THE IDEAL INFLATION INDEXED BOND AND IRVING FISHER'S IMPATIENCE THEORY OF INTEREST IN AN OVERLAPPING GENERATIONS WORLD

By

John Geanakoplos

July 2003

COWLES FOUNDATION DISCUSSION PAPER NO. 1429

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

YALE UNIVERSITY

Box 208281

New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

The Ideal Inflation Indexed Bond and Irving Fisher's Impatience Theory of Interest in an Overlapping Generations World

John Geanakoplos
Yale University*

July 18, 2003

Abstract

Irving Fisher long advocated inflation indexed bonds. I prove in the context of a multicommodity CAPM world that the best welfare improving bond pays the minimum money needed to achieve the same utility, and not the minimum needed to buy an ideal commodity bundle.

Irving Fisher also developed and advocated the impatience theory of interest. But in OLG economies, the rate of interest is determined by population growth, not impatience. I reconcile this contradiction by proving that in stationary OLG economies with land, the interest rate at the unique steady state does depend on impatience. Indeed, the proposition that greater impatience creates higher interest rates holds more generally in OLG with land than in Fisher's two-period model.

Keywords: Impatience, Theory of interest, Inflation indexed bond, Konüs index, Capital asset pricing, Efficiency, Overlapping generations, Land

JEL Classification: B22, B23, B31, D11, D52, D91, E31, E43, G12

Irving Fisher viewed the real rate of interest as the most important price in the economy, since it gives the relative value of consumption today in terms of consumption in the future. He proposed a theory of impatience to explain what the rate of interest should be in a finite horizon economy. He lamented that many people did not directly trade off consumption today for consumption tomorrow, but instead traded

*Presented on May 9, 1998 at the Conference in Celebration of Irving Fisher and the publication of **The Works of Irving Fisher**, sponsored by The Cowles Foundation for Research in Economics, Yale University.

money today for money tomorrow, thus enabling unexpected inflation to garble the real rate of interest. In this paper I examine whether Fisher's impatience theory of interest holds in an infinite horizon world with overlapping generations, and I derive the optimal indexed bond in a two-period economy with uncertainty and quadratic utilities.

Irving Fisher was profoundly interested in the correct measurement of inflation, and a leading proponent of inflation indexed bonds. He created an inflation index and meticulously published its values. He insisted his secretary sign an inflation indexed contract, linked to his index, and his company Rand Kardex issued an inflation indexed bond on the same day it officially opened for business under its new name.[1] He evidently regarded the proper inflation indexed bond as an important policy question with significant welfare implications. But he did not explain precisely what the welfare benefits are of choosing the right inflation indexed bond.

The problem is that as prices evolve over time, no household will maintain exactly the same consumption or the same utility. Indeed, no household will even maintain the same ratio of consumption goods; it will substitute goods that become relatively cheaper for goods that become relatively more expensive. Neither a bond that promises the money required to purchase the same commodity bundle, nor a bond that promises the money required to achieve the same utility, will be sufficient by itself to provide for the needs of the holder. On what theoretical basis is there to choose one over the other?

Practically speaking, it is much simpler to measure inflation by the cost of buying a fixed commodity bundle. As Fisher pointed out, then one does not have to worry about inferring what utility is, or how to deal with agents with heterogeneous utilities. And indeed, in actual practice in the United States and elsewhere, inflation indexed bonds make payments that guarantee the purchase of the same commodity bundle.

But current practice should not necessarily be the last word on the subject. From a theoretical point of view, however, it is not obvious what the right indexed bond is, or even what the criterion should be. (Fisher proposed 40 tests that an inflation index should satisfy.) Perhaps the best indexed bond should guarantee each agent the same marginal utility? Perhaps it should be tied to the growth of the economy? Fisher himself modified his view, and suggested as an "ideal" inflation index the geometric average of the Laspeyre and Paasche index of inflation. Since the Laspeyre index is greater than the Konüs index, and the Paasche index is smaller, their geometric average is likely to be near the Konüs index. The Konüs (1939) index is meant to measure the increase in cost necessary to achieve the same utility as prices change. Thus Fisher in the end advocated a practical index that could be used to create a bond that guarantees nearly the same utility. But that still leaves the theoretical question: what difference does it make?

Fisher's 40 criteria for the best index were mostly mechanical. For example, doubling all current prices ought to double the indexed price level. By contrast, I argue

[1] See Dimand (1999).

that the question of the correct indexed bond ought to become a portfolio welfare question: what additional asset will enable consumers operating in an economy with incomplete markets to best hedge the uncertainty caused by changes in relative prices and in future income? From this welfare point of view, I show that the ideal riskless bond should pay the Konüs index, that is it should have a monetary payoff in each state equal to the minimum cost of achieving a given utility for the representative consumer. I show that the welfare consequences of adding this asset are dramatically superior to those of adding the so-called inflation proof bonds we have in practice, even if every consumer's final equilibrium consumption necessarily gives a completely different utility in each state.

Irving Fisher was the inventor of the modern impatience theory of interest. Shortly after Fisher died, the great French economist Maurice Allais, followed by the great American economist Paul Samuelson, introduced the overlapping generations model, in which time goes on indefinitely into the infinite future. In that world there is always a steady state equilibrium in which the real rate of interest is equal to the rate of growth of the population, and has nothing to do with impatience. This apparent contradiction has often troubled me, though it does not seem to have been much discussed. The resolution I propose is simple: if an infinitely lived asset like land, which yields a steady dividend forever, is added to the model, the old lessons of Fisher are restored, and the equilibrium interest rate does depend on the rate of impatience. Indeed, Fisher's lessons can be shown to hold under weaker conditions in the setting of OLG and land than they do in his original two-period setting.

In Section 1 I describe an intertemporal model of stocks and bonds without uncertainty. I show, much like Fisher did in the appendix to his 1907 book on the rate of interest, that it can be reduced to a timeless Walrasian model. In that model it is easy to show that the real rate of interest increases with increased impatience, increases as the distribution of wealth shifts toward impatient people, increases as future endowments increase relative to current endowments, and increases as productivity improves, if aggregate demand for current consumption declines with the rate of interest and increases with wealth.

In Section 2 I describe an intertemporal model with uncertainty. I begin by specializing to the famous Capital Asset Pricing Model pioneered by Markowitz and Tobin, which has only one good per state. I show that in this model, as long as there is an asset $(1, ..., 1)$ that pays one unit of the good in every state, the equilibrium allocation is Pareto efficient, even if many assets are missing, and even if final consumption is very risky. (This result is due originally to Mossin (1977)). Without the $(1, ..., 1)$ asset, the final equilibrium would be dreadful. Then I review Tobin's famous mutual fund theorem (1958) that says that in this situation, everybody will hold the $(1, ..., 1)$ asset in his portfolio, together with only one other asset (the market). Thus the $(1, ..., 1)$ asset makes a dramatic difference to the welfare of the economy, and it is the unique asset that will do so. On this welfare basis, one could rigorously advocate introducing $(1, ..., 1)$ as the ideal riskless asset. But since there is only one good, the ideal riskless asset guarantees a constant commodity bundle and a constant utility.

To distinguish these two I add many goods to the CAPM model, as I did in my paper with Martin Shubik (1990). Now I find that there is still one asset that will bring the equilibrium to full efficiency. But it is not an asset that pays a fixed bundle in every state. Instead, it is a contingent bundle, calibrated to achieving the same utility at minimum cost, just as in the Konüs index. Once again it can be shown that all the agents will hold this in their portfolios, together with only one other asset.

Finally, in Section 3 I consider the overlapping generations model of Allais and Samuelson. In the conventional presentation of that model, all goods are perishable, and there is always a steady state equilibrium with rate of interest equal to the rate of population growth (and thus independent of impatience). I show that by adding land, or some other durable good that produces a fixed dividend in perpetuity, one gets only steady state equilibria in which the rate of interest does depend in the usual way on impatience, on the distribution of wealth and endowments, and on productivity, provided that agent utilities are additively separable.

1 A Model of Bonds and Stocks in the Spirit of Fisher

The Yale economist Irving Fisher (1867–1947) was the first person to develop a rigorous model of interest rates and the stock market. In order to do so he had to include stock-markets, bond-markets, commodity-markets, money, production and time in his conceptual model. He then reduced this model to a timeless model without assets, in which the rate of interest becomes simply a relative price between two goods. Asset prices can then be deduced from the present value of their dividends. In this simple model it is easy to do comparative statics, showing how the equilibrium interest rate is affected by changes in impatience, productivity, the growth of endowments, and the distribution of wealth between patient and impatient households. We give a modern version of Fisher's model, providing sufficient conditions to sign unambiguously the effects of these four factors on the real rate of interest.

1.1 The Intertemporal Economy

1.1.1 Time

In order to explain interest rates one needs a model with several time periods. For ease of exposition we assume that there are two times $s = 0, 1$. In each period there is a set $\mathcal{L}$ of goods traded; we denote the price of good ℓ traded at time s by $p_{s\ell} \geq 0$, and the vector of all goods traded at time s by $p_s \in \mathbb{R}^{\mathcal{L}}_{+}$. We assume that there are H households $h \in \mathcal{H}$ which live for both time periods. The households have time-separable utility functions of the form

$$U^h(x) = u^h(x_0) + \delta_h u^h(x_1)$$

where $u^h : \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}$ is increasing and concave.

1.1.2 Impatience

Following Fisher we assume that households are impatient, i.e., that $\delta_h < 1$ for all $h \in \mathcal{H}$. Fisher tells long stories of why this is a realistic assumption and why the level of impatience differs among households.

It is surprising how few people have challenged Fisher's view that by introspection we know that we are all impatient. One of the most interesting critiques of impatience was given by Yale student George Loewenstein, who argued in his dissertation (1985) that utility stemmed from anticipation, not from consumption. If that were true, people would tend to postpone pleasant experiences, to increase the excitement of anticipating them, and get bad things over with as soon as possible, both of which are the opposite of what Fisher would predict.

1.1.3 Endowments

We suppose that each household h is endowed with goods when young and when old:

$$e^h = (e_0^h, e_1^h) \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{L}}.$$

The greater is the endowment of households in the latter part of their lives relative to the early part of their lives, the greater we say the rate of growth of endowments.

Fisher believed there is a correlation between the size of the endowments and the rate of impatience across people. In short, he felt that patient types of people accumulated wealth, and passed both their wealth and their proclivity for patience on to their children.

1.1.4 Money

We suppose that the set of goods is $\mathcal{L} = \{0, 1, ..., L\}$. Good 0 represents money; it gives no utility to any agent, and is in zero supply: $e_{s0}^h = 0$ for all agents and time periods. A full-fledged model of money would require it to be in positive supply, and to play a special role in transactions and so on. Fisher never developed such a model, and we shall not do so either, in this paper. Nevertheless, even in our simple model, money takes on significance if assets promise payments in money.

1.1.5 Firms

We assume that there are F firms $f \in \mathcal{F}$ which are characterized by their production sets $Y^f \subset \mathbb{R}_-^L \times \mathbb{R}_+^L$. We write a production plan $y^f \in Y^F$ as $y^f = (y_0^f, y_1^f)$ and follow Debreu's convention that inputs are negative. Assuming that $Y^f \subset \mathbb{R}^L_- \times \mathbb{R}^L_+$ therefore implies that the firm has to invest in period 0 and produces in period 1, i.e., production takes time. The greater the outputs relative to the inputs, the more productive the economy is.

The firms are traded on a stock market and we denote by σ^h_f the amount of shares household h buys of firm f in period 0. This entitles the household to $\sigma^h_f y^f_1$ of the firm's output in period 1, and no dividend or obligation at time 0. This is in keeping with practice in American capital markets, where there is always an ex-dividend date. It is impossible that the instant a stock is purchased, all the dividends can be redirected to the new owner. Instead a reasonable amount of time must be left to transfer all the accounts, during which time the original owner maintains the responsibilities and benefits. Of course the price the stock sells for will reflect this fact; if a new buyer realizes he has bought too late to receive the next dividend check, he will pay less for the stock.

We denote the price of the share by π_f. Households have initial ownerships in firms $\bar{\sigma}^h_f$, where $\sum_h \bar{\sigma}^h_f = 1$. We assume that each firm is controlled by only one household, who chooses the production plan y^f, and we write $\mathcal{F}(h)$ for the set of all firms controlled by household h. We assume that the initial owners of firm f buy the inputs y^f_0 before the shares of the firm are traded on the stock market.

1.1.6 Bonds

A bond is a promise made at time $t = 0$ to deliver goods or money at time $t = 1$. We collect all bonds in a set $\mathcal{J}$. We assume that bond 0 promises to pay one dollar in the second period and denote its price by π_0. All other bonds $j \in \mathcal{J}$ are real bonds and promise to pay a bundle of goods $A^j \in \mathbb{R}^L_+$. We assume that the first bond pays one unit of every good.

We denote the amount of bond j household h holds by θ^h_j. If $\theta^h_j > 0$ the household purchases the bond, if $\theta^h_j < 0$ the household sells the bond. Since there is no government in the model, the bonds must be in zero net supply and there can only be trade in a bond j when some household sells it short, i.e., when $\theta^h_j < 0$ for some household h.

1.1.7 The Real Rate of Interest

With these definitions, the (gross) nominal interest rate $1 + \eta$ is just $1 + \eta = 1/\pi_0$. Under the simplifying assumption that there is only one good, $L = 1$, we have that the (gross) real interest rate $1 + r$, which measures how many of the good one gets in the second period by giving up one unit in the first period, is given by $1 + r = p_0/\pi_1$. Inflation is defined by $1+i = p_1/p_0$. When there is more than one good, the definition of inflation and the real rate of interest requires a theory of the proper index. We come to this in Section 2.

1.1.8 The Economy

The economy can now be described as a vector

$$E = (((e_0^h, e_1^h), U^h, \bar{\sigma}^h, \mathcal{F}(h))_{h\in\mathcal{H}}, (A^j)_{j\in\mathcal{J}}, (Y^f)_{f\in\mathcal{F}}).$$

Prices are given by $(p_0, p_1, (\pi_j)_{j\in\mathcal{J}}, (\pi_f)_{f\in\mathcal{F}})$ and each household h's choices are given by

$$(x_0^h, (\theta_j^h)_{j\in\mathcal{J}}, (\sigma_f^h)_{f\in\mathcal{F}}, x_1^h, (y_0^f, y_1^f)_{f\in\mathcal{F}(h)}).$$

1.1.9 The Budget Set

Agent h's budget set is defined as

$$\begin{aligned}
&B^h(p, \pi, (y^f)_{f\notin\mathcal{F}(h)}) \\
= &\{(x_0, x_1, (\sigma_f)_{f\in\mathcal{F}}, (\theta_j)_{j\in\mathcal{J}}, (y_0^f, y_1^f)_{f\in\mathcal{F}(h)}) \text{ s.t.} \\
&p_0 \cdot x_0 + \sum_{j\in\mathcal{J}} \pi_j\theta_j + \sum_{f\in\mathcal{F}} \pi_f\sigma_f - p_0 \cdot \sum_{f\in\mathcal{F}(h)} y_0^f \bar{\sigma}_f^h \leq p_0 \cdot e_0^h + \sum_{f\in\mathcal{F}} \pi_f \bar{\sigma}_f^h \\
&p_1 \cdot x_1 \leq p_1 \cdot e_1 + \sum_{j\in\mathcal{J}} \theta_j p_1 \cdot A^j + \sum_{f\in\mathcal{F}} p_1 \cdot y_1^f \sigma_f\}
\end{aligned}$$

1.1.10 Equilibrium

A Fisher equilibrium is a collection of prices and choices $(p, \pi, (x^h, \theta^h, \sigma^h)_{h\in\mathcal{H}}, (y^f)_{f\in\mathcal{F}})$ such that

- Markets clear:

  Good markets:

$$\sum_{h\in\mathcal{H}} x_0^h = \sum_{h\in\mathcal{H}} e_0^h + \sum_{f\in\mathcal{F}} y_0^f$$

$$\sum_{h\in\mathcal{H}} x_1^h = \sum_{h\in\mathcal{H}} e_1^h + \sum_{f\in\mathcal{F}} y_1^f$$

  Bond markets:

$$\sum_{h\in\mathcal{H}} \theta^h = 0$$

  Stock markets:

$$\sum_{h\in\mathcal{H}} \sigma^h = \sum_{h\in\mathcal{H}} \bar{\sigma}^h$$

- Agents maximize their utility functions subject to their budget sets.

1.1.11 Some Observations

Fisher made the following observations about the nature of intertemporal equilibrium.

1. Default: In the definition of equilibrium nobody is allowed to default. If a household buys an asset, or the shares of the firm, he thinks for sure he will get the promised payoffs. Similarly, if he sells an asset he never contemplates not paying what is owed, at least according to the budget constraint we have written. This is obviously unrealistic and the analysis should be extended. Fisher never constructed a profound theory of default, but he did observe that the promised interest rate would be higher for assets that had a higher chance of default.

2. Fisher's equation: If there is only one real good, then in equilibrium it must be true that

$$1 + \eta = (1 + r)(1 + i) = r + i + ri. \tag{1}$$

This follows from what is called the "absence of arbitrage." If Equation (1) did not hold we would have $1/\pi_0 \neq p_0/\pi_1 \cdot p_1/p_0$. However in this case households could buy the nominal bond and sell the real bond (or the other way around) and would make money today without having to pay anything on net tomorrow.

Note that in general one would expect $r \cdot i$ to be very small. In this case Equation (1) can be written as $\eta \equiv r + i$, which is what is usually called the Fisher equation.

3. Stock-prices: By the same logic as above, the absence of arbitrage opportunities also implies that $\pi_f = \pi_0 p_1 \cdot y_1^f = \frac{1}{1+\eta} p_1 \cdot y_1^f$. Therefore the price of a stock is nothing else than the discounted value of its dividends.

This holds incidentally for any bond as well. According to Fisher, the price of every asset is the discounted value of its dividends. Thus even if there are many goods in the model, if we take the ratio of the value of any bond's payoffs to its price, we will always get the same number $1 + \eta$. The nominal rate of interest is thus well-defined by the equilibrium. The real rate of interest is not well-defined by itself with many goods, because it depends on the rate of inflation. If potato inflation is much higher than apple inflation, then the potato cost today of one potato tomorrow will be higher than the apple cost today of one apple tomorrow. But once adjusted by the rate of inflation, applying Fisher's equation gives the same real interest rate.

4. Endogenous share prices: Since the price of a share depends on the firm's future output, it will change if the controller of the firm decides to change the production plan. This raises the question whether there is an easy way to characterize the production plan the controller of the firm chooses.

5. Separation Principle Inspecting the budget constraint reveals that every owner (who takes commodity prices as given) would like to see a production plan chosen that maximizes $p_0 y_0 + \pi_0 p_1 y_1$. As long as the controller is at least a partial owner, $\bar{\sigma}^h_f > 0$, he will therefore optimize his utility by choosing y^f to maximize total discounted profits, or profit for short, and it is in fact irrelevant for the production choices who controls the firm. This is Fisher's so-called principle of separation (between production choices and consumption choices).

1.2 A Simple Reformulation of the Fisher Model as a Walrasian Model

With the above insight that the owners of the firm will always choose production plans to maximize profits, we can simplify the model. Let us now define a Walrasian equilibrium for the economy

$$E = (((e^h_0, e^h_1), U^h, \bar{\sigma}^h)_{h\in\mathcal{H}}, (Y^f)_{f\in\mathcal{F}}).$$

We denote by q_1 the price at time $t = 0$ of goods consumed at time $t = 1$, but traded at time $t = 0$. We then have the following budget set for household $h \in \mathcal{H}$:

$$B^h(q) = \{x : q_0 \cdot x_0 + q_1 \cdot x_1 \leq q_0 \cdot e^h_0 + q_1 \cdot e^h_1 + \sum_{f\in\mathcal{F}} (q_0 \cdot y^f_0 + q_1 \cdot y^f_1)\bar{\sigma}^h_f\}.$$

An equilibrium $(q, (x^h)_{h\in\mathcal{H}}, (y^f)_{f\in\mathcal{F}})$ is then characterized by (1) goods-market-clearing, by (2) households maximizing utilities subject to their budget sets, and by (3) firms maximizing profits

$$\begin{aligned}
\sum_{h\in\mathcal{H}} x^h &= \sum_{h\in\mathcal{H}} e^h + \sum_{f\in\mathcal{F}} y^f \\
x^h &\in \arg\max_{x\in B^h(q)} U^h(x) \\
y^f &\in \arg\max_{y\in Y^f} q \cdot y
\end{aligned}$$

Notice that the Walrasian budget set has only one constraint. Furthermore, the Walrasian model and its equilibrium make no explicit mention of time (treating goods x_1 exactly symmetrically with goods x_0), no mention of bonds, no mention of who controls the firm, and no mention of trading shares of stock. These omissions make the model much simpler. Notice finally that the Debreu notation, measuring inputs as negative and outputs as positive, makes it very easy to express the idea of profit as a simple dot product $q \cdot y$, which further simplifies the exposition of the Walrasian model.

The following theorem proves that essentially nothing real is lost in this simplification, by showing that there is no difference between the production and consumption choices in the Fisher model and in the Walrasian model, and furthermore, that there is no difference in price ratios between contemporaneously traded objects in the two models. In particular, we can recover the bond and stock prices (relative to the price of some consumption good at time 0) from the Walrasian model, because stocks and time 0 goods are traded at the same time in the Fisher model. We cannot however recover the rate of inflation or the nominal rate of interest from the Walrasian model.

Theorem 1 *Consider any Fisher economy*

$$E = (((e_0^h, e_1^h), U^h, \bar{\sigma}^h, \mathcal{F}(h))_{h\in\mathcal{H}}, (A^j)_{j\in\mathcal{J}}, (Y^f)_{f\in\mathcal{F}})$$

and the corresponding Walrasian economy

$$E = (((e_0^h, e_1^h), U^h, \bar{\sigma}^h)_{h\in\mathcal{H}}, (Y^f)_{f\in\mathcal{F}}).$$

Suppose the Fisher economy has at least one asset or stock with nonzero price. Then, given a Fisher-equilibrium $(p, \pi, (x^h, \theta^h, \sigma^h)_{h\in\mathcal{H}}, (y^f)_{f\in\mathcal{F}})$ there is a Walrasian equilibrium $(q, (x^h)_{h\in\mathcal{H}}, (y^f)_{f\in\mathcal{F}})$ which has the same equilibrium consumption and production choices and where $q_0 = p_0$ and $q_1 = \pi_0 p_1$. Conversely, given a Walrasian equilibrium $(q, (x^h)_{h\in\mathcal{H}}, (y^f)_{f\in\mathcal{F}})$ there is a Fisher equilibrium $(p, \pi, (x^h, \theta^h, \sigma^h)_{h\in\mathcal{H}}, (y^f)_{f\in\mathcal{F}})$ that has the same consumption and production choices and such that $p_0 = q_0$ and such that for all $j \in \mathcal{J}, f \in \mathcal{F}$

$$\begin{aligned} \pi_j &= q_1 \cdot A^j \\ \pi_f &= q_1 \cdot y_1^f \\ \frac{p_0}{\pi_1} &= \frac{q_0}{q_1} = 1 + r \text{ (if } L = 1). \end{aligned}$$

The proof is left as an (easy) exercise.

Suppose there is no production. Then in this certainty model it makes no difference what the assets are, as long as there is at least one. We always get a reduction to the Walrasian model.

1.3 Four Determinants of the Interest Rate

We now show how impatience, the distribution of wealth between patient and impatient agents, the distribution of endowments between the present and future, and the productivity of the firms affect the real rate of interest in a one good economy. We start with an example, and then generalize it to the case where excess demand for current consumption is normal, and decreases as the rate of interest increases.

Production Example Suppose we have two time periods, 0 and 1, one good per period, and two agents A and B with utilities

$$\begin{aligned} W^A(x_0, x_1) &= \log x_0 + \delta^A \log x_1 \\ W^B(x_0, x_1) &= \log x_0 + \delta^B \log x_1 \\ 0 \le \delta^B &< \delta^A \le 1 \end{aligned}$$

and with endowments

$$\begin{aligned} e^A &= (e_0^A, e_1^A) = (3/4, 1/4) \\ e^B &= (e_0^B, e_1^B) = (3/8, 1/8) \end{aligned}$$

Suppose there is a firm, owned entirely by agent A with production function

$$\begin{aligned} y_1 &= C(-y_0)^\alpha \\ C &\ge 0 \\ 0 &< \alpha < 1 \end{aligned}$$

Let the prices be given by $(q_1, q_2) = (1, q)$. Irving Fisher was the first one to recognize clearly that one could think of q as the discount rate,

$$q = \frac{1}{1+r}$$

where r is the real rate of interest.

For simplicity of notation, let us replace $-y_0$ with k. In equilibrium we must have that the firm maximizes profit

$$\pi(q, k) = qf(k) - k$$

where $f(k) = Ck^\alpha$, giving us the condition that marginal revenue product equals price

$$q\alpha Ck^{\alpha-1} = 1$$

This in turn gives

$$\begin{aligned} k^{\alpha-1} &= 1/q\alpha C \\ k^{1-\alpha} &= q\alpha C \\ k(q) &= q^{\frac{1}{1-\alpha}}(\alpha C)^{\frac{1}{1-\alpha}} \end{aligned}$$

giving for maximum profit

$$\begin{aligned} \Pi(q) &= qC\left[q^{\frac{1}{1-\alpha}}(\alpha C)^{\frac{1}{1-\alpha}}\right]^\alpha - q^{\frac{1}{1-\alpha}}(\alpha C)^{\frac{1}{1-\alpha}} \\ &= \alpha^{\frac{\alpha}{1-\alpha}} q^{\frac{1}{1-\alpha}} C^{\frac{1}{1-\alpha}} - q^{\frac{1}{1-\alpha}}(\alpha C)^{\frac{1}{1-\alpha}} \\ &= (\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}) q^{\frac{1}{1-\alpha}} C^{\frac{1}{1-\alpha}} \end{aligned}$$

The excess demand for the time 0 good is given by

$$z_0(q) \equiv \frac{1}{1+\delta^A}\left[\frac{3}{4}+\frac{1}{4}q+\Pi(q)\right]+\frac{1}{1+\delta^B}\left[\frac{3}{8}+\frac{1}{8}q\right]+k(q)-\left(\frac{3}{4}+\frac{3}{8}\right).$$

When $C = 0$ and there is no production, one can easily see that $z_0(q)$ is monotonically increasing in q.[2] In general the excess demand with two agents could be arbitrary (and thus nonmonotonic in q), as Debreu's decomposition theorem assures us.

When we add production, taking $C > 0$, we can confirm from our formulae that $z_0(q)$ is still increasing in q. Even without the formulae, it is evident that $\Pi(q)$ increases as q increases, giving the firm's owner more wealth. The firm's owner A spends more on good 0 when his wealth increases. And clearly the firm's demand for good 0, $k(q)$, is increasing in q, since the marginal revenue product increases in q.[3]

1.3.1 Some Comparative Statics

Suppose we have an economy with one good today and tomorrow, for which aggregate excess demand for current consumption, $z_0(q)$, is strictly increasing in $q = q_1/q_0$, that is, strictly decreasing in the real interest rate. Suppose also that all individual demands for current consumption are normal, so that increased wealth increases the demand if prices are held constant. These hypotheses will hold if all the consumers have Cobb–Douglas utilities, and if production is strictly concave, as we saw in our example. In this situation we can derive unambiguous comparative statics.

[2] Thus in equilibrium we would have an interest rate that lies between what would have prevailed with just agent A and the interest rate which would have prevailed had the economy consisted only of agent B, but closer to the former because A is richer.

[3] Although it is not important for us, it is also true that demand for good 1 is decreasing in q. The excess demand for the second good is given by

$$z_1(q) \equiv \frac{\delta^A}{1+\delta^A}\left[\frac{3}{4}+\frac{1}{4}q+\Pi(q)\right]\Big/ q+\frac{\delta^B}{1+\delta^B}\left[\frac{3}{8}+\frac{1}{8}q\right]\Big/ q-C\left[q^{\frac{1}{1-\alpha}}(\alpha C)^{\frac{1}{1-\alpha}}\right]^{\alpha}-\left(\frac{1}{4}+\frac{1}{8}\right)$$

Note that the extra term coming from production satisfies

$$\begin{aligned}\frac{\delta^A}{1+\delta^A}\Pi(q)/q-f(k(q)) &= \frac{\delta^A}{1+\delta^A}[qf(k(q)-k(q)]/q-f(k(q)) \\ &= \frac{-1}{1+\delta^A}f(k(q))-\frac{\delta^A}{1+\delta^A}k(q)/q \\ &= \frac{-1}{1+\delta^A}f(k(q))-\frac{\delta^A}{1+\delta^A}q^{\frac{1}{1-\alpha}}(\alpha A)^{\frac{1}{1-\alpha}}/q\end{aligned}$$

which is clearly decreasing in q. Thus excess demand in our example satisfies gross substitutes. But for some other production functions, $z_1(q)$ would not have been decreasing in q, and gross substitutes would fail. However, with Cobb–Douglas utilities, no matter what concave, smooth production function we chose, we would find that $z_0(q)$ increases in q.

- When any agent, such as A in the example, becomes more impatient, the real rate of interest rises.

  At the old prices the excess demand for good 0 will become positive. Therefore to clear the market, q must fall. Therefore the real interest rate $1/q - 1$ must increase. Intuitively, we can describe this comparative statics result by saying that if people care less about the future, they will demand a higher interest in the bank in order to save. A shift in population attitudes toward more impatience (the Now Generation supposedly ushered in by Reagan) ought to raise interest rates.

- If the distribution of wealth shifts from an impatient agent to a patient agent, such as from B to A in the example, i.e., if e_0^B decreases but e_0^A increases by the same amount, then the real rate of interest goes down.

  The argument is as above, but in reverse. At the old equilibrium prices, there must now be less demand for good 0 because agent A will spend $3/4$ of his new money on the good, but agent B will reduce his expenditure on the good by $4/5$ of the money he lost. The only way to clear the market is to increase q. This lowers the real rate of interest. In words, we can say that if wealth in the economy shifts from people who are impatient to people who are patient, the interest rate the banks must give will go down. Fisher believed that the wealthy tended to be more patient, so during a period when the rich get richer, as has happened since 1980 in the US, the real interest rate should go down, according to Fisher.

- If future endowments increase, all else equal, the rate of interest will rise.

  At the old equilibrium prices, excess demand for good 0 must go up (since wealth has gone up and current consumption is a normal good). But to restore equilibrium, q must go down. Hence the interest rate goes up.

- If productivity and marginal productivity increase, the real rate of interest will rise.

  If the firm becomes more profitable at the old prices, income to its owners will rise and so demand for period 0 goods by its owners goes up (by agent A in the example). If marginal productivity goes up, then at the old prices the firm itself will also demand more input at the same prices. Together this creates excess demand for good 0 at the old prices. Hence q must go down.

It is worth mentioning that even if all utilities are of the form $U^h(x) = u^h(x_0) + \delta_h u^h(x_1)$, excess demands may not be monotonic in q, because price substitution effects and income effects may go in opposite directions. Thus Fisher's comparative statics conclusions depend on additional restrictions on utilities.

2 Time and Uncertainty: The Ideal Inflation Proof Bond

2.1 The GEI Model

Now we add uncertainty to the model. To keep things simple, we stick with pure exchange. The analogue of the Walrasian model from last section is the Arrow-Debreu model, in which agents face only one budget constraint. The analogue of the Fisher temporal model is the GEI (general equilibrium with incomplete markets) model. The canonical GEI model $((U^h, e^h)_{h\in H}, A)$ has two time periods, period 0 and period 1, and S different states of the world $(s = 1, ..., S)$ in period 1.

Agent utilities and endowments $(U^h, e^h)_{h\in H}$ are as in the Arrow–Debreu model. In case agents have von Neumann–Morgenstern preferences, we can write the utilities as

$$U^h(x) = u_0^h(x_0) + \sum_{s=1}^{S} \gamma_s^h u^h(x_s)$$

where γ_s^h is the subjective probability agent h attributes to the state s. A matrix of asset payoffs $A \in \mathbb{R}_+^{SJL}$ is also given. The payoff $A_{s\ell j}$ represents the quantity of good ℓ promised for delivery in state s by asset j. As in the Fisher model, it will often be convenient to assume that agents are impatient, and to embody that with the hypothesis that $u^h(\cdot) = \delta^h u_0^h(\cdot)$, with the discount factor satisfying $0 < \delta^h < 1$.

The model preserves the same methodological premises of the Fisher intertemporal model and the Arrow–Debreu wor1d: agent optimization, market clearing, rational expectations (in the sense that everybody anticipates what price will arise in the different states of the world) and perfect competition (in the sense that nobody can affect the prices by whatever he does). But it adds uncertainty, something Fisher never formally did in a state space setting.

The GEI economy is analogous to the Fisher economy (without production), but with many states of nature. It cannot be reduced to an Arrow-Debreu economy however, because in the Arrow–Debreu model, agents can exchange commodities at time zero for any one of the state contingent commodities in the future, whereas in the GEI model, agents are restricted to exchange commodities at time zero for assets, and the assets are limited to payoffs specified by the matrix A. Of course by buying and selling assets an agent can influence wealth in different states. But if there are only a very few assets, the agent can arrange very limited wealth transfers. For example, if there is only one asset, which pays 1 apple in all states at time one, then by buying this asset an agent can effectively loan money (giving money today in exchange for a repayment tomorrow), while by selling the asset he can borrow money, getting money today but paying in the future. However that single asset does not provide insurance, because it does not help to transfer wealth from, say, state 1 to state 5.

To keep our notation consistent with the standard GEI literature, we shall drop money from the set of explicitly named commodities, leaving us with $\mathcal{L} = \{1, ..., L\}$.

The budget set for agent h is given by

$$B^h(p,\pi) = = B(p,\pi,e^h,A) = \{(x^h,\theta^h) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}^J : \ p_0 \cdot (x_0 - e_0) + \pi \cdot \theta^h = 0$$
$$\text{and } \forall s = 1, ..., S, \ p_s \cdot (x_s - e_s) = p_s \cdot \sum_j A_s^j \theta_j^h\}.$$

Individuals choose among nonnegative consumption plans x, specifying the consumption of each commodity $\ell \in L$ in each state $s \in S^* = \{0, 1, ..., S\}$ at time zero and in the future. A consumption plan can be chosen only if it can be financed by a portfolio of asset holdings, θ, which can be positive or negative. An agent who wants to consume beyond his endowment in state 0 can sell assets; this will require him to make deliveries in the future. We are still assuming that everybody keeps his promises so that default is not allowed. There are $S + 1$ budget constraints in the GEI model instead of just one budget constraint as in the Arrow–Debreu model.

Equilibrium is defined by:

$$(p, \pi, (x^h, \theta^h)_{h \in H}) \text{ is a GEI iff}$$

$$\sum_{h=1}^{H} (x^h - e^h) = 0 \tag{2.1}$$

$$\sum_{h=1}^{H} \theta^h = 0 \tag{2.2}$$

$$(x^h, \theta^h) \in B^h(p,\pi), \ h = 1, ..., H \tag{2.3}$$

$$(x,\theta) \in B^h(p,\pi) \Rightarrow U^h(x) \leq U^h(x^h), \ h = 1, ..., H. \tag{2.4}$$

As it is clear from (2.1)–(2.4), equilibrium is a price vector for commodities (at all date-event pairs), a price vector for the assets, and then plans for individuals that are optimal in their budget sets, such that supply equals demand for goods and assets.

(1)–(2) translate aggregate feasibility, (3) says that each agent chooses plans in his budget set, and (4) adds that there is nothing in the budget set that is better than the selected plan.

2.1.1 Some Notation

Let us introduce a few pieces of notation. Given a vector $x \in \mathbb{R}^{S^* L}$, we take $\tilde{x}$ to be the (uncertain) period 1 components of x

$$p = (p_0, \underbrace{p_1, ..., p_S}_{\tilde{p} \in \mathbb{R}^{SL}}) = (p_0, \tilde{p})$$
$$y = (y_0, \underbrace{y_1, ..., y_S}_{\tilde{y} \in \mathbb{R}^{SL}}) = (y_0, \tilde{y})$$

The box notation $\tilde{p}\boxdot\tilde{y}$ means the money value of the bundle y in every state (therefore, it is an S-dimensional vector).

$$\tilde{p} \boxdot \tilde{y} \equiv (p_1 \cdot y_1, ..., p_S \cdot y_S)' \in \mathbb{R}^S.$$

For the whole matrix $A \in (\mathbb{R}^{SL})^J$ we can take

$$\tilde{p} \boxdot A \equiv (\tilde{p} \boxdot A^1, ..., \tilde{p} \boxdot A^J) \in \mathbb{R}^{SJ},$$

where A^j is the jth column of A.

Each column j of $\tilde{p} \boxdot A$ corresponds to the money pay-off in each state for asset j; each column j of $\tilde{p} \boxdot A$ collapses the SL dimensions of column j of A into S dimensions.

Let us introduce a piece of notation to designate an inner product which will be used in the sequel. If $\mu \in \mathbb{R}^S$ and $z \in \mathbb{R}^S$ and γ is a probability measure on S, then we write

$$\mu \cdot_\gamma z \equiv \sum_{s=1}^{S} \gamma_s \mu_s z_s.$$

It goes without saying that all the usual projection theorems hold for this inner product. Given an arbitrary $S \times J$ matrix V, and an arbitrary vector $b \in \mathbb{R}^J$, and an arbitrary strictly positive probability measure γ on S, there is a unique vector $\mu^* \in \text{span}[V] = \{V\theta : \theta \in \mathbb{R}^J\}$ such that $\mu^* \cdot_\gamma V^j = b_j$ for each column j of V. In particular, given an arbitrary vector $\mu \in \mathbb{R}^S$, the projection of μ onto span$[V]$ is the unique vector $\mu^* \in \text{span}[V]$ such that $\mu^* \cdot_\gamma V^j = \mu \cdot_\gamma V^j$ for each column j of V.

2.1.2 Linear Pricing Lemma for GEI

The following lemma is fundamental in the theory of incomplete markets.

Unique Linear Pricing Lemma *Suppose* $(p, \pi, (x^h, \theta^h)_{h\in H})$ *is a* GEI *equilibrium for the* GEI *economy* $((U^h, e^h)_{h\in H}, A)$ *such that for some agent* $h \in H$, *the utility* U^h *is differentiable, and such that* $p_s \cdot x_s^h > 0$ *for all* $s \in S^*$. *Let* γ *be a strictly positive probability measure on* S. *Then there is a* unique *vector* $\mu^* \in \text{span}[\tilde{p} \boxdot A]$ *such that* $\pi_j = \mu^* \cdot_\gamma [\tilde{p} \boxdot A^j]$ *for all assets* j.

Proof We know that for each state $s \in S^*$, there is some commodity $\ell_s \in L$ such that $x_{s\ell} > 0$. Define

$$\mu_s^h = \frac{1}{\gamma_s} \frac{1}{p_{s\ell_s}} \frac{\partial U(x^h)}{\partial x_{s\ell_s}}$$

(where we take $\gamma_0 = 1$). From the first order conditions of agent h's optimization, we must have

$$\pi_j = \frac{1}{\mu_0^h}\tilde{\mu}^h \cdot_\gamma [\tilde{p} \boxdot A^j] \text{ for all } j \in J$$

Define μ^* as the projection of $\frac{1}{\mu_0^h}\tilde{\mu}^h$ into span$[\tilde{p} \boxdot A]$. Then μ^* has the right inner product with each asset payoff, and it is the unique vector to do so, since any other vector in span$[\tilde{p} \boxdot A]$ would necessarily make a different inner product with at least one vector. Since the prices of the assets do not depend on the agent h, the projection μ^* also does not depend on the agent h. ■

2.1.3 Riskless Rate of Interest with Uncertainty

When there is uncertainty, and more than one state of nature in the second period, how should we define the riskless rate of interest? The answer of course depends on how we define a riskless asset. The traditional answer, given by Fisher and others before him, is to look for an asset that always gives the same purchasing power. If there is only one good, and agents have common priors and von Neumann–Morgenstern utilities, the answer seems straightforward. The riskless asset should simply pay off enough money in each state to enable the holder to buy one unit of the commodity in every state. Such an asset would always give the same purchasing power, by construction. It also gives the same utility in each state to an agent for whom it is the sole source of income.

Why is it important that it give the same utility in each state to any owner who holds no other asset, since no consumer will hold only that single asset? Typically no consumer will end up with exactly the same utility in each state. Why shouldn't the ideal asset pay enough money to guarantee the same marginal utility in each state? Or why shouldn't the ideal asset pay a constant fraction of aggregate income (i.e., of GNP)?

Thus even for the simple case of one consumption good it is essential to make clear why it is important to have a riskless asset. This can be done in the capital asset pricing model, which we now review.

2.2 CAPM with One Commodity

We are going to show that when there is only one consumption good, the asset which pays off 1 unit of the good in every state does play a special role, at least under the conditions of the capital asset pricing model (CAPM). Let $\tilde{1} = (1, ..., 1) \in \mathbb{R}^S$. We shall show that:

1) As long as every agent can sell off his future income today, the presence of the $(1, ..., 1)$ asset guarantees that interior equilibria are fully Pareto optimal.

2) Everybody does indeed hold the $(1, ..., 1)$ asset in his portfolio, and in fact just one other asset, even though final consumption may be very risky.

We shall then show that even when there are multiple goods per state, there is still an asset that satisfies properties (1) and (2).

Capital Asset Pricing Model

Assumptions

$$L = 1 \tag{1}$$

$$U^h(x) = u_0^h(x_0) + \sum_{s=1}^{S} \gamma_s^h u^h(x_s) = u_0^h(x_0) + \sum_{s=1}^{S} \gamma_s^h \left(x_s - \tfrac{1}{2}\alpha_h x_s^2\right) \tag{2}$$

$$\gamma_s^h = \gamma_s \quad \forall h, s \tag{3}$$

$$1 - \alpha_h e_s > 0 \ \ \forall h, s \text{ where } e_s = \sum_{h \in H} e_s^h \tag{4}$$

$$\tilde{e}^h \in \text{span}[A] \quad \forall h \in \mathcal{H} \tag{5}$$

$$\tilde{1} \in \text{span}[A] \tag{6}$$

(1) says that there is only one good (we are also dropping money). In fact it is only important that there is one good for each $s \in S$ in period one. There could be many goods in period zero. (2) says that people have von Neumann–Morgenstern quadratic utilities, and (3) says that agents have objective probabilities γ. Quadratic utilities guarantee that agents only care about the mean and variance of their consumption. That could also be achieved by assuming normality of endowments and asset payoffs. But in that context the Pareto efficiency theorem below is not true.

Assumption (4) guarantees monotonicity. Since quadratic utility eventually declines with more consumption, we must suppose that the utilities are chosen such that agents still want to eat more even if they consume the whole endowment.

The fifth, crucial assumption, is that agents' initial endowments are in the span of the assets (they are tradeable); this is a non-generic assumption that is necessary to ensure Pareto optimality. This is perhaps not realistic, in view of real-life asymmetries of information and moral hazard issues that normally prevent people from selling their future income. We assume away all these problems.

Finally, (6) says that the $(1, ..., 1)$ asset is tradeable in our economy. In this one-good model, the natural definition of a riskless asset is that of an asset which pays one unit of the good in every state of nature, and we shall see that the presence of $\tilde{1}$ in the economy is very important.

Let us state the following Pareto efficiency theorem. This was first proved by Mossin (1977), but we give the proof provided in Geanakoplos–Shubik (1990).

CAPM Efficiency Theorem *Let* $x^h \gg 0$ *at a* CAPM GEI *satisfying assumption* (1)–(6), *including that* $(1, ..., 1) \in \text{span}[A]$. *Then* $(x^h)_{h \in H}$ *is* Pareto optimal.

This theorem says that even if the number of assets is much smaller than the number of the states (so that it is impossible to insure most risks), equilibrium allocations are fully Pareto optimal, if the $(1, ..., 1)$ asset is tradeable, and if every agent can sell his future income. Without the $(1, ..., 1)$ asset, equilibrium allocations may be terrible. By adding the $(1, ..., 1)$ asset, equilibrium jumps to the Pareto frontier, even though there are many other missing asset market that have not been added.

Proof Let the marginal utilities of consumption (unweighted by probabilities) be denoted, as in Lemma 1, by

$$\left(\frac{\partial u^h(x_s)}{\partial x_s}\right)_{s \in S} = \tilde{\mu}^h = \tilde{1} - \alpha^h \tilde{x}^h$$

Then from the first order conditions of agent h, we know that

$$\pi_j = \frac{1}{\mu_0^h}\tilde{\mu}^h \cdot_\gamma [\tilde{p} \boxdot A^j] \text{ for all } j \in J$$

But

$$\begin{aligned} & \tilde{1} \in \text{span}[A] \\ & \tilde{x}^h = \tilde{e}^h + A\theta^h \in \text{span}[A] \\ \Rightarrow\ & \frac{1}{\mu_0^h}\tilde{\mu}^h \in \text{span}[A] \\ \Rightarrow\ & \frac{1}{\mu_0^h}\tilde{\mu}^h = \mu^* \quad \forall h \in H. \end{aligned}$$

The last implication follows from the unique linear pricing lemma proved in the last section. But now we know all individuals' marginal utilities of income across all states are equal, up to a proportionality coefficient. Hence the equilibrium is fully Pareto efficient. ■

The next theorem shows that the $(1, ..., 1)$ asset will not only be held by every consumer, but that in fact it is virtually the only thing that every consumer holds. This mutual fund theorem was first proved by Tobin (1958). The proof presented here follows Geanakoplos–Shubik (1990).

CAPM Mutual Fund Theorem *Let* $\tilde{x}^h \gg 0$ *at a* CAPM *equilibrium arising from a* GEI *economy satisfying* (1)–(6). *Then*

$$\tilde{x}^h \in \text{span}[\tilde{1}, \tilde{e}] \quad \forall h \in H$$

where

$$\tilde{e} = \sum_{h=1}^{H} \tilde{e}^h.$$

This theorem says that at least in the CAPM model, even though all individual consumptions $\tilde{x}^h$ might turn out to be risky, every agent will hold a combination of the market portfolio and the $(1, ..., 1)$ asset. The $(1, ..., 1)$ asset is thus indispensable to agent optimization.

Proof From the last theorem, we know there is a scalar $\lambda^h = 1/\mu_0^h > 0$ such that

$$\begin{aligned} \lambda^h(\tilde{1} - \alpha^h \tilde{x}^h) &= \mu^* \quad \forall h \\ \Rightarrow \tilde{x}^h &= \frac{1}{\alpha^h}\tilde{1} - \frac{1}{\alpha^h \lambda^h}\mu^*. \end{aligned}$$

So, everybody's final consumption is a combination of just two pay-off vectors. Summing over all the agents we get:

$$\begin{aligned} &\tilde{e} \equiv \sum_{h=1}^{H} \tilde{e}^h = \sum_{h=1}^{H} \tilde{x}^h = \sum_{h=1}^{H} \left[\frac{1}{\alpha^h}\tilde{1} - \frac{1}{\alpha^h \lambda^h}\mu^*\right] = \alpha\tilde{1} - \beta'\mu^*, \ \alpha > 0, \ \beta' > 0 \\ &\Rightarrow \mu^* = (1/\beta')(\alpha\tilde{1} - \tilde{e}) \in \text{span}[\tilde{1}, \tilde{e}] \\ &\Rightarrow \tilde{x}^h \in \text{span}[\tilde{1}, \tilde{e}]. \end{aligned}$$

■

Before moving to the next section, we note in passing that we can give a more gripping interpretation of the GEI linear pricing theorem in the special case of CAPM. The following theorem is originally due to Sharpe and Lintner.

CAPM Security Market Line *Let* $\tilde{x}^h \gg 0 \ \forall h$ *at a* CAPM GEI. *Then* $\exists r > -1$ *and* $\delta > 0$ *such that* $\forall j \in J$,

$$\pi_j = \frac{E_\gamma A^j}{1+r} - \beta \text{Cov}_\gamma(\tilde{e}, A^j).$$

More generally, if $z \in \text{span}[A]$

$$\pi(z) = \frac{E_\gamma z}{1+r} - \beta \text{Cov}_j(\tilde{e}, z).$$

Proof Letting $\beta = 1/\beta' > 0$, and recalling that $\alpha > 0$,

$$\begin{aligned}
\pi_j &= \mu^* \cdot_\gamma A^j \\
&= \beta(\alpha\tilde{1} - \tilde{e}) \cdot_\gamma A^j, \ \alpha, \beta > 0 \\
&= \beta\alpha\tilde{1} \cdot_\gamma A^j - \beta\tilde{e} \cdot_\gamma A^j \\
&= \beta\alpha E_\gamma A_j - \beta \mathrm{Cov}_\gamma(\tilde{e}, A^j) \\
&= \beta(\alpha - E_\gamma \tilde{e}) E_\gamma A_j - \beta \mathrm{Cov}_\gamma(\tilde{e}, A^j)
\end{aligned}$$

Similarly,

$$\pi(z) = \beta(\alpha - E_\gamma \tilde{e}) E_\gamma z - \beta \mathrm{Cov}_\gamma(\tilde{e}, z). \text{ Since } \pi(\tilde{e}) > 0, \ \alpha - E_\gamma \tilde{e} > 0,$$

hence letting $1/(1+r) = \beta(\alpha - E_\gamma \tilde{e})$,

$$\pi(z) = \frac{E_\gamma z}{1+r} - \beta \mathrm{Cov}_\gamma(\tilde{e}, z).$$

■

2.3 Multicommodity CAPM

The main lesson for us of CAPM is that if we just add the asset $(1, ...1)$, everybody is as well off as possible. The form this asset takes gives us a blueprint for what we should mean by a *riskless* asset. Not surprisingly, when there is only one good, the ideal riskless asset pays the same quantity of the real good in each state of nature. (This does rule out paying the same marginal utility in each state.) The surprise is that even when everybody ends up with risky consumption, everybody trades almost exclusively in the riskless asset, and social welfare is so dramatically improved.

Once we go over to a multicommodity world, the form of the ideal riskless asset becomes much less obvious. Should it allow for the purchase of the same "ideal" consumption bundle in every state? Or should it allow each agent to achieve the same utility in every state? If every agent regarded consumption as twice as pleasurable in state 1 as in state 2, should the ideal riskless asset pay off the same bundle in both states, or twice as much in state 1, or half as much in state 1? We shall answer these questions by investigating whether in the multicommodity world, under some circumstances, there is again a special asset whose introduction guarantees the Pareto optimality of equilibrium. If there is, we shall take its form to define what we mean by the ideal riskless asset.

The practical world has already spoken on this issue. Inflation proof bonds almost invariably take the form of guaranteeing the same consumption bundle in every state. Thus in the US the so called TIPs (Treasury inflation protected bonds) are designed to enable holders to purchase the same CPI commodity bundle. But I shall demonstrate that an asset that guarantees the same utility is much more helpful. To see why, consider the following multicommodity CAPM model.

Multiple Commodity CAPM We now move to a world with many goods in each state, $L > 1$. If we allow for completely general utilities, it is hard to say anything concrete. So we specialize to a simple generalization of CAPM, keeping as many of the assumptions (2)–(5) as possible. A crucial assumption will be made about utilities:

Assumption 2′ Let $U^h(x) \equiv u_0^h(x_0) + \sum_{s=1}^{S} \gamma_s^h(v_s(x_s) - \frac{1}{2}\alpha_h[v_s(x_s)]^2)$ where $v_s : \mathbb{R}^L \to \mathbb{R}$ is smooth, concave, homogeneous of degree 1 $\forall s \in S$, $\forall h \in H$ (v_s is the same across agents).

This utility is the composition of an (idiosyncratic) quadratic function with a (common) homogenous of degree one function. We will be relying very heavily on the hypothesis that there is a common v_s. This allows us to define one asset whose payoffs give each agent an equal utility across every state.

Because of the quadratic aspect, this utility also declines eventually with more and more consumption. So we need the natural counterpart of old assumption (4) in this new set-up:

Assumption 4′ $1 - \alpha^h v_s(e_s) > 0$.

Having made the previous assumption we can formally prove the following theorem. This theorem was first stated in Geanakoplos–Shubik (1990). The proof here is new.

Multiple Commodity CAPM Efficiency Theorem *Consider a* GEI *satisfying* (2′), (3), (4′) *and* (5). *Let* $(p, \pi, (x^h, \theta^h)_{h\in H})$ *be a* GEI *for which* $x^h \gg 0$ $\forall h \in H$. *Suppose there is an asset* $r \in \text{span}[A]$ *of the form*

$$r = \left(\frac{1}{v_1(e_1)}e_1, ..., \frac{1}{v_S(e_S)}e_S\right).$$

Then $(x^h)_{h\in H}$ *is Pareto Optimal.*

Notice that by homogeneity of degree 1,

$$v_s\left(\frac{1}{v_s(e_s)}e_s\right) = \frac{v_s(e_s)}{v_s(e_s)} = 1$$

so that the "ideal" asset r yields the same utility of consumption $(1 - \frac{1}{2}\alpha_h)$ in every state to any agent h who consumes exclusively its payoff. But what is more important, it gives the cheapest possible bundle in each state of achieving that utility, given the prices prevailing in that state. A fixed commodity bundle also gives the same utility in every state (assuming v_s does not depend on s), but it is not the cheapest bundle to achieve that utility, given changing prices. Thus the appropriate riskless asset does reflect what is available, i.e., it is not a fixed bundle. In general, no fixed commodity bundle would do as well.

Proof Let us begin by examining the given multi-commodity CAPM equilibrium $(p, \pi, (x^h, \theta^h)_{h \in H})$. In equilibrium we know that in each state prices will be proportional to each agent's marginal utilities:

$$p_{s\ell} = \lambda_s^h \frac{\partial U^h(x)}{\partial x_{s\ell}} = \lambda_s^h \gamma_s \left[\frac{\partial v_s(x_s^h)}{\partial x_{s\ell}} - \alpha_h v(x_s) \frac{\partial v_s(x_s^h)}{\partial x_{s\ell}} \right]$$
$$= \mu_s^h \frac{\partial v_s(x_s^h)}{\partial x_{s\ell}}$$

where $\mu_s^h = \lambda_s^h \gamma_s [1 - \alpha_h v_s(x_s)] > 0$.

Since v_s is homogeneous of degree 1 and common to all h, it follows from a standard theorem in aggregation that in equilibrium every x_s^h is proportional to e_s (see Deaton–Mullbauer (1980)). Since the derivatives of v_s must be homogeneous of degree 0, $\partial v_s(x_s^h)/\partial x_{s\ell} = \partial v_s(e_s)/\partial x_{s\ell}$ does not depend on h. Furthermore, since scaling p_s up or down by itself does not affect equilibrium, we might as well assume $p_s = Dv_s(e_s)$.

Finally, by homogeneity, we conclude that $p_s \cdot e_s = v_s(e_s)$ and for any scalar multiple of e_s, such as any x_s^h, $p_s \cdot x_s^h = v_s(x_s^h)$. For any y_s with $p_s \cdot y_s = p_s \cdot x_s^h$, we know from the fact that x_s^h is equilibrium consumption that $v_s(y_s) \leq v_s(x_s^h) = p_s \cdot x_s^h = p_s \cdot y_s$. By homogeneity, $v_s(y_s) \leq p_s \cdot y_s \; \forall s \in S, \forall y_s \in \mathbb{R}_+^L$.

We shall now prove the theorem by reducing the economy to a one-commodity CAPM in which CAPM assumptions (1)–(6) hold.

Now we define a related one-good CAPM economy $((\hat{U}^h, \hat{e}^h)_{h \in H}, \hat{A})$ where we maintain the same number of states and agents as in the multi-commodity CAPM. Let

$$\hat{U}^h(x_0, x_1, ..., x_s) = u_0^h(x_0) + \sum_{s=1}^{S} \gamma_s [x_s - \tfrac{1}{2} \alpha_h x_s^2] \quad \forall h \in H$$

where the u_0^h and α_h parameters are the same as in the multi-commodity CAPM utility. Define

$$\hat{e}^h = (e_0^h, p_1 \cdot e_1^h, ..., p_S \cdot e_S^h) \in \mathbb{R}_{++}^{L+S} \quad \forall h \in H$$

where the p_s are taken from the multi-commodity equilibrium. Define

$$\hat{A}_{sj} = p_s \cdot A_{sj}, \; \forall s \in S, \; j \in J.$$

Observe that since $r \in \text{span}[A]$, $(1, ..., 1) = \tilde{p} \boxdot r \in \text{span}[\hat{A}]$. More generally, for any vector $y \in \mathbb{R}_+^{L(S+1)}$, define the vector $\hat{y} \in \mathbb{R}_+^{L+S}$ by $\hat{y}_0 = y_0$ and $\hat{y}_s = p_s \cdot y_s$, $\forall s \in S$. Conversely, for any vector $\hat{y} = (\hat{y}_0, \hat{y}_1, ..., \hat{y}_s) \in \mathbb{R}_+^{L+S}$, define the vector $\vec{y} \in \mathbb{R}_+^{L(1+S)}$ by

$$\vec{y}_0 = \hat{y}_0; \; \vec{y}_s = \frac{\hat{y}_s}{p_s \cdot e_s} e_s \text{ for all } s \in S.$$

Note that $v_s(\vec{y}_s) = \hat{y}_s v_s(e_s/p_s e_s) = \hat{y}_s$, and $p_s \cdot \vec{y}_s = \hat{y}_s$.

Define the price vector q in the CAPM economy by $q_0 = p_0$, and $q_s = 1$ for all $s \in S$. First let us see that $(q, \pi, (\hat{x}^h, \theta^h)_{h\in H})$ is an equilibrium for the economy $((\hat{U}^h, \hat{e}^h)_{h\in H}, \hat{A})_{h\in H}$, where the π and θ^h are as in the original multi-commodity equilibrium. Since x_s^h is proportional to e_s for all h, $\hat{x}_s^h = p_s \cdot x_s^h = v_s(x_s^h)$. Thus $U^h(x^h) = \hat{U}^h(\hat{x}^h)$ for all $h \in H$.

Clearly, for all vectors (y^h, φ^h), $(y^h, \varphi^h) \in B^h(p, \pi, e^h, A) \Leftrightarrow (\hat{y}^h, \phi^h) \in B^h(q, \pi, \hat{e}^h, \hat{A}) \Leftrightarrow (\vec{y}^h, \phi^h) \in B^h(p, \pi, e^h, A))$. Thus $(\hat{x}^h, \theta^h) \in B^h(q, \pi, \hat{e}^h, \hat{A})$. Moreover, if some $(\hat{x}, \hat{\theta}) \in B^h(q, \pi, \hat{e}, \hat{A})$, then $(\vec{x}, \hat{\theta}) \in B^h(p, \pi, e^h, A)$. By virtue of the fact that (x^h, θ^h) are equilibrium demands, $\hat{U}^h(\hat{x}^h) = U^h(x^h) \geq U^h(\vec{x}) = \hat{U}^h(\hat{x})$, so $(\hat{x}^h, \theta^h)$ are equilibrium demands in the one-good CAPM.

Our CAPM economy satisfies all assumptions (1)–(6). Hence by our one-good Pareto efficiency theorem, $(\hat{x}^h)_{h\in H}$ is fully Pareto efficient in the one-good CAPM.

If $(y^h)_{h\in H}$ Pareto dominates $(x^h)_{h\in H}$ in the multicommodity CAPM, and if $\sum_{h\in H} y^h = \sum_{h\in H} e^h$, then $(\hat{y}^h)_{h\in H}$ must Pareto dominate $(\hat{x}^h)_{h\in H}$ and still be feasible in the one good CAPM, a contradiction. ■

Multicommodity CAPM Mutual Fund Theorem *Under the assumptions of the last theorem, let $\tilde{x}^h \gg 0$ at a multicommodity* CAPM GEI. *Then*

$$\tilde{x}^h \in \text{span}[r, \tilde{e}] \quad \forall h \in H$$

where r is the appropriate ideal asset defined above.

The proof is given by combining the one-commodity CAPM proof with the proof given above for the multi-commodity CAPM efficiency theorem. Once again we are able to prove that there is a crucial asset which is necessary and sufficient for the economy to achieve full efficiency (assuming every agent can sell off his future endowment).

2.4 The Ideal Inflation Proof Bond

In the context of the multicommodity CAPM we can positively describe the ideal inflation proof bond. It does not pay the same amount of money in each state. It does not guarantee the same consumption in each state. It does not guarantee the same marginal utility in each state. On the contrary, *it guarantees the same utility in each state*. If agents feel twice as good about consumption in state 1 as in state 2, the ideal riskless asset will pay half as many goods in state 1.

2.4.1 What would Fisher's Ideal Index Give?

The Konüs (1939) index measures inflation as the ratio of the cost of achieving the same utility between this period and a previous base period. Our "ideal" bond corre-

sponds to a Konüs inflation indexed bond. In Fisher's proposal of 1925 for an indexed bond, he suggested using a fixed commodity bundle chosen in some previous base period. This corresponds to the Laspeyre index of inflation. As Fisher recognized clearly, using a fixed bundle to measure inflation tends to overestimate it (assuming the bundle chosen in the base year was utility maximizing) because, when the price of some commodities go up, agents substitute into other cheaper commodities. His recommendation for the ideal inflation index was more sophisticated than the Laspeyre index. He suggested taking the geometric average of the Laspeyre index and the Paasche index as an "ideal" index. The Paasche index is formed by taking the ratio of the cost of buying today's consumption bundle at today's prices to the cost of buying today's consumption bundle at yesterday's prices. Assuming that today's bundle was utility maximizing at today's prices, the Paasche index always underestimates the Konüs inflation. Averaging the two can give an index that is close to the Konüs index. It can be shown that if the functions v_s are all equal to Cobb–Douglas utilities with exponents $1/L$ for every commodity, then the Fisher index reduces precisely to the Konüs index (see Diewert (1976)). The Fisher index would then provide for Pareto efficient trade under the multicommodity CAPM assumptions of this section.
.

2.4.2 An Open Question

This result leaves open the question about what the ideal riskless bond should be in case there is more agent heterogeneity. For example, if v_s also depends on h, there may be no single asset which provides every agent with a state independent utility. Further analysis is required to see whether in this case it is helpful to have more than one inflation proof bond, or whether there is indeed any particular advantage to introducing a bond linked to some consumer price index.

3 Overlapping Generations Economies (OLG)

Irving Fisher argued that the rate of interest depends on impatience, on productivity, on the distribution of endowments between today and tomorrow, and on the distribution of wealth between patient and impatient agents. In 1947, Maurice Allais of France introduced a model of overlapping generations in which he showed that neither impatience nor any of the other aforementioned factors has anything to do with the rate of interest, at least for some equilibria. He found that the rate of interest is equal to the rate of growth of population. In 1958 Paul Samuelson of MIT rediscovered the OLG model and added a new twist. He showed that there could be assets whose prices exceed the present discounted value of all their future dividends, thus contradicting another central tenet of Fisher. Samuelson concentrated on just one such asset, namely green pieces of paper called fiat money, which provided no utility whatsoever and yet sold for a positive price. The situation in which an asset sells for more than its "fundamental value" is called a bubble.

OLG models have an infinite number of agents and an infinite number of goods. Each agent lives for two periods, when he is 'young' and 'old'. In the simplest version of the model, which we shall concentrate on here, there is one good in each time period.

In Samuelson's original paper, he gave an endowment of 1 to the young and nothing to the old in each generation. He also assumed one agent per generation. We can present the structure as:

$\ddots$						
	1	0				
		1	0			
			1	0		
					$\ddots$	
time (past)	0	1	2	3		time (future)

Notice that at a given date, t, the old have nothing to offer the young, so according to Samuelson there could be no trading. Samuelson ascribed the inefficiencies we will see in this model to a lack of a "double coincidence of wants." He suggested that one way to deal with this was to introduce money.

We will begin to analyze the OLG model as if it could be thought of as a Walrasian or Arrow–Debreu model with one budget constraint per agent, and infinity in a few places. (That immediately rules out the lack of double coincidence of wants explanation for any of the OLG properties, since in Arrow–Debreu all goods and agents are directly linked.) This is exactly analogous to the transformation of the temporal Fisher economy into the timeless Walrasian economy that we saw in Section 1. We shall find that without durable goods, there are typically two steady state equilibria. One we call **F** after Fisher, since the rate of interest there depends on the impatience of the consumers, though the equilibrium real rate of interest is negative, and the equilibrium is inefficient. In the other steady state equilibrium **S**, as Allais and Samuelson claimed, the rate of interest is equal to the growth rate of the population (namely zero in our example), completely independent of the impatience of the consumers.

After examining the Allais–Samuelson case, we add land and find that now there is only one steady state equilibrium, which is efficient, and in which the rate of interest depends exactly as before on impatience and the other factors mentioned earlier in Fisher's two period model.

3.1 Description of the Basic OLG Model

The set of commodities and agents are now both infinite: $\mathcal{L} = \mathbb{Z} = \{..., -1, 0, 1, ...\}$; $\mathcal{H} \equiv \{th : (t, h) \in \mathbb{Z}\times H\}$, where we think of one commodity per period and H agents per generation. Each period is regarded as the time of a generation (say 30 years). Each agent lives for two generations, and has an endowment $e^{th} = (..., e^{th}_t, e^{th}_{t+1}, ...)$, where e^{th}_s is the endowment of an agent born at t in period s. Thus e^{th}_t is the endowment of agent th when young and e^{th}_{t+1} is the endowment of agent th when old. In the basic model without land, we assume there is no endowment other than these two. Payoffs depend only on consumption while alive $U^{th}(..., x_t, x_{t+1}, ...) = U^{th}(x_t, x_{t+1})$. We shall assume that U^{th} is strictly concave and strictly increasing in goods corresponding to periods in which the agent is alive, and similarly that the endowment when young are positive, $e^{th}_t > 0$.

A *Walrasian OLG equilibrium* is a price vector $q = (..., q_{-1}, q_0, q_1, ...)$ and an allocation $\{x^{th} = (..., x^{th}_t, x^{th}_{t+1}, ...) : th \in \mathcal{H}\}$ such that

1) market clearing: $\sum_{th\in\mathcal{H}} x^{th}_s = \sum_{h\in H} x^{s-1,h}_s + \sum_{h\in H} x^{sh}_s = \sum_{th\in\mathcal{H}} e^{th}_s, \ \forall s$

2) value of individual endowments are finite: $\sum_{-\infty}^{+\infty} q_s e^{th}_s < \infty, \ \forall th \in \mathcal{H}$

3) $x^{th} \in B^{th}(q) = \{x : \sum_{-\infty}^{+\infty} q_s x_s \leq \sum_{-\infty}^{+\infty} q_s e^{th}_s\}, \ \forall th \in \mathcal{H}$

4) $x^{th} \in B^{th}(q) \Rightarrow U^{th}(x) \leq U^{th}(x^{th}), \ \forall th \in \mathcal{H}$

In case there is no land, $\sum_{th\in\mathcal{H}} e^{th}_s = \sum_{h\in H} e^{s-1,h}_s + \sum_{h\in H} e^{s,h}_s$, and $\sum_{-\infty}^{+\infty} q_s e^{th}_s = q_t e^{th}_t + q_{t+1} e^{th}_{t+1}$, so condition (2) is trivial. Actually, condition (2) follows from conditions (1), (3) and (4) for economies in which agents are only interested in a finite number of goods, for if an agent had infinite wealth, he would still have money left over to spend after buying the whole of the aggregate endowment of the goods that he liked, and then he would want to buy still more, contradicting market clearing.

As in the two-period Walrasian equilibrium described earlier, the prices q_t represent the price that would be paid at time 0 to obtain one unit of consumption at time t.

In all our four examples we shall assume there is only one agent per generation, and drop the superscript h. In our theorems, however, we shall allow for multiple agents per generation.

Example 1 Let $U^t(..., x_{t-1}, x_t, x_{t+1}, ...) = \log x_t + \log x_{t+1}$, and let $e^t_t = 3, e^t_{t+1} = 1$, and $e^t_s = 0$ otherwise, for all $t \in H$.

It can easily be seen that there are two stationary equilibria, where q_{t+1}/q_t is the same for all t. In one $q_t = 3^t$, $-\infty < t < \infty$, so $q_{t+1}/q_t = 3$. Each agent optimizes by consuming her endowment. Thus, $x^t_t = 3$, $x^t_{t+1} = 1$, for all $t \in \mathcal{H}$, and clearly the markets clear. We call this equilibrium **F**, after Fisher.

In the other stationary equilibrium, $q_t = 1$, $-\infty < t < \infty$. This clears all the markets because demand becomes $(x_t^t, x_{t+1}^t) = (2, 2)$ for all $t \in \mathcal{H}$. We call this the **S** equilibrium, after Samuelson.

Since the consumers' utility function is concave, consumers are better off smoothing consumption

$$U^t(2, 2) > U^t(3, 1)$$

Thus Equilibrium **F** is **not** Pareto efficient.

In the **F** equilibrium $1 + r_t = q_t/q_{t+1} = 1/3 \Rightarrow r_t = -2/3$.While in the **S** equilibrium,

$$r_t = 0$$

equal to the rate of population growth.

Example 2 Consider again the welfare functions in Example 1, $U^t(x_t, x_{t+1}) = \log x_t + \log x_{t+1}$, but now let the endowments for each generation t be $(e_t^t, e_{t+1}^t) = (6, 1)$. It is easy to find two steady-state equilibria as before. In Equilibrium **2F**, we let $q = (..., 1, 6, 36, ...)$ and $(x_t^t, x_{t+1}^t) = (6, 1) = (e_t^t, e_{t+1}^t)$. The interest rate is $1+r = 1/6$, $r = -5/6$, so an increase in endowments when young decreases the rate of interest, just as Irving Fisher predicted. On the other hand, there is also an Equilibrium **2S** in which $q = (..., 1, 1, 1, ...)$ and $(x_t^t, x_{t+1}^t) = (3\frac{1}{2}, 3\frac{1}{2})$ for all t. In this "Samuelson equilibrium," the interest rate is still 0 despite the change in endowments. The Samuelson equilibrium confirms the view of Allais that the distribution of endowments across the life-cycle will not change the interest rate.

Example 3 Suppose now that $U^t(x_t, x_{t+1}) = \log x_t + 0.5 \log x_{t+1}$, and that $(e_t^t, e_{t+1}^t) = (3, 1)$. Compared to Example 1, the agents have gotten more impatient. Again we can calculate that there are two stationary equilibria. In the Fisher equilibrium, $q = (..., 1, \frac{3}{2}, \frac{9}{4}, ...)$ and $(x_t^t, x_{t+1}^t) = (3, 1)$ for all t. The interest rate has risen to $1 + r_t = \frac{2}{3}$, $r_t = -\frac{1}{3}$. But in the other Samuelson equilibrium, $q = (..., 1, 1, 1, ...)$ and $(x_t^t, x_{t+1}^t) = (\frac{8}{3}, \frac{4}{3})$. Again the interest rate remains at 0 despite the increase in impatience of every agent.

In all three examples, the real interest rate in the Samuelson equilibrium was equal to the population growth rate, namely zero.[4] Samuelson actually considered a variation of the economy we have been working with, in which time has a beginning at $t = 1$. He added a positive endowment of fiat money to the endowment of old agents at time $t = 1$. Then he showed that what we called the S-equilibrium could be realized as an equilibrium in his truncated model by giving money a positive value.

[4] Another important point is that in all these economies the Fisher equilibrium was not Pareto efficient. Welfare is improved if every young person gives the contemporaneous old $\Delta > 0$ goods, much as happens in the US pay-as-you-go social security system. Samuelson once regarded this as a powerful argument in favor of the US social security system.

Money is a durable good that yields no dividends and provides no utility, yet in the Samuelson model it has a positive value, contradicting another one of Fisher's central tenets.

3.2 Adding Land to the OLG Model

Land is meant to be an asset which yields a steady dividend forever. The owner of the land effectively owns all its dividends, and hence has an endowment stream that goes on forever, even though he will die after at most two periods. We start, as Samuelson suggested, at time 1, with the old agents from time 0 and the young agents born at time 1, but with no money, and with all the land in the possession of agents born at time 0. The definition of Walrasian OLG equilibrium given above also applies with land, except that now we restrict to $t \geq 0$ and $s \geq 1$. The only other difference is that in the basic OLG economy, each agent owns goods in at most two periods, while with land, some agents $0h$ might own endowments of goods in every period. In fact we shall always assume that in OLG with land, at least one agent's endowment includes a nonnegligible fraction of the aggregate endowment of *every* good.

Example 4 We assume individual endowments of fruit are $(3, 1)$ when young and old, as in examples 1 and 3.

1				
3	1			
	3	1		
			$\ddots$	
1	2	3	4	time

Now we add an infinitely durable good to the OLG model that yields real dividends forever. In addition to the other endowments, we assume that the person born in time 0 owns a durable good called land or trees that yields one unit of the perishable consumption good per period, forever. We can write the effective endowments of the perishable goods as

2	1	1	1	
3	1			
	3	1		
		3	1	
1	2	3	4	time

To solve the for the Walrasian OLG equilibrium, we write the budget sets:

$$B^0(q) = \{x_1^0 : q_1 x_1^0 \leq q_1 + q_1 + \underbrace{\sum_{t=2}^{\infty} q_t}_{\substack{\text{value of} \\ \text{the land}}} \}$$

$$B^t(q) = \{(x_t^t, x_{t+1}^t) : q_t x_t^t + q_{t+1} x_{t+1}^t \leq 3q_t + q_{t+1}\} \text{ for all } t \geq 1$$

Market clearing now requires that $x_t^{t-1} + x_t^t = e_t^{t-1} + e_t^t + 1 = 1 + 3 + 1 = 5$, $t = 1, 2, ...$ We defer computing the stationary equilibrium until the next section.

Note that consumer 0 solves: $\max \log x_1^0$ s.t. $x_1^0 \in B^0(p)$, so $x_1^0 = (q_1 + \sum_{t=1}^{\infty} q_t)/q_1$. We know that $x_1^0 \leq 5$ (to satisfy market clearing), so

$$\sum_{t=1}^{\infty} q_t \leq 4q_1 < \infty.$$

Thus in any equilibrium in example 4, the value of the aggregate endowment $5 \sum_{t=1}^{\infty} q_t$ must be finite. In examples 1–3, it was infinite in every equilibrium.

3.2.1 Temporal OLG Equilibrium with Land

We have described the OLG model with land by adding the dividends of the land to the endowments of the time 0 generation, in accordance with how much land each of them owns to start. In the Walrasian (or Arrow–Debreu) equilibrium described in Section 3.1, consumption goods in every period are traded simultaneously at some mythical place and time where the souls of the unborn and dead all meet. With durable goods like land, we can define a more realistic *temporal* OLG equilibrium by allowing trade to occur only between living agents. At any given time, two generations are living, and they can exchange fruit (consumption goods) and land, just as in the two period temporal equilibrium described in Section 1. And again we shall see that the temporal equilibrium is identical to the Walrasian equilibrium.

Define:

$\mathbb{1}K_0^{0h} =$ acres of land owned initially by consumers $0h$.

$K_s^{th} =$ acres of land purchased in year $s \geq 1$ by consumer th. We allow $K_s^{sh} < 0$, but require $K_s^{s-1,h} \geq 0$.

$x_s^{th} =$ number of fruit consumed in year s by consumer th. We require $x_s^{th} \geq 0$.

$p_t =$ price of a time t fruit *paid at time t*.

$\Pi_t =$ price of a time t acre of land *paid at time t*.

$f =$ output of fruit per acre of land in each period.

$\bar{f} = f \sum_{h \in H} K_0^{0h}$ = total output of fruit from all land, in each period.

In the Walrasian setting, q_t was the price paid at time 0 for fruit at time t. That was the present value price, whereas p_t here is the current value price. Allowing $K_s^{sh} < 0$ is analogous to allowing short sales of assets in the two-period model. Insisting on $K_s^{s-1,h} \geq 0$ is necessary, since we do not allow for default, and the agent cannot pay back after he is dead.

Recalling that the owner of land at time t has the right to consume the fruit it bears at time $(t+1)$, we define the temporal budget constraints for each agent $0h$ and th, $h \in H, t \geq 1$:

$$B^{0h}(p,\Pi) = \{(x_1^{0h}, K_1^{0h}) : p_1 x_1^{0h} + \Pi_1 K_1^{0h} \leq \underbrace{p_1 e_1^{0h}}_{\text{endowed fruit}} + \underbrace{p_1 K_0^{0h} f}_{\text{fruit from endowed land}} + \underbrace{\Pi_1 K_0^{0h}}_{\text{endowed land}}\}$$

$$B^{th}(p,\Pi) = \{(x_t^{th}, K_t^{th}, x_{t+1}^{th}, K_{t+1}^{th}) : p_t x_t^{th} + \Pi_t K_t^{th} \leq p_t e_t^{th} \text{ and } p_{t+1} x_{t+1}^{th} + \Pi_{t+1} K_{t+1}^{th} \leq p_{t+1} e_{t+1}^{th} + p_{t+1} K_t^{th} f + \Pi_{t+1} K_t^{th}\} \tag{2}$$

A consumer born in year $t \geq 1$ has two budget constraints. The first limits his purchase of fruit and land in year t. The second limits his purchase of fruit and land in year $(t+1)$. Increasing K_t^{th} has no direct effect on the consumer's utility, but allows him to increase x_{t+1}^{th}. Increasing K_{t+1}^{th} does not benefit the consumer in any way, so in equilibrium, $K_{t+1}^{th} = 0$.

A temporal OLG equilibrium with land $((p_t, \Pi_t)_{t \geq 1}, (x^{th}, K^{th})_{th \in \mathcal{H}})$ satisfies

1) market clearing in commodities: $\sum_{th \in \mathcal{H}} x_s^{th} = \sum_{h \in H} x_s^{s-1,h} + \sum_{h \in H} x_s^{sh} = \sum_{th \in \mathcal{H}} e_s^{th} + \bar{f},\ \forall s \geq 1$

2) market clearing in land: $\sum_{th \in \mathcal{H}} K_s^{th} = \sum_{h \in H} K_0^{0h},\ \forall s \geq 1$

3) $(x^{th}, K^{th}) \in B^{th}(p,\Pi), \forall th \in \mathcal{H}$

4) $(x, K) \in B^{th}(p,\Pi) \Rightarrow U^{th}(x) \leq U^{th}(x^{th}), \forall th \in \mathcal{H}$

In our next theorem we shall prove that temporal OLG is identical to Walrasian OLG. The idea is that the return on land in the temporal equilibrium implicitly defines the trade-off between consumption at times t and $t+1$, and thus the rate of interest at time t:

$$1 + r_t = \frac{p_{t+1} f + \Pi_{t+1}}{p_{t+1}} \Bigg/ \frac{\Pi_t}{p_t}.$$

The denominator describes how much land can be acquired with one unit of fruit at time t, and the numerator describes how much fruit can be acquired at time $t+1$ from the dividends and sale of the land.

Walrasian-Temporal Equilibrium Equivalence Theorem for OLG with Land *Let* $E = ((U^{th}, e^{th})_{th\in\mathcal{H}}, (K_0^{0h})_{h\in H}, f)$ *be an* OLG *economy with productive land,* $\bar{f} = f\sum_{h\in H} K_0^{0h} > 0$*. Then there is a temporal* OLG *equilibrium* $((p_t, \Pi_t)_{t\geq 1}, (x^{th}, K^{th})_{th\in\mathcal{H}}))$ *for* E *if and only if there is a Walrasian* OLG *equilibrium* $(q_t, (x^{th})_{th\in\mathcal{H}})$ *for* E*, where for all* $t \geq 1$

$$\frac{q_t}{q_{t+1}} = \frac{p_{t+1}f + \Pi_{t+1}}{p_{t+1}} \Big/ \frac{\Pi_t}{p_t}$$
$$\frac{\Pi_t}{p_t} = \frac{1}{q_t}\sum_{\tau=t+1}^{\infty} q_\tau f.$$

Proof We prove first that the two equations above are equivalent. Suppose first that $\Pi_t/p_t < \infty$ is defined for all $t \geq 1$ by the bottom equation. Then

$$\frac{p_{t+1}f + \Pi_{t+1}}{p_{t+1}} \Big/ \frac{\Pi_t}{p_t} = \frac{f + \frac{1}{q_{t+1}}\sum_{\tau=t+2}^{\infty} q_\tau f}{\frac{1}{q_t}\sum_{\tau=t+1}^{\infty} q_\tau f} = \frac{q_t}{q_{t+1}} \frac{q_{t+1}f + \sum_{\tau=t+2}^{\infty} q_\tau f}{\sum_{\tau=t+1}^{\infty} q_\tau f} = \frac{q_t}{q_{t+1}}$$

Conversely, suppose the top equation holds for all $t = s \geq 1$. Rewriting it and iterating it back to $t < s$, we get

$$\frac{\Pi_s}{p_s} = = \frac{q_{s+1}}{q_s}\left(f + \frac{\Pi_{s+1}}{p_{s+1}}\right)$$
$$\frac{\Pi_{s-1}}{p_{s-1}} = \frac{q_s}{q_{s-1}}\left(f + \frac{q_{s+1}}{q_s}\left(f + \frac{\Pi_{s+1}}{p_{s+1}}\right)\right) = \frac{1}{q_{s-1}}\left(q_s f + q_{s+1}f + q_{s+1}\frac{\Pi_{s+1}}{p_{s+1}}\right)$$
$$\frac{\Pi_t}{p_t} = \frac{1}{q_t}\left[\sum_{\tau=t+1}^{s+1} q_\tau f\right] + \frac{1}{q_t}\left[q_{s+1}\frac{\Pi_{s+1}}{p_{s+1}}\right].$$

Letting $s \to \infty$ shows that $\sum_{t=2}^{\infty} q_t < \infty$. Hence $q_{s+1}(\Pi_{s+1}/p_{s+1}) = \sum_{\tau=s+2}^{\infty} q_\tau f \to 0$, giving the bottom equation.

Next we show that given these two equations, the temporal and Walrasian budget sets contain exactly the same feasible consumption vectors for any individual. Observe that doubling both Π_t and p_t for any t does not affect any agent's budget set. Hence we can scale up Π_t and p_t so that $\Pi_t = p_{t+1}f + \Pi_{t+1}$. Then from the top equation, $q_t/q_{t+1} = p_t/p_{t+1}$. Adding consumer th's budget constraints

$$p_t x_t^{th} + \Pi_t K_t^{th} \leq p_t e_t^{th}$$
$$p_{t+1}x_{t+1}^{th} + \Pi_{t+1}K_{t+1}^{th} \leq p_{t+1}e_{t+1}^{th} + p_{t+1}K_t^{th} f + \Pi_{t+1}K_t^{th}$$

gives

$$p_t x_t^{th} + p_{t+1}x_{t+1}^{th} \leq p_t e_t^{th} + p_{t+1}e_{t+1}^{th}.$$

This shows that if $(x, K) \in B^{th}(p, \Pi)$, then $x \in B^{th}(q)$ for every agent with $t \geq 1$. Conversely, if $x \in B^{th}(q)$, then $\frac{q_t}{q_{t+1}}(e_t^{ht} - x_t^{ht}) = (x_{t+1}^{ht} - e_{t+1}^{ht})$. Define

$$K_t = (e_t^{ht} - x_t^{ht}) \Big/ \frac{\Pi_t}{p_t}.$$

Using the first equation, the additional consumption this allows in period $t+1$ is

$$\frac{p_{t+1} f + \Pi_{t+1}}{p_{t+1}} K_t = \frac{p_{t+1} f + \Pi_{t+1}}{p_{t+1}} (e_t^{ht} - x_t^{ht}) \Big/ \frac{\Pi_t}{p_t} = \frac{q_t}{q_{t+1}} (e_t^{ht} - x_t^{ht}) = (x_{t+1}^{ht} - e_{t+1}^{ht}).$$

Hence $(x, K) \in B^{th}(p, \Pi)$. For agents $0h$, simply apply the bottom equation for Π_1/p_1. ■

Corollary *Let* $((p_t, \Pi_t)_{t \geq 1}, (x^{th}, K^{th})_{th \in \mathcal{H}}))$ *be a temporal equilibrium for an* OLG *economy* $((U^{th}, e^{th})_{th \in \mathcal{H}}, (K_0^{0h})_{h \in H}, f)$ *with productive land,* $\bar{f} = f \sum_{h \in H} K_0^{0h} > 0$*. Then* $(x^{th})_{th \in \mathcal{H}}$ *is Pareto efficient, and land is always priced as the discounted value of its future dividends.*

Proof From the previous theorem, the temporal equilibrium is equivalent to a Walrasian equilibrium. From condition (2) of Walrasian OLG equilibrium, when there is land the requirement that the income of every individual agent be finite also guarantees that the total value of the *aggregate* endowment will be finite, since the latter is no more than a finite multiple of the former for some agent born at time 0. Hence the usual proof of the Pareto efficiency of competitive equilibrium goes through, as was pointed out by Wilson (1981).[5] By introducing a durable good whose payoff is a nonnegligible fraction of aggregate endowment, we have assured that there are no Pareto inefficient competitive equilibria.[6] Furthermore, the proof of the last theorem also showed that the price of land must be equal to the present discounted value of its dividends. ■

In fact it can easily be shown that if we add other durable goods, like paper money, to the OLG economy with land, then all of these will be priced at the present

[5]This proof works by arguing that if all agents could be made better off by an alternative allocation, this alternative allocation would cost more to each agent than his endowment, otherwise he would have bought it. Adding over all agents, the cost of the aggregate consumption of the new allocation must be more than the aggegate value of endowments, if both numbers are finite, contradicting the feasibility of the new allocation. When aggregate endowments have infinite value, this proof fails. (In the equilibria of examples 1–3, the value of the aggregate endowment was infinite).

[6]Consequently the rationale for social security must come from elsewhere.

discounted value of their dividends.[7] This gives us a first indication that the OLG model with land will behave much more like the Fisher two-period model.

3.2.2 Stationary OLG Equilibrium with Land

If $U^{th}(x_t, x_{t+1}) = U^h(x_1, x_2)$ and $(e_t^{th}, e_{t+1}^{th}) = (e_1^h, e_2^h)$ do not depend on t, for $t \geq 1$, and if $\bar{f} > 0$, then we have a stationary OLG economy with (productive) land. A *stationary* OLG equilibrium is an OLG equilibrium where the one-period rate of interest remains constant over time, $q_1 = 1, q_2 = q, q_3 = q^2 \cdots$. We call $q = 1/(1+r)$ the market discount rate and r the one-period rate of interest.

For each agent $th \in \mathcal{H}$, let $(x_1^{th}(q), x_2^{th}(q)) \in \arg\max\{U^{th}(x_1, x_2) : x_1 + qx_2 \leq e_t^{th} + qe_{t+1}^{th}, x_1, x_2 \geq 0\}$. Given our hypotheses on utilities, we must have that $(x_1^{th}(q), x_2^{th}(q))$ is a uniquely defined, continuous function, and $x_1^{th}(q) + x_2^{th}(q) \to \infty$ as $q \to 0$, for all $t \geq 1$.[8] Let $(z_1^{th}(q), z_2^{th}(q)) = ((x_t^{th}(q) - e_t^{th}), (x_{t+1}^{th}(q) - e_{t+1}^{th}))$ be the excess demand of agent th, and let $(z_1^t(q), z_2^t(q)) = (\sum_{h \in H}(z_1^{th}(q), z_2^{th}(q)))$ be the aggregate excess demand of generation $t \geq 1$. For stationary economies we can drop the t and write $(z_1^h(q), z_2^h(q))$ and $(z_1(q), z_2(q))$.

Stationary OLG Equilibrium Existence Theorem *Every stationary equilibrium for any* OLG *economy with productive land $\bar{f} > 0$ has a strictly positive interest rate,* $0 < q < 1$. *Moreover, every stationary* OLG *economy with productive land $\bar{f} > 0$ such that U^h is strictly concave and strictly monotonic, and such that $e_1^h > 0$,* for all *$h \in H$, has a stationary equilibrium.*

Proof From the last theorem we know that the value of land is finite, so $0 < \frac{1}{q_1} \sum_{\tau=2}^{\infty} q^\tau f < \infty$. Hence $0 < q < 1$.

In stationary equilibrium we must have that

$$z_1(q) + z_2(q) = \bar{f}$$

where $z_1(q)$ is the excess demand of one generation's young and $z_2(q)$ is the excess demand of the previous generation's old. For any q, each agent will spend all his money, and hence the budget set inequality will be an equality and $z_1(q) + qz_2(q) = 0$. Hence when $q = 1$, $z_1(q) + z_2(q) = 0 < \bar{f}$. But as $q \to 0$, $z_1(q) + z_2(q) \to \infty$. By continuity there is some $0 < q < 1$ with $z_1(q) + z_2(q) = \bar{f}$.

This clears all markets for time $t \geq 2$. What about $t = 1$? Luckily, we do not need to check anything more. Since the value of the aggregate endowment is finite,

[7] Otherwise the owners of the asset would spend more on consumption goods than the value of their endowments of consumption goods. Adding over all agents, we would get that total expenditure on consumption goods is greater than the total value of consumption goods, since the latter is finite, contradicting market clearing.

[8] The latter is true because as the price of the second good goes to zero, holding fixed the price of the first good at 1, income is still bounded away from zero. Hence the monotonic agent will demand arbitrarily more of the second good, while the demand for the first good must remain nonnegative.

and since every agent spends all his money, total expenditures must equal the total value of endowments. But from the foregoing, for each good $t \geq 2$ total expenditures equal total value. Hence also for the one remaining good $t = 1$ we must have total expenditures equal total value. But since $q_1 = q^{1-1} = 1$, that means demand equals supply for time 1 goods as well. ■

Computing Stationary Equilibrium for Example 4 At each date $t \geq 2$, the young and the old must have demands that sum to all the goods in the economy, namely their joint endowments plus the output of apples from the land. Consider the case $t = 2$. Then the present value (i.e., as of time 1) of income of the old (born at time 1) is $3 + q$, while for the young born at time 2 the present value of income (as of time 1 again) is $3q + q^2$. Recalling the formula for Cobb–Douglas demand, and recalling that the total supply of goods in every time period is $3 + 1 + 1 = 5$, we can solve

$$\begin{aligned} x_t^{t-1} + x_t^t &= 5 \\ \Rightarrow \frac{\frac{1}{2}[3+q]}{q} + \frac{\frac{1}{2}[3q+q^2]}{q} &= 5 \end{aligned}$$

Solving the quadratic equation gives two solutions, only one of which has $0 < q < 1$: $\{q = 0.55,\ (x_t^t, x_{t+1}^t) = (1.78, 3.22),\ \text{for all } t \geq 1\}$. It must therefore be the stationary equilibrium.

By stationarity the same solution clears the markets for all time $t \geq 2$. It also clears the market for $t = 1$, as can easily be verified by noting that $q/(1-q) = 1.22$. We must add the demand of the agent who is born at time 1 and the demand of the agent who is old at time 1:

$$\begin{aligned} x_1^1 &= \tfrac{1}{2}(3+q) = \tfrac{3}{2} + \tfrac{1}{2}q \\ x_1^0 &= 1 + 1 + \underbrace{[q + q^2 + q^3 + \cdots]}_{\text{value of land}} = 2 + \tfrac{q}{1-q} \text{ (assuming } q < 1) \end{aligned}$$

$$\begin{aligned} q &= 0.55 \\ x_1^0 &= 2 + \frac{q}{1-q} = 3.22 \\ x_t^t &= \tfrac{1}{2}(3+q) = 1.78 \\ x_{t+1}^t &= \tfrac{1}{2q}(3+q) = 3.22 \\ r &= 1/q - 1 = (1/.55) - 1 = 82\% \end{aligned}$$

Taking advantage of the Fisher capital value equation that the land price must be equal to the present value of its dividends, we can immediately translate this Walrasian equilibrium into a temporal OLG equilibrium: $(p_t, \Pi_t) = (1, 1.22)$ for all $t \geq 1$.

3.2.3 Land Restores Many Two-Period Lessons

Adding land solves several problems. First, Pareto optimality is restored. Second, the value of every durable good is equal to the present value of its dividends. Third, we can imagine that trades take place between living agents, i.e., we do not need a mythical market where all of the generations of consumers trade all of the goods simultaneously. Fourth, the interest rate is positive. And last, but not least, the rate of interest is determined by impatience, and by the other factors mentioned earlier. (This last observation seems to be new).

3.3 What Determines the Rate of Interest

If in example 4 we make each generation more impatient, the real rate of interest will go up. Taking $U^t(x_t, x_{t+1}) = \log x_t + 0.5 \log x_{t+1}$ we get $r = 139\%$. If we increase the productivity of the tree, the rate of interest will go up. For example, if we take $\bar{f} = 2$, then $r = 154\%$. If we increase the endowment when young, the rate of interest will go down. For example, if we take $e_t^t = 6$, we get $r = 38\%$.

Thus the determinants of the rate of interest that we saw already in the two period model affect the rate of interest in same way in OLG with land. We can prove this in a more general situation.

Before stating the next theorem, let us observe that in any stationary OLG equilibrium with productive land, the old agents at time 1 will consume their endowments of goods, plus all the dividends of land (since they own all the land), plus the goods obtained by selling off their land. From stationarity, it follows that in every period the old collectively will eat more than the sum of their old endowments.

3.3.1 Impatience Theorem in OLG with Land or Trees

Theorem *Consider any stationary* OLG *economy with productive land $\bar{f} > 0$ such that $U^{th}(x_t, x_{t+1}) = u^h(x_t) + \delta_h u^h(x_{t+1})$, where u^h is strictly concave, increasing, and twice differentiable, for all $th \in \mathcal{H}$. Suppose that in a stationary equilibrium every old agent consumes more than his old endowment, and every young agent consumes something positive. Then more impatience (decreasing any δ_h) implies a higher interest rate, as does higher productivity of land (greater $\bar{f}$), as does increasing any old endowment e_2^h,* or lowering any young endowment *e_1^h.*

Proof In stationary equilibrium, we must have that

$$z_1(q) + z_2(q) - \bar{f} = 0,$$

where the functions $(z_1(q), z_2(q))$ also implicitly depend on the $(\delta_h, e_1^h, e_2^h)_{h\in H}$. Under the hypotheses of the theorem, we shall show that around the equilibrium, the left hand side is differentiable in $(q, (\delta_h, e_1^h, e_2^h)_{h\in H})$ and decreasing in q. The implicit function theorem then tells us that any perturbation to the above equation will move equilibrium q in the same direction, and move equilibrium $r = 1/q - 1$ in the opposite direction.

From elementary consumer demand theory, we know that the individual excess demands $(z_1^h(q), z_2^h(q))$ are differentiable in all the parameters at the equilibrium q (since they are positive there and since the utilities are twice differentiable). From Slutsky's equation, we know that we can decompose the derivative of any individual excess demand into a Hicksian term and an income effect term

$$\frac{\partial z_1^h(q)}{\partial q} = \frac{\partial z_1^{hH}(q)}{\partial q} - \frac{\partial z_1^h(q)}{\partial I} z_2^h(q)$$
$$\frac{\partial z_2^h(q)}{\partial q} = \frac{\partial z_2^{hH}(q)}{\partial q} - \frac{\partial z_2^h(q)}{\partial I} z_2^h(q)$$

where the first term on the right hand sides $\partial z_i^{hH}(q)/\partial q$ is the Hicksian or compensated demand of agent h, and $\partial z_i^h(q)/\partial I$ is the income effect term. We know that Hicksian own effects are negative, so $\partial z_2^{hH}(q)/\partial q < 0$. We also know that Hicksian demand keeps utility constant, so

$$\frac{\partial z_1^{hH}(q)}{\partial q} + q\frac{\partial z_2^{hH}(q)}{\partial q} = 0$$

Since in stationary equilibrium $q < 1$, it follows that

$$\frac{\partial z_1^{hH}(q)}{\partial q} + \frac{\partial z_2^{hH}(q)}{\partial q} < 0.$$

From the fact that utility is additively separable, we know that demands are normal, $\partial z_i^h(q)/\partial I > 0$, for $i = 1, 2$. By hypothesis, we have that the old consume more than their endowments, so $z_2^h(q) > 0$. Therefore

$$\frac{\partial z_1^h(q)}{\partial q} + \frac{\partial z_2^h(q)}{\partial q} = \frac{\partial z_1^{hH}(q)}{\partial q} + \frac{\partial z_2^{hH}(q)}{\partial q} - \frac{\partial z_1^h(q)}{\partial I} z_2^h(q) - \frac{\partial z_2^h(q)}{\partial I} z_2^h(q) < 0$$

when evaluated at equilibrium q. Adding over all the agents in H, it follows that

$$\frac{\partial z_1(q)}{\partial q} + \frac{\partial z_2(q)}{\partial q} < 0.$$

It follows from the implicit function theorem that increasing $\bar{f}$ necessarily decreases $q = 1/(1 + r)$ and thus increases the interest rate.

Increasing the impatience of some consumers makes them want to eat more today. But at the same prices, that means their total consumption over time must decrease, since the value of future consumption is less ($q < 1$). Hence to restore equilibrium, q must fall and interest rates must rise.

Finally, switching young endowment to old endowment reduces income at the old prices (since $q < 1$), thus reducing demand when young and old, but not affecting total supply (which is the sum of endowment when young and old). Hence to restore market clearing, q must again fall and interest rates rise. A similar argument can be given if endowments when young or old are changed separately. ■

The proof that more impatience raises the interest rate can be proved under weaker hypotheses in the case of OLG with land than it was in the two period case. Far from contradicting Fisher's impatience theory of interest, OLG economies confirm it more emphatically.

References

[1] Allais, M. (1947). **Éconmie et Intérêt**. Paris: Imprimerie Nationale et Librairie des publications Officielles, 800 p., in two volumes.

[2] Deaton, A. and J. Muellbauer (1980). **Economics and Consumer Behavior**. Cambridge: Cambridge University Press.

[3] Dimand, R. (1999). "Letter on Fisher and Indexed Bonds," *Journal of Economic Perspectives*, 13(1): 224–225.

[4] Diewert, W.E. (1976). "Exact and Superlative Index Numbers," *Journal of Econometrics,* 4(2): 115–145.

[5] Fisher, I. (1907). **The Rate of Interest**. New York: MacMillan.

[6] Fisher, I. (1918). "Stabilizing the Dollar. In W.J. Barber (ed.), **The Works of Irving Fisher**, Vol. 6. London: Pickering & Chatto.

[7] Fisher, I. (1925). "The Stabilized Bond — A new Idea in Finance," *The Annalist*, as revised in Fred R. Fairchild and Ralph T. Compton (eds.), **Economic Problems: A Book of Selected Readings**. New York: Macmillan, 1928. Reprinted in W. J. Barber (ed.), **The Works of Irving Fisher**, Vol. 8. London: Pickering & Chatto.

[8] Fisher, I. (1930). **The Theory of Interest**. New York: MacMillan.

[9] Fisher, I. (1997). "The Making of Index Numbers." In W.J. Barber (ed.), **The Works of Irving Fisher**, Vol. 7. London: Pickering & Chatto.

[10] Geanakoplos, J.D. (1987). "Overlapping Generations Model of General Equilibrium." In J. Eatwell, M. Milgate, P. Newman (eds.), **The New Palgrave, A Dictionary of Economics**, Vol. 3. London and Basingstoke, The Macmillan Press Limited, pp. 767–779.

[11] Geanakoplos, J.D. and M. Shubik (1990). "The Capital Asset Pricing Model as a General Equilibrium with Incomplete Markets," *The Geneva Papers on Risk and Insurance Theory*, 15(1): 55–71.

[12] Geanakoplos, J.D. (1990). "An Introduction to General Equilibrium With Incomplete Asset Markets," *Journal of Mathematical Economics*, 19: 1–38.

[13] Konüs A.A. (1939) "The Problem of the True Index of the Cost of Living,", *Econometrica*, 7: 10–29.

[14] Lintner, J. (1965). "The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budget," *Review of Economics and Statistics* 47: 13–37.

[15] Loewenstein, G. (1985). "Expectations and Intertemporal Choice," Yale PhD dissertation.

[16] McCulloch, J. Huston (1980). "The Ban on Indexed Bonds," *American Economic Review*, 70(5): 1018–1021.

[17] Mossin, J. (1977). **The Economic Efficiency of Financial Markets**. Lexington Books.

[18] Samuelson, P.A. (1958). "An Exact Consumption Loan Model of Interest, with or without the Social Contrivance of Money," *Journal of Political Economy*, 66(5): 467–482.

[19] Sharpe, W. (1964). "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, *Journal of Finance,* 19: 425–442.

[20] Tobin, J. (1958). "Liquidity Preference as Behavior Towards Risk," *The Review of Economic Studies*, 67: 65–86.

[21] Tobin, J. (1987). "Irving Fisher." In J. Eatwell, M. Milgate, P. Newman (eds.), **The New Palgrave, A Dictionary of Economics**,Volume 2. London and Basingstoke, The Macmillan Press Limited, pp. 369–376.

[22] Wilson, C. (1981). "Equilibrium in Dynamic Models of Pure Exchange," *Journal of Economic Theory*, 24(1): 95–111.