FROM NASH TO WALRAS VIA SHAPLEY-SHUBIK

By

Pradeep Dubey and John Geanakoplos

April 2002

COWLES FOUNDATION DISCUSSION PAPER NO. 1360

From Nash to Walras via Shapley–Shubik

Pradeep Dubey and John Geanakoplos

Abstract

We derive the existence of a Walras equilibrium directly from Nash's theorem on noncooperative games. No price player is involved, nor are generalized games. Instead we use a variant of the Shapley–Shubik trading-post game.

1 Introduction

The paper of Nash [12] on the existence of equilibrium points in noncooperative games was historically critical for Walrasian analysis. In order to prove existence of Walras equilibrium, Arrow and Debreu [2], [5], [6] extended Nash's model to "generalized games" and added a fictitious price player (whose payoff was "the value of excess demand") to the economy. Walras Equilibria (WE) were than obtained as Nash Equilibria (NE) of a generalized game that included the price player.

We show that WE exist without stepping outside the original framework of Nash. In fact, WE are the limits of NE of a sequence of games $\Gamma(M)$. No price player is involved nor are generalized games. The games $\Gamma(M)$ adhere completely to the standard format laid out by Nash: each player has a compact, convex strategy-set; and a continuous payoff function which is concave in his own strategy. We do replace each original agent in the economy by a type consisting of a continuum of identical agents.[1] But since we restrict to type-symmetric (TS) strategies, all measure-theoretic technicalities are avoided. By an analysis identical to that of Nash, we verify the existence of TSNE, and hence of a WE.

Our game is a variant of the Shapley–Shubik trading-post game ([17], [18], [19]). It is defined by a continuous map from agents' "bids" (strategies) to prices and feasible reallocations (outcomes). To our mind, the most salient feature of the game-theoretic approach is that there *is* such a map: no matter what strategies agents choose, a feasible outcome is always engendered. In Walrasian analysis we are left in the dark as to what happens out of equilibrium.

The search for game-theoretic foundations of Walras equilibrium has a long and rich history. In his famous paper of 1838, Cournot [4] introduced a basic model in

---

[1] One might even argue that the notion of an NE becomes more viable with a continuum, for then unilateral deviations of any agent are drowned in market aggregates and are not detectable. Each agent naturally assumes that others will hold their strategies fixed if he deviates. (The continuum also eases the transition from NE to WE, at the technical level. But, for completeness' sake, we add a last Section 6 which shows that our analysis remains intact when a finite set of agents is replicated and the number of replicas goes to infinity.)

which firms compete at a single market by strategically choosing how much output to produce and sell. The Cournot game is the archetypal example of a Nash game with continuous, concave payoffs. Equilibrium always exists, and Cournot himself observed that as the number of agents increases, price-taking behavior is induced and WE is achieved in the limit. Bertrand [3] defined another game in which prices are the strategic variables, and showed that NE coincide with WE, provided there are at least two competing firms. The Bertrand game, however, has discontinuous payoffs and so is not in the Nash format.

The single market analyses of Cournot and Bertrand were extended by many authors to come to grips with multiple markets in general equilibrium. Following the Bertrand tradition naturally led to discontinuous games. In [8], [10], [16], coincidence of NE and WE was established, but the authors relied on the existence of WE to show the existence of their NE, rather than the other way around.

The extension of the Cournot tradition to general equilibrium was pioneered by Lloyd Shapley and Martin Shubik [17], [18], [19]. The difficulty to overcome is that an agent might want to sell in one market and buy in another. How can the Walras budget set be incorporated in the game? One possibility is to introduce a market for every pair of commodities, as in [1]. But this does not necessarily yield consistent prices, and so there may be NE which are very far from WE; thus the existence of NE does not imply that of WE. A different approach, taken by Shapley and Shubik, was eloquently described by Shapley [17].

> The decisive step was to meet the problem of money head on — to accept the proposition that, in the world of buying and selling, money is "real." Granting this, the rest falls into place with remarkably few other generality-restricting assumptions.

Shapley and Shubik explicitly introduced money as the stipulated medium of exchange. Their model was carried forward by several others, who took up the theme of showing that Cournot–Nash equilibria converge to Walras equilibria. Curiously, however, the most direct route from Nash to Walras seems to have been missed. In [7], [17], [18], and [19], the money is one of the intrinsically valuable commodities, and convergence only obtains under special conditions on the endowments and preferences of the economy. In general, there may not be enough of the commodity money to sustain all the WE trades. This suggests the introduction of fiat money, and in particular of "inside" fiat money, which can be borrowed by all agents and must be repaid after trade. But, with borrowing, the possibility of default must also be reckoned with. In [11], [13], and [14], there is inside fiat money and NE converge to WE, but the games do not conform to the Nash format: harsh, discontinuous penalties are imposed on those who default (e.g., confiscation of all consumption), converting the game — at bottom — into a generalized game. In [7] and [20] genuine Nash games are presented, which also entail the coincidence of NE and WE. But once again the existence of NE is inferred from that of WE. It seems worthwhile to us (at the very least from a pedagogical point-of-view) to describe a simple variant of the Shapley–Shubik model which adheres to the Nash format, and whose NE can easily

2

be shown to converge to WE, under the standard assumptions on the underlying economy.

In our game $\Gamma(M)$, inside fiat money is the sole medium of exchange, and a "trading-post" is set up for each commodity $\ell \in L$ in the economy. Every agent t puts up his entire[2] endowment e_ℓ^t for sale at post ℓ. Agents initially have no money, but can borrow up to M at zero interest from a bank. They then choose how much money to bid at each post for purchases. An external agent also bids one dollar at each post to trigger trade there. The bank and the external agent and the trading-posts are all strategic dummies. They have no choices to make and so, unlike the price player in Debreu, they do not optimize. Prices p_ℓ form at each post ℓ as the ratio of total money bid to total commodity received at post ℓ.

An agent t who bids β_ℓ units of money at post ℓ receives $x_\ell^t = \beta_\ell/p_\ell$ units of commodity ℓ in return. He also obtains $p_\ell e_\ell^t$ units of money as sales revenue. This describes how prices mediate trade. Notice that the trading-posts always clear and generate a feasible reallocation of the endowments, no matter what the agents bid. (Thus in our scenario markets always clear. If, in addition, agents optimize, we obtain an NE. By contrast, in Walrasian analysis, agents always optimize, and markets clear only at WE.)

Every agent t in our game obtains the payoff

$$u^t(x^t) - \max\left\{0, \sum_{\ell \in L}\beta_\ell - \sum_{\ell \in L} p_\ell e_\ell^t\right\}.$$

The max term reflects the fact that h gets no utility from consuming fiat money, but is penalized for defaulting on his loan.

The game $\Gamma(M)$ depends on the borrowing limit M which compactifies agents' strategy spaces. By Nash's theorem, $\Gamma(M)$ has a TSNE. As $M \to \infty$, limits of the TSNE yield WE. This is so even though the strategy sets and the default penalties are defined completely independently of the characteristics of the agents in the economy. We prove that if some agents default, other agents must be bidding less than their sales revenue. The defaulters in effect rob from those who end up with surplus. This drives the latter to borrow and spend up to the limit M, pushing prices $p(M)$ higher as $M \to \infty$. Eventually the default penalty does bite and chokes off real default.

To the best of our knowledge, the only genuine games whose NE were shown to exist and to converge to WE in general were given by Dubey–Geanakoplos [9], Sahi–Yao [15], and (inspired by Sahi–Yao) by Sorin [21]. In [9] the model introduces outside money, along with inside money, and an endogenous positive interest rate. By eschewing these, the model we present now is simpler and reveals a more direct route from Nash to Walras. The models in [15] and [21] are an order of magnitude more complex than ours, and also impose somewhat stronger conditions on the underlying economy.

---

[2]This simplifies the analysis. We could have made the more realistic assumption that agents sell what they want. The game would then get more complicated but without affecting the result.

2 Walras Equilibrium

Let $H = \{1, ..., H\}$ be the set of households and $L = \{1, ..., L\}$ the set of commodities. For $h \in H$, $e^h \in \mathbb{R}^L_+$ is the endowment and $u^h : \mathbb{R}^L_+ \to \mathbb{R}$ the utility of consumption of household h. We assume, for all $h \in H$,

(i) $e^h \gg 0$

(ii) u^h is continuous, concave and weakly monotonic (i.e., $x \gg y$ implies $u^h(x) > u^h(y)$)

(For relaxations of (i) and (ii) see Section 5.)

Recall that $(p, (x^h)_{h \in H}) \in \mathbb{R}^L_+ \times \mathbb{R}^{LH}_+$ is a *Walras equilibrium* (WE) of the economy $(e^h, u^h)_{h \in H}$ if

$$\sum_{h \in H} x^h \leq \sum_{h \in H} e^h;$$

and

$$x^h \in \arg\max\{u^h(y) : y \in B^h(p)\}$$

for all $h \in H$, where $B^h(p) \equiv \{y \in \mathbb{R}^L_+ : p \cdot y \leq p \cdot e^h\}$.

3 Nash Equilibrium

Consider a continuum of agents $(0, H]$ made up of types $(h - 1, h]$, $h \in H$. Each $t \in (h-1, h]$ has endowment $e^t = e^h$ and utility $u^t = u^h$. The underlying population measure is Lebesgue.

For every commodity $\ell \in L$, there is a "trading-post" to which agents send commodity ℓ for sale and (fiat) money for purchase. For simplicity we suppose that they put up their entire endowment for sale so that the post receives[3] $\int_0^H e^h_\ell dt = \sum_{h \in H} e^h_\ell \equiv \bar{e}_\ell$ units of commodity ℓ. We further suppose then an external agent puts up 1 dollar of fiat money at each post in order to trigger trade there. Any agent $t \in (0, H]$ can borrow money b^t_ℓ, $\ell \in L$, at zero interest to bid at the posts. But he is not allowed to borrow more than M in total. Thus the strategy-set of each $t \in T$ is $S(M) = \{b^t \in \mathbb{R}^L_+ : \sum_{\ell \in L} b^t_\ell \leq M\}$. Suppose the choice of strategies $b = \{b^t\}_{t \in T}$ constitutes a measurable function. Then prices $p \equiv p(b) \in \mathbb{R}^L_{++}$ form at the trading-posts according to the rule:

$$p_\ell = \frac{\bar{b}_\ell + 1}{\bar{e}_\ell} > 0$$

for $\ell \in L$. Each post ℓ clears at its price $p_\ell(b)$, so that $t \in (0, H]$ obtains the consumption bundle $x^t \equiv x^t(p(b), b^t) \in \mathbb{R}^L_+$ with components

$$x^t_\ell = \frac{b^t_\ell}{p_\ell(b)}$$

---

[3]We write $\int_{h-1}^h f^t dt \equiv \bar{f}^h$ and $\int_0^H f^t dt \equiv \bar{f}$. Of course, $\bar{e}^h = e^h$ in our setup.

and also obtains $p_\ell(b)e_\ell^t$ units of money as revenue from the sale of his endowment e_ℓ^t, leaving him with the "net deficit"

$$d^t \equiv d^t(p(b), b^t) = \sum_{\ell \in L} b_\ell^t - \sum_{\ell \in L} p_\ell(b)e_\ell^t.$$

The payoff to t from the outcome (x^t, d^t) is

$$\Pi^t(b) = u^t(x^t) - d_+^t = u^t\left(\left(\frac{b_\ell^t}{p_\ell(b)}\right)_{\ell=1}^L\right) - \left[\sum_{\ell=1}^L (b_\ell^t - p_\ell(b)e_\ell^t)\right]_+$$

where $d_+^t = \max\{0, d^t\}$. The max term reflects the fact agents gain no utility from consuming fiat money, but incur disutility from defaulting on their loans.

Observe that the trading posts simply redistribute everything they receive. Since the external agent sends L dollars and no commodities to the posts, and receives commodities but no money in return, we must have

$$L + \sum_{\ell \in L} \bar{b}_\ell = \sum_{\ell \in L} p_\ell(b)\bar{e}_\ell = \sum_{h \in H} \sum_{\ell \in L} p_\ell(b)e_\ell^h$$

$$\sum_{h \in H} \bar{x}_\ell^h \leq \bar{e}_\ell \quad \forall \ell \in L$$

for any $b : (0, H] \to S(M)$. Thus the trading post mechanism guarantees the feasibility of the final allocation to agents, no matter what they bid, and regardless of whether or not they are optimizing.

A measurable choice $b_* : (0, H] \to S(M)$ is said to be a *Nash equilibrium* (NE) of $\Gamma(M)$ if, for a.a.$t \in (0, H]$

$$b^t \in \arg\max_{\beta \in S(M)} \Pi^t(b|_t\beta)$$

where $b|_t\beta$ is the same as b except that b^t is replaced by β.

A choice $b : (0, H] \to S(M)$ is called *type-symmetric* if $b^t = b^h$ whenever $t \in (h-1, h]$, and $h \in H$. In this case we write $b \equiv (b^1, ..., b^H)$. A type-symmetric NE will be denoted TSNE.

4 From Nash to Walras

4.1 Existence of Nash Equilibrium

For any type-symmetric choice of strategies $b = (b^1, ..., b^H) \in (S(M))^H$ consider the set $\varphi^t(b)$ of best replies of any agent $t \in (h-1, h]$ to b:

$$\varphi^t(b) = \arg\max_{\beta \in S(M)} \Pi^t(b|_t\beta) = \arg\max_{\beta \in S(M)} \left\{ u^h\left(\left(\frac{\beta_\ell}{p_\ell(b)}\right)_{\ell \in L}\right) - \left[\sum_{\ell \in L} \beta_\ell - \sum_{\ell \in L} p_\ell(b)e_\ell^h\right]_+ \right\}$$

By symmetry, $\varphi^t(b)$ is the same for all $t \in (h-1, h]$ and we denote it $\varphi^h(b)$. From the fact that β does not affect $p(b)$, i.e., $p(b) = p(b|_t\beta)$, it follows that $x^t(p(b), \beta)$ and

$d^t(p(b), \beta)$ are linear in β (for fixed b), and hence $\Pi(b|_t \beta)$ is concave in β. Since $S(M)$ is convex, we conclude that so is $\varphi^h(b)$.

We view Π as a function from $(S(M))^H \times S(M)$ to $\mathbb{R}$. Clearly Π is continuous in $(b^1, ..., b^H, \beta)$. Hence by the maximum principle, the correspondence $\varphi^h : (S(M))^H \rightrightarrows S(M)$ is compact valued and upper semi-continuous. Therefore, so is the correspondence $\varphi : (S(M))^H \rightrightarrows (S(M))^H$ defined by $\varphi = (\varphi^1, ..., \varphi^H)$. By Kakutani's theorem it has a fixed point which is easily seen to be a TSNE.

4.2 Walras Equilibria as Limits of Nash Equilibria

For each integer $M \geq 1$, let $b(M) = (b^h(M))_{h \in H}$ be a TSNE in $\Gamma(M)$, with prices $p(M)$ and outcomes $(x^h(M), d^h(M))_{h \in H}$. Since commodities are conserved by the trading posts, $\sum_{h \in H} x^h(M) \leq \bar{e}$ (with the external agent receiving $\bar{e} - \sum_{h \in H} x^h(M)$). In particular, each $x^h(M)$ is uniformly bounded. Furthermore, since any agent of type h always has the option of spending and consuming nothing, $u^h(\bar{e}) - d^h_+(M) \geq u^h(x^h(M)) - d^h_+(M) \geq u^h(0)$.

Hence $d^h_+(M)$ is also uniformly bounded above by $u^h(\bar{e}) - u^h(0)$. Let $\delta(M) = \{h \in H : d^h(M) > 0\}$ be the set of agents who are running a deficit, and let $\sigma(M) = \{h \in H : d^h(M) < 0\}$ be the set of agents who are running a surplus. Then of course $\sum_{h \in H} d^h(M) = \sum_{h \in \delta(M)} d^h(M) + \sum_{h \in \sigma(M)} d^h(M)$. Since money is conserved by the trading posts

$$L - \sum_{h \in \sigma(M)} (-d^h(M)) + \sum_{h \in \delta(M)} d^h_+(M) = L + \sum_{h \in H} d^h(M) = 0,$$

so $\sum_{h \in \sigma(M)} -d^h(M) = L + \sum_{h \in \delta(M)} d^h_+(M)$ is uniformly bounded above, and so each $-d^h(M)$ is also uniformly bounded above. Hence $d^h(M)$ is uniformly bounded.

Thus we may pass to a convergent subsequence with $x^h(M) \to x^h$, $d^h(M) \to d^h$, for all $h \in H$, and $p(M)/||p(M)|| \to p$ (where $||y|| \equiv \sum_{i=1}^L |y_i|$). We shall show that $(p, (x^h)_{h \in H})$ is a Walras equilibrium.

Observe first that $\sigma(M) \neq \phi$ because $L > 0$. Every agent of type $h \in \sigma(M)$ must be spending up to his limit M, for otherwise he could spend a little more at each post, consuming strictly more of every commodity, without incurring any default, contradicting that he is optimizing at the TSNE. Since he is running a surplus, $p(M) \cdot e^h > M$, which shows $||p(M)|| \to \infty$ as fast as M.

Since surpluses and deficits are uniformly bounded in M, it follows immediately that $d^h(M)/||p(M)|| = p(M)/||p(M)|| \cdot (x^h(M) - e^h) \to 0$, and thus $p \cdot (x^h - e^h) = 0$ for all $h \in H$. Since we already have $\sum_{h \in H} x^h \leq \sum_{h \in H} e^h$, it only remains to verify that if $y \in B^h(p) = \{y \in \mathbb{R}^L_+ : p \cdot y \leq p \cdot e^h\}$ then $u^h(y) \leq u^h(x^h)$.

Take $0 < \lambda < 1$. Then $p \cdot \lambda y \leq \lambda p \cdot e^h = \lambda p \cdot x^h < p \cdot x^h$ and so for large M, $[p(M)/||p(M)||] \cdot \lambda y < [p(M)/||p(M)||] \cdot x^h(M)$, and so $p(M) \cdot \lambda y < p(M) \cdot x^h(M) \leq M$. Hence letting $\beta^h_\ell(M) = p_\ell(M)\lambda y_\ell$, for $\ell \in L$, we have $\beta^h(M) \in S(M)$, and $[\sum \beta^h_\ell(M) - p(M) \cdot e^h]_+ \leq [p(M) \cdot x^h(M) - p(M) \cdot e^h]_+ = d^h_+(M)$. Since $b^h(M)$ gives a payoff at least as high as $\beta^h(M)$, yet incurs at least as much penalty, $u^h(x^h(M)) \geq u^h(\lambda y)$. Passing to the limit, $u^h(x^h) \geq u^h(\lambda y)$. But $\lambda < 1$ was arbitrary. Hence $u^h(x^h) \geq u^h(y)$. ∎

5   From Nash to Walras in Greater Generality

5.1   Without Concavity

Extending the game $\Gamma(M)$ to nonconcave u^h would seem to be folly. For example, if $u^h : \mathbb{R}_+ \to \mathbb{R}$ is defined by $u^h(x) = x^2$, then all agents of type h will wish to go infinitely bankrupt. But things are not so bad after all. They cannot borrow or bid more than M. We shall see that an NE always exists in $\Gamma(M)$, though agents of the same type may choose different (but indifferent!) strategies. Passing to the limit as $M \to \infty$, we again obtain a WE.

We retain all the other hypotheses except for the concavity of u^h.

To see that an NE of $\Gamma(M)$ exists, replace the correspondence $b \mapsto \mathsf{X}_{h \in H}\varphi^h(b)$ by $b \mapsto \mathsf{X}_{h \in H}\mathrm{Co}(\varphi^h(b))$ where $\mathrm{Co}(\varphi^h(b))$ denotes the convex hull of $\varphi^h(b)$. The conditions of Kakutani's theorem are still met, so there exists $\tilde{b}^1, ..., \tilde{b}^H$ such that $\tilde{b}^h \in \mathrm{Co}(\varphi^h(\tilde{b}))$. By Caratheodory's theorem, there exist $L + 1$ points $\tilde{b}^{h1}, ..., \tilde{b}^{h(L+1)}$ in $\varphi^h(\tilde{b})$ and $L + 1$ weights $\lambda^{h1}, ..., \lambda^{h(L+1)}$ in $\mathbb{R}_+$ such that $\sum_{i=1}^{L+1} \lambda^{hi} = 1$ and $\tilde{b}^h = \sum_{i=1}^{L+1} \lambda^{hi}\tilde{b}^{hi}$. Partition $(h - 1, h]$, from left to right into $L + 1$ consecutive intervals hi of lengths $\lambda^{h1}, ..., \lambda^{h(L+1)}$. Let the agents in the hith interval choose the strategy $\tilde{b}^{hi}$. This gives us an NE in which each type indulges in at most $L + 1$ different bids.

For each integer $M \geq 1$ take a NE $(p(M), ((b^{hi}(M), x^{hi}(M), d^{hi}(M), \lambda^{hi}(M))_{i=1}^{L+1})_{h \in H}$. Passing to a subsequence, we may assume that $\lambda^{hi}(M) \to \lambda^{hi}$ for all h and i.

Let $N = \{hi : \lambda^{hi}(M) \to 0\}$ and let $G = \{hi : \lambda^{hi}(M) \nrightarrow 0\}$. Then $\sum_{\{i:hi\in G\}} \lambda^{hi} = 1$ for all $h \in H$. Furthermore, by feasibility in $\Gamma(M)$, $x^{hi}(M)$ is uniformly bounded across M for $hi \in G$. Hence we can take convergent subsequences $x^{hi}(M) \to x^{hi}$ for all $hi \in G$. Also (by the old argument) $d_+^{hi}(M)$ is uniformly bounded for all $hi \in G$.

Our previous proof that $p(M) \to \infty$ as fast as M remains intact. Observe next that for any agent $t \in (0, H]$, his deficit $d^t(M)$ is bounded above and below independent of t (though no longer of M), i.e.,

$$-p(M) \cdot \bar{e} \leq d^{hi}(M) \leq M, \text{ for all } hi$$

It follows that

$$\frac{\lambda^{hi}(M)d^{hi}(M)}{||p(M)||} \to 0 \text{ for all } hi \in N.$$

Since $L + \sum_{hi \in N \cup G} \lambda^{hi}(M)d^{hi}(M) = 0$ and $||p(M)|| \to \infty$, we get $\sum_{hi \in G} \lambda^{hi}(M)[d^{hi}(M)/||p(M)||] \to 0$. But $d_+^{hi}(M)$ is uniformly bounded above for all $hi \in G$, so we conclude that $d^{hi}(M)/||p(M)|| \to 0$ for all $hi \in G$.

Treating the types $hi \in G$ exactly as our H types earlier, the proof proceeds as before. We obtain a Walras equilibrium in the limit.

5.2   With Quasi-Concavity

Assume that the utilities are quasi-concave. Let $x : (0, H] \to \mathbb{R}_+^L$ be a WE allocation with prices p, as in Section 5.1 above. Define $\bar{x}^h = \int_{h-1}^h x^t dt$. Since x^t maximizes

$u^t \equiv u^h$ on $B^h(p)$, for $t \in (h-1, h]$, it follows from the quasi-concavity of u^h that so does $\bar{x}^h$. Thus $(p, (\bar{x}^h)_{h \in H})$ is a WE.

5.3   Without Positivity of Endowments

Let us return to the case of concave utilities, but now replace $e^h \gg 0$ with $e^h > 0$ and $\sum_{h \in H} e^h \gg 0$. Were we to assume irreducibility, we could easily modify our argument to obtain WE in the limit. Without such an assumption, what do we get?

It is easy to see that TSNE still exist in $\Gamma(M)$, since we never used $e^h \gg 0$ in that part of the argument. Indeed the convergence of TSNE as $M \to \infty$ proceeded all the way to a limit $(p, (x^h)_{h \in H})$ with $\sum_{h \in H} x^h \leq \sum_{h \in H} e^h$ and $p \cdot x^h = p \cdot e^h$ for all $h \in H$ without invoking $e^h \gg 0$. We only needed $e^h \gg 0$ to show that $u^h(x^h) \geq u^h(y)$ for all $y \in B^h(p)$. Without the hypothesis $e^h \gg 0$, we can show that this is so for $y \in B_*^h(p)$, where $B_*^h(p)$ is the set of *stratified* budget-feasible consumption bundles.

Here commodities are partitioned into sets (strata) $L = L_1 \cup \cdots \cup L_k$ and it is understood that commodities in $L_{\gamma+1}$ are infinitely more valuable than those in L_γ (i.e., $L_{\gamma+1}$ is a higher "strata" than L_γ); or, to put it another way, the money involved in the trade of $L_{\gamma+1}$ has an exchange rate of infinity, relative to the money of L_γ. Let $\gamma(h) \equiv \max_\gamma \{e_\ell^h > 0 \text{ for some } \ell \in L_\gamma\}$. Then $B_*^h(p) = \{x \in \mathbb{R}_+^\ell : x_\ell = 0 \text{ for } \ell \in \bigcup_{\gamma=\gamma(h)+1}^k L_\gamma \text{ and } \sum_{\ell \in L_{\gamma(h)}} p_\ell(x_\ell - e_\ell^h) \leq 0\}$.

Assume we have a convergent subsequence of TSNE, with $p_\ell(M)/p_j(M)$ converging possibly to zero or infinity. Define ℓ to be in the same strata as j if $0 < \lim_{M \to \infty} p_\ell(M)/p_j(M) < \infty$; and to be in a higher strata than j if $\lim_{M \to \infty} p_\ell(M)/p_j(M) = \infty$. Then we obtain a stratification of the commodities. For each strata γ and $\ell \in L_\gamma$, define $p_\ell = \lim[p_\ell(M)/\sum_{\ell \in L_\gamma} p_i(M)]$. It is easy to see that, with these prices, we have a stratified WE.

6   Finitely Many Agents

We assumed a continuum of agents to ease the transition from NE to WE. But our argument works equally well when there is a finite set of agents converging to a continuum. First, NE exist in the finite case, once again by Nash's proof. (The only thing to check is concavity of payoffs. But $x_\ell^h = (b_\ell^h/[1 + \sum_{h' \in H} b^{h'}])\bar{e}_\ell$ is concave in b_ℓ^h, and then so is $u^h(x^h(...b^h...))$ in b^h, using the fact that u^h is both concave *and monotonic*.) We move towards the continuum by considering k "replicas" for each $h \in H = \{1, ..., H\}$, all of whom have the same endowments and preferences as h. This yields the game $\Gamma_k(M)$ with kH agents. By the standard symmetrization of Nash's argument, a TSNE exists in $\Gamma_k(M)$. We take $k = M$ and consider a sequence of TSNE of $\Gamma_M(M)$, $M \to \infty$. As before, some type must run a surplus, and all M agents of that type bid M (since outcome is strictly monotonic in bids). So total expenditures are of the order of $M \cdot M = M^2$, and since total expenditures equal $L + p(M) \cdot M \sum_h e^h$, it follows that $||p(M)|| \to \infty$ as fast as M. Moreover the defaults $d^h(M)$ stay bounded as before, which implies that there exists a subsequence of TSNE whose consumptions and normalized prices $((x^h(M)_{h \in H}, p(M)/||p(M)||)$ converge to

$((x^h)_{h \in H}, p)$, with $p \cdot x^h = p \cdot e^h$ for all $h \in H$.

It only remains to check that if $p \cdot y \leq p \cdot e^h$, then $u^h(y) \leq u^h(x^h)$. This, in turn, will follow if we can show that αy is feasible[4] for any agent of type h in $\Gamma_M(M)$ without incurring default, for any $0 < \alpha < 1$ and large enough M. Let $L_+ = \{\ell \in L : p_\ell > 0\}$ and $L_0 = \{\ell \in L : p_\ell = 0\}$. For all $\ell \in L_+$, the total bid on ℓ is growing like M^2. So an agent of type h has negligible (percentage) influence on $p_\ell(M)$ by unilaterally varying his bid between 0 and M on any commodity $\ell \in L_+$ (for large enough M). Fix $1 > \alpha'' > \alpha' > \alpha$, and let the agent bid $\alpha' y_\ell p_\ell(M)$ for every $\ell \in L_+$ in $\Gamma_M(M)$. For large M, he will acquire *at least* αy_ℓ via this bid. Moreover he will be left with *at least* $(1 - \alpha'')p(M) \cdot e^h$ more money to bid on $\ell \in L_0$, without defaulting. (This is so since he was bidding $p(M) \cdot x^h(M)$ with default $d^h(M)/||p(M)|| \to 0$, and since $p(M) \cdot x^h(M)/p(M) \cdot e^h \to 1$). For each $\ell \in L_0$, $p_\ell(M)/M \to 0$. Hence the total bid $\bar{b}_\ell(M)$ on ℓ satisfies $\bar{b}_\ell(M)/(M\bar{e}_\ell)M \to 0$. By bidding $\beta_\ell(M) = (1/\#L_0)(1 - \alpha'')p(M) \cdot e^h$ (which is of the order of M), the agent acquires at least $z_\ell(M) \equiv \beta_\ell(M)/[\beta_\ell(M) + \bar{b}_\ell(M)]M\bar{e}_\ell$. Since $\beta_\ell(M)M$ is of order M^2, $\beta_\ell(M)M/\bar{b}_\ell(M) \to \infty$. Thus $z_\ell(M) \to \infty$ as $M \to \infty$, which is certainly greater than αy_ℓ. This shows that the agent of type h can acquire consumption at least as big as αy (for large enough M), without incurring default, proving that $((x^h)_{h \in H}, p)$ is a WE.

---

[4] I.e., feasible via unilateral deviation at the TSNE of $\Gamma_M(M)$.

References

[1] Amir, R., S. Sahi, M. Shubik, and S. Yao, 1990. "A Strategic Market Game with Complete Markets," *Journal of Economic Theory*, **51**(1).

[2] Arrow, K.J. and G. Debreu, 1954. "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, **55**: 265–90.

[3] Bertrand, J., 1883. "Théorie Mathématique de la Richesse Sociale," *Journal de Savants*, 499–508.

[4] Cournot, A., 1838. *Recherches sur les Principes Mathématiques de la Théorie des Richesses.* English ed., N. Bacon, ed., *Researches into the Mathematical Principles of the Theory of Wealth.* New York: Macmillan, 1897.

[5] Debreu, G., 1952. "A Social Equilibrium Existence Theorem," *Proceedings of the National Academy of Sciences,* **38**: 886–93.

[6] Debreu, G., 1962. "New Concepts and Techniques for Equilibrium Analysis," *International Economic Review,* **3**: 257–73.

[7] Dubey, P. and L.S. Shapley, 1994. "Non-cooperative General Exchange with a Continuum of Traders: Two Models," *Journal of Mathematical Economics* **23**: 253–293.

[8] Dubey, P., 1982. "Price-Quantity Strategic Market Games," *Econometrica* **50**: 111–126.

[9] Dubey, P. and J. Geanakoplos, 2001. "Inside and Outside Money, Gains to Trade and IS-LM," Cowles Foundation Discussion Paper No. 1257R, Yale University, forthcoming in *Economic Theory.*

[10] Hurwicz, L., 1979. "Outcome Functions Yielding Walrasian and Lindhal Allocations at Nash Equilibrium Points," *Review of Economic Studies,* **46**, 217–27.

[11] Jaynes, J., M. Okuno, and D. Schmeidler, 1978. "Efficiency in an Atomless Economy with Fiat Money," *International Economic Review,* **19**: 149–57.

[12] Nash, John F., 1950. "Equilibrium Points in N-Person Games," *Proceedings of the National Academy of Sciences,* **36**: 48–49.

[13] Peck, J. and K. Shell, 1985. "Market Uncertainty: Sunspot Equilibria in Imperfectly Competitive Economies," CARESS Working Paper W.P. #85-21.

[14] Postlewaite, A. and D. Schmeidler, 1978. "Approximate Efficiency of Non-Walrasian Nash Equilibria," *Econometrica,* **46**, 127–37.

[15] Sahi, S. and S. Yao, 1989. "The Noncooperative Equilibria of a Trading Economy with Complete Markets and Consistent Prices," *Journal of Mathematical Economics*, **18**: 325–346.

[16] Schmeidler, D., 1980. "Walrasian Analysis via Strategic Outcome Functions," *Econometrica,* **48**(7): 1585–93.

[17] Shapley, L.S., 1976. "Non-cooperative General Exchange," in S.A.Y. Lin (ed.), *Theory of Measurement of Economic Externalities*. New York: Academic Press.

[18] Shapley, L.S. and M. Shubik, 1977. "Trade Using One Commodity as a Means of Payment," *Journal of Political Economy,* **85**: 937–68.

[19] Shubik, M., 1973. "Commodity Money, Oligopoly, Credit, and Bankruptcy in a General Equilibrium Model," *Western Economic Journal,* **11**: 24–38.

[20] Shubik, M. and C. Wilson, 1977. "The Optimal Bankruptcy Rule in a Trading Economy Using Fiat Money,' *Zeitschrift für Nationalökonomie*, 337–354.

[21] Sorin, S., 1996, "Strategic Market Games with Exchange Rates," *Journal of Economic Theory* **69**: 431–446.