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ECONOMETRIC ANALYSIS OF FISHER'S EQUATION

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Abstract

Fisher's equation for the determination of the real rate of interest is studied from a fresh econometric perspective. Some new methods of data description for nonstationary time series are introduced. The methods provide a nonparametric mechanism for modelling the spatial densities of a time series that displays random wandering characteristics, like interest rates and inflation. Hazard rate functionals are also constructed, an asymptotic theory is given and the techniques are illustrated in some empirical applications to real interest rates for the US. The paper ends by calculating Gaussian semiparametric estimates of long range dependence in US real interest rates, using a new asymptotic theory that covers the nonstationary case. The empirical results indicate that the real rate of interest in the US is (fractionally) nonstationary over 1934–1997 and over the more recent subperiods 1961–1985 and 1961–1997. Unit root nonstationarity and short memory stationarity are both strongly rejected for all these periods.

Key words and phrases: fractional integration; hazard rate; long range dependence; real rate of interest; semiparametric estimation; sojourn time; spatial density.

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1. Introduction

Since Irving Fisher (1986,1930) formalized[1] the notion of a real rate of interest, the concept has played a significant role in the formulation of a wide range of economic models. These include individual agent decision making regarding investment, savings, and portfolio allocations, options pricing models in finance and the modern theory of inflation targeting in macroeconomics, to name but a few. Naturally enough, in the light of the role that the real rate plays in economic theory models, a good deal of attention has been devoted in the literature, especially in macroeconomics, to the measurement of the real rate and to the characterization of its temporal dependence properties. Prima facie, this task seems like a simple exercise in time series econometrics. However, the empirical analysis is complicated by two factors: (i) the apparent nonstationary behavior of the series involved, particularly interest rates but also sometimes inflation; and (ii) the fact that the ex ante real rate of interest depends on inflation expectations and is therefore not directly measured. Perhaps, because of these complicating factors, no consensus seems to have emerged about the time series properties of the real rate of interest, in spite of intensive empirical study. In particular, while economic theory models routinely assume that the real rate of interest is a constant, or fluctuates in a stationary way about a constant mean, the empirical work indicates that this is not so or at best holds only over short regimes.

Figure 1 shows monthly data for the ex post real interest rate in the US over the period 1961:1–1985:12. The series is calculated by taking the US 90-day Treasury Bill rate for the nominal interest rate and by using the US monthly CPI (all commodities, with no adjustment for housing costs) to compute 3-month inflation rates. The figure also shows subgroup means calculated over the subperiods 1961:1–1973:1, 1973:2–1982:1, and 1982:2–1985:12. The data cover the same period as that studied recently by Garcia and Perron (1994), who used regime shift methods to estimate the ex ante real rate over approximately these subperiods. These authors concluded that the ex ante real rate of interest was effectively constant, but subject to occasional mean shifts over 1961–1985. They found two mean shifts over this time period and gave results very similar to those obtained by subgroup means that are displayed in Figure 1.For these data, at least, the conclusion does not seem unreasonable.

Figures 2 and 3 graph the ex-post real interest rate series calculated in the same way over the longer periods 1961–1997 and 1934–1997. Over the 1961–1997 period the graphs shows subsample means for the additional two subperiods 1990–1993 and 1994–1997. Apparently, there is a need to allow for continuing regime shifts in the mean level if this approach to modeling the real rate of interest is to give acceptable results. For the longer period, an even larger number of mean shifts is needed to accommodate this approach and the results seem much less satisfying — we merely

[1] Fisher (1896) credited Marshall (1895) for making the distinction between real and nominal interest. It appears the idea that expected inflation affects interest rates can be traced to earlier political speeches and political economy pamphlets. Howitt (1992) and Laidler (1991) provide some further information about the history of the concept and the distinction between real and nominal rates.Fisher seems to have been the first to conduct a sustained study and to explore the matter in serious empirical research.

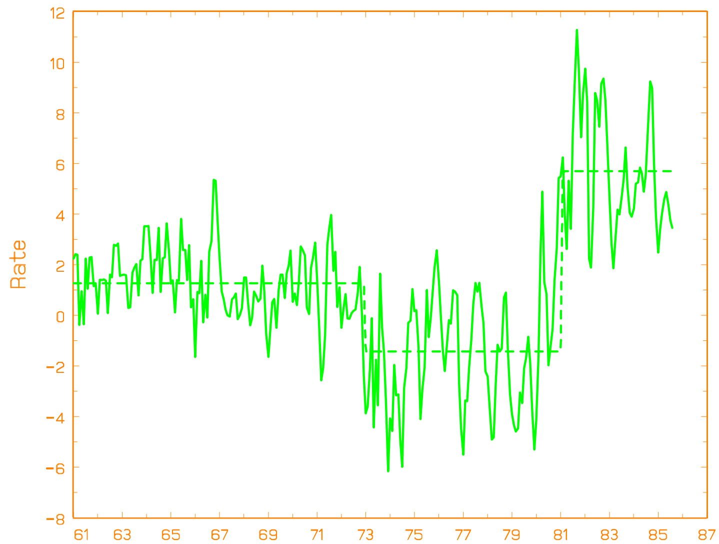

Figure 1: Ex-Post Real Interest Rate: 1961–1985

illustrate what is involved in Figure 3. It is hard to conclude that one is doing any more than simply curve fitting in such exercises and then possibly providing ex post rationalizations for the mean shifts. Note, however, that the data in Fig. 3 clearly do support the conclusion reached in Fama's (1975) influential study that the real rate has a constant mean over the period 1953–1971. Notwithstanding the apparent constancy over 1953–1971, so many mean shifts are needed to model the data this way that the Garcia–Perron conclusion that the ex ante real rate of interest fluctuates about a constant mean, subject to occasional mean shifts, seems much less reasonable over the longer period 1934–1997 than it does for 1961–1986. Indeed, over longer periods such as this, most formal tests (e.g., Rose, 1988, and Walsh, 1987) support a conclusion that is the opposite of stationarity and favor unit root nonstationary. One of the goals of the present paper is to determine whether there are other hypotheses that are more reasonable than these two alternatives.

Another goal of the paper is to contribute some new methods to assist in the econometric analysis of data of this type. The methods given here furnish a new way of describing and characterizing data like interest rates and inflation that appear to have nonstationary elements. More specifically, the paper proposes a nonparametric spatial density estimate as a new descriptive tool for nonstationary time series. Many series like the interest rates shown in Figures 1–3 behave as if they have no fixed mean. The random wandering characteristic of these series is hard to describe quantitatively. What a spatial density does is provide useful quantitative information about the spatial location characteristics of a time series, in just the same way as a probability density can be used to characterize stationary time series. As will be shown, in

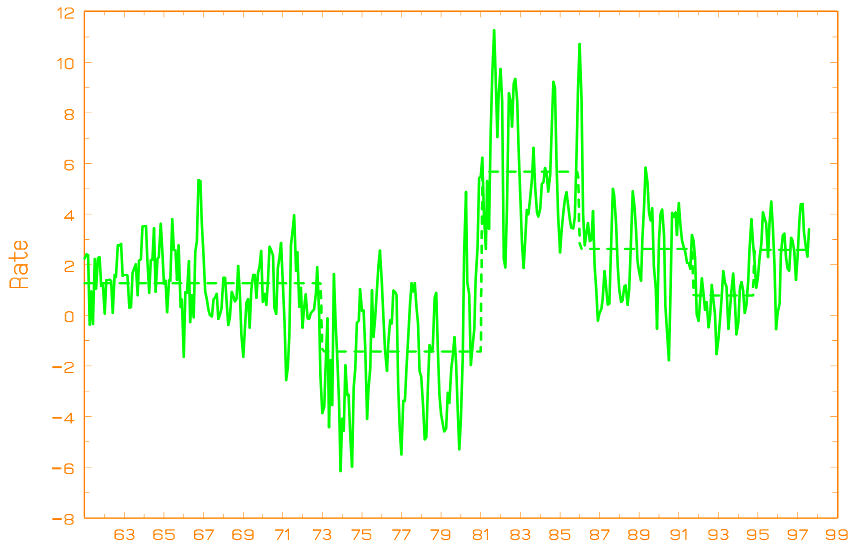

Figure 2: Ex-Post Real Interest Rate: 1961-1997

contrast to a probability density, the spatial density is a random process. However, it turns out that we can still obtain consistent estimates of spatial densities using nonparametric techniques, like those of kernel methods, which have proved useful in studying iid and strictly stationary time series. We outline these new procedures here and provide an asymptotic theory that characterizes their large sample properties and facilitates inference.

Once nonparametric spatial density estimates have been obtained, we can use them in similar ways to that of a probability density — to study and quantify the locational characteristics of the time series. We illustrate these ideas by looking at nonparametric estimates of the spatial location of the ex post real rate of interest in the US. This exercise provides some nonparametric evidence on recent empirical findings about real interest rates and the Fisher effect. In addition, we show how to construct a new type of hazard function for nonstationary time series. Hazard rates are particularly helpful in studying financial series, as they can be used to quantify the hazards of certain interest rate and inflation rate levels, for example. The methods can also be used to study how empirical hazard rates evolve over time. Again, we illustrate the techniques in some empirical applications to the ex post real rate of interest series for the US economy shown in Figures 1–3.

Finally, we attempt to model the real rate of interest directly as a potentially nonstationary long memory process. This involves the econometric estimation of the long memory parameter (d), without making any assumptions about the stationary components in the data generating process. The procedure we use is a Gaussian semiparametric estimator, whose properties have only recently been rigorously explored

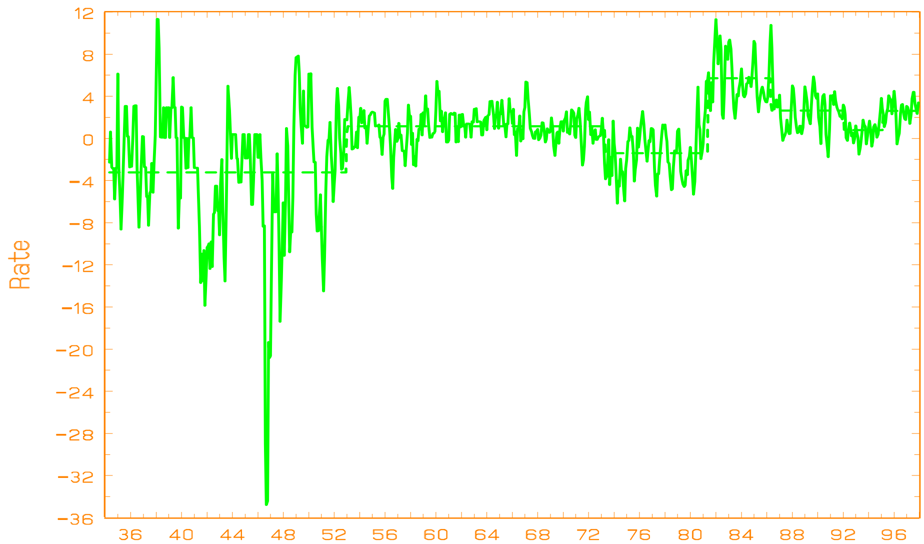

Figure 3: Ex-Post Real Interest Rate: 1934–1997

in the stationary case (Robinson, 1995) where d is restricted to $|d| < \frac{1}{2}$. The author (1998) has now developed an asymptotic theory for this estimator in the nonstationary case where $d > \frac{1}{2}$, which we will briefly report in Section 6 and use to estimate d and to provide confidence intervals, thereby enabling us to answer the question of whether the preferred model for data on the real rate of interest is stationary or nonstationary.

The paper is organized as follows. The Fisher effect and associated regression equations are discussed in Section 2. The econometric ideas that underlie the spatial modelling apparatus that we introduce are laid out in Sections 3 and 4. Section 5 provides an empirical application of the methods to US data over the period 1934–1997. Section 6 reports our new asymptotics and empirical estimates of the long memory parameter. Section 7 concludes the paper and discusses some related issues. Proofs are collected together in Section 8 with some of the new asymptotic theory that is introduced in the paper.

2. The Fisher Effect

Irving Fisher formulated the concept of the (ex ante) real rate of interest (r_t^e) to provide a rate of interest which accounted for the value of loan repayments in real dollar terms. That is, a nominal interest rate of i_t will assure a real rate of r_t^e when the anticipated price change is π_t^e provided $i_t = r_t^e + \pi_t^e + r_t^e \pi_t^e$, thereby adjusting the compensation to the lender for the anticipated losses in purchasing power in the principal and the interest. The term $r_t^e \pi_t^e$ is usually ignored because it is of smaller order, so that the Fisher equation is commonly written as

$$i_t = r_t^e + \pi_t^e. \tag{1}$$

Equation (1) is sometimes made more specific by indexing the interest rate by the time period (m) of the bond to maturity and by using m-period ahead inflation expectations, leading to the alternate formulation

$$i_t^m = r_t^{e,m} + \pi_t^{e,m}. \tag{2}$$

The empirical literature surrounding equation (1) is vast and we will not attempt to provide a review here. Fisher (1930) himself began the empirical work and considered the problems surrounding the relationship between prices and interest rates to be

> *"... of such vital importance that I have gone to much trouble and expense to have such data as could be found compiled, compared, and analyzed." (op.cit., p. 399)*

The main object of Fisher's work was

> *"... to ascertain to what extent, if at all, a change in the general price level actually affects the market rates of interest." (op. cit., p. 399)*

He conducted correlational studies between inflation and interest rates with annual data primarily for the US and the UK, concluding as follows:

> *"Our first correlations seemed to indicate that the relationship between P' (inflation) and i (interest rate) is either very slight or obscured by other factors. But when we make the much more reasonable supposition that price changes do not exhaust their effects in a single year but manifest their influence with diminishing intensity over long periods which vary in length with the conditions, we find a very significant relationship, especially in the period which includes the World War, when prices were subject to violent fluctuations." (op. cit., p.423)*

Since this initial work by Fisher, a substantial amount of empirical work has been conducted with data from many countries, covering different periods of time and maturities. However, little consensus seems to have emerged from these studies about the nature of the Fisher effect. In particular, there seem to be little agreement about the statistical properties of the real rate r_t^e.

Two recent studies that make significant methodological departures in studying this problem are by Mishkin (1992), and the already cited paper by Garcia and Perron (1994). These papers bring some modern nonstationary time series and regime shift methods to bear in analyzing the Fisher equation. Using residual based cointegration tests, Mishkin finds support in the data for a 'long run' Fisher effect in which inflation and interest rates share a common stochastic trend. Observe that if π_t^m is the m-period inflation rate then (2) gives the following relation between the ex post real rate of interest ($r_t^m = i_t^m - \pi_t^m$) and the ex ante rate of interest $r_t^{e,m}$

$$r_t^m = r_t^{e,m} + (\pi_t^{e,m} - \pi_t^m) = r_t^{e,m} + \varepsilon_t \tag{3}$$

Under rational expectations, where agents or the market use all information efficiently in forecasting inflation, the forecast error $\varepsilon_t = \pi_t^{e,m} - \pi_t^m$ will be a martingale difference and can be assumed to be stationary, or integrated of order zero ($I(0)$). Under this hypothesis, the ex post and ex ante real rates differ by a stationary component and therefore have the same long run time series properties. Thus, following Mishkin (1992), the ex post real rate can only be $I(1)$ if $r_t^{e,m}$ is $I(1)$. Hence, a test for a unit root in r_t^m against stationary alternatives can be interpreted as test for a unit root in $r_t^{e,m}$ against a stationary ex ante real rate. Put another way, cointegration between i_t^m and π_t^m is the alternative hypothesis in a test for a unit root in the ex post real rate of interest. The Fisher effect then corresponds to the hypothesis that the ex ante real rate of interest is stationary, so that, under rational expectations and stationary forecast errors for inflation, the Fisher effect implies a stationary ex post real rate of interest.

In spite of their apparent simplicity, equations (1) and (3) present a host of econometric difficulties arising from the fact that the variables r_t^e and π_t^e are unmeasured latent variables, and from the time series difficulties in modelling apparently nonstationary series like interest rates and inflation.

While Fisher (1930) analyzed the relationship between interest rates and inflation using correlational techniques, modern approaches rely on regression methods — sometimes, as in Summers (1983), in the frequency domain where low frequencies can be used to emphasize long run properties. Depending on the properties of the real rate r_t^e, equation (3) suggests a regression link between i_t and π_t^e. In particular, the Fisher effect asserts that the coefficient b should be unity in a regression of the form

$$i_t = c + b\pi_t^e + u_t$$

and the residuals u_t should be stationary. Under this hypothesis, we can write the ex post real rate of interest as

$$r_t = i_t - \pi_t = c + b\,(\pi_t^e - \pi_t) + u_t = c + w_t, \tag{4}$$

which implies stationary fluctuations about a constant level c.

In a celebrated study mentioned earlier, Fama (1975) found empirical evidence of a constant real rate of interest over the period 1953–1971. Mishkin (1981) subsequently rejected constancy in the real rate in a more extensive study covering the longer periods 1953–1979 and 1931–1952. Later investigations by Rose (1988) and Walsh (1987) found evidence in support of unit root nonstationarity in ex post real rates. Most recently, Garcia and Perron (1994) reanalyzed data over the period 1961–1986 using regime shift techniques and found support for a constant real rate of interest, subject to infrequent changes in the constant c. In short, the empirical evidence gives a mixed picture about the statistical properties of the real rate of interest, and it is probably fair to say that the generating mechanism for the real rate is very imperfectly understood. Although the bulk of the econometric evidence now points against the hypothesis of a pure Fisher effect in which the real rate is stationary about a constant mean, this does not rule out modified Fisher effects such as those supported in the Garcia–Perron study.

We now propose to take a very different approach to studying these data. The essence of the new approach is descriptive, but the asymptotic theory that we have developed for the quantities involved enables us to use them in an inferential framework as well. This helps us to corroborate and assess earlier empirical findings.

3. Spatial Densities for Nonstationary Series

Our proposal is to study the spatial characteristics of a time series rather than focus on their temporal dependence properties, as one would normally do in a stationary time series analysis. The starting point is to assume that, when appropriately normalized and transformed into a random function on the interval $[0, 1]$, the time series trajectories converge weakly to those of a continuous stochastic process on the same interval. Such an assumption applies under a very wide variety of possible conditions and it seems a very weak requirement if one is to make any headway in the development of asymptotic methods. Accordingly, we may suppose that the limit process, $M(r)$ say, of the normalized series is a continuous semi-martingale (e.g., see Protter, 1990). This requirement would then include the huge class of time series for which functional central limit theorems are known to apply, leading for example to Brownian motions and diffusion processes as special cases (e.g., see Phillips and Solo, 1992). While this class of time series is substantial, one case that is excluded by the semimartingale requirement is time series that upon normalization tend to fractional processes like fractional Brownian motion (e.g., Mandlebrot and van Ness, 1967 and Taqqu, 1975). This case does seem to be important in applications of our ideas to nonstationary series, because there is increasing evidence that economic time series are well modelled by long memory processes with fractional Brownian motion limits — for some recent macroeconomic time series evidence see Gil-Alana and Robinson (1997). Fortunately, there is a way to include fractional processes within our theory and this is indicated later (see (12) below).

For a semimartingale $M(r)$ it is known that there exists an increasing stochastic process[2] (increasing in the argument r, that is), called the local time of M at s, and denoted $L_M(r, s)$ which represents the amount of time that the limit process spends in the spatial vicinity of the point s. The local time process is defined as

$$L_M(r, s) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^r 1(|M(t) - s| < \varepsilon) d[M]_t, \tag{5}$$

[2] The reader is referred to Revuz and Yor (1994) for much of the underlying stochastic process theory used here. Good introductions are Chung and Williams (1990) and Karatzas and Shreve (1992).

where $[M]_t$ is the quadratic variation process of M. Notice that this definition of 'time spent in the vicinity of s' is expressed in units of variation, as measured by $d[M]_t$. In effect, $L_M(r, s)$ measures the contribution to the quadratic variation $[M]_r$ over the interval $[0, r]$ that comes from variation in $M(t)$ around the level s.

It turns out that we can also write down an inverse relation of the form

$$[M]_t = \int_{-\infty}^{\infty} L_M(t, s) ds, \tag{6}$$

which gives a decomposition of the quadratic variation into contributions to the conditional variance that come from fluctuations in the process that occur in the neighborhood of different spatial points $s \in [-\infty, \infty]$. In a sense, we can think of (6) as the spatial equivalent for a continuous nonstationary random process of the decomposition of the variation of a stationary time series into contributions from different frequencies, i.e.,

$$\sigma^2 = \int_{-\pi}^{\pi} f_{xx}(\lambda) d\lambda$$

where $f_x(\lambda)$ is the spectral density of X_t and $\sigma^2 = \text{var}(X_t)$. In this formula, λ is a continuous variable representing the frequency of the oscillations into which the time series variation is being decomposed. However, in (6), the variable s represents spatial points in the real line, where the process spends some time. Fluctuations in the process around the point s, which can occur at various times in the time interval $[0, t]$, then contribute to the density $L_M(t, s)$ of the process at this spatial point.

Equation (6) is particularly interesting in the special case where $M(r)$ is the Brownian motion $B(r)$ with variance ω^2. Here, the parameter ω^2 arises as the long run variance of the shocks that drive the unit root process which converges weakly to $B(r)$. In this case, we have

$$\omega^2 = \int_{-\infty}^{\infty} L_B(1, s) ds,$$

giving a decomposition of the long run variance ω^2 into components that reflect the density of the fluctuations in the process around all spatial points $s \in (-\infty, \infty)$.

The local time process $L_M(r, s)$ is known to satisfy the following equation

$$|M(r) - s| = |M(0) - s| + \int_0^r \text{sgn}(M(t) - s) dM(t) + L_M(r, s), \tag{7}$$

where $\text{sgn}(x) = 1, -1$ for $x > 0$, $x \leq 0$. In fact, the process $L_M(r, s)$ is sometimes defined in this manner — e.g., see Revuz and Yor (1991). Equation (7) gives a development of the function $|M(r) - s|$ about its value at $r = 0$ and thereby provides a mechanism of generalizing the Ito stochastic calculus of functions that are continuously differentiable to the second order ($\mathbb{C}_2$ functions) to functions which are not everywhere differentiable and smooth, and, in particular, to convex functions, which we will show below. These extensions look to be particularly valuable in theoretical models where convex functions play a big role, but have not to my knowledge been used yet.

As is well known (e.g., Protter, 1990, p. 74), if $f \in \mathbb{C}_2$ we have the Ito formula for continuous semi-martingales given by

$$f(M(r)) = f(M(0)) + \int_0^r f'(M(t))dM(t) + \frac{1}{2}\int_0^r f''(M(t))d[M]_t \tag{8}$$

On the other hand, when f is convex we have, based on (7)

$$f(M(r)) = f(M(0)) + \int_0^r f'_-(M(t))dM(t) + \frac{1}{2}\int_{-\infty}^{\infty} f''_g(p)L_M(r,p)dp \tag{9}$$

where f'_- is the left derivative of f and f''_g is the second derivative of f in the generalized function sense. When $f \in \mathbb{C}_2$ we have $f'_-(M(t)) = f'(M(t))$, and $f''_g(p) = f''(p)$, the second derivative in the usual sense, and the occupation time formula (e.g., Revuz and Yor, 1991, p. 209) gives

$$\frac{1}{2}\int_{-\infty}^{\infty} f''(p)L_M(r,p)dp = \frac{1}{2}\int_0^r f''(M(t))d[M]_t, \tag{10}$$

so that (9) reduces to (8) in this special case.

Observe that (10) shows the sense in which $L_M(r,p)$ is a spatial density for the process $M(r)$, recording the amount of time (measured in units of the quadratic variation $[M]_t$) that the process has spent in the immediate vicinity of p over the time interval $t \in [0,r]$.

We now propose to use these concepts in studying the spatial properties of time series like interest rates and inflation. Our first task is to estimate the local time process $L_M(r,p)$ for a particular time series. Just as the spectral density of a stationary time series is estimated by nonparametric methods, the local time $L_M(r,p)$ can estimated in a nonparametric manner as follows. Let X_t be a time series for which satisfies a functional law of the form

$$\frac{1}{n^\alpha}X_{[nr]} \Rightarrow M(r) \tag{11}$$

where $M(r)$ is a continuous semi-martingale for $r \in [0,1]$. For instance, when X_t is a integrated process of order one, standard functional central limit theorems (e.g., Phillips and Solo, 1992) lead to (11) with $\alpha = \frac{1}{2}$ and $M(r) = B(r)$, a Brownian motion with variance $\omega^2 = 2\pi f_{\Delta x}(0)$. When X_t is near integrated, so that the quasi differences $\Delta_c X_t = X_t - (1+\frac{c}{n})X_{t-1}$ are stationary with positive spectral density at the origin, then $\alpha = \frac{1}{2}$ and $M(r) = J_c(r)$, a linear diffusion process (e.g., Phillips, 1987). When X_t is fractionally integrated and $(1-L)^d X_t = u_t$ is stationary with positive spectrum $\omega^2 > 0$ at the origin and with $d > \frac{1}{2}$, then we have the functional law (cf. Taqqu, 1975)

$$\frac{1}{n^{d-\frac{1}{2}}}X_{[nr]} \Rightarrow B_{d-1}(r) = \frac{\omega}{\Gamma(d)}\int_0^r (r-s)^{d-1}dW(s),$$

where $B_{d-1}(r)$ is fractional Brownian motion. In this case the limit process $M(r) = B_{d-1}(r)$ is not a semi-martingale. Nevertheless, the stochastic integral that occurs in (7) can still be defined in the L_2 sense, as discussed recently in Chan and Terrin (1996). In consequence, we offer the following definition of the local time of fractional Brownian motion based on equation (7) and using as

$$L_{B_{d-1}}(r,s) = |B_{d-1}(r) - s| - |B_{d-1}(0) - s| - \int_0^r \operatorname{sgn}(B_{d-1}(t) - s) dB_{d-1}(t). \quad (12)$$

These examples seem to cover most cases of empirical interest that arise in the present literature on nonstationary economic time series. In the development that follows we will assume we are working with an integrated process X_t for which $n^{-\frac{1}{2}}X_{[nr]} \Rightarrow B(r)$. Or, if we want to allow for distant initial conditions (at time $-[n\kappa]$ for some $\kappa \geq 0$) in the origination of X_t then we can use the limit $n^{-\frac{1}{2}}X_{[nr]} \Rightarrow B(r) + B_0(\kappa)$, where B and B_0 are independent Brownian motions, with the process B_0 extending in a reverse (negative) direction reflecting the effect of distant initial conditions at some fraction κ of the sample size n. Extensions to the fractional Brownian motion case can be made by appropriate changes in normalization. This will cover a wide class of interesting practical cases. Extensions of the theory to the more general case of (11) are also possible but are more difficult and will call upon strong approximation versions of (11) which, at least to the author's present knowledge, do not seem to be available in the probability literature as yet for a general class of processes. A complete development of the asymptotic theory in our case to cover situations as general as (11) is therefore beyond the scope of the present paper.

A natural candidate for estimating the local time of the limit process of $n^{-\frac{1}{2}}X_{[nr]}$ at s is the scaled kernel estimate

$$\frac{\widehat{\omega}^2}{nh_n}\sum_{t=1}^{n} K\left(\frac{s - X_t}{h_n}\right) = \frac{\widehat{\omega}^2}{nh_n}\sum_{t=1}^{n} K\left(c_n\left\{\frac{s}{\sqrt{n}} - \frac{X_t}{\sqrt{n}}\right\}\right), \quad c_n = \frac{\sqrt{n}}{h_n} \quad (13)$$

where h_n is a bandwidth parameter, $K(\cdot)$ is a kernel function and $\widehat{\omega}^2$ is a consistent estimate of $\omega^2 = 2\pi f_{\Delta x}(0)$. Upon restandardization of this estimate we have the following extension of a result obtained recently in Phillips and Park (1998).

3.1 Theorem *Suppose $\widehat{\omega}^2 \to_p \omega^2 = 2\pi f_{\Delta x}(0)$. If $s = s_0 + \sqrt{n}a$, with s_0 and a fixed, and if assumptions 8.1-8.4 in the Appendix hold, then as $n \to \infty$*

$$\widehat{L}_B\left(r, \frac{s}{\sqrt{n}}\right) = \frac{\widehat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]} K\left(\frac{s - X_t}{h_n}\right) \to_p L_B(r, a - B_0(\kappa)). \quad (14)$$

The definition of $\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$ involves scaling the conventional kernel estimator given in (13) by $\sqrt{n}$. The reason for this restandardization is that in the nonstationary case the process X_t wanders away from the location s at the rate $\sqrt{n}$ and, for such departures from s, $K(h_n^{-1}(s - X_t))$ is negligibly small. In effect, the stochastic trend property of X_t reduces the order of magnitude of the kernel estimate compared with the stationary case.

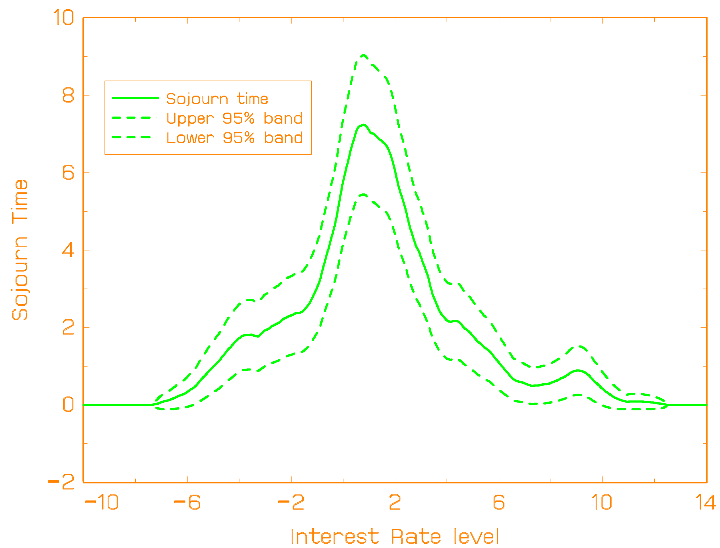

Figure 4: Spatial Density of Ex Post Real Rate: 1961–1985

Note that the estimate $\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$ evaluates the local time at a spatial point $n^{-\frac{1}{2}}s$ that is affected by the rate of convergence of $n^{-\frac{1}{2}}X_{[nr]}$ to its Brownian motion limit process. Hence, if s is fixed and initial conditions are $O_p(1)$, i.e., $\kappa = 0$, as $n \to \infty$, then the estimate $\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$ is consistent for the local time of the limit process at the origin, i.e., $L_B(r, 0)$. When $\kappa > 0$, the random initialization shifts the spatial point of evaluation of the local time estimate so that it is centered around $B_0(\kappa)$. So, initial conditions generally play a role in the spatial density of the process, as we might well expect for a nonstationary series.

Our next step is to construct confidence regions for the density estimate $\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$. This can be done using the limiting distribution theory for $\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$ given in Theorem 3.2 below. The limit distribution turns out to be mixed normal (denoted MN in what follows) with a mixing variate that is proportional to the spatial density itself. This limit theory justifies the construction of confidence intervals in the usual manner.

3.2 Theorem *If $s = s_0 + \sqrt{n}a$, with s_0 and a fixed, if assumptions 8.1-8.4 in the Appendix hold, and if* $\sqrt{c_n}(\widehat{\omega}^2 - \omega^2) = o_p(1)$ where $c_n = h_n^{-1}\sqrt{n}$, *then as* $n \to \infty$

$$\begin{aligned}
&\sqrt{c_n}\left[\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right) - L_B(r, a - B_0(\kappa))\right] \\
\Rightarrow\ & 2\int_{-\infty}^{\infty} K(p)Q(L_B(r, a - B_0(\kappa)), p)dp \\
=_d\ & 4L_B(r, a - B_0(\kappa))^{\frac{1}{2}}\int_0^{\infty} K(p)W(p)dp \\
=_d\ & 4L_B(r, a - B_0(\kappa))^{\frac{1}{2}}N(0, K_2) \\
\equiv\ & MN(0, 16K_2L_B(r, a - B_0(\kappa)))
\end{aligned}$$

where $Q(a,b)$ is a standard Brownian sheet, $W(p)$ is a standard Brownian motion and

$$K_2 = \int_0^{\infty}\int_0^{\infty} K(p)(p \wedge t)K(t)dpdt.$$

This result enables us to construct confidence intervals for $L_B(r, a - B_0(\kappa))$ for each spatial point. Thus,

$$\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right) \pm 1.96\left(\frac{16K_2\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right)}{c_n}\right)^{\frac{1}{2}}$$

is a 95% confidence interval for $L_B(r, a - B_0(\kappa))$. When $K(\cdot)$ is a normal kernel, some calculations show that K_2 takes the value $K_2 = \pi^{-\frac{1}{2}}\left(2^{\frac{1}{2}} - 1\right)$.

Note that we are measuring spatial departures from the origin in units of $\sqrt{n}$ in the limit. So, if we want a confidence interval for the spatial density of the process at a point like $\sqrt{n}a^0$, we set $s = \sqrt{n}a^0 + X_0$ and compute

$$\widehat{L}_B\left(r, a^0 + n^{-\frac{1}{2}}X_0\right) \pm 1.96\left(\frac{16K_2\widehat{L}_B\left(r, a^0 + n^{-\frac{1}{2}}X_0\right)}{c_n}\right)^{\frac{1}{2}}. \tag{15}$$

It is interesting to compare local time confidence intervals like (15) above with the confidence intervals for a probability density from kernel estimates of a probability density. By traditional theory here (e.g., Silverman, 1986) we have the following 95% confidence interval for a density $f(x)$,

$$\widehat{f}(x) \pm 1.96\left(\frac{k_2\widehat{f}(x)}{nh}\right)^{\frac{1}{2}} \tag{16}$$

where

$$k_2 = \int_{-\infty}^{\infty} K(r)^2dr. \tag{17}$$

The differences between (15) and (16) involve: (i) the scale factors k_2 and $16K_2$, where the difference is due to the temporal dependence in the trajectory of X_t (so that the covariance kernel enters the definition of K_2) and definitional differences between local time and a probability density; and (ii) the rate of convergence — $\sqrt{c_n}$ in the case of the spatial density, compared with $\sqrt{nh}$ in the case of the probability density. In other respects, (15) simply extends our existing theory of nonparametric density estimation to spatial density estimation for stochastic processes.

4. Hazard Rates for Nonstationary Time Series

One advantage of a nonparametric treatment of spatial density is that we can define interesting functionals derived from the density that help us to shed light on the nature of variation in the data. While there are many obvious notions that arise in this way, one that we will pay attention to here is the idea of a spatial hazard function. We define the spatial hazard function $H_M(t,a)$ associated with a given spatial density $L_M(t,s)$ as follows

$$H_M(t,a) = \frac{L_M(t,a)}{\int_a^\infty L_M(t,s)ds}. \tag{18}$$

The form of (18) is analogous to that of the hazard rate $\theta(x)$ associated with a probability density $f(x)$ as

$$\theta(x) = \frac{f(x)}{\int_x^\infty f(s)ds} = \frac{f(x)}{1-F(x)} = \frac{f(x)}{\overline{F}(x)},$$

where $F(x)$ is the cdf of the distribution and $\overline{F}(x) = 1 - F(x)$ is the survivor function. Such conventional hazard rates are now extensively used in empirical econometric work to help model and understand phenomena like unemployment duration and quits in the labor market and have widespread use in other fields, such as the statistical analysis of medical phenomena relating to the contraction of diseases.

How do we interpret such hazard rates in the case of nonstationary time series? Suppose the time series under study X_t is inflation and $M(r)$ is the weak limit process of $n^{-\alpha}X_{[nr]}$, with local time $L_M(r,s)$. The spatial hazard $H_M(t, a = \frac{s}{n^\alpha})$ measures the conditional risk over the period $[0,t]$ (which is expressed in standardized units of fractions of the overall sample) of an inflation rate of s, given that inflation is at least as great as s. We can then study the form of this hazard as a function of the inflation rate s, just as we might look at unemployment duration in the conventional hazard analysis of independent data. Thus, we can see whether the hazard declines, increases or stays constant as we increase s. We can also look at the hazard as a function of the length of the time interval t, and examine what happens to the hazard rate as the time period evolves and new data is introduced. So we can see whether the hazard rate of a certain rate of inflation rises or falls over time. Of course, we might ultimately contemplate modelling such hazard rates through the use of covariates or policy interventions. These possibilities exploit the dual argument property of the

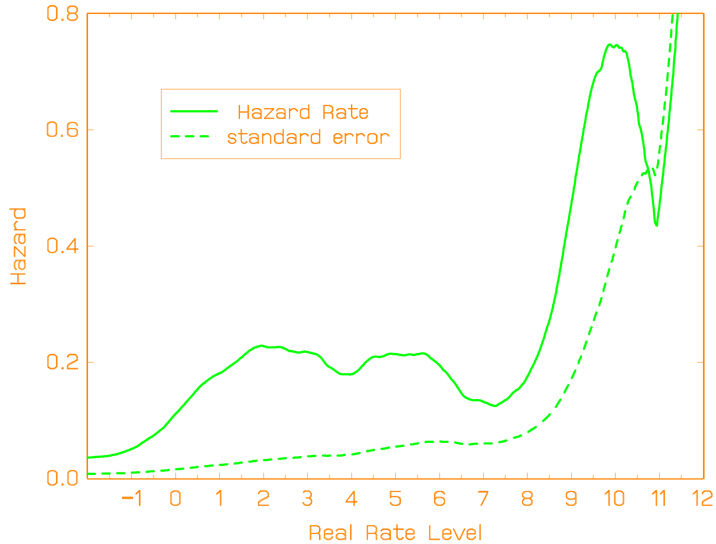

Figure 5: Hazard Function for Ex Post Real Rate: 1961–1985

spatial density $L_M(r,s)$ and hazard $H_M(t,a)$, so that it becomes possible to analyze time series effects on hazard rates. Like $L_M(r,s)$, the function $H_M(t,a)$ is a random process in its two arguments, but unlike conventional hazard rates which depend on the unknown but estimable probability distribution of the data, the hazard $H_M(t,a)$ is an unknown but estimable path dependent stochastic process.

Now take the case where $M(r)$ is Brownian motion $B(r)$. The spatial hazard function $H_B(t,a)$ is empirically estimable using the nonparametric spatial density estimate $\widehat{L}_B(t,\frac{s}{\sqrt{n}})$ discussed in the last section of the paper. For $s = s_0 + \sqrt{n}a$, we construct the estimator

$$\widehat{H}_B(t,a) = \frac{\widehat{L}_B(t,a)}{\int_a^\infty \widehat{L}_B\left(t,\frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}}}.$$

This estimator is consistent for $H_M(t,a)$ as the following theorem shows, where the result is given in the case of $\sqrt{n}$ convergence to a Brownian motion limit process.

4.1 Theorem *Suppose* $\widehat{\omega}^2 \to_p \omega^2 = 2\pi f_{\Delta x}(0)$. *If* $s = s_0 + \sqrt{n}a$, *with* s_0 *and* a *fixed, and if Assumptions* 8.1–8.4 *in the Appendix hold, then as* $n \to \infty$

$$\widehat{H}_B\left(t,\frac{s}{\sqrt{n}}\right) = \frac{\widehat{L}_B\left(t,\frac{s}{\sqrt{n}}\right)}{\int_{\frac{s}{\sqrt{n}}}^\infty \widehat{L}_B\left(t,\frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}}} \to_p H_B(t, a - B_0(\kappa))$$

where $\widehat{L}_B\left(r,\frac{s}{\sqrt{n}}\right) = \frac{\widehat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]} K\left(\frac{s-X_t}{h_n}\right)$.

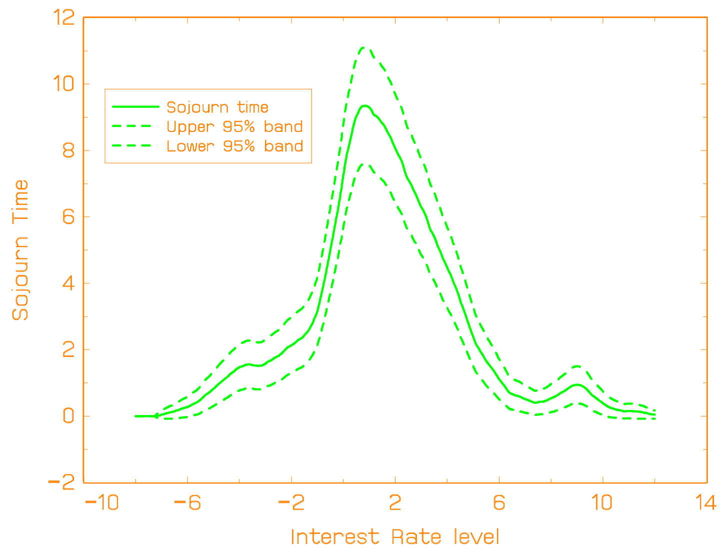

Figure 6: Spatial Density of Ex Post Real Rate: 1961–1997

As in the case of the spatial density, when the limit process is reached at a $\sqrt{n}$ rate of convergence, we measure spatial departures from the origin in units of $\sqrt{n}$. So, if we want to estimate the hazard of the process at a point like a^0, we set $s = \sqrt{n}a^0 + X_0$ and compute $\widehat{H}_B\left(t, a^0 + \frac{X_0}{\sqrt{n}}\right)$.

The function $F_M(t,a) = \int_a^\infty L_M(t,s)ds$ can be called the survivor function corresponding to $L_M(t,s)$, and is analogous to the survivor function $\overline{F}(a) = \int_a^\infty f(s)ds$ of a probability density f. Since we are using kernel estimates of $L_M(t,s)$ we can estimate $F_M(t,a)$ using

$$\begin{aligned}\widehat{F}_M\left(t, \frac{s}{\sqrt{n}}\right) &= \int_{\frac{s}{\sqrt{n}}}^{\infty} \widehat{L}_M\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}} \\ &= \frac{\widehat{\omega}^2}{\sqrt{n}}\sum_{t=1}^{[nr]}\overline{\mathcal{K}}\left(\frac{s - X_t}{h_n}\right)\end{aligned}$$

where

$$\mathcal{K}(b) = \int_{-\infty}^{b} K(x)dx, \quad \overline{\mathcal{K}}(b) = 1 - \mathbb{K}(b).$$

For most kernels, although not the Gaussian, $\overline{\mathcal{K}}(b)$ is available in closed form, simplifying calculations for the hazard and survivor functions. For example, if K is the Epanechnikov kernel we have

$$K(s) = \begin{cases} \frac{3}{4\sqrt{5}}\left(1 - \frac{s^2}{\sqrt{5}}\right) & -\sqrt{5} \le s \le \sqrt{5} \\ 0 & \text{otherwise} \end{cases},$$

and

$$\mathcal{K}(s)=\begin{cases}0 & s<-\sqrt{5}\\ \frac{3}{4\sqrt{5}}\left(s+\sqrt{5}\right)-\frac{1}{45^{\frac{3}{2}}}\left(s^3+5^{\frac{3}{2}}\right) & -\sqrt{5}\leq s\leq\sqrt{5}\\ 1 & s>\sqrt{5}\end{cases}.$$

An asymptotic theory for the estimated hazard function $\widehat{H}_B\left(t,\frac{s}{\sqrt{n}}\right)$ is given in the following theorem.

4.2 Theorem *If* $s=s_0+\sqrt{n}a$, *with* s_0 *and* a *fixed, if Assumptions* 8.1–8.4 *in the Appendix hold with* $h_n\to 0$, *and if* $\sqrt{c_n}(\widehat{\omega}^2-\omega^2)=o_p(1)$ where $c_n=h_n^{-1}\sqrt{n}$, *then as* $n\to\infty$

$$\begin{aligned}&\sqrt{c_n}\left[\widehat{H}_B\left(r,n^{-\frac{1}{2}}s\right)-H_B(r,a-B_0(\kappa))\right]\\ \Rightarrow\ &2\int_{-\infty}^{\infty}K(p)Q\left(\frac{L_B(r,a-B_0(\kappa))}{\left(\int_{a-B_0(\kappa)}^{\infty}L_M((r,s)ds\right)^2},p\right)dp\\ \equiv\ &4MN\left(0,K_2\frac{H_B(r,a-B_0(\kappa))^2}{L_B(r,a-B_0(\kappa))}\right)\end{aligned}$$

where $Q(a,b)$ *is a standard Brownian sheet.*

This result delivers an asymptotic standard error for the hazard $\widehat{H}_B\left(r,n^{-\frac{1}{2}}s\right)$, viz.

$$\left(\frac{16K_2\widehat{H}_B\left(r,n^{-\frac{1}{2}}s\right)^2}{c_n\widehat{L}_B\left(r,n^{-\frac{1}{2}}s\right)}\right)^{\frac{1}{2}},$$

which can be used to assess significance of the hazard estimates in practical applications. Interestingly, this formula is analogous to that obtained in traditional hazard analysis for a probability density, which takes the well known form (cf. Silverman, 1986, p. 148)

$$\left(\frac{k_2\theta(x)^2}{nhf(s)}\right)^{\frac{1}{2}}$$

The differences arise only from differences in the rate of convergence to the spatial density and the probability density and the scale constants $16K_2$ and k_2.

5. The Empirical Density of Real Interest Rates

We now illustrate the use of these concepts in analyzing the empirical spatial density of the real rate of interest. Hopefully, this will help to shed some new light on the nature of the Fisher effect and provide corroborative evidence for other studies that use conventional econometric methods.

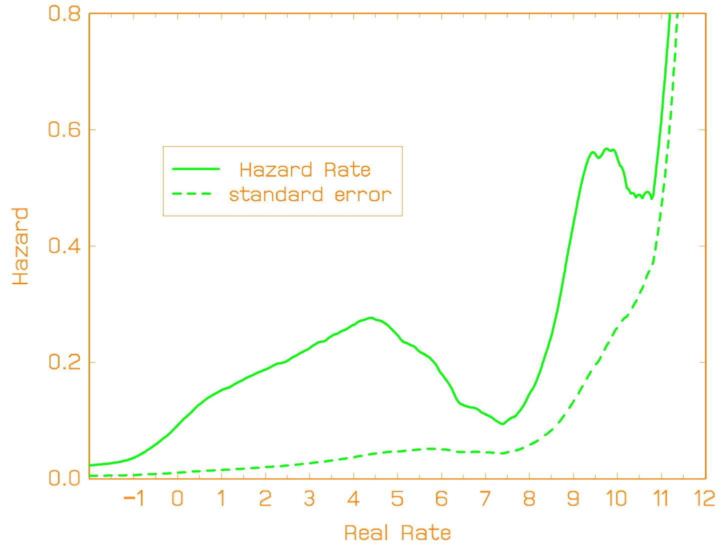

Figure 7: Hazard Function for Ex Post Real Rate: 1961–1997

We start with the period 1961–1985 studied by Garcia and Perron (1994). Figure 4 gives the spatial density estimate[3] for the ex-post real rate of interest shown in Figure 1. The estimated spatial density shows a dominant mode around the level 1.5% and evidence for four minor modes around –4%, –2%, 4% and 9%. Only the modes at 1.5% and 9% appear to be statistically significant. These results provide partial support for Garcia and Perron's conclusion in favor the hypothesis that the real rate of interest over 1961–1985 fluctuated about three constant levels around –2%, 1.5% and 4%. However, our nonparametric analysis indicates that there are also secondary modes around –4% and 9% and that these modes, especially the latter appear to be more significant than those around –2% and 4% which were the only secondary levels identified in the Garcia–Perron switching regimes approach. The significant mode around 9% seems to have been missed by Garcia and Perron.

The estimated hazard rates for the same period of data are shown in Figure 5. The graph shows that there are two peak levels of hazard for the real rate — levels around 1.5%–3% and 4.5%–6% — and a further peak around 9%. The sharply rising hazard at higher levels is not significant and can be ignored. Roughly speaking, these results tell us that, if we are to have high real rates of interest, the risk rises with the real rate to a plateau when rates are around 1.5%–3%, tapers off and then rises again to a plateau in the 4.5%–6% region before falling off again and rising to a further peak around 9%. Note that the very high peak at 9% is not significant, as the standard error is rising quickly at this point. Moreover, the peak at 9% indicates that if the real rate is to be as high as this, then it is very likely that it will be in the region of

[3] In these and in our other calculations we used a bandwidth of $h_n = n^{-\frac{1}{5}}$.

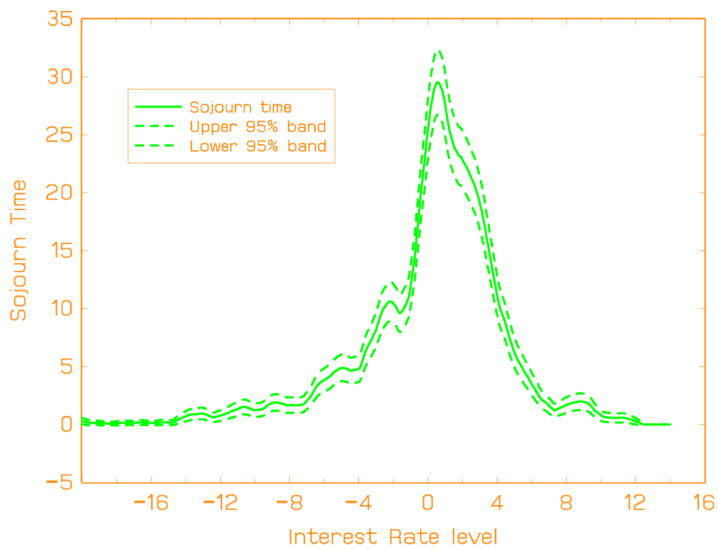

Figure 8: Spatial Density of Real Rate: 1934–1997

9%, given the observed data over the period 1961–1985.

The spatial density of the real rate changes as we add data for the period through to 1997 and back to 1934. First, consider the effect of the data through to 1997. As is apparent in Figure 6, there are now only three modes — around –4%, 1.5% and 9% in the spatial density. Only the modes around 1.5% and 9% appear significant, as before. The data over 1985–1997 has smoothed out the spatial density in the region between 2%–6% and there is no evidence of modes around 4% and -2%, at least for the bandwidth choice $h_n = n^{-\frac{1}{5}}$ that we are using here. Next, as we add data for the period back to 1934, we see from Figure 8 that modes around –5%, –2% have reappeared, although there is no evidence of a mode around 4%. At least for this data set, the spatial density computed over the full period 1934–1997 indicates that the real interest rate has regions of sojourn time around –5%, –2%, and 9% and the dominant mode is around 1.5%.

Hazard rates over the period 1961–1997 are shown in Figure 7 and in Figure 9 for 1934–1997. In both cases, there are now only two peaks in the hazard function — around 3–4% and 9%. Both of these appear to be more significant than they are for the shorter data set. Looking at Figure 9, it seems clear that the substantial region of hazard for positive real rates is in the 1%–6% zone, and for high rates, a zone around 9%.

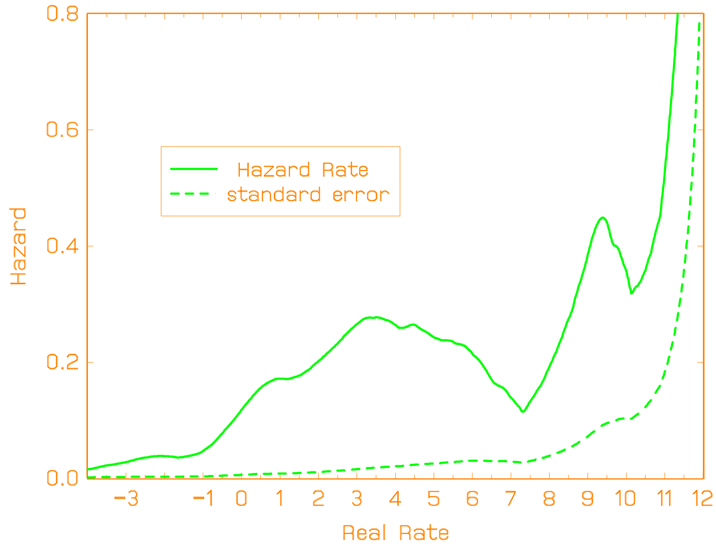

Figure 9: Hazard Function for Real Rate: 1934–1997

6. Is the Real Rate Nonstationary?

Tests for nonstationarity in the real rate of interest have in the past focussed on comparing unit root nonstationarity with stationary alternatives. We now consider a broader range of alternatives accommodated by allowing for fractional integration.

Our approach is semiparametric, so that we can retain as much generality as possible regarding the generating mechanism for the real rate of interest r_t. In particular, we consider a model for the ex post real rate of the form

$$(1-L)^d r_t = u_t, \tag{19}$$

where u_t is a zero mean stationary process with spectral density $f_{uu}(\lambda)$ and is assumed to satisfy Assumption 8.1 in the Appendix. By virtue of (19), the spectrum of r_t has the following asymptotic form in the vicinity of the origin

$$f_{rr}(\lambda) \sim \frac{f_{uu}(0)}{\lambda^{2d}}, \quad \lambda \sim 0. \tag{20}$$

Several semiparametric estimation procedures for d are available and a brief review is given in Robinson (1994). We propose to estimate d in (19) by maximizing a local Gaussian likelihood constructed under the assumption that u_t is Gaussian, but not requiring this condition, and relying on frequencies λ in the neighborhood of the origin. Following Künsch (1986) and motivated by (20), we use a Gaussian objective function, defined in terms of the parameter d and $G = f_{uu}(0)$

$$Q_m(G,d) = \frac{1}{m}\sum_{j=1}^{m}\left[\log\left(G\lambda_j^{-2d}\right) + \frac{\lambda_j^{2d}}{G} I_r(\lambda_j)\right] \tag{21}$$

where $I_r(\lambda_j) = w_r(\lambda_j)w_r(\lambda_j)^*$ is the periodogram, $\lambda_j = \frac{2\pi j}{n}$ $j = 0, 1, ..., n-1$ are the harmonic frequencies, and $w_r(\lambda_j) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} r_t e^{i\lambda_j}$ is the discrete Fourier transform of r_t. The integer m in (21) is less than n and defines the number of frequencies in the vicinity of the origin that are being used in the estimation of the parameter d.

The local Gaussian estimates of G and d are obtained by minimizing $Q_m(G,d)$, so that

$$(\widehat{G}, \widehat{d}) = \arg \min_{0<G<\infty,\ d>0} Q_m(G,d),$$

which involves numerical optimization. It will be convenient in what follows to distinguish the true values of the parameters by the notation (G_0, d_0).

Concentrating (21) with respect to G, we find that the estimate $\widehat{d}$ satisfies

$$\widehat{d} = \arg\min_{d} R(d)$$

where

$$R(d) = \log \widehat{G}(d) - 2d\frac{1}{m}\sum_{j=1}^{m}\log\lambda_j, \quad \widehat{G}(d) = \frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d} I_r(\lambda_j). \tag{22}$$

Recently, Robinson (1995) analyzed the above estimators in the stationary case where $d_0 \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Under rather weak regularity conditions on the smoothness of $f_{uu}(\lambda)$ as $\lambda \to 0+$, the innovations in the Wold representation of u_t and an expansion rate condition on m, which requires that $m \to \infty$ but $\frac{m}{n} \to 0$ as $n \to \infty$, Robinson showed that $\widehat{d} \to_p d_0$ and $\widehat{G}(\widehat{d}) \to_p G_0$. Under a slight strengthening of these conditions, Robinson also established that $\widehat{d}$ is asymptotically normally distributed with the limit distribution

$$m^{\frac{1}{2}}(\widehat{d} - d_0) \to_d N\left(0, \tfrac{1}{4}\right). \tag{23}$$

This limit theory makes possible statistical testing and the construction of confidence intervals for d_0 in the stationary case.

At present, there is no limit theory in the literature that covers semiparametric estimation of d_0 in the nonstationary case. In order to complete the present study, the author has developed a limit theory for the nonstationary case. The details are lengthy and will be reported in a separate paper (Phillips, 1998) to which the reader is referred. The main results from this investigation are that under regularity conditions that are broadly similar to those of Robinson (1995), the author has established that when $d_0 \in (\frac{1}{2}, 1]$ and $n \to \infty$

$$\widehat{d} \to_p d_0, \quad \widehat{G}(\widehat{d}) \to_d G_0 + C(d_0) \tag{24}$$

where

$$C(d_0) = \left(\frac{1}{\Gamma(-d_0)}\right)^2 \int_{r=0}^{1}\int_{a=r}^{1}(1-r)^{d_0-\frac{1}{2}}\frac{1}{a^{1+d_0}}$$

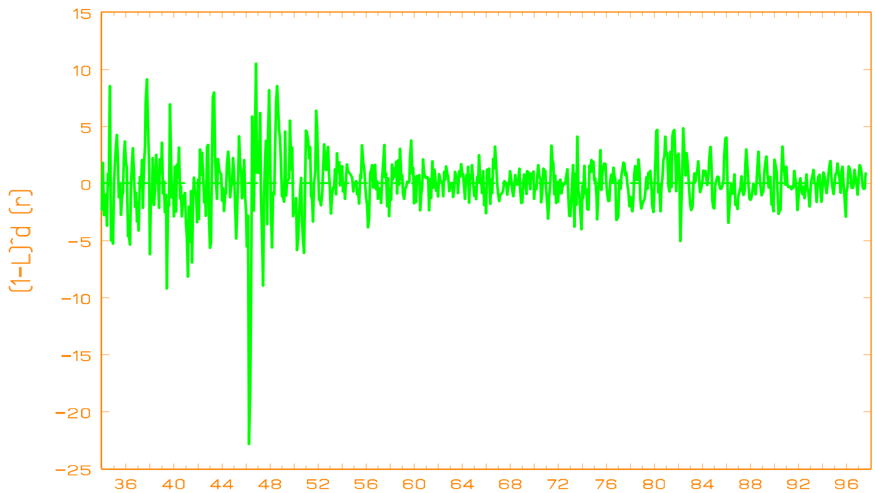

Figure 10: $(1-L)^{\widehat{d}} r_t$ for 1934–1997

$$\times \int_{s=0}^{1} (1-s)^{d_0 - \frac{1}{2}} \frac{1}{(a-r+s)^{1+d_0}} ds da dr B_{d_0 - 1}(1)^2, \quad d_0 \neq 1$$

$$C(d_0) = B(1)^2, \quad d_0 = 1$$

Here, $B_{d_0-1}(1)$ denotes fractional Brownian motion and is defined by the stochastic integral

$$B_{d-1}(r) = \frac{\omega}{\Gamma(d)} \int_0^r (r-s)^{d-1} dW(s)$$

where W is standard Brownian motion.

Observe from (24) that while $\widehat{d}$ is consistent in the nonstationary case, $\widehat{G}(\widehat{d})$ is an inconsistent estimator of G_0 and converges in distribution to a random limit.

In addition, it is shown in Phillips (1998) that the limit distribution of $\widehat{d}$ is as follows

$$m^{\frac{1}{2}}(\widehat{d} - d_0) \Rightarrow MN\left(0, \frac{1}{4} \frac{G_0^2}{(G_0 + C(d_0))^2}\right). \tag{25}$$

Thus, $\widehat{d}$ has a different limit distribution in the nonstationary case from (23) above. First, the distribution is mixed normal rather than normal. Second, $\widehat{d}$ is more efficient asymptotically when $d_0 > \frac{1}{2}$ than when $d_0 < \frac{1}{2}$, since $C(d_0) > 0$ and the random mixing variate in (25) therefore satisfies

$$\frac{G_0^2}{(G_0 + C(d_0))^2} < 1. \tag{26}$$

To use the limit theory in (25) to test hypotheses about d we need a consistent

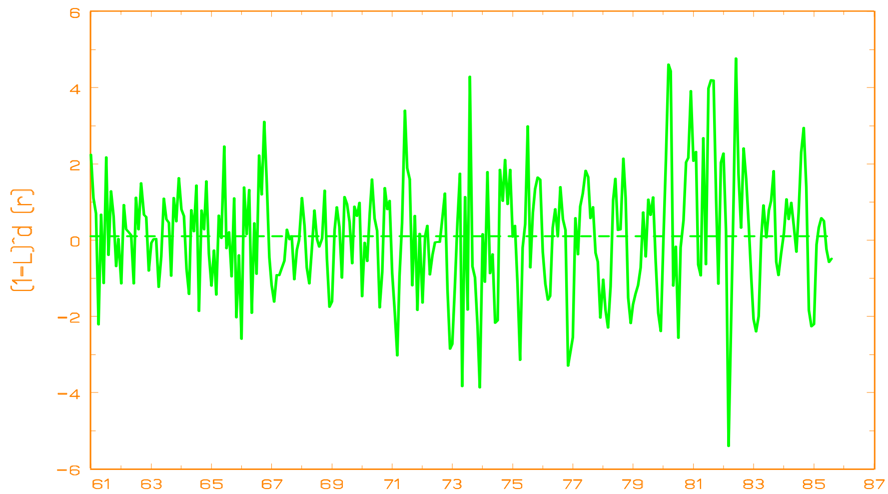

Figure 11: $(1-L)^{\widehat{d}} r_t$ for 1961–1985

estimate of the variance. This can be accomplished as follows. Compute the residuals

$$\widehat{u}_t = (1-L)^{\widehat{d}} r_t = \sum_{k=0}^{t} \frac{(-\widehat{d})_k}{k!} r_{t-k} \tag{27}$$

and form the estimate

$$\widehat{G}_0 = \frac{1}{m} \sum_{j=1}^{m} I_{\widehat{u}}(\lambda_j)$$

in place of $\widehat{G}(\widehat{d})$ in (22). Then, Wald tests on d_0 can be conducted using the t-ratio $t_d = \widehat{d}/s_{\widehat{d}}$, where the standard error is calculated using the formula

$$s_{\widehat{d}}^2 = \frac{1}{4m} \left(\frac{\widehat{G}_0}{\widehat{G}(\widehat{d})} \right)^2$$

This estimate has the advantage that it is consistent for both the stationary and nonstationary cases. An LM test could also be constructed using the hypothesized value d_0 in the construction of $\widehat{G}$ and $\widehat{G}_0$.

To form confidence intervals it is most convenient to use intervals of the form

$$\widehat{d} \pm 1.96 \left(\frac{1}{4m} \right)^{\frac{1}{2}}$$

which, in view of (26), will be conservative, and which will apply in both the stationary and nonstationary zones of d.

With this methodology and using $m = n^{\frac{3}{4}}$ frequency ordinates, we found semiparametric estimates of d in model (19) for the ex post real rate for the three time periods 1961–1985, 1961–1997 and 1934–1997. The results are given in Table 1.

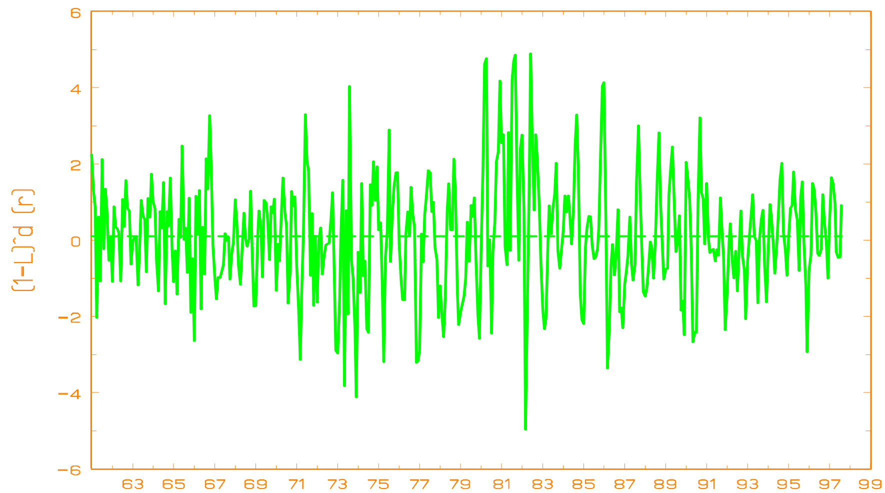

Figure 12: $(1-L)^{\widehat{d}} r_t$ for 1961–1997

Table 1: Empirical Estimates of d : Ex post Real Rate			
Period	$\widehat{d}$	$s_{\widehat{d}}$	95% Confidence Interval
1961–1985	0.589	0.0570	$[0.472, 0.705]$
1961–1997	0.508	0.0500	$[0.407, 0.607]$
1934–1997	0.527	0.0404	$[0.446, 0.608]$

For each period the estimates of d are greater than 0.5 and indicate that the ex post real rate of interest is nonstationary. Moreover, the estimates of d are all close, which is interesting and perhaps surprising given that the sample path of r_t varies substantially over the full time period and that other approaches require multiple regime shifts to model this data even over shorter periods. The confidence intervals show that unit root nonstationarity and short memory are both clearly rejected. However, in every case the confidence intervals for d include some long memory stationary alternatives with d less than 0.5 but greater than 0.4.

Thus, our estimates of d provide empirical evidence that supports the conclusion of Rose (1988) and Walsh (1987) that the real rate is nonstationary. But we reject unit root nonstationarity and the estimates of d are in every case not significantly greater than 0.5, so there is also some support from our estimates for the hypothesis that the real rate is marginally stationary but with very long memory.

Fig.'s 10–12 show the effect of d-differencing the real rate of interest as in (27) using the relevant estimates $\widehat{d}$ for each of the subperiods. The figures also display the mean value of the residual series $\widehat{u}_t = (1-L)^{\widehat{d}} r_t$ for each period. As is apparent from the figures, in each case $\widehat{u}_t$ appears to be stationary with short memory.

Finally, Fig. 13 gives spatial density estimates for the differenced series $\widehat{u}_t = (1-L)^{\widehat{d}} r_t$ calculated for each subperiod. The densities are calculated here by normalizing

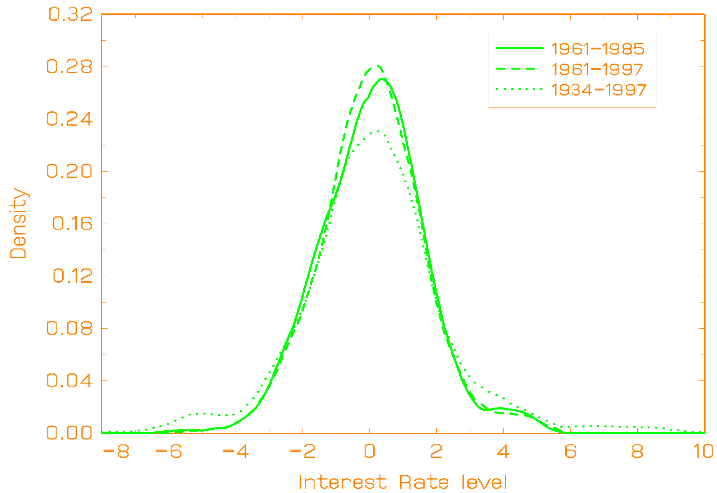

Figure 13: Densities of $(1-L)^{\widehat{d}} r_t$

the spatial densities in the same way for each subperiod – that is by $\frac{1}{nh}$ in place of $\frac{\widehat{\omega}^2}{\sqrt{n}h}$ in formula (14) – so that they are directly comparable and correspond to conventional kernel estimates for a stationary density. As is apparent from the figure, the fitted densities are symmetric and quite close, although there is apparently greater dispersion in the case of the longer data set 1934-1997.

Table 2: Empirical Estimates of d : Nominal Rate

Period	$\widehat{d}$	$s_{\widehat{d}}$	95% Confidence Interval
1961–1985	0.865	0.050	$[0.748, 0.981]$
1961–1997	0.924	0.048	$[0.824, 1.024]$
1934–1997	0.950	0.035	$[0.869, 1.032]$

In a similar way, we found semiparametric estimates of d for the nominal rate and inflation over the three time periods 1961–1985, 1961–1997 and 1934–1997. The results are given in Tables 2 and 3. The results in Table 2 corroborate many earlier findings that the nominal rate of interest is nonstationary. Unit root nonstationarity is included in the confidence band for d over the longer periods 1961-1997 and 1934-1997. The point estimates of d increase from 0.86 to 0.95 as the time period lengthens.

Table 3: Empirical Estimates of d : Inflation

Period	$\widehat{d}$	$s_{\widehat{d}}$	95% Confidence Interval
1961–1985	0.698	0.056	$[0.581, 0.814]$
1961–1997	0.581	0.049	$[0.481, 0.681]$
1934–1997	0.530	0.040	$[0.449, 0.611]$

The estimates reported in Table 3 show that inflation appears to be nonstationary over 1961-1985, but long memory stationary dependence in inflation is not rejected for the longer periods 1961-1997 and 1934-1997. For all time periods, unit root nonstationarity and short memory dependence are rejected. These estimates therefore do not support results like those in Miskin (1992) where inflation is treated as an $I(1)$ variable. The point estimates of d decrease as the time period lengthens, although in each case these exceed 0.5.

Figure 14: Nominal 90-day TB Rate

Comparing the results in Table 1 with those in Tables 2-3, the point estimates of d appear to more stable over the three subperiods for the real rate than for either the nominal rate or inflation. Also, there is some incompatibility between the three sets of estimates. In a model with three fractionally variables $y_t \equiv I(d_y)$, $x_t \equiv I(d_x)$, $z_t \equiv I(d_z)$ satisfying the simple linear relation $y_t = x_t - z_t$, the behaviour of y_t is necessarily dominated by the dominant component of x_t and z_t. Thus, if $d_x > d_z$, then $d_y = d_x$. In the present case, this suggests that the fractional integration of the nominal rate should dominate the real rate. However, the separate empirical estimates indicate that the fractional integration of the real rate is significantly less than that of the nominal rate. There are several possible explanations for this type of empirical incompatibility, including model misspecification and scale differences between the variables. It seems likely in the present case that the greater volatility of inflation over the nominal rate of interest is a contributing factor in these differences. Similar phenomena have been found to arise in the context of exchange rate data, where there are major differences in scale between spot returns and the forward premium - see Maynard and Phillips (1998). The data are graphed in Fig.'s 14-15 and seem to support this explanation.

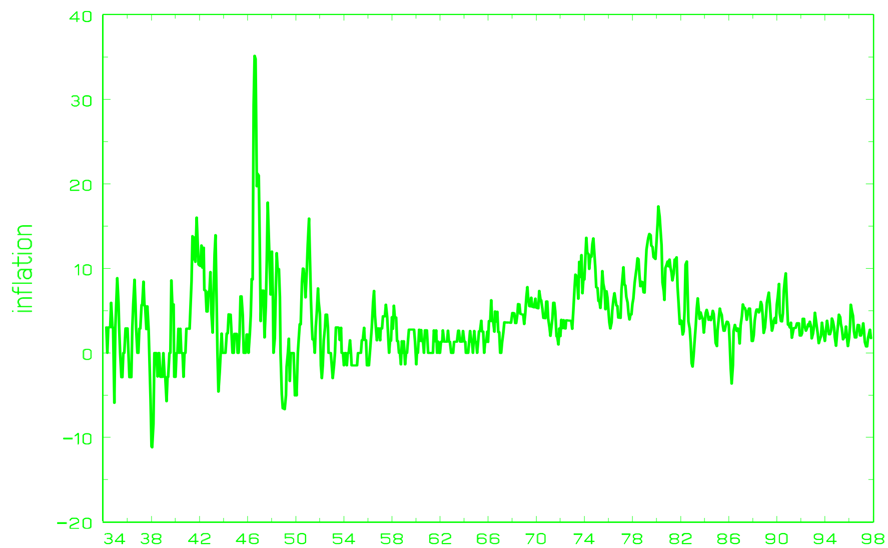

Figure 15: CPI 3 Month Inflation Rate

7. Conclusion

This paper introduces some new statistical procedures for describing and analyzing nonstationary data and applies these techniques to the study of Fisher's equation for the real rate of interest. Under weak conditions that assure some form of functional limit theorem for standardized forms of the data, we construct asymptotically valid spatial density and hazard function measures for the series, both of which take the form of random processes rather than nonrandom functions as in the case of stationary time series. Consistent techniques for estimating these quantities are given, together with a limit theory that enables the measures to be used in inference.

Using these spatial density techniques, we analyze the ex post real rate of interest in the US over the period 1934–1997. Results over the subperiod 1961–1985 provide some corroborating evidence to support the conclusion of a recent study by Garcia and Perron (1994) that the real rate fluctuates about constant levels that change over regimes. However, over the longer period 1934–1997, the regime change approach requires many change points, seems artificial, and lacks parsimony.

An alternative semiparametric model is considered which allows for fractional integration in the real rate, including unit root nonstationary and stationary long range dependence as special cases, to account for various types of long run behavior, and which retains generality with respect to the modelling of the short run component of the series. Through the fractional integration parameter in this model the extent of nonstationarity in the real rate can be directly measured. Gaussian local likelihood estimation of the fractional integration parameter is performed using recently obtained asymptotic results that apply in the nonstationary case. Empirical estimates of the fractional differencing parameter are computed for the ex post rate over 1934–1997 and two shorter subperiods. The results confirm evidence of nonstationarity in the real rate for all time periods, and therefore generally support the conclusion reached by Rose (1988). Our estimates also confirm Mishkin's (1992) rejection of unit root nonstationarity in the real rate of interest. However, although unit root nonstationarity and short memory can be rejected and point estimates indicate nonstationarity, confidence intervals for the fractional differencing parameter are in the region $[0.4, 0.6]$ and therefore do not completely rule out the possibility of a stationary real rate of interest with very long range dependence.

The fractional integration model for the real rate of interest seems to be successful in transforming the data to stationarity over all three periods 1934-1997, 1961–1985 and 1961-1997, all with a very similar fractional integration parameter. In this respect, the model is at once more parsimonious and more generally applicable than a model with many regime shifts.

8. Technical Appendix and Proofs

Our approach to an asymptotic theory for spatial density and hazard rate estimation relies on recent work in Phillips and Park (1998) — hereafter P^2 — for kernel density estimation and regression for nonstationary time series. We only sketch the derivations we need here.

Suppose X_t is a unit root time series with differences $\Delta X_t = u_t$ and initialization X_0 that satisfy Conditions 8.1 and 8.2 below. Distant initial conditions are permitted in 8.2 and play a role in the spatial density asymptotics. Conditions 8.3 and 8.4 relate to the kernel function and restrictions on the bandwidth h and they are used in P^2. Note the important difference between 8.4 and conventional assumptions about bandwidth in kernel density estimation - here the bandwidth can be of the form $h_n = cn^k, \quad k \in \left[-\frac{1}{4}+\delta, \frac{1}{12}-\delta\right]$, so that bandwidths that increase with n are permissible. The bandwidth cannot decrease too fast or increase too fast as $n \to \infty$. Condition 8.1 allows for differences u_t that follow a linear process and is standard (Phillips and Solo, 1992). The higher moment condition in 8.1 is useful in assuring the validity of a strong approximation to partial sums of u_t.

8.1 Assumption

(a) *u_t is a linear process $u_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$ with $C(1) \neq 0$ and $\sum_{j=0}^{\infty} j^{\frac{1}{2}} |c_j| < \infty$.*

(b) ε_t is iid$(0, \sigma^2)$ *with $E(|\varepsilon_j|^p) < \infty$, for some $p > 2$.*

8.2 Assumption *The initial conditions of X_t are set at $t = 0$ and X_0 has the following general form allowing for effects in the distant past*

$$X_0 = u + \sum_{j=0}^{[n\kappa]} u_{-j}, \text{ for some } \kappa \geq 0$$

where u is an $O_{\text{a.s.}}(1)$ random variable with $E(|u|^p) < \infty$ for some $p > 2$.

8.3 Assumption *The kernel $K(\cdot)$ is a symmetric and nonnegative density with integrable characteristic function φ_K and satisfies the following conditions for some $r > 2$:*

$$\int_{-\infty}^{\infty} K(s) = 1, \int_{-\infty}^{\infty} s^{2r} K(s)\, ds < \infty, \ \sup_s K(s) < \infty;$$

8.4 Assumption $n^{1-\delta} h_n^4 \to \infty$, *and* $h_n / n^{(1-\delta)/12} \to 0$ *for some* $\delta > 0$.

8.5 Proof of Theorem 3.1 The proof follows the same lines as that of Theorem 3.1 of P^2. As in that theorem, we need to augment the probability space so that a strong approximation to the limit Brownian motion $B(r)$ of $n^{-\frac{1}{2}} X_{[nr]}$ can be used. The only changes in the proof are:

(i) Since u_t is a linear process we need to rely on extended versions of the preliminary Lemmas 5.5 and 5.7 in P^2 that apply for linear processes instead of *iid* random variables. These extensions follow in precisely the same way, but make use of Lemma 4 rather than proposition 1 of Akonom (1993), which applies for linear processes

(ii) We use the consistent scale estimate $\widehat{\omega}^2$ of the long run variance $\omega^2 = 2\pi f_{\Delta x}(0)$ in the definition of the spatial density estimate. This is not needed in Theorem 3.1 of P^2 because chronological local time $\overline{L}_B(r, a) = \omega^{-2} L_B(r, a)$ is used in P^2 in place of local time.

(iii) Since $\widehat{\omega}^2 \to_p \omega^2$, the resulting limit holds in probability rather than almost surely.

8.6 Proof of Theorem 3.2 From Lemma 2.9(d) of P^2 we have

$$2^{-1} \lambda^{1/2} \left\{ L_B \left(t, r + \tfrac{s}{\lambda} \right) - L_B(t, r) \right\} \xrightarrow{d} Q(L_B(t, r), s),$$

where $Q(a,b)$ is a standard Brownian sheet. Then, using Theorem 3.1 of P[2], we have

$$
\begin{aligned}
& \sqrt{c_n}\left[\widehat{L}_B\left(r, n^{-\frac{1}{2}}s\right) - L_B(r, a - B_0(\kappa))\right] \\
= \; & \sqrt{c_n}\left[\frac{\widehat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]} K\left(\frac{s - X_t}{h_n}\right) - L_B(r, a - B_0(\kappa))\right] \\
= \; & \sqrt{c_n}\left[c_n\omega^2 \int_0^r K\left(c_n\left\{\frac{s - X_0}{\sqrt{n}} - B(g)\right\}\right) dg - L_B(r, a - B_0(\kappa))\right] + o_p(1) \\
= \; & \sqrt{c_n}\left[c_n \int_{-\infty}^{\infty} K\left(c_n\left\{\frac{s - X_0}{\sqrt{n}} - p\right\}\right) L_B(r, p)dp - L_B(r, a - B_0(\kappa))\right] + o_p(1) \\
= \; & \sqrt{c_n}\int_{-\infty}^{\infty} K(q)\left[L_B\left(r, \frac{s - X_0}{\sqrt{n}} - \frac{q}{c_n}\right) - L_B(r, a - B_0(\kappa))\right] dq + o_p(1) \\
= \; & \int_{-\infty}^{\infty} K\left(q\right)\sqrt{c_n}\left[L_B\left(r, \frac{s - X_0}{\sqrt{n}} - \frac{q}{c_n}\right) - L_B(r, a - B_0(\kappa))\right] dq + o_p(1) \\
\to_d \; & 2\int_{-\infty}^{\infty} K(q) Q(L_B(r, a - B_0(\kappa)), q)dq \\
=_d \; & 2L_B(r, a - B_0(\kappa))^{\frac{1}{2}} \int_{-\infty}^{\infty} K(q)Q(1, q)dq \\
=_d \; & 4L_B(r, a - B_0(\kappa))^{\frac{1}{2}} \int_0^{\infty} K(q)W(q)dq \\
=_d \; & 4L_B(r, a - B_0(\kappa))^{\frac{1}{2}} N\left(0, \int_0^{\infty}\int_0^{\infty} K(q)(q \wedge p)K(p)dqdp\right),
\end{aligned}
$$

as required.

8.7 Proof of Theorem 4.1 By virtue of Theorem 3.1 and since $s = s_0 + \sqrt{n}a$ we have

$$
\begin{aligned}
\widehat{H}_B\left(t, \frac{s}{\sqrt{n}}\right) &= \frac{\widehat{L}_B\left(t, \frac{s}{\sqrt{n}}\right)}{\int_{\frac{s}{\sqrt{n}}}^{\infty} \widehat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}}} \\
&\to_p \frac{L_B(t, a - B_0(\kappa))}{\int_a^{\infty} L_B(t, b - B_0(\kappa))db} \\
&= \frac{L_B(t, a - B_0(\kappa))}{\int_{a - B_0(\kappa)}^{\infty} L_B(t, c)dc} = H_B(t, a - B_0(\kappa))
\end{aligned}
$$

giving the stated result.

8.8 Proof of Theorem 4.2 We have

$$
\sqrt{c_n}\left[\widehat{H}_B\left(r, n^{-\frac{1}{2}}s\right) - H_B(r, a - B_0(\kappa))\right]
$$

$$= \sqrt{c_n}\left[\frac{\widehat{L}_B\left(t, \frac{s}{\sqrt{n}}\right)}{\int_{\frac{s}{\sqrt{n}}}^{\infty} \widehat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}}} - \frac{L_B(t, a - B_0(\kappa))}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc}\right] \tag{28}$$

Observe that

$$\begin{aligned}
\int_{\frac{s}{\sqrt{n}}}^{\infty} \widehat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}} &= \frac{\widehat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\int_{\frac{s}{\sqrt{n}}}^{\infty} K\left(\frac{p - X_t}{h_n}\right)\frac{dp}{\sqrt{n}} \\
&= \frac{\widehat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\int_{\frac{s}{\sqrt{n}}}^{\infty} K\left(c_n\left\{\frac{p - X_t}{\sqrt{n}}\right\}\right)\frac{dp}{\sqrt{n}} \\
&= \frac{\widehat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\int_{a}^{\infty} K\left(c_n\left\{b - \frac{X_t}{\sqrt{n}}\right\}\right)db \\
&= \frac{\widehat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\frac{1}{c_n}\overline{\mathcal{K}}\left(c_n\left\{a - \frac{X_t}{\sqrt{n}}\right\}\right) \\
&= \frac{\widehat{\omega}^2}{n}\sum_{t=1}^{[nr]}\overline{\mathcal{K}}\left(c_n\left\{a - \frac{X_t}{\sqrt{n}}\right\}\right) \qquad (29)
\end{aligned}$$

Now, under the assumption that $\sqrt{c_n}(\widehat{\omega}^2 - \omega^2) = o_p(1)$, using the strong approximation $\frac{X_{[nr]}}{\sqrt{n}} = B_0(\kappa) + B(r) + o_{\text{a.s.}}\left(n^{-\frac{1}{2}+\frac{1}{p}}\right)$ as in P[2], and proceeding as in the proof of Theorem 3.1 of P[2], we get

$$\begin{aligned}
\frac{\widehat{\omega}^2}{n}\sum_{t=1}^{[nr]}\overline{\mathcal{K}}\left(c_n\left\{a - \frac{X_t}{\sqrt{n}}\right\}\right) &= \omega^2\int_0^r \overline{\mathcal{K}}(c_n\{a - B_0(\kappa) - B(s)\})ds + o_p\left(\frac{1}{\sqrt{c_n}}\right) \\
&= \omega^2\int_0^r \overline{\mathcal{K}}(c_n\{a - B_0(\kappa) - B(s)\})ds + o_p\left(\frac{1}{\sqrt{c_n}}\right) \\
&= \omega^2\int_{-\infty}^{\infty} \overline{\mathcal{K}}(c_n\{a - B_0(\kappa) - c\})L_B(r,c)dc + o_p\left(\frac{1}{\sqrt{c_n}}\right) \\
&= \omega^2\int_{a-B_0(\kappa)}^{\infty} L_B(r,c)dc + o_p\left(\frac{1}{\sqrt{c_n}}\right) \qquad (30)
\end{aligned}$$

since

$$\begin{aligned}
\overline{\mathcal{K}}(c_n\{a - B_0(\kappa) - c\}) &= \int_{c_n\{a-B_0(\kappa)-c\}}^{\infty} K(s)ds \\
&= \begin{cases} O\left(\frac{1}{c_n^{2r-1}}\right) & \text{for } c < a - B_0(\kappa) \\ 1 + O\left(\frac{1}{c_n^{2r-1}}\right) & \text{for } c > a - B_0(\kappa) \end{cases}
\end{aligned}$$

from the given tail behavior of the kernel $K(s)$ and where $r > 2$ (see Assumption 8.3).

It follows from (28)-(29) and Theorem 3.2 that

$$
\begin{aligned}
&\sqrt{c_n}\left[\widehat{H}_B\left(r, n^{-\frac{1}{2}}s\right) - H_B(r, a - B_0(\kappa))\right] \\
&= \sqrt{c_n}\left[\frac{\widehat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc + o_p\left(\frac{1}{\sqrt{c_n}}\right)} - \frac{L_B(t, a - B_0(\kappa))}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc}\right] \\
&= \frac{1}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc}\sqrt{c_n}\left[\widehat{L}_B\left(t, \frac{s}{\sqrt{n}}\right) - L_B(t, a - B_0(\kappa))\right] \\
&\Rightarrow \frac{2}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc}\int_{-\infty}^{\infty} K(q)Q(L_B(r, a - B_0(\kappa)), q)dq \\
&= \frac{4L_B(r, a - B_0(\kappa))^{\frac{1}{2}}}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc} MN(0, K_2) \\
&\equiv 4MN\left(0, K_2\frac{L_B(r, a - B_0(\kappa))}{\left(\int_{a-B_0(\kappa)}^{\infty} L_M((r,s)ds\right)^2}\right),
\end{aligned}
$$

giving the required result.shifts.

9. Notation

$\rightarrow_{\text{a.s.}}$	almost sure convergence	$\Rightarrow, \rightarrow_d$	weak convergence
$=_d$	distributional equivalence	$[\cdot]$	integer part of
$:=$	definitional equality	$r \wedge s$	$min(r, s)$
$o_{\text{a.s.}}(1)$	tends to zero almost surely	$\equiv$	equivalence
$\rightarrow_p$	convergence in probability	$o_p(1)$	tends to zero in probability
$(a)_k$	$(a)(a+1)\ldots(a+k-1)$		

10. Data Sources

(a) **Consumer Price Index**

Not seasonally adjusted
Area: US city average (ie urban)
Items: All items
Base: 1982-84=100
Source: Bureau of Labor Statistics, *Monthly Labor Review*
Code: CUUR0000SA0

(b) **3-Month Treasury Bill Rate**

Secondary Market

Average of Daily Closing Bid
Annualized using a 360-day year for bank interest
Quoted on a discount basis
Source: Board of Governors of the Federal Reserve System, *Federal Reserve Bulletin*
Code: TB3MS

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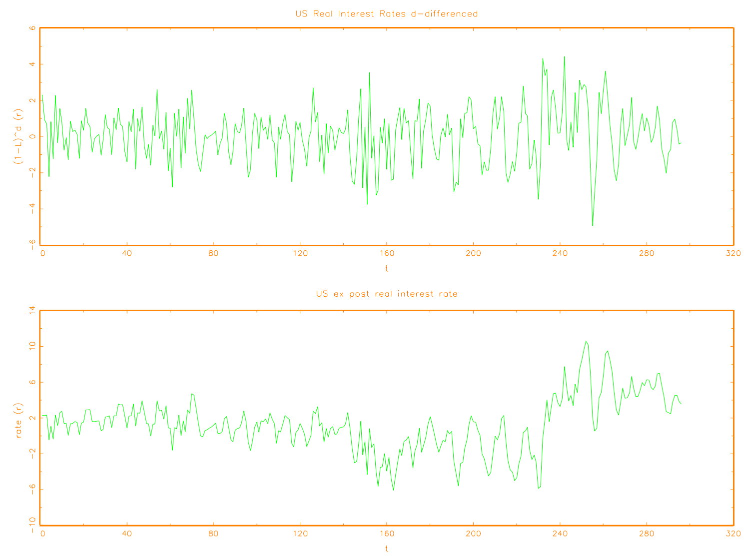

Figure 16: US Real Interest Rate and Fractionally Differenced

1 Extra Notes

Irving Fisher viewed inflation expectations as being dependent on the distant as well as the recent past and he devised techniques for fitting distributed lags to capture this dependence over long horizons. In carrying out some correlational studies between inflation and interest rates, Fisher's (1930) concluded:

> *"Our first correlations seemed to indicate that the relationship between P' (inflation) and i (interest rate) is either very slight or obscured by other factors. But when we make the much more reasonable supposition that price changes do not exhaust their effects in a single year but manifest their influence with diminishing intensity over long periods which vary in length with the conditions, we find a very significant relationship, especially in the period which includes the World War, when prices were subject to violent fluctuations."*

According to Fisher's (1930) equation, the stochastic properties of the real rate of interest depend on those of nominal interest rates and the history of inflation as manifested in the distributed lag relation that defines inflation expectations. Empirical evidence indicates that nominal interest rates and inflation have different temporal dependence properties, and balancing these variables seems to be needed if a conventional econometric relation between them is to be established. This paper seeks to do so by positing a potential long memory relation between the real rate of interest and inflation. This approach is in line with Fisher's original idea, but uses the modern theory of strong dependence rather than straight line or polynomial distributed lags. Some empirical work is conducted that explores these ideas using some of Fisher's original data and some other long time series of interest rates and inflation.

Hence, Rose's conclusions about CCAPM model invalidity apply and validity of modern inflation targeting models like those in Svensson in question — check Svensson's paper. May still be cointegration between inflation and interest rate, but not strong enough to validate a real rate that fluctuates about a constant mean. More generally could ask if there is a linear filter that is cointegrating with stationary error — this would be in line with Fisher who originally thought that one needed very long distributed lags to get correlational effects between inflation and interest rates. However, this may not be the correct way to construct inflation expectations and may invalidate rational expectations.

Beyond the behavior of understanding Since are quite limited. is not as straightforward as it is for stationary series. While covariance stationary series have second moment properties that are completely characterizes by their autocovariances or spectral density, and may first appear. While some progress was made in setting up nonstationary spectra in the 1960's, it has not been used much - never in economics. We propose a radically different type of descriptive decomposition of variance - based on spatial occupation times.

Spatial density approach. How much time does the data spend in the neighborhood of a particular spatial point.

Hazards: what is the risk that inflation will be greater than $x\%$. That the real rate of interest will exceed $x\%$. Has the occupation time of the real rate around $x\%$ been declining over time.

Long range dependence and Fisher's measurement of inflation expectations. Estimating d. Why VAR's not useful here, cf. Litterman and Weiss.

Is there a cointegrating relation between interest rates and price inflation? Does it involve long range dependence. Does it violate rational expectations.

that they fully explain what happens in regressions of this sort and why it happens, there is room for considerable debate about the implications of these results for empirical research. One viewpoint was clearly stated by a referee of the paper in the following way:

> "We use the term spurious regression in contrast to say the concept of cointegrated regressions, i.e.e the possibility that certain sets of variables explain the trend of the dependent process in an economic sensible way. The fact that trending time series have valid representations in terms of other independent processes or deterministic functions of time is not of much interest from an economic viewpoint, unless it helps separate the wheat from the chaff."

In commenting on this orthodox view, I will make only two points here and leave it to future debate to take the discussion further. First, it needs to be emphasized that cointegrating regressions do not explain trends. Instead, they relate trends in multiple time series and thereby pass the trending behavior along to secondary variables that are usually also endogenous, leaving the trends themselves to be explained by unit roots, time polynomials and trend breaks. As this paper shows, the trends themselves can be validly modelled in a variety of ways. Thus, the central issue addressed in this paper remains present in modern cointegration-based models of nonstationary time series. The second point is that the nature of trending mechanisms in economics is little understood and econometricians have little guidance from economic theory models about meaningful economic specifications. Were this not so, we would not be as heavily dependent as we presently are on unit root models, time polynomials, trend breaks, kernel regression fits and such like in capturing trends in empirical research. Against this background and with the current class of nonstationary models used in econometrics, it is virtually inevitable that the trending processes that appear in econometric models have little intrinsic economic meaning, even though the trends themselves may be of considerable economic interest. This paper shows that, even in the impoverished class of trending mechanism that we currently employ in empirical research, a limit theory of the trending process is possible and that it will often be based, in part at least, on a 'limit theory of the sample period fit'. This limit theory brings with it attendant qualifications such as those in the Introduction about the use of these mechanisms in a predictive context.

The results presented here have some implications for unit root modelling and testing. In recent years much of that literature has emphasized the importance of setting up a general maintained hypothesis that includes 'alternative' specifications to a unit root model, such as deterministic trends and trend breaks. The results of this paper show that such specifications are not necessarily alternatives to a unit root model at all. Since unit root processes have limiting representations entirely in terms of these functions, it is apparent that we can mistakenly 'reject' a unit root model in favor of a trend 'alternative' when in fact that alternative model is nothing other than an alternate representation of the unit root process itself. A development of the asymptotic theory in this case and a study of the impact of such considerations on empirical work are left for a future paper.