and definitions, such as those of superadditivity and balancedness, and also provides a more coherent geometrical picture for the mind's eye (see figure below).



For several reasons, it is most convenient not to work with the maximal surface alone, but to include all points that lie "below" it. The conventions can all be summed up by the following condition: for any two payoff vectors α and β, if $\alpha \in V(S)$ and if $\alpha_i \geq \beta_i$ for each $i \in S$, then $\beta \in V(S)$.

We always define $V(S)$ so that it is a closed set. Its interior is denoted by $D(S)$; intuitively $D(S)$ is the set of all payoff vectors that the coalition S can improve upon. In many applications $V(S)$ and $D(S)$ will be convex sets, but this is not a requirement. Similarly, most applications superadditivity will hold; this condition takes the following simple form:

$$V(S \cup T) \supseteq V(S) \cap V(T) \text{ if } S \cap T = \phi.$$

If we try to express a transferable-utility game in this notation, the resulting sets $V(S)$ have an especially simple form. Transferability means that with any payoff vector α in $V(s)$ we must include all other vectors $\alpha + \pi$ where $\sum_{i \in S} \alpha_i = 0$.

Thus $V(S)$ is a half-space whose upper boundary (which is all that really matters) is a hyperplane oriented in a direction that depends only on S. The equation of this hyperplane has the form of $\sum_{i \in S} \alpha_i =$ constant, and the constant is the number $v(S)$ of the transferable theory.

2 The Edgeworth Core

The Edgeworth core, both in its side-payment and no-side-payment version is based on the characteristic function of a "c-game," it is a "game of consent" or a "game of orthogonal coalitions," i.e., a game in which we more or less believe the plausibility of the story of how it is derived.

As the investigation of the Edgeworth core is well known only one *caveat* is noted. The modeling problems in extending the exchange economy to one with production or in trying to capture conditions concerning features such as non-symmetric information pose great difficulties. Many attempts have been made and there does not yet appear to be a clear consensus on these extensions.

In the discussion here all of the remarks are limited to games associated with an exchange economy.

3 The Cournot Core

Given the body of literature on Market Games and on Strategic Market Games, a natural question to consider is whether there are any worthwhile ties which can further link these somewhat different modes of modeling and analysis. A head-on approach to dealing with the cooperative or coalitional theory link of Strategic Market Games to Market Games is to accept the strategic structure of the former as the underlying primitive concept and to apply the appropriate operators to it in order to derive an acceptable characteristic function.

The type of behavior operators which one might contemplate in attempting to produce a satisfactory coalition form must be able to reflect realistically the possibilities of threat. In the study of oligopolistic behavior among few firms, predatory behavior and threats have often been taken into consideration. A problem with modeling threats is how does one reflect the costs and plausibility of the threat. Military doctrine distinguishes between "capabilities and intentions." One can adopt a "maxmin" approach where the assumption is that the complementary coalition $\bar{S}$ will try to do as much damage as possible to S; but this is highly pessimistic. A more reasonable way to handle the valuation of threats in a damage-exchange rate mode as is suggested by Nash [5] and extended by Harsanyi and Selten.

Although this approach appears to be feasible and I conjecture that one can obtain a plausible theorem yielding the convergence of the Cournot core towards the set of competitive equilibria of an associated trading economy, if one is concerned primarily with the behavior of the core in large games, the Walrasian core which reflects the law of one price and can be formulated in a way to avoid the threat problems appears to be more fruitful for several purposes.

4 The Walrasian Core

Rather than becoming enmeshed with the problems of the adequate representation of threats in the Cournot core, yet at the same time modifying the Edgeworth core to reflect a law of one price, let $\omega^i$and x^i stand respectively for the initial and final bundle of goods held by an individual i. An S-allocation is defined by Qin, Shapley and Shimomura [6] as "*Walras-feasible* if, in addition to satisfying *Edgeworth-feasibility"* it is supportable by a price schedule, that is, if there exists a price schedule $p \in P$ such that $p \cdot x^i = p \cdot \omega^i$ holds for all $i \in S'$.

The feasible set of outcomes attainable by a coalition S that is "*Walras-feasible* will be a subset of those satisfying the *Edgeworth-feasible* conditions. They will be contained in the "cylinders" noted in the figure. Qin, Shapley and Shimomura establish the convergence of the Walras core to the competitive equilibria[4] along with a detailed comparison of the differences between the Edgeworth and Walrasian cores.

5 Some Further Extensions

A basic motivation in the development of strategic market games was to be able to encompass the cooperative and non-cooperative game theory approaches to the problems of exchange and prices in an exchange economy in a way that their limiting behavior could be examined and compared. In essence, the replicated cooperative solutions described trajectories on a (type symmetry adjusted) Pareto optimal surface. The nature of the strategic form is that it requires far more detail in description than the cooperative form. In essence, the specification of the strategy sets provides

[4]Existence follows immediately from the observation that the feasible set for any coalition S under the Walrasian definition is contained in the feasible set under the Edgeworth definition. Thus from the Edgeworth results the core will exist but be "looser."

the existence of minimal institutions as reflected in the basic rules of the game which serve as the carriers of process. The investigation of the core is carried out at a higher level of abstraction than that of the strategic market game. In a sense, it is less institutional and more concerned with properties which are invariant to the details of changes in institutional form that often have considerable influence in noncooperative analysis.

The three variations of the core discussed above are natural relatives to the class of underlying exchange economies. Of the three variants the Walrasian core appears to be amenable to further extension and modification which, in turn, may reflect not only aspects of exchange and price but the roles of money and credit in an exchange economy. In further work in progress Qin, Shapley and Shubik are considering the extensions.

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