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EXISTENCE OF WALRAS EQUILIBRIUM WITHOUT A PRICE PLAYER OR GENERALIZED GAME*

by

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1. INTRODUCTION

The paper of Nash ([9]) on the existence of equilibrium points in noncooperative games was historically critical for Walrasian analysis. In order to prove existence of Walras equilibrium, Arrow and Debreu ([1], [2]) extended Nash's model to "generalized games" and added a fictitious price player (whose payoff was "the value of excess demand") to the original agents in the economy. Walras Equilibria (W.E.) were then obtained as the Nash Equilibria (N.E.) of a generalized game that included the price player.

But W.E. can be shown to exist without stepping outside the original framework of Nash. In fact, W.E. are N.E. of a strategic market game introduced by Shapley and Shubik. No price player is involved nor are generalized games. The model adheres completely to the standard format laid out by Nash: each player $1 \leq \alpha \leq n$ has a compact, convex strategy–set S^{α}; and a continuous payoff function $u^{\alpha}: S^1 \times \cdots \times S^n \to \mathbb{R}$, which is concave on S^{α}, for every fixed choice of strategies of the other players. To obtain W.E. as N.E. we do need to replace each player α by a type α, consisting of a continuum of identical agents. But since we restrict to type–symmetric strategies, all measure-theoretic technicalities are avoided. By an analysis identical to that of Nash, we verify the existence of an N.E., and hence of a W.E.

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In the Shapley–Shubik model, money is introduced as a medium of exchange, and a "trading–post" set up for each commodity in the economy ([15], [16], [17]). Agents (strategically) send quantities of commodities for sale and/or money for purchase to these posts. Prices form in a natural manner at each post as the ratio of total money to total quantity received, and they mediate trade. Thus the posts function independently of each other, and there is a form of "decentralization" at the level of the market trading mechanism itself. The money in the model, as was pointed out by Shapley and Shubik, may be one of the commodities, or paper that is made legal tender; and a variety of rules concerning its availability are possible, in particular whether it can be borrowed and what the penalty is for agents who go bankrupt.

This model contains within it a straight proof of existence of W.E. as N.E., but though the model has been much studied this fact has gone surprisingly unnoticed. For those variants of the model in which W.E. and N.E. coincide (in the continuum), existence of N.E. is proved either via generalized games[1] or else by invoking the known existence of W.E. ([8], [10], [11]). In the variants which adhere to the Nash framework ([4], [15], [16], [17]) N.E. exist by Nash's argument, but they do not coincide with W.E. except under very special conditions. In this note we show that there is a variant "in between." It is in the Nash framework and yet its N.E. coincide with W.E. Thus we obtain an existence proof of W.E., without a price player or generalized games. It seems to us useful, from a pedagogical point–of–view, to describe this direct route from Nash to Walras.

Other models of strategic games of the Bertrand type, which use both prices and quantities as strategies, have been developed ([5], [7], [14]). They show coincidence of W.E. and N.E., but rely upon the known existence of W.E. to infer that N.E. also exist.

[1]The notion of "generalized game" is not easy to interpret in terms of how the game is to be played. What happens if the independent choices of the players turn out to be jointly infeasible? One can, of course, assign an arbitrarily large and negative payoff to each player in this case, so that he prefers any feasible choice to any infeasible one. At the formal level a game is reinstated, but with ad hoc and unrealistic "penalties," which have no relation to the "size" of the crime. The payoffs in such a game are discontinuous and so do not fit into the Nash framework.

The one exception is provided by a paper of Sahi and Yao ([12]), who work with a model due to Shapley, prove the existence of N.E., and show that N.E. converge to W.E. under replication of the agents. But the model is an order–of–magnitude more complex than what we present here. (Also in [12] somewhat stronger conditions are imposed on utilities.)

The result of our note is embedded in [6], but we thought it useful to extract it on its own (without all the complications that arise in [6]).

2. THE EXCHANGE ECONOMY

Consider an exchange economy with n agents and ℓ commodities. Let $e^\alpha \in \mathbb{R}_+^\ell$ be the endowment, and $u^\alpha : \mathbb{R}_+^\ell \to \mathbb{R}$ the utility function, of $1 \le \alpha \le n$. We assume

(i) $$\sum_{\alpha=1}^{n} e^\alpha >> 0$$

(ii) $e^\alpha \neq 0$ for $1 \le \alpha \le n$

(iii) u^α is continuous, concave and strongly monotonic (i.e. strictly increasing in each variable) for $1 \le \alpha \le n$.

(For relaxations of (iii), see Remark 1.)

Recall that $(x^1, \ldots, x^n; p)$, where each $x^\alpha \in \mathbb{R}_+^\ell$ and $p \in \mathbb{R}_{++}^\ell$, is a *Walras Equilibrium* (W.E.) if

$$\sum_{\alpha=1}^{n} x^\alpha = \sum_{\alpha=1}^{n} e^\alpha ,$$

and

$$x^\alpha = \operatorname{argmax}\{u^\alpha(y) : y \in \mathbb{R}_+^\ell, p \cdot y = p \cdot e^\alpha\} \text{ for } 1 \le \alpha \le n .$$

3. THE GAMES Γ_M

We consider a continuum of traders $T = [0,n)$, made up of n types $T_\alpha = [\alpha-1, \alpha)$, where $e^t = e^\alpha$ and $u^t = u^\alpha$ for $t \in T_\alpha$. For $M > 0$, let $S_M = \{y \in \mathbb{R}^\ell : \frac{1}{M} \leq y_j \leq 2M$ for $1 \leq j \leq \ell\}$. Then S_M is the strategy-set of each $t \in T$ in Γ_M. An integrable choice $b : T \to S_M$ produces outcomes $(x^t, c^t) \in \mathbb{R}^\ell_+ \times \mathbb{R}$ for all $t \in T$, where x^t is the final bundle of t and c^t is his "net credit." The payoff Π^t to t of (x^t, c^t) is $u^t(x^t) - c^t_+$, where $c^t_+ = \max\{0, c^t\}$. Here $-c^t_+$ is a "bankruptcy penalty" imposed on t if he owes $c^t > 0$ to the "bank" at the end of trade.

It remains to define the strategy–to–outcome map. Denote $b(t)$ by b^t and $\sum_{j=1}^{\ell} b^t_j$ by $\overline{b}^t$. We interpret $\overline{b}^t$ as an "I.O.U." note sent by t to obtain money from the bank. The supply of bank money in the game Γ_M is M, hence an interest rate $\sigma \in \mathbb{R}$, with $1+\sigma \geq \frac{n}{M^2} > 0$, is formed by the rule (assuming b integrable)

$$(1) \qquad 1 + \sigma(b) = \frac{\int_T \overline{b}^t}{M}.$$

The money at hand for t is then $\overline{b}^t/(1+\sigma)$. The amount $b^t_j/(1+\sigma)$ is sent by t for purchase of commodity j at the j^{th} "trading–post" $(1 \leq j \leq \ell)$. We suppose for simplicity that all the endowments are up for sale. Thus the price $p_j > 0$ is given by

$$(2) \qquad p_j(b) = \frac{\int_T b^t_j}{(1 + \sigma(b))\int_T e^t_j} = \frac{\int_T b^t_j}{(1 + \sigma(b))(\sum_{\alpha=1}^{n} e^\alpha_j)},$$

for $1 \leq j \leq \ell$. Trade is mediated by these prices, so t winds up with (x^t, c^t) where

$$(3) \qquad x^t_j(b) = \frac{b^t_j}{(1+\sigma(b))p_j(b)}, \quad 1 \leq j \leq \ell$$

(4) $$c^t(b) = \overline{b}^t - \sum_{j=1}^{\ell} p_j(b) e_j^t ,$$

and obtains the payoff

(5) $$\Pi^t(b) = u^t(x^t(b)) - c_+^t(b) .$$

A *Nash Equilibrium* (N.E.) is an integrable b such that, for all $t \in T$ and $a^t \in S_M$,

$$\Pi^t(b|a^t) \leq \Pi^t(b)$$

where $(b|a^t)$ is the same as b, except that b^t has been replaced by a^t. If, furthermore, $b^t = b^{t'}$ for $t, t' \in T_\alpha$ and $1 \leq \alpha \leq n$, we say that b is a type–symmetric play and that the N.E. is *type–symmetric* and denote it by T.S.N.E. In this case b is written as $\{b^\alpha\}_{\alpha=1}^n$ and (x,c) as $\{x^\alpha, c^\alpha\}_{\alpha=1}^n$ without confusion.

We confine attention to type–symmetric plays b, and deviations to $a^t \in S_M$ by a single player t in the continuum. Thus both b and $(b|a^t)$ are automatically integrable. Moreover all measure-theoretic technicalities are avoided.

4. EXISTENCE OF WALRAS EQUILIBRIUM AS NASH EQUILIBRIUM

THEOREM 1: *Under assumptions (i), (ii), (iii), a T.S.N.E. exists for* Γ_M *if* $M > 0$. *Moreover as* $M \to \infty$, *"limits" of T.S.N.E. outcomes of* Γ_M *exist; and each such limit is a W.E.*

The proof of Theorem 1 follows from Claims 1–4 below.

CLAIM 1: A T.S.N.E. exists for Γ_M, any $M > 0$.

PROOF: This follows from the theorem of Nash. We recall the main steps. Let $b = \{b^\alpha\}_{\alpha=1}^n$ be a type–symmetric play. Then, for fixed b, the map $(b|a^t) \to (x^t, c^t)$ is a linear function of $a^t \in S_M$ since a^t cannot affect σ or p. Hence $\Pi^t(b|a^t) = u^t(x^t) - c_+^t$ is continuous and concave for $a^t \in S_M$. Define $S_M^\alpha(b) \subset S_M$ by

$S_M^\alpha(b) = \operatorname{argmax}\{\Pi^t(b|a^t) : a^t \in S_M, t \in T_\alpha\}$. (Clearly the definition is invariant of the choice of $t \in T_\alpha$.) Consider the map

$$b = \{b^\alpha\}_{\alpha=1}^n \longrightarrow S_M^1(b) \times \dots \times S_M^n(b)$$

from $(S_M)^n$ to subsets of $(S_M)^n$. By Kakutani's theorem it has a fixed point, which is easily verified to be a T.S.N.E. □

CLAIM 2: $\sigma \geq 0$ at any T.S.N.E. of Γ_M.

PROOF: If $\sigma < 0$ then $c^t < 0$ for all $t \in T_\alpha$ and some α. Any such t could purchase more of every commodity without going bankrupt, and by (weak) monotonicity of utilities his payoff would increase, a contradiction. □

CLAIM 3: Let $\sigma(M)$ be the interest rate produced at some T.S.N.E. of Γ_M. Then $\sigma(M) \to 0$ as $M \to \infty$.

PROOF: Suppose $\sigma(M) > K > 0$ for all M along a subsequence of M. Let $\{x^\alpha(M), c^\alpha(M)\}_{\alpha=1}^n$ be the T.S.N.E. outcomes. Then $c^\alpha(M) > \frac{KM}{n}$ for some type α. If $-\frac{KM}{n} + u^\alpha(\sum_{r=1}^n e^r) < \min_{1 \leq \beta \leq n} e^\beta(0) - \frac{\ell}{M}$, each $t \in T_\alpha$ would have improved by choosing $a^t = (1/M, \dots, 1/M)$ in S_M, a contradiction. □

Choose a subsequence $b(M)$ of T.S.N.E. of Γ_M, $M \to \infty$, with outcomes $\{x^\alpha(M), c^\alpha(M)\}_{\alpha=1}^n$, prices $p(M)$ and interest rate $\sigma(M)$ such that $x^\alpha(M) \to x^\alpha$, for $1 \leq \alpha \leq n$; and $\frac{p_j(M)}{\sum_{k=1}^{\ell} p_k(M)} \to p_j$ for $1 \leq j \leq \ell$.

CLAIM 4: $(x^1, \dots, x^n; p)$ is a W.E.

PROOF: Note that at a T.S.N.E., the net credit $c^t \geq 0$ for each t, therefore, $p(M) \cdot x^\alpha(M) \geq \frac{p(M) \cdot e^\alpha}{1 + \sigma(M)}$ for all M, and all $\alpha = 1, \dots, n$. Since $\sigma(M) \to 0$ by Claim

3, $p \cdot x^\alpha \geq p \cdot e^\alpha$ for all α. Since $\sum_{\alpha=1}^{n} x^\alpha(M) = \sum_{\alpha=1}^{n} e^\alpha$, we also have that $\sum_{\alpha=1}^{n} x^\alpha = \sum_{\alpha=1}^{n} e^\alpha$, and that $p \cdot x^\alpha = p \cdot e^\alpha$ for all α.

We now show that $p >> 0$, and hence that $p \cdot e^\alpha > 0$ for all $\alpha = 1, \ldots, n$. Suppose for some j and i, $p_j(M)/p_i(M) \to \infty$. Choose α with $x_j^\alpha > 0$ (such α must exist since $\sum_{\alpha=1}^{n} x_j^\alpha = \sum_{\alpha=1}^{n} e_j^\alpha >> 0$). By strong monotonicity, $u^\alpha(y) > u^\alpha(x^\alpha)$, where $y_k = x_k^\alpha$ for $k \neq i$, and $y_i = x_i^\alpha + \epsilon$ for $\epsilon > 0$. By continuity, if $\eta > 0$ is sufficiently small, $u^\alpha(z) > u^\alpha(x^\alpha)$, where $z_k = y_k$ for $k \neq j$, but $z_j = y_j - \eta$. If $p_j(M)/p_i(M) \to \infty$, then for sufficiently large M and $t \in T_\alpha$, there is $c < c^\alpha(M)$ and $a^t \in S_M$ with (z,c) the outcome to t of $(b(M)|a^t)$, a contradiction. Thus $p >> 0$.

Let $y \in \mathbb{R}_+^\ell$ and $p \cdot y < p \cdot e^\alpha$. Then since $p \cdot e^\alpha > 0$, there exists $z \in \mathbb{R}_+^\ell$ with $y << z$ and $p \cdot z < p \cdot e^\alpha$. But $\sigma(M) \to 0$ as $M \to \infty$; thus, once again, for sufficiently large M there exists $c < 0$ and $a^t \in S_M$ such that the outcome of $(b(M)|a^t)$ to $t \in T_\alpha$ is (z,c). So $u^\alpha(z) \leq u^\alpha(x^\alpha(M))$ for all sufficiently large M. By continuity of u^α, $u^\alpha(z) \leq u^\alpha(x^\alpha)$. By *weak* monotonicity of u^α, $u^\alpha(y) < u^\alpha(z) \leq u^\alpha(x^\alpha)$. Since y was arbitrary, we get (invoking the continuity of u^α again) that $u^\alpha(y) \leq u^\alpha(x^\alpha)$ for all $y \in \mathbb{R}_+^\ell$ with $p \cdot y \leq p \cdot e^\alpha$. This proves Claim 4. □

5. VARIATIONS ON THE THEME

We are done with the existence of W.E. in Theorem 1. But it may be useful to record some related results.

Theorem 1 does not quite exhibit a game whose N.E. coincide with W.E. (only the limits, as $M \to \infty$, of N.E. of Γ_M are W.E.). But if a boundary condition is satisfied by utilities, then for large but finite M, we get coincidence (Theorem 2). There is also a variant of the model in which N.E. of Γ_M are W.E. (Theorem 3).

THEOREM 2: *Suppose, in addition to (i), (ii), (iii), that each* (u^α, e^α) *satisfies (iv) (the boundary condition) if* $u^\alpha(y) \geq u^\alpha(e^\alpha)$ *and* $y \leq \sum_{\beta=1}^{n} e^\beta$ *, then* $y >> 0$ *, i.e. the indifference surfaces through the initial endowments do not intersect the boundary of* $\mathbb{R}_+^\ell$ *in the relevant region. Then for some finite* M *, the T.S.N.E. outcomes of* Γ_M *coincide with the W.E. of the economy.*

PROOF: To show that any T.S.N.E. outcome is a W.E., it suffices to show that for any sequence as in Claim 4, for finite M, $\sigma(M) = 0$ and $b_j^\alpha > 1/M$ for all $\alpha = 1, \ldots, n$, $j = 1, \ldots, \ell$. Observe that by the boundary condition, the limiting $x^\alpha >> 0$ for all $\alpha = 1, \ldots, n$. Since $\sigma(M) \geq 0$ for all M, $\sum_{\alpha=1}^{n} \overline{b}^\alpha(M) \geq M \to \infty$, and so $p_j(M) \to \infty$ for all j, since as shown in Claim 4, the relative prices are bounded. But then every agent must be bidding $b_j^\alpha(M) > 1/M$ in order to purchase $x_j^\alpha(M)$ near $x_j^\alpha > 0$. By concavity and strong monotonicity, there is r sufficiently large such that for any $\alpha = 1, \ldots, n$ and $j = 1, \ldots, \ell$, and $z \geq \frac{1}{2}x^\alpha >> 0$, if $y_k = z_k$ for $k \neq j$, and $y_j = z_j - \epsilon\frac{1}{r}$, then $u^\alpha(y) > u^\alpha(z) - \epsilon$ for all sufficiently small ϵ. If $\sigma(M) > 0$, then some agent α is going bankrupt $c^\alpha(M) > 0$. Such an agent could always bid $\epsilon < c^\alpha(M)$ less on some commodity j, getting $y_j(M) = x_j^\alpha(M) - \epsilon\frac{1}{p_j(M)}$, $y_k(M) = x_k^\alpha(M)$ if $k \neq j$, saving ϵ of bankruptcy penalty. For large M, $p_j(M) > r$, and this contradicts the optimality of the outcome $(x^\alpha(M), c^\alpha(M))$ to players of type α.

Conversely, let $(x^1, \ldots, x^n, p)$ be a W.E. with $\sum_{j=1}^{\ell} p_j = 1$. Then from the concavity of u^α, and the Kuhn–Tucker conditions, for each α there is a smallest $\lambda^\alpha > 0$ such that $u^\alpha(x) - \lambda^\alpha p\cdot(x - x^\alpha) \leq u^\alpha(x^\alpha)$ for all $x \in \mathbb{R}_+^\ell$. Since the set of W.E. is compact, with prices bounded away from 0,, there is some λ bigger than any of the λ^α arising from a W.E. Choose M so large that $M > \lambda \sum_{\alpha=1}^{n} \sum_{j=1}^{\ell} e_j^\alpha$. Given the W.E. $(x^1, \ldots, x^n, p)$, choose $q = \mu p$ for some scalar $\mu > 0$ so that $q\cdot \sum_{\alpha=1}^{n} e^\alpha = M$. Let

$b_j^\alpha = p_j x_j^\alpha$, for all $\alpha = 1, \ldots, n$, $j = 1, \ldots, \ell$. Since by construction, $\mu > \lambda > \lambda^\alpha$, $q > \lambda^\alpha p$ and so $u^\alpha(x) - \sum_{j=1}^{\ell} q_j [x_j - x_j^\alpha]^+ \leq u^\alpha(x^\alpha)$, no agent will want to go bankrupt, or play differently. □

By choosing a variant of the game Γ_M we can establish the coincidence of W.E. and T.S.N.E. without the boundary condition. Consider the game $\Gamma_M(\epsilon)$ defined exactly as Γ_M , except that the common strategy set is now $\Gamma_M(\epsilon) = \{y \in \mathbb{R}_+^\ell \mid \epsilon \leq y_j \leq 2M,$ $j = 1, \ldots, \ell\}$, for $\epsilon \geq 0$. If $\epsilon = 0$, we must also specify the strategy to outcome map in case $\sigma = -1$, or some $p_j = 0$. We do so by defining division by zero to be zero wherever it occurs. The game $\Gamma_M(0)$ is now well–defined, though not continuous. Define an *active* T.S.N.E. of $\Gamma_M(0)$ to be a T.S.N.E. at which $\sigma > -1$ and $p_j > 0$ for all $j = 1, \ldots, L$.

LEMMA: *Fix any* $M > 0$. *Under hypotheses (i), (ii), (iii), any limit* b *of T.S.N.E. strategies* $b(\epsilon)$ *for the games* $\Gamma_M(\epsilon)$ *as* $\epsilon \to 0$ *is an active T.S.N.E. of* $\Gamma_M(0)$.

PROOF: This follows from the fact that the prices $p(\epsilon)$ resulting from $b(\epsilon)$ in $\Gamma_M(\epsilon)$ always satisfy $p(\epsilon) \cdot \sum_h e^h = M$. By the argument from Claim 4, we cannot have $p_j(\epsilon)/p_i(\epsilon) \to \infty$, hence $\lim_{\epsilon \to 0} p(\epsilon) \to p >> 0$. □

THEOREM 3: *Under conditions (i), (ii), (iii), for all* M *sufficiently large, the active T.S.N.E. outcomes of* $\Gamma_M(0)$ *are identical to the W.E.*

PROOF: The proof is along the same lines as Theorem 2. □

REMARK: We can replace the strong monotonicity in hypothesis (iii) by (iii)(a) *weak monotonicity,* (iii)(b) *irreducibility.*[2] Assuming weak monotonicity, the only place strong monotonicity played a role was to insure that $p \cdot e^\alpha > 0$ for all $\alpha = 1, \ldots, n$. With only weak monotonicity Claims 1–4 show that there exists a "quasi–equilibrium" (see [3]). But

[2]For a definition see [3].

it is well–known that irreducibility guarantees that all quasi-equilibria are genuine equilibria.

To weaken (iii) further, suppose now that the u^a are only quasi–concave. The only place concavity was used in the proof of Theorem 1 was in Claim 1, to establish the concavity of the payoff function $u^a(x) - c^a_+$. However, by an extension of the arguments used in [13], a (pure strategy) N.E. still exists, just from the continuity of $u^a(x) - c^a_+$. Passing to limits as in Theorem 1 yields an N.E. allocation x which is a W.E., though x need not be type symmetric. Replace x^t by $\int_{a-1}^{a} x^t dt \equiv x^a$, for each $t \in T_a$. By quasi-concavity of utilities, $\{x^a\}_{a-1}^{n}$ is also a W.E. allocation.

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