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AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

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THE LINEAR EXCHANGE MODEL AND INDUCED WELFARE OPTIMA

Rolf R. Mantel

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by

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0. Introduction

Gale (1957) was first in analyzing the pure trade model of general competitive equilibrium in which consumers have linear utility functions. Using Kakutani's fixed point theorem, he showed that an equilibrium exists if the economy is irreducible. He also showed that in the latter case the utility levels achieved by the traders are uniquely determined by the equilibrium conditions.

Eaves (1975) demonstrated that Lemke's algorithm can be used to compute an equilibrium for this model in a finite number of steps. The finiteness is explained by an argument of Gale that a solution exists which is a rational function of the coefficients of the system of inequalities defining an equilibrium.

It is the purpose of the present paper to show that the fixed point methods employed by Gale and Eaves are not necessary for the linear utility pure trade model. This will be done by presenting a concave, monotone, social welfare function which has the property that maximizing feasible utility allocations are competitive equilibrium utility allocations, so that the equilibrium problem is effectively reduced to a concave nonlinear programming problem with linear constraints. This perhaps surprising result follows quite easily from the special structure of the problem, using the tools for equilibrium analysis set forth in Mantel (1965, 1966); the relation of those investigations with the present one is pursued more closely in other work of the author (1976).

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1. The linear utility model

Following Eaves (1975) in its description, with some changes in notation, let the linear utility pure trade model be described by two positive $n \times m$ matrices

$$C = (c^1,\dots,c^m) \quad \text{and} \quad W = (w^1,\dots,w^m) .$$

The i-th consumer, $i \in \{1,\dots,m\} = M$, has an endowment vector w^i. His utility is given by $u_i = c^i . x^i$, where x^i, the i-th column of the nonnegative allocation $n \times m$ matrix X, is a vector of quantities of commodities, his consumption bundle.

An equilibrium for (C,W) is a pair (p, X) such that the price vector p is in $P = R^n_+ \setminus \{0\}$, $X \geq 0$, and

i. for every i, x^i maximizes his utility on the budget set $\{x \geq 0 \mid p.x \leq p.w^i\}$,

ii. $\Sigma\, x^i \leq \Sigma\, w^i$

Condition i. states the usual preference maximizing behavior of consumers, wheras ii. is the requirement of market balance, so that demand does not exceed supply.

<u>Remark</u>: The assumption that the tastes matrix C and the endowments W be both positive is too strong for the results in the sequel to hold. These would be true if these matrices were assumed to be nonnegative and the economy irreducible, meaning that for all $I \in M$ and $J \in N = \{1,\dots,n\}$ whenever $w^i_j = 0$ for all $(i,j) \in I \times J$, there exists a pair $(i,j) \in M\setminus I \times N\setminus J$ such that $c^i_j \neq 0$. The added generality does not seem to be warranted, since the proofs would be considerably less transparent, and the final result could

be achieved anyhow by slightly perturbing the original matrices.

By an ingenious transformation, Eaves converts the equilibrium conditions into a linear complementarity problem. He shows how the solution can be computed by Lemke's algorithms.

2. The existence of equilibrium in the linear utility model

Gale (1957) showed the existence of equilibrium using Kakutani's fixed point theorem. Eaves (1975) showed existence by an application of Lemke's algorithm. It is known, and has been shown by Scarf (1973) that a modification of the latter can be used to prove Kakutani's fixed point theorem, so that even though Eaves procedure allows a relatively simple computation of an equilibrium solution, he did not achieve a conceptual simplification over Gale's existence proof.

The existence of rational solutions allows one to hope for more, especially since it is known that the equilibrium utility levels are unique. The problem is of relatively simple structure. Due to the characteristics of the utility functions, one would expect consumers to specialize in the consumption of a single commodity, were it not for the need to satisfy the budget restrictions, in the same way as in the Ricardian theory of comparative advantage constant costs induce countries to specialize in the production of a single commodity. Note that for each consumer all commodities are perfect substitutes in the interior of the positive orthant of commodity space, becoming complements on the boundary. Each excess demand correspondence has a value equal to a simplex of maximal dimensionality for a single price vector. For almost all prices--in the sense of Lebesgue measure--is the excess demand a singleton.

In order to exploit the special structure of the problem hinted at in the previous lines, consider the social welfare function

$$b(u) = \max \{ \min_{i,j} \{u_i \, p_j \, / \, c^i_j \, (w^i.p) \} \mid p \in P \} ,$$

defined for $u \in R^m_+$. It is easily checked that b is well defined, since it is sufficient to let p vary on the compact unit simplex $S = \{p \in P \mid e.p = 1 \}$, where e represents an n-vector whose coordinates are equal to unity.

The existence of a social welfare function which is increasing in each trader's utility level, and which has the property that the set of maximizing feasible utility allocations coincides with the set of competitive utility allocations, is shown by Mantel (1976a). The function b defined above is derived by a straightforward application of the results in that reference.

The fact that b turns out to be concave follows from the very special structure of the problem. As argued by Mantel (1976b), this will be so whenever the economy is perfectly balanced, in the sense that the sets of utility outcomes to the coalitions of traders are the values of an additive function of individual endowment levels.

Lemma 1. The social welfare function b has the following properties, for all $u, v \in R^m_+$ and all $\lambda \in R_+$.

1. (Monotonicity) $u > v$ implies $b(u) > b(v) \geq 0$
2. (Homogeneity) $b(\lambda u) = b(u)$
3. (Superadditivity) $b(u+v) \geq b(u) + b(v)$
4. (Continuity) b is continuous at u .

Proof. 1. From the definition of b it is obvious that b is nonnegative if u is nonnegative, positive if u is positive. Thus, if $b(v) = 0$ the conclusion follows. Otherwise there exists some $p \in S$ such that for all i,

$$\begin{aligned} 0 < b(v) &\leqslant v_i \; p_j \; / \; c^i_j \; (w^i \; .p) \\ &< u_i \; p_j \; / \; c^i_j \; (w^i \; .p) \\ &\lessgtr b(u) \end{aligned}$$

where the first inequality has been assumed, the second follows from the definition of b, the third from the fact that the first two imply that $p > 0$, and the last again from the definition of b , and holds of course only for the minimizing pair (i,j) .

2. Follows immediately from the definition of b .

3. If $b(u)$ or $b(v)$ is zero, the conclusion follows trivially from 1.; thus assume $t = b(u)$ and $s = b(v)$ are positive. There exist p, q in S such that for all i, j

$$t \leq u_i \, p_j \, / \, c_j^i \, (w^i . p)$$

$$s \leq v_i \, q_j \, / \, c_j^i \, (w^i . q) \, .$$

Obviously both price vectors are positive. Set

$$r_j = (p_j{}^t \, q_j{}^s)^{1/(t+s)} .$$

Since the arithmetic mean exceeds the geometric mean when the weights are equal, it is easily checked that for all k, j, unless $p_k/p_j = q_k/q_j$,

$$r_k/r_j < [t(p_k/p_j) + s(q_k/q_j)] \, / \, (t+s).$$

Consequently one obtains for all i, j , unless $p/p_j = q/q_j$

$$(u_i + v_i) \, / \, (t+s) \geq \left[t \, c_j^i \, (w^i . p) \, / \, p_j + s \, c_j^i \, (w^i . q) \, / \, q_j \right] / \, (t+s)$$

$$> c_j^i \, (w^i . r) \, / \, r_j$$

so that

$$t+s \leq \min_{i,j} \, (u_i + v_i) \, r_j \, / \, c_j^i \, (w^i . r) \leq b(u+v) \, .$$

This relation 3. holds with strict inequality whenever $p \neq q$.

4. The function $f(u,p) = \min_{i,j} \, u_i \, p_j \, / \, c_j^i \, (w^i . p)$ is continuous on $R_+^m \times S$. Its maximum with respect to p is continuous on R_+^m . ||

Remark: 2. Properties 2. and 3. of b in the previous lemma imply that b is concave on R^m_+ .

Consider now the following concave nonlinear maximization problem with linear constraints.

(P) max $b(u)$, subject to

(1) $u_i - c^i . x^i \leqslant 0$ $i \in M$

(2) $X e \leqslant W e$

$(u,X) \geqslant 0$

where the utility vector u and the allocation matrix X will be called feasible if they satisfy the constraints. Note that (1) gives bounds on each consumer's utility, and (2) is the requirement that markets be in balance.

Note that this is a standard concave programming problem with very simple constraints, hence should be solvable by any of the currently standard procedures.

The following upper bound on feasible social welfare levels is easily obtained.

Lemma 2. If the pair (u,X) is feasible for (P) , then $b(u) \leqslant 1$.

Proof.

Since the inequality is obviously satisfied if $b(u) = 0$, we may restrict our attention to the case in which $b(u) > 0$. There exists a positive p in S such that for all i, j

$$b(u) \leqslant u_i \, p_j \, / \, c^i_j \, (w^i . p)$$

or

$$b(u) \, c^i_j \, (w^i . p) \leqslant u_i \, p_j \, .$$

Multiplying by x_j^i and summing over j, one obtains for all i

$$b(u)\,(c^i . x^i)\,(w^i . p) \leqslant u_i\,(p . x^i)\ .$$

Since $u > 0$, considering inequalities (1) one has

$$(3) \qquad b(u)\,(w^i . p) \leqslant (p . x^i)$$

so that adding over i,

$$b(u)\,(p . We) \leqslant p . Xe \leqslant p . We\ ,$$

where the last inequality follows from (2) when multiplied by p. Since $p . We$ is positive, the conclusion follows.

The following result is fundamental for the proof of the existence of competitive equilibrium.

Lemma 3. There exists a feasible pair (u^*, X^*) such that $b(u^*) = 1$.

Proof. The constraint set of the nonlinear programming problem (P) is obviously compact, and b is continuous by 4. of Lemma 1. Thus there exists a pair (u^*, X^*) which solves (P). It remains to show that for this maximum $b(u^*) = 1$.

Define the Lagrangean function L by

$$L(u,X,\alpha,q) = b(u) + \Sigma\, \alpha_i(c^i . x^i - u_i) + q.(We \dot{-} Xe)\ .$$

Note that for a convenient choice of (u,X) the constraints of (P) will be satisfied everywhere with strict inequality, so that Slater's constraint qualification holds. An application of Uzawa's (1959) theorem provides two vectors (α^*, q^*) of Lagrange multipliers such that the Lagrangean L has a nonnegative saddle point at $(u^*, X^*, \alpha^*, q^*)$. This means that

$L(u,X,\alpha^*,q^*) \leqslant L(u^*, X^*, \alpha^*, q^*) \leqslant L(u^*, X^*, \alpha, q)$ for all nonnegative (u,X,α,q). The right hand saddle point inequality implies that the constraints of (P) are satisfied by (u^*,X^*); so are also the complementary slackness conditions

$$\alpha_i^* (c^i . x^{i*} - u_i^*) = 0 , \quad i \in M$$

$$q^* (We - X^*e) = 0 .$$

Thus $L(u^*, X^*, \alpha^*, q^*) = b(u^*)$.

From the left hand saddle point inequality one obtains, by considering arbitrarily large X ,

(4) $$q^* \geqslant c^i \alpha_i^* , \qquad i \in M$$

For $X = 0$ one can then deduce the complementary slackness conditions

(5) $$q^* . x^{i*} = \alpha_i^* (c^i . x^{i*}) , \qquad i \in M$$

It is easily verified that there exists a feasible utility vector which is strictly positive, so that $b(u^*) > 0$, and as a consequence $u^* > 0$.

From the left hand saddle point inequality one obtains, by varying u , that

(6) $$b(u) - \alpha^*.u \leqslant b(u^*) - \alpha^*.u^* = 0$$

for all nonnegative u . By lemma 1, b is homogeneous, so that the right hand side of this inequality is necessarily zero, and u^* maximizes the left hand side. Since b takes on positive values and is homogeneous, α^* cannot be zero. But then (4) implies that $q^* > 0$. Since $u^* > 0$, (1) implies that x^{i*} is not zero. Therefore, the left hand side of (5) is positive, and so must be α^* . Note that $x^{i*} \neq 0$ means, because of (5), that (4) does not hold with strict inequality, so that for all $i \in M$

(7) $$\alpha_i^* = \min_j \; q_j^* / c_j^i$$

Define for all i,

(8) $$u_i = (q^*.w^i) / \alpha_i^*$$

Then, from (6)

$$b(u^*) \geq b(u) + \alpha^*.(u^* - u) \;;$$

from (8),

$$= b(u) + \alpha^*.u^* - q^*.We;$$

from (1), (2), and complementary slackness,

$$= b(u) + \Sigma[\alpha_i^* (c^i.x^{i*}) - q^*.x^{i*}] \;;$$

from (5), $= b(u)$;

from the definition of b ,

$$\geq \min_{i,j} u_i \, q_j^* / c_j^i (w^i.q^*) \;;$$

from (8) ,

$$= \min_i \min_j q_j^* / (c_j^i \, \alpha_i^*) \;;$$

and from (7),

$$= 1 \;.$$

||

<u>Theorem 1.</u> An optimal solution to (P) provides a competitive allocation X^* , with corresponding utility levels u^* .

<u>Proof.</u> Let p^* in S maximize $f(u^*,p)$. By lemma 3., $b(u^*) = 1$, so that (3) becomes

$$w^i.p^* \leq x^{i*}.p^* \;,$$

which together with (2) implies that the allocation X^* satisfies all consumer's budget restrictions with equality. Furthermore, the definition of b implies

$$b(u^*) = 1 \leq u_i^* \, p_j^* \;/\; c_j^i (w^i .p^*)$$

for all i, j, so that for all i

(9) $$p^* \geq c^i \alpha_i^*$$

with α_i^* defined by

(10) $$\alpha_i^* = (w^i .p^*) / u_i^*$$

Consequently, if x is nonnegative and satisfies the budget restriction

$$p^* . x \leq p^* . w^i \text{ , one has}$$

$$c^i . x \leq (p^* . x)\, u_i^* \,/\, (p^* . w^i) \leq u_i^* \text{ .}$$

Therefore, each consumer is maximizing his preferences subject to the budget constraint. ||

Corollary 1. An optimal solution (u^*, X^*) to (P) maximizes $\alpha^* . u$ subject to the restrictions of (P), with α^* as defined in (10).

Proof. Consider the following sequence of inequalities satisfied by any feasible pair (u, X).

$$\begin{aligned} \alpha^* . u &\leq \Sigma\, \alpha_i^* (c^i . x^i) && \text{by (1)} \\ &\leq \Sigma\, p^* . x^i && \text{by (9)} \\ &\leq \Sigma\, p^* . w^i && \text{by (2)} \\ &= \alpha^* . u^* && \text{by (10)} \end{aligned}$$

Corollary 2. Let (p^*, X^*) be an equilibrium, and let u^* be the corresponding utility vector. The pair (u^*, X^*) solves (P).

Proof. Obviously, (u^*, X^*) is feasible for (P). In view of lemma 2., it is only necessary to demonstrate that $b(u^*) = 1$. Since all consumers have monotone preferences, equilibrium prices must be positive. Since the i-th consumer maximizes his preferences subject to the budget constraint, which is satisfied by the bundle $(p^* . w^i)\, e^j / p_j^*$, where e^j is the j-th unit vector, one has

$$u_i^* \geq c_j^i \,(w^i . p^*) \,/\, p_j^*$$

for all i, j. Consequently, from the definition of b,

$$b(u^*) \geq \min_{i,j} u_i^* \; p_j^* / c_j^i (w^i.p^*) \geq 1$$ ||

Corollary 3. Equilibrium prices, p^*, utility levels u^*, and welfare weights α^* are unique.

Proof. At equilibrium, for each i one has $p^*.x^{i*} = p^*.w^i > 0$, so that there exists some j for which $x_j^{i*} > 0$. From relation (9), by complementary slackness it is know that $p_j^* = c_j^i \alpha_i^*$; therefore it follows that

$$\alpha_i^* = \min_j p_j^* / c_j^i$$

so that equilibrium prices uniquely determine the welfare weights. From equations (10) it then follows that equilibrium prices uniquely determine the utility levels. It only needs to be shown now that equilibrium prices are unique.

Let two asterisks distinguish the second equilibrium. By corollary 2., and lemmas 2. and 3., $b(u^*) = b(u^{**}) = 1$. Since by remark 2 b is concave, whereas the constraint set of (P) is convex, lemma 2. implies that $b[(u^* + u^{**})/2] = 1$. The sentence at the end of the proof of 3. of lemma 1, implies then that $p^* = p^{**}$. ||

Corollary 4.

Equilibrium prices p^*, utility levels u^*, and welfare weights α^* are rational functions of the elements of the matrices C and W.

Proof: Eaves (1975) shows that the equilibrium problem is equivalent to a linear complementarity problem, with coefficients which are either zero or unity or elements of C or W. The variables of the two problems are linked by rational relations.

Linear complementarity problems have rational solutions; the conclusion follows from corollary 3.

Remark 3. The conclusion to be drawn from these results is that the linear utility model can be solved for its equilibrium solution by concave programming methods, due to the existence of a concave welfare function consistent with the social ordering implied by the initial distribution of individual endowments. Note that the constraints of (P) are linear, though the welfare function need not be linear. It can be shown that by a change in the variables the problem can be cast into a form in which the objective function is linear, and the constraints, if not linear, are described by additively separable concave functions.

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