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OPTIMAL OUTPUT AND PRICE POLICIES FOR GENERAL DISTRIBUTED LAG DEMAND EQUATIONS

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By

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I. Introduction

Assume that a monopolist sells k products that are related in demand. A classical problem of economic statics is to determine the outputs and prices which maximize net revenue at a given point of time. Considerably harder is the problem of finding a sequence of outputs and prices over time which maximize the present value of net revenue when the related demand equations depend on lagged as well as current values of prices and outputs.[2] The solution of this problem is the subject of this paper.

Assuming the k demand equations are linear difference equations of arbitrary but finite order n , we give necessary and sufficient conditions for the existence of a finite present value, algorithms for calculating the optimal paths, and some results on the nonnegativity of the quantity and price sequences. In our formulation of the problem we work with a discount factor and an infinite horizon.

Many of our results pertain to the nature of general linear decision rules.[3] Our formal problem is that of maximizing a quadratic form subject to linear constraints expressed by a system of difference equations and certain nonnegativity conditions. Although we could give our results for this abstract problem, it would be at the cost of losing certain intuitive insights. The relevance of our results to the general problem of linear decision rules will be obvious in due course.

Our problem naturally divides itself into the study of the simpler one product case and then the more complicated multi-product case. It is gratifying that many of the results for the simple case apply to k products after suitable modifications.

II. The One-Product Case

The monopolist is assumed to face a demand equation as follows:

(1) $$q_t + a_1 q_{t-1} + \cdots + a_n q_{t-n} = f_t - b p_t , \quad b > 0 .$$

q_t = quantity demanded during period t

p_t = price during period t

f_t = an arbitrary function of time which may represent seasonals, trends, real income, etc.

This demand equation could arise because, for instance, customers hold inventories so that current purchases depend on previous purchases. It is convenient to adopt a more compact notation. We use the linear lag operator L which has the property that

(2) $$L x_t = x_{t-1}$$

and its inverse, the forward shift operator $E = L^{-1}$. The operator

(3) $$A(L) = 1 + a_1 L + \cdots + a_n L^n , \quad a_n \neq 0$$

is an n^{th} degree polynomial in the lag operator L. With its help the demand equation becomes

$$(4) \qquad A(L)\, q_t = f_t - b\, p_t$$

Let c_t denote total cost at time t. Assume that c_t is a quadratic function of current quantity, q_t, and a finite number of lagged quantities. Thus net revenue is

$$r_t = p_t\, q_t - c_t$$

and the present value (P.V.) of net revenue is

$$(5) \qquad \text{P.V.} = \sum_{0}^{\infty} \beta^t\, r_t = \Sigma\, \beta^t\, p_t q_t - \Sigma\, \beta^t\, c_t$$

From now on we adopt the convention that where the indexes of summation are omitted they shall be understood to run from zero to ∞. The discount factor, β, is the reciprocal of 1 plus the discount rate so that

$$(6) \qquad 0 < \beta < 1 .$$

In certain formal expressions, we may allow $\beta = 1$.

Clearly, the cost and gross revenue terms in the expression for present value have the same formal structure because both are quadratic expressions in current and lagged quantities. Therefore, nothing substantive is lost by confining attention to the gross revenue term in (5) and completely neglecting c_t.

We seek necessary and sufficient conditions on the polynomial operator $A(L)$ so that sequences $\{p_t\}$ and $\{q_t\}$ can be found such that P.V. has a finite maximum where P.V. is given by

$$\text{(7)} \qquad \text{P.V.} = \Sigma\, \beta^t\, p_t\, q_t$$

and the constraints given by (4) are satisfied for all t .

Nonnegativity constraints are more difficult. The nonnegativity of the optimal sequences $\{p_t\}$ and $\{q_t\}$ clearly depends not only on the nature of the demand operator $A(L)$ and the optimal operator to be derived below but also on the properties of the forcing function f_t . This greatly complicates the necessary and sufficient conditions that guarantee non-negativity. The results given below depend only on the properties of the operators and not on the specific properties of the forcing function f_t other than its nonnegativity.

We now turn to the necessary and sufficient conditions for the existence of a finite maximum present value. Although the problem has several avenues of approach, the one we give generalizes most easily to the multi-product case. Because there are an infinite number of time periods, the conditions for a finite optimum are most securely based on first principles. Suppose that $\{q_t\}$, a bounded sequence of scalars, subject to the condition that $q_t = 0$ for all $t < 0$, is the optimal quantity path. Consider the bounded sequence $\{q_t + d_t\}$ where also $d_t = 0$ for $t < 0$. Without losing generality, we may set $b = 1$.[4] In an obvious notation,

$$\text{P.V. } \{q_t + d_t\} = \Sigma\, \beta^t (q_t + d_t) \Big(f_t - A(L)\,(q_t + d_t)\Big)$$

$$\text{(8)} \qquad \text{P.V. } \{q_t + d_t\} = \Sigma\, \beta^t\, q_t \Big(f_t - A(L)\Big)\, q_t + \Sigma\, \beta^t\, d_t \Big(f_t - A(L)\Big)\, q_t$$

$$- \;\Sigma\, \beta^t\, q_t\, A(L)\, d_t - \Sigma\, \beta^t\, d_t\, A(L)\, d_t$$

The following lemma allows the regrouping of terms.

Lemma 1:

(9) $$\Sigma \beta^t q_t A(L) d_t = \Sigma \beta^t d_t A(\beta E) q_t$$

provided $0 \leq \beta < 1$ and the sequences $\{q_t\}$ and $\{q_t + d_t\}$ are such that the series (9) are absolutely convergent.

Proof: The case $n = 2$ illustrates the general case sufficiently well. The left expression is

$$q_o d_o + \beta q_1 (d_1 + a_1 d_o) + \beta^2 q_2 (d_2 + a_1 d_1 + a_2 d_o) + \ldots$$

This may be rearranged as

$$d_o (q_o + \beta a_1 q_1 + \beta^2 a_2 q_2) + \beta d_1 (q_1 + \beta a_1 q_2 + \beta^2 a_2 q_3) + \ldots$$

provided both series converge absolutely as assumed by hypothesis.
Lemma 1 applied to (8) yields

(10) $$\text{P.V. } \{q_t + d_t\} = \text{P.V. } \{q_t\} + \Sigma \beta^t d_t [f_t - A(L) q_t - A(\beta E) q_t] - \Sigma \beta^t d_t A(L) d_t$$

Since $\{d_t\}$ may be chosen arbitrarily, we obtain the first theorem:

Theorem 1: In order that the bounded sequence $\{q_t\}$ maximize P.V. uniquely, it is necessary and sufficient that q_t be the bounded solution of

(11) $$[A(L) + A(\beta E)] q_t = f_t$$

which satisfies the condition that $q_t = 0$ for $t < 0$ and that

(12) $$\Sigma \beta^t d_t A(L) d_t > 0$$

for any sequence $\{d_t\}$ not identically zero.

At this point several remarks are pertinent. First, it is clear that the restriction to bounded quantity paths is unnecessarily severe since the present value is finite even when quantity grows over time provided the growth factor is less than $\beta^{-t/2}$.[5] Second, we have so far imposed no restrictions on the demand operator $A(L)$ and in particular we have said nothing about its stability. Finally, the forcing function f_t can grow over time and it is clear from (11) that this imparts a growth to quantity. In fact exogenously given growth factors serve to determine a discount factor which makes the present value finite.

It is also instructive to write (12) in matrix form and the case $n = 2$ is sufficiently general. Thus

$$(13) \qquad \Sigma\, \beta^t d_t A(L) d_t = (d_0\ d_1\ d_2 \ldots) \begin{bmatrix} 1 & 0 & 0 & \ldots \\ \beta a_1 & \beta & 0 & \ldots \\ \beta^2 a_2 & \beta^2 a_1 & \beta^2 & \ldots \\ & \ldots & & \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \ldots \end{bmatrix}$$

By modifying the infinite triangular matrix of the quadratic form in (13) in a simple way, it becomes possible to apply the Herglotz Lemma and obtain necessary and sufficient conditions on the demand operator $A(L)$ that will tell when condition (12) is satisfied. It is helpful to prove a lemma.

<u>Lemma 2</u>:

$$\Sigma\, \beta^t d_t A(L) d_t > 0 \quad \text{if and only if}$$

$$\Sigma\, \beta^{t/2} d_t [A(\beta^{1/2} L) + A(\beta^{1/2} E)]\, \beta^{t/2} d_t > 0 .$$

Proof:

$$\Sigma \beta^t d_t A(L) d_t = \Sigma \beta^{t/2} d_t \beta^{t/2} A(L) d_t = \Sigma \beta^{t/2} d_t A(\beta^{1/2} L) \beta^{t/2} d_t$$

Apply Lemma 1 to the right side of (12) so that

$$\begin{aligned} \Sigma \beta^t d_t A(L) d_t &= \Sigma \beta^t d_t A(\beta E) d_t \\ &= \Sigma \beta^{t/2} d_t A(\beta^{1/2} E) \beta^{t/2} d_t \end{aligned}$$

Hence

$$(14) \qquad \Sigma \beta^t d_t A(L) d_t = 1/2 \left(\Sigma \beta^{t/2} d_t [A(\beta^{1/2} L) + A(\beta^{1/2} E)] \beta^{t/2} d_t \right)$$

and the lemma readily follows.

The matrix associated with the right side of the form in (14) has a special character as illustrated for $n = 2$.

$$\begin{bmatrix} 2 & \beta^{1/2} a_1 & \beta a_2 & 0 & \cdots \\ \beta^{1/2} a_1 & 2 & \beta^{1/2} a_1 & \beta a_2 & \cdots \\ \beta a_2 & \beta^{1/2} a_1 & 2 & \beta^{1/2} a_1 & \cdots \end{bmatrix}$$

It is a band symmetric matrix to which the Herglotz Lemma applies.[6]

The Herglotz Lemma asserts that the sum

$$(16) \qquad \sum_{u,v=0}^{S} g(u - v) h(u) \overline{h(v)} > 0$$

for all finite integers S if and only if

$$(17) \qquad g(u) = \int_{-\pi}^{\pi} e^{iux} dF(x) , \qquad -\pi \leq x \leq \pi , \quad u = 0, \pm 1, \pm 2, \ldots$$

where $F(x)$ is a non-decreasing, bounded, nonnegative function of the real variable x . Thus $F(x)$ may be thought of as a density function and $g(u)$ as its characteristic function. Notice the bar over $h(v)$ which denotes the complex conjugate. Hence the form in (16) is to be positive definite over the complex numbers.

To see how this Lemma gives the desired conditions on $A(L)$, we begin by defining

$$(18) \qquad a_{-u} = a_u \quad \text{for} \quad u = 1, \ldots, n \quad \text{and} \quad a_o = 1$$

where a_u is the coefficient of L^u in the polynomial $A(L)$. Suppose we allow the d_t's in (14) to be complex. Evidently if the form in (14) is positive over complex numbers then it will be positive over the real numbers. Thus write

$$(14') \qquad \Sigma\, \beta^t\, \bar{d}_t\, A(L)\, d_t = 1/2 \;\; \Sigma\, \beta^{t/2}\, \bar{d}_t\, [A(\beta^{1/2}\, L) + A(\beta^{1/2}\, E)]\, \beta^{t/2}\, \bar{d}_t$$

and make the correspondences as follows:

$$(18) \qquad \begin{aligned} g(u-v) &= a_{u-v}\, \beta^{|u-v|/2} \quad u\ ,\ v = 0\ ,\ 1\ ,\ 2\ , \ldots \\ h(u) &= \beta^{u/2}\, d_u \end{aligned}$$

For any finite integer S , the right side of (14') becomes

$$(19) \qquad \sum_{u,v=0}^{S} \beta^{u/2}\, \bar{d}_u\, \beta^{|u-v|/2}\, a_{u-v}\, \beta^{v/2}\, d_v$$

It is easy to check (19) by premultiplying (15) by the row vector whose components are $\beta^{v/2}\, d_v$ and postmultiplying by the column vector whose components are $\beta^{u/2} d_u$. The Herglotz Lemma, however, applies to finite sums whereas the sum in (14') is over all of the positive integers t . Therefore, we need the following:

<u>Lemma 3</u>:

$$\Sigma \beta^{u/2} d_u \beta^{|u-v|/2} a_{u-v} \beta^{v/2} d_v > 0 \tag{20}$$

for all sequences $\{\beta^{u/2} d_u\}$ bounded in modulus if and only if the finite sum (19) is positive for all finite S .

Proof:

Assume (20) is positive for all bounded sequences $\{\beta^{u/2} d_u\}$. Obviously, the finite sums in (19) must also be positive for otherwise there would be a subsequence of the infinite sum which would be negative contrary to the hypothesis. Conversely, assume that the finite sum is positive for all bounded sequences. Since the infinite sum is absolutely convergent, there is an integer S sufficiently large that allows us to approximate the infinite sum by a finite sum as closely as we please. Therefore, if the finite sums are positive for all choices of S then the infinite sums must also be positive for all bounded sequences.

Now we can prove

<u>Theorem 2</u>:

$\Sigma \beta^t \bar{d}_t A(L) d_t > 0$ for all sequences $\{\beta^{t/2} d_t\}$ bounded in modulus and $0 < \beta \leq 1$ if and only if

$$A(e^{ix}) + A(e^{-ix}) > 0 \quad \text{for all} \quad x \in [-\pi, \pi] \ . \tag{21}$$

Proof:

According to (16) and the correspondence set up in (18),

$$\beta^{|u|/2} a_u = \int_{-\pi}^{\pi} e^{iux}\, dF(x) \ , \quad u \neq 0 \quad 2a_o = \int_{-\pi}^{\pi} dF(x) \tag{22}$$

Since $\Sigma \, |a_u| \beta^{|u|/2}$ is finite, the function $F(x)$ is differentiable.[7] Therefore,

$$F'(x) = \frac{1}{2\pi} [1 + \sum_{u=1}^{n} \beta^{|u|/2} a_{-u} e^{iux} + 1 + \sum_{u=1}^{n} \beta^{|u|/2} a_u e^{-iux}]$$

or, more compactly,

$$F'(x) = \frac{1}{\pi} [A(\beta^{1/2} e^{ix}) + A(\beta^{1/2} e^{-ix})] \tag{23}$$

The right side of (14') is positive for all bounded sequences $\{\beta^{t/2} d_t\}$ and $0 < \beta \leq 1$ if and only if (20) holds. Appealing to Lemma 3, the Herglotz Lemma, and (23), inequality in (20) holds if and only if $F'(x) > 0$ for all $0 < \beta \leq 1$ and all $-\pi \leq x \leq \pi$. Since

$$A(\beta^{1/2} e^{ix}) + A(\beta^{1/2} e^{-ix}) = 2\text{Re}\, A(\beta^{1/2} e^{ix}) \ ,$$

the left side is a harmonic function defined on the closed unit disk.[8] Therefore it attains its extrema on the unit circle where $\beta = 1$. This completes the proof of the theorem.

It is worth noting that since $a_u = a_{-u}$, (21) is equivalent to the following:

$$1 + a_1 \cos x + \ . \ . \ . + a_n \cos nx > 0 \ . \tag{21'}$$

We illustrate the conditions imposed on $A(L)$ for $n = 2$. In this case the a's must be such that

$$\varphi(x) = 1 + a_1 \cos x + a_2 \cos 2x > 0 \quad \text{for all} \quad -\pi \leq x \leq \pi . \tag{24}$$

This condition is met if and only if $\min \varphi(x) > 0$. Since

$$\varphi'(x) = - \sin x (a_1 + 4\, a_2 \cos x) ,$$

$$\varphi'(x) = 0 \quad \text{for} \quad x = 0 , \pi , \quad \text{and} \quad \cos x = - \frac{a_1}{4\, a_2} \text{ provided } |a_1| < 4\, |a_2| .$$

Hence

$$\varphi(0) = 1 + a_1 + a_2 > 0$$

$$\varphi(\pi) = 1 - a_1 + a_2 > 0 , \quad \text{and}$$

$$\varphi\left(\cos^{-1} \left(- \frac{a_1}{4\, a_2}\right)\right) = \frac{2 - a_1^2 - 8\,(a_2 - 1/2)^2}{8\, a_2} > 0$$

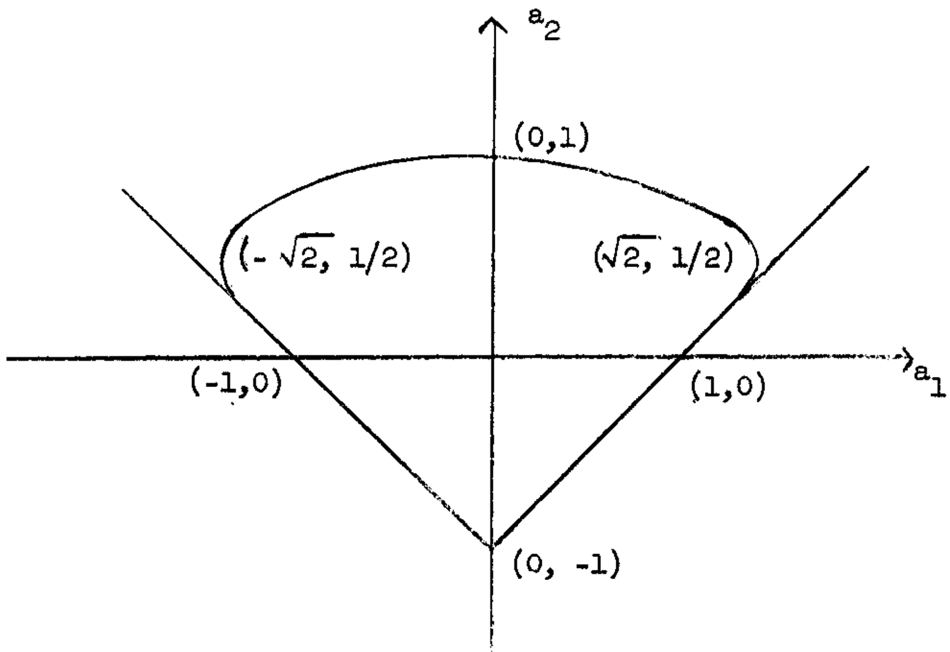
The admissible region in the (a_1, a_2)-space is shown in Figure 1. Any pair of points in the cone-shaped region fulfills the requirement of (24) and the associated matrix (15) will be positive definite.

Condition (21) permits further simplification in terms of the reciprocal equation

$$A(z) + A(z^{-1}) = 0 .$$

This equation plays a fundamental role in much of the development. For $|z| = 1$, there is an obvious correspondence between $A(z) + A(z^{-1})$ and $A(e^{ix}) + A(e^{-ix})$ because $z^{-1} = e^{-ix}$ when $z = e^{ix}$.

Figure 1



<u>Theorem 3</u>: Let

$$A(z) = 1 + a_1 z + \ldots + a_n z^n \,, \quad a_n \neq 0 \,,$$

have real coefficients. The reciprocal equation,

$$A(z) + A(z^{-1}) = 0 \tag{25}$$

has n root inside the unit circle if and only if

$$A(z) + A(\bar{z}) > 0 \quad \text{for} \quad |z| = 1 \,. \tag{26}$$

Proof:

Assume $A(z) + A(\bar{z}) > 0$ for $|z| = 1$. Since $z^{-1} = \bar{z}$ on the unit circle, the reciprocal equation (25) cannot have any roots on the unit circle. Therefore, it has n roots inside the unit circle (and n reciprocal roots outside the unit circle).

Assume that (25) has n roots inside the unit circle. Form

$$A(z) + A(z^{-1}) = z^{-n} g(z) \tag{27}$$

so that $g(z)$ is a reciprocal polynomial with real coefficients of degree $2n$. Thus it is possible to represent $g(z)$ as follows:

$$g(z) = a_n \prod_{j=1}^{n} (z - r_j)(z - r_j^{-1}) \tag{28}$$

where the r_j's are the roots of $g(z) = 0$ inside the unit circle, i.e., $|r_j| < 1$. Define the polynomial

$$B(z) = \Pi\,(z - r_j) = z^n + b_1 z^{n-1} + \ldots + b_n \tag{29}$$

If any root r_j is complex, then its conjugate is also a root inside the unit circle and included in (29). Therefore, $B(z)$ has real coefficients. Note that

$$b_n = (-1)^n \Pi r_j \tag{30}$$

$$z^{-n} g(z) = a_n B(z) \Pi (1 - z^{-1} r_j^{-1})$$

Therefore,

$$A(z) + A(z^{-1}) \equiv a_n/b_n \; B(z) B(z^{-1}) \tag{31}$$

It follows from (31) and the fact that the b's are real that

$$2 = a_n/b_n \, (1 + b_1^2 + \ldots + b_n^2) \, .$$

This implies that

$$a_n/b_n > 0 \, .$$

Therefore, for $|z| = 1$,

$$A(z) + A(\bar{z}) = a_n/b_n \, |B(z)|^2 > 0 \tag{32}$$

which completes the proof of the theorem. The factor $B(z)$ is uniquely determined by the condition that all of its roots are contained inside the unit disk. We remark that Theorem 3, but not the elementary proof, generalizes to hold for any nonnegative integrable function on the unit circle.[9] The importance of this theorem lies in the fact that it becomes possible to ascertain whether a symmetric band matrix is positive definite in a finite number of steps. One needs only to calculate the roots of the reciprocal polynomial and see whether any have modulus 1 . With the aid of this theorem we can prove an important corollary:

Corollary: If the reciprocal equation (25) has no roots on the unit circle, then the polynomial $A(z) = 0$ has no roots in the closed unit disk.

Proof: If $A(z) = 0$ had a root z_o on the unit circle then $A(z_o) = 0$ and $A(z_o^{-1}) = 0$ so that (25) would vanish for z_o . Since (26) is a harmonic function, it does not vanish inside the unit circle if it is always positive on the unit circle. Therefore, $A(z) = 0$ has no roots in the closed unit disk.

Theorem 3 completes the analysis of (12). We now direct our attention to (11) so that we may obtain algorithms for computing the optimal quantity and price paths. Since (11) is a difference equation of order $2n$, its analysis presents no difficulties. It has a solution

$$q_t = [A(L) + A(\beta E)]^{-1} f_t + \hat{q}_t \tag{33}$$

where

$$[A(L) + A(\beta E)] \hat{q}_t = 0 . \tag{34}$$

Solutions of the homogeneous difference equation (34) are linear combinations of the roots of the characteristic polynomial

$$A(z^{-1}) + A(\beta z) = 0 . \tag{35}$$

Make the change of variable

$$z = w/\beta^{1/2} \tag{36}$$

and obtain the reciprocal equation

$$(37) \qquad A(\beta^{1/2} w) + A(\beta^{1/2} w^{-1}) = 0 .$$

The roots of (35) and (37) are related by the transformation (36). Obviously, (37) has roots inside as well as outside the unit circle. What is less obvious is whether (35) has any roots inside the unit circle. Indeed if all of the roots of (37) lie outside of the disk with radius $\beta^{1/2}$, then there are no roots of (35) inside the unit circle. This in turn would mean that the only bounded solution of (34) is the trivial one, $\hat{q}_t = 0$ for all t. It is, therefore, comforting to have the following:

Theorem 4:

If $A(e^{ix}) + A(e^{-ix}) > 0$ for $-\pi \leq x < \pi$, then (35) has n roots inside the unit circle.

Proof: It is clear from (36) and (37) that we must show that the reciprocal equation (37) has n roots inside the disk with radius $\beta^{1/2}$.

Suppose (37) had a root w_o such that $\beta^{1/2} < |w_o| < 1$. This implies

$$\beta^{1/2}|w_o| < 1 \quad \text{and} \quad \beta^{1/2}|w_o^{-1}| < 1 .$$

Then there would be points $z_o = \beta^{1/2} w_o$ and $z_1 = \beta^{1/2} w_o^{-1}$ inside the unit circle such that $\operatorname{Re} A(z_o)$ and $\operatorname{Re} A(z_1)$ differ in sign. At some point on a line joining them, $\operatorname{Re} A(z) = 0$ which would contradict the hypothesis that the harmonic function $\operatorname{Re} A(z) > 0$ for $|z| = 1$.

At this stage it becomes appropriate to discuss the stability of the difference equations.

Definition:

$$A(L)\, q_t = 0$$

a linear difference equation of order n, is said to be stable if all of the roots of the characteristic polynomial

$$A(z) = 0$$

are outside the unit circle.

The definition implies that a homogeneous difference equation has a bounded non-trivial solution satisfying n arbitrary initial conditions if and only if it is stable. The Corollary to Theorem 3 shows that if the present value has a finite maximum then the demand operator $A(L)$ must be stable because all of the roots of $A(z) = 0$ are outside the unit circle. Similarly, Theorems 3 and 4 imply that the optimal operator $A(L) + A(\beta E)$ has a stable factor. Let

$$C(z) = z^n B(z^{-1}) = 1 + b_1 z + \ldots + b_n z^n \tag{38}$$

so that all of the roots of $C(z) = 0$ are outside the unit circle. According to Theorem 3,

$$A(z) + A(z^{-1}) = (a_n/b_n)\, C(z)\, C(z^{-1})$$

Therefore, substituting the forward shift operator E for z and the lag operator L for z^{-1}, the optimal operator factors as follows:

$$A(E) + A(L) = (a_n/b_n)\, C(E)\, C(L) .$$

Upon introducing the factor β , one readily verifies that

(39) $$A(\beta E) + A(L) = (a_n/b_n)\ C(L)\ C(\beta E)$$

With the aid of (39), the difference equation which the optimal output path must satisfy becomes

(40) $$C(L)q_t = (a_n/b_n)\ C(\beta E)^{-1}\ f_t$$

Since $C(\beta E)^{-1}$ is an infinite series in positive powers of βE , (40) shows that the optimal path responds to the "present value" of the forcing function f_t .

The simplest way of solving (40) exploits the recursive structure of a difference equation. The recursive algorithm has the additional advantage of taking account of the initial conditions in a simple way. When $A(L)$ is n/th order, the optimal quantity path depends on n initial q's which are in general nonzero. This is easily illustrated for $n=2$, a sufficiently general case which avoids cumbersome notation. Let the forcing function f_t have the representation as follows:

$$f_t = h_{1,t} + h_{2,t}$$

where $h_{1,t}$ is an arbitrary function of time of smaller order of magnitude than $\beta^{-t/2}$, that is,

$$h_{1,t} = o\ (\beta^{-t/2})$$

and

$$h_{2,t} = \begin{cases} -b_1 q_{-1} - b_2\ q_{-2} & t=0 \\ -b_2\ q_{-1} & t=1 \\ 0 & t>1 \end{cases}$$

Thus $h_{2,t}$ accounts for the initial levels of quantity. For brevity set

$$s_t = C(\beta E)^{-1} f_t$$

The difference equation (40) can be rewritten as a set of linear equations:

$$q_o = s_o$$

$$q_1 + b_1 q_o = s_1$$

$$q_t + b_1 q_{t-1} + b_2 q_{t-2} = s_t \qquad t > 1$$

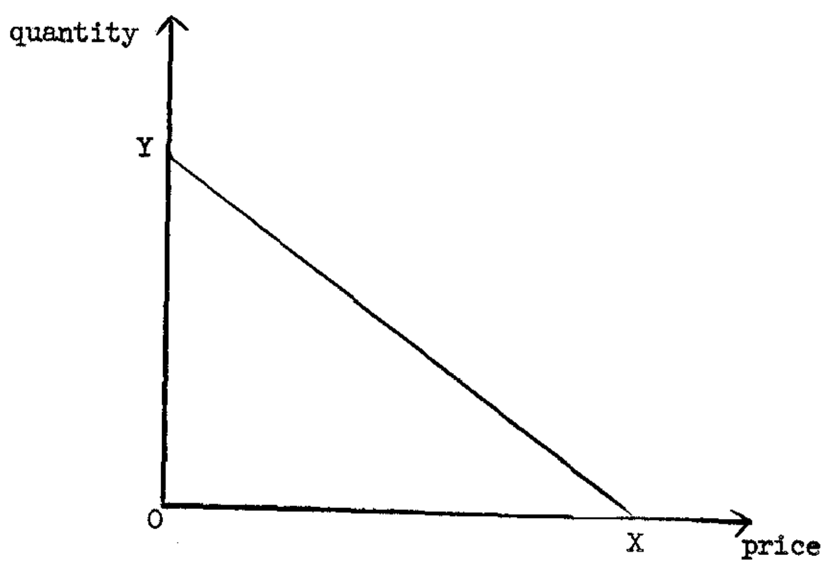
These are readily solved recursively for the q's .

Another advantage of the recursive algorithm is that it facilitates the analysis of nonnegativity constraints on the strategic variables, the q's and p's . It is helpful to begin the discussion of nonnegativity with the original demand equation

$$(41) \qquad A(L)\, q_t = f_t - b\, p_t = y_t$$

In the static case, this function reduces to a linear relation between p and q which is illustrated in Figure 2. The ordinate OY corresponds to f_t and gives the quantity that would be purchased at a zero price. The abscissa OX gives the lowest price that is just high enough to reduce the quantity demanded to zero. A positive quantity is purchased at any positive price in the interior of the interval (0,OX) . We shall assume that these properties also apply to dynamic demand equations such as (41). In other words we assume that if the price at time t is in the interval $0 \leq p_t \leq (1/b)\, f_t$ then the corresponding quantity demanded is nonnegative. Thus we shall require the demand operator A(L) to convert nonnegative inputs into nonnegative outputs. Formally, we state

Figure 2



<u>Definition</u> 2: Let $A(L)\ x_t = y_t$ be a linear difference equation of n^{th} order. $A(L)$ is an unconditionally nonnegative operator if and only if for any bounded nonnegative sequence $\{y_t\}$ there is a bounded nonnegative sequence $\{x_t\}$ which satisfies the difference equation.

Obviously, there are operators which transform <u>some</u> sequences of nonnegative inputs into sequences of nonnegative outputs but which fail to have this property for <u>all</u> nonnegative input sequences. Such operators are conditionally nonnegative. The adjective "conditionally" is a reminder that whether the output sequence is nonnegative depends on the properties of the specific input sequence $\{y_t\}$. For a given operator $A(L)$ it is possible in principle to characterize the space of input sequences that would produce nonnegative output sequences. Such a characterization would lead to the study of the properties of the input sequences $\{y_t\}$ for the given $A(L)$ which would give rise to nonnegative outputs. For some operators $A(L)$ there may not be any input sequences which would yield nonnegative outputs. Our analysis poses a different problem. Given the set of all bounded nonnegative sequences, it seeks to characterize the class of operators $A(L)$ which yield bounded nonnegative outputs. On economic grounds this appears to be the more relevant problem. The requirement that bounded inputs produce bounded outputs implies that the operator must be stable.

<u>Lemma 4</u>: If $A(L)$ is unconditionally nonnegative (u.n.n.) then it is stable.

Proof:

Consider the special input sequence $\{\delta_{ot}\}$ where $\delta_{ot} = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise} \end{cases}$

By hypothesis the sequence $\{x_t\}$ which satisfies

(42) $$A(L)\, x_t = \delta_{ot} \,, \quad t \geq 0$$

is bounded and nonnegative. Define the power series

(43) $$A(z)^{-1} = \Sigma\, \gamma_t\, z^t \,.$$

Obviously, $A(z)\, \Sigma\, \gamma_t\, z^t = 1$ so that the coefficients of (43) must satisfy (42). That is,

$$x_t = \gamma_t$$

Since γ_t is a linear combination of all of the roots of $A(z^{-1}) = 0$ and is bounded, all of the roots of $A(z^{-1}) = 0$ must lie inside the unit circle. Therefore, all of the roots of $A(z) = 0$ are outside the unit circle and $A(L)$ is stable.

In the course of this proof we have virtually shown

Lemma 5: The operator $A(L) = 1 + a_1\, L + \ldots + a_n\, L^n$ is unconditionally nonnegative if and only if γ_t is nonnegative, where γ_t is the coefficient of z^t in (43).

Proof:

Since as we have seen

(44) $$A(L)\, \gamma_t = \delta_{ot} \,,$$

if $A(L)$ is u.n.n. then $\gamma_t \geq 0$. Sufficiency is obvious.

It would be desirable to have an algorithm for calculating whether a given operator $A(L)$ is u.n.n. in a finite number of steps. The two preceding lemmas suggest that the reciprocal $A(z)^{-1}$ should be examined for any negative coefficients. However, this would require inspection of an infinite number of coefficients. Since γ_t satisfies (44), it would be better to have criteria of nonnegativity in terms of the n roots of $A(z) = 0$, the n coefficients, $a_1, \ldots, a_n$ or a judicious combination. Fortunately, there are some useful necessary conditions that can shorten the labor of determining whether a given operator is u.n.n. To derive these necessary conditions we exploit the relation between unconditional nonnegativity and characteristic functions of discrete valued random variables.

Definition 3: The characteristic function of the discrete valued random variable ξ where

$$P(\xi = u) = d_u$$

$$d_u > 0 \quad \text{and} \quad \Sigma\, d_u = 1$$

is defined by

$$f(x) = \Sigma\, e^{iux}\, d_u$$

where x is a real variable.[10]

A theorem of Bochner asserts that $f(x)$ is a characteristic function if and only if it is a nonnegative function. That is, the hermitian matrix

$$[f(x_u - x_v)]\ , \quad u, v = 1, 2, \ldots, m$$

is positive semi-definite for any points $x_1, \ldots, x_m$.[11] Thus a characteristic function has properties as follows:

(i) $f(0) = 1$

(ii) $f(-x) = \overline{f(x)}$

(iii) $|f(x)| \leq f(0)$

Lemma 6: $A(L)$ is unconditionally nonnegative if and only if

$$f(x) = [\sigma\ A(e^{ix})]^{-1} \tag{45}$$

is the characteristic function of a discrete valued random variable ξ such that

$$P(\xi = t) = \sigma\ \gamma_t \geq 0 \tag{46}$$

for some $\sigma > 0$.

Proof:

Assume that $A(L)$ is u.n.n. By Lemma 4, $A(z) = 0$ has no roots inside the closed unit disk and by Lemma 5 $\gamma_t \geq 0$ for all $t \geq 0$. Therefore, $A(z)^{-1}$ converges on the closed unit disk. Set $\sigma = A(1) > 0$ and $f(x)$ is the desired characteristic function. The converse is obvious since $\sigma\ \gamma_t \geq 0$ and $\sigma > 0$ imply $\gamma_t \geq 0$. We note also that $\sigma < 1$ since $\gamma_o = 1$ and $\Sigma\ \gamma_t > 1$.

An unconditionally nonnegative operator $A(L)$ possesses several important properties and we collect these in the following:

Theorem 5: If $A(L)$ is unconditionally nonnegative then (i) the smallest root of $A(z) = 0$ must be positive, (ii) $a_1 < 0$, (iii) $|A(e^{ix})| > A(1)$, (iv) $0 < A(1) < 1$, (v) strict inequality obtains in (iii) unless $x = 0$.

Proof:

(i) The asymptotic behavior of γ_t where γ_t is the coefficient of z^t in the power series $A(z)^{-1}$ is governed by the reciprocal of the roots of smallest modulus of $A(z) = 0$. If the smallest root (or roots) of $A(z) = 0$ were negative then γ_t would change sign contradicting the hypothesis that $A(L)$ is u.n.n. Next suppose that the roots of the smallest modulus of $A(z) = 0$ were complex. Thus there would be a pair of roots

$$r_1 = \rho\, e^{ix} \quad \text{and} \quad \bar{r}_1 = \rho\, e^{-ix} \quad -\pi \leq x \leq \pi .$$

For large enough t,

$$\gamma_t \sim \rho^t \,[e^{i(tx + \theta)} + e^{-i(tx + \theta)}]$$

or in real terms,

$$\gamma_t \sim 2\rho^t \cos\,(tx + \theta)\,.$$

In order that $\cos(tx + \theta) \geq 0$ for all t, we must have

$$-\pi/2 + 2m\,\pi \leq tx + \theta \leq \pi/2 + 2m\,\pi$$

where m depends on t and is integral.

Therefore, x must satisfy the following:

$$2\,\pi\, m/t - (1/t)\,(\pi/2 + \theta) \leq x \leq (1/t)(\pi/2 - \theta) + 2\,\pi\, m/t \quad \text{for all } t = 1, 2, \ldots$$

Therefore, $2\,\pi\, m/t \leq x \leq 2\,\pi\, m/t$. This is satisfied only by $x = 0$. Thus the roots must be real and hence positive.

(ii) Since $\gamma_1 = -a_1 > 0$, this proves (ii). Cf. (44).

(iii) By Lemma 6, the hypothesis implies that $A(e^{ix})^{-1}$ is proportional to a characteristic function so that $|A(e^{ix})^{-1}| \leq A(1)^{-1}$. Therefore,

$$|A(e^{ix})| \geq A(1) \quad \text{for all} \quad -\pi \leq x \leq \pi .$$

(iv) Obviously, $A(1) > 0$. In Lemma 6 we observed that

$$\Sigma \gamma_t > 1$$

But $A(1)^{-1} = \Sigma \gamma_t$. Hence $A(1) < 1$.

(v) The random variable for which $A(e^{ix})^{-1}$ is the characteristic function, takes on integral values with probabilities proportional to γ_t. Therefore, the span of the distribution is 1 and strict inequality obtains in (iii) for all $x \neq 0$ in the interval $-\pi \leq x \leq \pi$.[12] This completes the proof.

One of the implications of Theorem 5 is that at most a finite number of terms in the reciprocal of $A(z)$ requires examination to ascertain unconditional nonnegativity because eventually γ_t is dominated by the largest root of $A(z^{-1}) = 0$ which must be positive. Unfortunately, no conditions are known to us which characterize completely either the roots of $A(z) = 0$ or the coefficients for an unconditionally nonnegative operator. It is however easy to verify two important sufficient conditions. First, if $a_1, a_2, \ldots, a_n$ are all negative then $A(L)$ is unconditionally nonnegative. This follows easily from (44). Second, if all of the roots of $A(z) = 0$ are positive, then $A(L)$ must be u.n.n. This follows from the more general proposition that the product of characteristic functions is a characteristic function. If all of the roots of $A(z) = 0$ are positive then

$$A(z) = \Pi (1 - r_j z) \qquad 0 < r_j < 1$$

and each factor $(1 - r_j z)$ is proportional to the characteristic function of a simple random variable distributed according to a binomial law so that the product must also be a characteristic function.

The discussion of unconditional nonnegativity refers so far to the demand operator $A(L)$ which contains only positive powers of L. Hence the reciprocal $A(z)^{-1}$ converges in the closed unit disk. The optimal operator, $[A(L) + A(\beta E)]$ refers to the difference equation which the optimal output path satisfies.

(20) $$[A(L) + A(\beta E)]\, q_t = f_t$$

The corresponding function $[A(z^{-1}) + A(\beta z)]$, where z^{-1} replaces L and z replaces E contains both negative as well as positive powers of z so that its reciprocal converges in an annulus where it can be represented by a Laurent series. The definition of unconditional nonnegativity can be extended to apply to the optimal operator. Since it is assumed that the optimal operator is such that the maximum present value is finite, the optimal output path satisfies

(40) $$C(L)\, q_t = (a_n/b_n)\, C(\beta E)^{-1} f_t$$

Definition 2 applies to $C(L)$ and the operator $C(L)$ is unconditionally nonnegative if and only if every coefficient of $C(\beta E)^{-1}$ is nonnegative. Moreover, Theorem 5 applies to $C(L)$ if it is u.n.n.

It is convenient at this point to defend our emphasis on u.n.n. operators. The optimal quantity path is required to be nonnegative because the firm is not allowed to repurchase goods from its customers. It would be more natural perhaps to maximize the present value of receipts subject to the nonnegativity constraints on q_t explicitly. If this is done then (11) requires only slight modification since for $q_t = 0$, the permissible variation of d_t must be positive which implies that the optimal output path must satisfy

$$f_t - [A(L) + A(\beta E)]\, q_t \leq 0$$

in order that neighboring sequences have a lower P.V. It would, therefore, become necessary to analyze these inequalities to determine whether there is a nonnegative sequence which would satisfy them. It is obvious that there is always a solution when $f_t \geq 0$ and $f_t = 0$ for $t > T$. This path of study would lead to an investigation of those sequences $\{f_t\} \geq 0$ which would yield solutions to the difference inequalities. However, if $C(L)$ is u.n.n. then for any nonnegative f_t there exists a nonnegative sequence $\{q_t\}$ such that (40) is satisfied. In other words, unconditional nonnegativity is a sufficient condition for the existence of nonnegative solutions. Indeed, it is sufficient to ensure that equality and not merely inequality holds Therefore, u.n.n. of $C(L)$ implies that a linear decision rule maximizes P.V. Unfortunately, even the analysis of unconditional nonnegativity is quite difficult and we can only report partial results on the relations between u.n.n. of the demand operator $A(L)$ and the optimal operator $C(L)$.

First, there is the following useful theorem:

Theorem 6: If $A(L)$ is unconditionally nonnegative and $A(e^{ix}) + A(e^{-ix}) > 0$ for all $-\pi \leq x \leq \pi$ then

$$|C(z)| < \sqrt{\frac{\Sigma b_u^2}{\Sigma a_u}}\, |A(z)| \quad \text{for all } |z| < 1 . \tag{47}$$

and

$$0 < \sum_{o}^{n} a_u < \sum_{o}^{n} b_u^2 , \quad a_o = b_o = 1 \tag{48}$$

where $C(z)$ is defined in (38).

Proof:

The hypothesis ensures that $A(z) = 0$ has no roots in the closed unit disk by the corollary to Theorem 3 and that by Theorem 3

$$A(z) + A(z^{-1}) = (a_n/b_n)\, C(z)\, C(z^{-1})$$

is a valid representation such that $C(z) = 0$ has no roots in the closed unit disk. By Lemma 6 $A(e^{ix})^{-1}$ is proportional to a characteristic function. Hence

$$A(e^{ix})^{-1} = \Sigma\, \gamma_t\, e^{ixt} , \quad \gamma_t \geq 0 \quad \text{for all} \quad t = 0, 1, \ldots$$

Therefore,

$$A(e^{ix})^{-1} + A(e^{-ix})^{-1} = [A(e^{ix}) + A(e^{-ix})] \,/\, |A(e^{ix})|^2 \tag{49}$$

is also a characteristic function subject to multiplication by a certain positive normalizing constant. Applying the factorization and property (iii) of characteristic functions,

$$A(e^{ix})^{-1} + A(e^{-ix})^{-1} = (a_n/b_n)\, |C(e^{ix})|^2 / |A(e^{ix})|^2 \leq 2/A(1) \tag{50}$$

Since

$$(a_n/b_n)\ (1 + b_1^2 + \ldots + b_n^2) = 2 \quad \text{and} \quad A(1) = \Sigma\, a_u$$

we obtain

$$|C(e^{ix})| < \sqrt{\frac{\Sigma b_u^2}{\Sigma a_u}}\ |A(e^{ix})| \ . \tag{51}$$

Since both $A(z)$ and $C(z)$ are zero-free in the closed unit disk, both $\log |A(z)|$ and $\log |C(z)|$ are harmonic functions in the closed unit disk. Hence, we may represent $\log |A(z)|$ as follows:

$$\log |A(z)| = (1/2\pi) \int \operatorname{Re}\left(\frac{e^{ix} + 2}{e^{ix} - 2}\right) \log |A(e^{ix})|\, dx \tag{52}$$

for all $|z| < 1$, and similarly for $\log C(z)$. Since $\operatorname{Re}\left(\frac{e^{ix} + 2}{e^{ix} - 2}\right)$ is positive, the inequality in (51) extends to the entire unit disk and we obtain (47).[13]

It follows that

$$\log |C(z)| < (1/2)\ m + \log |A(z)|$$

where $m = (\Sigma b_u^2)\ /\ (\Sigma a_u) > 0$

Since $C(0) = A(0) = 1$, $\log m > 0$, $m > 1$ and (48) holds.

This theorem establishes a connection between the demand and optimal polynomials on the hypothesis that the demand operator is unconditionally nonnegative and permits a finite maximum present value. If in addition $C(L)$ is u.n.n., then Theorems 5 and 6 together are helpful in deciding for a given problem whether the nonnegativity conditions are automatically satisfied.

The following rather special theorem serves the purposes of delineating the difficulties of generalizing about unconditional nonnegativity.

Theorem 7: Given that

$$A(L) + A(E) = (a_2/b_2)\ C(L)\ C(E)$$

where both $A(L)$ and $C(L)$ are quadratics with real coefficients, unconditional nonnegativity of $C(L)$ implies u.n.n. of $A(L)$.

Proof:

Denote the roots of $C(z) = 0$ by s_j^{-1} and of $A(z) = 0$ by r_j^{-1} where $|s_j| < 1$ and $|r_j| < 1$, $j = 1, 2$. Since $C(L)$ is u.n.n., its largest root is positive and therefore both roots are real.

$$A(z) + A(z^{-1}) = (a_2/b_2)\ C(z)\ C(z^{-1})$$

we can equate coefficients of like powers of z thereby obtaining

$$(1/a_2)\ a_1 = (1/b_2)\ [b_1 + b_1\ b_2] = -\ [(s_1 + s_2)/\ s_1\ s_2 + (s_1 + s_2)]$$

Therefore,

$$a_1/a_2 = -\ [(1/s_1) + (1/s_2) + s_1 + s_2]$$

In addition,

$$2/a_2 = (1/b_2)\ [1 + b_1^2 + b_2^2]$$

Since $C(L)$ is u.n.n., both roots of $C(z^{-1}) = 0$ must be real and their sum is positive. Therefore, there are either two positive roots or a positive and a smaller (in absolute value) negative root. First assume there are two positive roots so that $b_2 > 0$ and $b_1 < 0$. Clearly,

$$\Sigma\, 1/s + \Sigma\, s > 0 \quad \text{so that} \quad a_1/a_2 < 0 .$$

Since a_2 has the same sign as b_2, $a_1 < 0$ and $A(z^{-1}) = 0$ has two positive roots. Hence $A(L)$ is u.n.n. Next assume that $b_2 < 0$ so that one of the roots of $C(z) = 0$ is negative and necessarily $b_1 < 0$. In this case,

$$\Sigma\, 1/s + \Sigma\, s < 0$$

since it is dominated by the smaller and negative root of $C(z^{-1}) = 0$. Hence $a_1/a_2 > 0$, $a_2 < 0$, and $a_1 < 0$. Therefore, $A(z^{-1}) = 0$ has two real roots whose sum is positive which implies that in this case also $A(L)$ is u.n.n.

The converse to Theorem 7 is false. To see why consider this simple counter-example. Let $A(L) = (1 - r\,L)^2$ where $0 < r < 1$. Obviously

$$(1 - rz)^2 + (1 - r\,z^{-1})^2 = 0$$

has no real roots so that $C(L)$ cannot be u.n.n. because $C(z) = 0$ has a pair of complex conjugate roots.

It is also not possible to extend Theorem 7 to polynomials of higher degree. For example, consider cubics. As before we denote the roots of $A(z^{-1}) = 0$ by r_j and of $C(z^{-1}) = 0$ by s_j where the r's and s's are all inside the unit disk. Equate coefficients of equal powers of z using

$$A(z) + A(z^{-1}) = (a_3/b_3)\; C(z)\; C(z^{-1})$$

and obtain the following:

$$a_2/a_3 = (1/b_3)\ [b_2 + b_1\ b_3] = -\ [\Sigma\ s_1\ s_2\ /\ s_1\ s_2\ s_3 + \Sigma\ s]$$
$$= -\ [\Sigma\ 1/s + \Sigma\ s]$$

($\Sigma\ s_1\ s_2$ means the sum over all possible distinct pairs)

$$a_1/a_3 = (1/b_3)\ [b_1 + b_1\ b_2 + b_2\ b_3]$$
$$= \Sigma\ 1/s_1\ s_2 + \Sigma\ s_1\ s_2 + \Sigma\ s\ \Sigma\ 1/s$$
$$2/a_3 = 1/b_3\ [1 + b_1^2 + b_2^2 + b_3^2]$$

Assume that $C(L)$ is u.n.n. and has three real roots the largest of which is positive. Therefore, b_1 , b_2 , and $b_3 < 0$. Since a_3 has the same sign as b_3 , $a_3 < 0$. If the two negative roots are small then

$$\Sigma\ 1/(s_1\ s_2) + \Sigma\ s_1\ s_2 + \Sigma\ s\ \ \Sigma\ 1/s\ < 0 .$$

Therefore, $a_1/a_3 < 0$ and $a_1 > 0$ so that $A(L)$ cannot be u.n.n.

Even if all of the roots of $C(z) = 0$ are positive, it is possible to construct an example in which only one of the roots of $A(z) = 0$ is positive, the other two roots are complex conjugates, and $A(L)$ is not u.n.n. Hence there do not seem to be any neat general results linking unconditional nonnegativity of the optimal and demand operators.

As noted above, nonnegativity of the output path is required because the firm is not allowed to repurchase the commodity from its customers. That is, if the solution gave rise to some negative q_t's , this would imply that repurchases were required and without analysis of the customers' inventory levels, this procedure would not make any sense. Hence only nonnegative output paths can be optimal. However, the optimal price sequence could include some negative prices. Negative prices would mean that the firm finds it desirable to make some sales at prices below cost in order to maximize its long run profits. To see how this can come about we proceed to obtain the optimal price path. The demand equation is given by

$$A(L)\, q_t = f_t - b\, p_t$$

and the optimal output path is the bounded solution of

$$[A(L) + A(\beta E)]\, q_t = f_t \ ,$$

Therefore,

$$A(\beta E)\, q_t = b\, p_t$$

Eliminating q_t ,

$$A(\beta E)\, A(L)^{-1}\, [f_t - b\, p_t] = b\, p_t$$

which reduces to

$$(53) \qquad [A(L) + A(\beta E)]\, p_t = (1/b)\, A(\beta E)\, f_t$$

Hence the optimal price path is the bounded solution of (53). In factored form,

$$(53') \qquad C(L)\, p_t = (a_n/b_n)\, (1/b)\, A(\beta E)\, C(\beta E)^{-1}\, f_t \ .$$

Even if $C(L)$ is u.n.n. so that every coefficient of $C(\beta E)^{-1}$ is nonnegative, and even if $A(L)$ is u.n.n., at least one coefficient of $A(\beta E)$, namely $a_1 < 0$, so that the right side of (53') may be negative for some t. Hence the optimal price path may require some prices to be negative. This is more likely to be true the closer the discount factor is to one and the larger in absolute value is a_1. The latter condition could arise when the largest and positive root of $A(z^{-1}) = 0$ is close to one so that the customers of the firm have a strong propensity to repeat their purchases of the good.

The next theorem compares the stabilizing behavior of the demand and optimal operators.

<u>Theorem 8</u>: Let s_1 denote the largest root in modulus inside the unit circle of the reciprocal equation $A(z) + A(z^{-1}) = 0$ and assume that the reciprocal equation has no roots on the unit circle. If s_1 is real there is a real root of $A(z^{-1}) = 0$ denoted r_1 such that

$$|s_1| < |r_1| < 1$$

Proof:

By hypothesis, $A(s_1) + A(s_1^{-1}) = 0$. Since the reciprocal equation has no roots on the unit circle, $A(z)$ has no roots in the closed unit disk by the corollary to Theorem 3. Therefore, $A(z) > 0$ for all real z such that $|z| < 1$. Therefore,

$$A(s_1) > 0 \quad \text{and} \quad A(s_1^{-1}) < 0$$

Since $A(1) > 0$, there must be a real root r_1^{-1} of $A(z) = 0$ such that

$$1 < |r_1^{-1}| < |s_1^{-1}|$$

which proves the theorem.

This theorem has several important implications. First, if the optimal operator is u.n.n., then s_1 is positive so that the demand operator must also have a positive root r_1 such that $0 < s_1 < r_1$. This implies that the transient component of the optimal output path damps out more quickly than the transient component of the demand equation. Thus if the optimal operator is u.n.n. it is asymptotically stabilizing relative to the demand operator. It should be noted that it is not necessarily true that the largest root of $A(z^{-1}) = 0$ is real since the theorem merely asserts that there is at least one real root of this equation which exceeds the largest real root of the reciprocal equation inside the unit circle.

It can be shown by a counter-example that if the largest root of the reciprocal equation inside the unit circle is complex then its modulus can exceed that of the largest root of $A(z^{-1}) = 0$. Thus unconditional non-negativity of the optimal operator is a necessary but not a sufficient condition for the asymptotic stability of the optimal operator relative to the demand operator.

In concluding the analysis of the single product problem, it is appropriate to discuss the discount factor and the stability of the demand operator. At the outset the stipulation of a finite present value of profits limited the class of admissible forcing functions, f_t, to those whose growth rate did not exceed $\beta^{1/2}$. In money terms this implies that profits may increase over time at a rate less than the discount factor. It makes more economic sense to accept as a given the growth rate of f_t and then assume that this determines the least discount factor that will bring about a finite present value. Thus the rate of growth of f_t can be completely arbitrary, since it fixes the discounting factor. However, there is no discount factor which will convert an unstable demand operator into a stable one. In other words we assume that the internal response of the system, that is the demand operator $A(L)$, must be stable and we allow growth only via the forcing function f_t.

III. The Multi-Product Case

We now assume that the monopolist wishes to maximize the present value of the revenue arising from the sale of k products for which the demand is represented by a system of k linear difference equations of order n. Thus write the demand equations as follows:

$$p_t = f_t - [A_o q_t + A_1 q_{t-1} + \dots + A_n q_{t-n}] \tag{1}$$

where

$p_t = k \times 1$ vector of prices

$q_t = k \times 1$ vector of quantities

$f_t = k \times 1$ vector of arbitrary functions of time

$A_u = k \times k$ matrix of real constants, $u = 0, 1, \dots, n$.

The set of equations represented by (1) corresponds to (II.1) (Section II, equation 1). Using the lag operator L, (1) becomes

$$p_t = f_t - A(L)\, q_t \tag{2}$$

where $A(L)$ is a polynomial matrix and each of the k^2 elements of $A(L)$ is a polynomial in L of at most degree n. That is,

(3) $A(L) = [f_{hj}(L)]$, $h,j = 1, \ldots, k$, and $f_{hj}(L)$ is a polynomial.[14]

Another way of representing $A(L)$ shows more clearly its polynomial character.

(4) $$A(L) = A_0 + A_1 L + . . . + A_n L^n .$$

Since each coefficient of $A(L)$, A_u , is a $k \times k$ matrix, the polynomial matrix $A(L)$ is said to be of order k . Hence $A(L)$ is a polynomial matrix of order k and degree n . The degree is determined by the highest power of L which has a non-null matrix coefficient. In addition we define the polynomial matrix in the complex <u>scalar</u> z as follows:

(5) $$A(z) = A_0 + A_1 z + . . . + A_n z^n .$$

<u>Definition 4</u>: The matrix polynomial $A(z)$ given by (5) is of full rank if and only if $\det A(z)$ does not vanish identically.

It follows that if $A(z)$ is of full rank then the polynomial equation

(6) $$\det A(z) = 0$$

has a finite number of roots and the polynomial matrix is singular only for those values of z that equal the roots of (6).

From now on all of the matrix polynomials considered are assumed to be of order k and of full rank.

A matrix power series is an expression of the form

(7) $$M(z) = \Sigma M_u z^u$$

where each of the k^2 elements converges in some disk. The disk of convergence of $M(z)$ is determined by the smallest disk of convergence of each of the k^2 elements of $M(z)$.

Definition 5: A recurrent matrix power series of the form (7) is a power series whose matrix coefficients satisfy a recurrence relation of the form

$$(8) \qquad M_{t+n} + R_1 M_{t+n-1} + \ldots + R_n M_t = 0 , \quad t \geq 0 .$$

Although systems of difference equations have much in common with single difference equations, they present certain complications which deserve our attention. The solution of a system of difference equations is handled most expeditiously by means of rational matrix functions. Let

$$(9) \qquad P(z) = P_o + P_1 z + \ldots + P_{n-1} z^{n-1}$$

Definition 6: $Y(z)$ is said to be a proper rational matrix function of the complex scalar z if it can be represented as follows:

$$(10) \qquad Y(z) = P(z) A(z)^{-1}$$

provided both A_o and P_o are nonsingular and $A(z)$ and $P(z)$ are matrix polynomials of degree n and at most $n-1$ defined, respectively, in (5) and (9). A rational matrix function is the sum of a polynomial matrix and a proper rational matrix function. We now prove that every rational matrix can be represented as a recurrent power series.[15]

Lemma 7: A necessary and sufficient condition that a matrix power series in z should be a recurrent matrix power series is that it should be the expansion of a proper rational matrix function of z.

Proof: Sufficiency.

$$A(z)^{-1} = \text{adj } A(z) / \det A(z)$$

$$= \text{adj } A(z) / \Pi (1 - r_j z)$$

where adj $A(z)$ means the adjoint of $A(z)$ and r_j^{-1} denotes the j/th root of $\det A(z) = 0$. The adjoint of $A(z)$ is an ordinary matrix polynomial and $\det A(z)$ is a scalar polynomial. Therefore, $A(z)^{-1}$ can be represented in a matrix power series by means of a partial fraction expansion of the denominator $\det A(z)$. The power series is absolutely convergent in the disk

$$|z| < |r_1^{-1}| \leq |r_j^{-1}|$$

Hence $P(z)\, A(z)^{-1}$ is also a power series which converges in the same disk as $A(z)^{-1}$ and

(11) $$Y(z) = \Sigma\, Y_t\, z^t = P(z)\, A(z)^{-1}$$

is a formal identity in z . Form

(12) $$P(z) = Y(z)\, A(z) ,$$

equate the matrix coefficients of equal powers of z and obtain

(13)
$$\begin{aligned} Y_o A_o &= P_o \\ Y_1 A_o + Y_o A_1 &= P_1 \\ &\;\;\cdots \\ Y_{n-1} A_o + \ldots + Y_o A_{n-1} &= P_{n-1} \end{aligned}$$

(14) $$Y_{t+n} A_o + \ldots + Y_t A_n = 0 , \quad t > 0 .$$

The latter, written more concisely is

(14') $$A(L)\, Y_{t+n} = 0$$

This proves that $Y(z)$ is a recurrent power series.

Conversely, if the coefficients of the power series

$$Y(z) = \Sigma\, Y_t\, z^t$$

satisfy a recurrence relation of the form (14), then form the product

$$Y(z)\, A(z)$$

and observe that all of the terms of degree $t > n$ vanish by virtue of (14) so that $Y(z)$ has a representation (12) and is, therefore, a proper rational matrix function.

The system (14) unlike (1) is a difference equation in the matrices Y_t. Hence it includes as special cases difference equations in vectors such as (1). We may now prove

Theorem 9: If $A(z)$ is nonsingular for all $|z| \leq 1$ then

$$A(L)\, Y_{t+n} = 0\,, \quad t \geq 0$$

has a non-trivial bounded solution such that Y_t is a certain linear combination of the powers of the distinct roots of

$$\det A(z^{-1}) = 0$$

Proof: By hypothesis, the matrix power series

$$Y(z) = \Sigma\, Y_t\, z^t$$

obeys a recurrence relation so that there is a representation of $Y(z)$

$$Y(z) = P(z)\, A(z)^{-1} = P(z)\, \operatorname{adj} A(z) / \Pi\, (1 - r_j\, z)$$

A partial fraction expansion of the right side shows that Y_t is a certain linear combination of powers of r_j which will be more or less complicated depending on whether the roots are repeated or simple. The solution can be bounded only if

$$|r_j| < 1$$

for all j so that $Y(z)$ converges in the disk $|z| < |r_1^{-1}| \leq |r_j^{-1}|$ which includes the unit circle.

Just as for a scalar difference equation, a homogeneous difference equation which has a bounded non-trivial solution is said to be stable. Hence the hypothesis of the preceding theorem could have required $A(L)$ to be a stable matrix operator in place of $A(z)$ being nonsingular on the closed unit disk. The general difference equation

$$A(L)\, Y_t = M_t \tag{15}$$

where M_t is a sequence of matrices has a solution given by

$$Y_t = A(L)^{-1}\, M_t \tag{16}$$

The simplest computational procedure is to use the recursions displayed in (13).

For our subsequent development we shall need certain additional facts about matrix polynomials. The first is too simple to require formal statement in a theorem. The polynomial

$$\det A(z) = 0$$

has a root $z = 0$ if and only if A_o is singular. This is clear since

$$\det A(0) = \det A_o = 0$$

and the second equality holds if and only if A_o is singular.

Theorem 10: A necessary and sufficient condition for a k-order n/th degree polynomial matrix to have kn singularities is that the matrix coefficient A_n, be nonsingular.[16]

Proof: Sufficiency. Assume A_n is nonsingular so that

$$A_n^{-1} A(z) = M_o + M_1 z + \ldots + M_{n-1} z^{n-1} + I z^n$$

Therefore,

$$A_n^{-1} A(z) = [g_{hj}(z)]$$

where every element on the diagonal is of degree n and every element off the diagonal is of degree n-1 at most. Hence

$$\det A_n^{-1} A(z) = 0$$

is a polynomial equation of degree kn and A(z) has kn singularities.

To prove necessity assume that $\det A(z) = 0$ has kn roots and suppose that A_n is singular of rank $r < k$. There exist nonsingular scalar matrices P and Q such that

$$P A_n Q = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where I_r is the $r \times r$ identity matrix. Therefore,

$$P A(z) Q = [g_{hj}(z)]$$

has r polynomials on the diagonal of degree n and $k-r$ on the diagonal of at most degree $n-1$. All of the off-diagonal polynomials are at most of degree $n-1$.

$$\det P\,A(z)\,Q = \det A(z) = 0$$

is a polynomial of degree at most

$$rn + (k-r)(n-1) = kn - (k - r) < kn$$

which contradicts the hypothesis that $\det A(z) = 0$ has kn roots.

In the course of proving this theorem it was shown that if the rank of A_n is $r \leq k$, then the maximum degree of the polynomial $\det A(z)$ equals

$$kn - (k - r)$$

If in addition A_o is nonsingular, then the minimum degree is kr . These results can be improved by taking into consideration the ranks of the other matrix coefficients but the complications are considerable and the matter is not worth pursuing. The degree of the $\det A(z)$ gives the number of arbitrary constants which enter the solution of the system of difference equations. It should be noted, however, that

$$\text{(i) } \det A(z) = 0 \text{ , and (ii) } \det A(z^{-1}) = 0$$

do not necessarily have the same degree. If A_o is nonsingular then (ii) will be of degree kn as shown by Theorem 10. However, (i) will not be of degree kn unless A_n is nonsingular. If A_n is singular then some of the roots of (ii) will equal zero. Therefore, (i) and (ii) will be of the same degree if and only if A_n is nonsingular or, equivalently, all of the roots of (ii) are nonzero.

These brief remarks about systems of difference equations must suffice and we return to the main problem of maximizing the present value of revenue for the multi-product monopolist. The present value of revenue is defined by

$$(17) \qquad P.V. = \Sigma\, \beta^t\, q_t'\, p_t$$

This differs from the corresponding expression for the single product case only in that the revenue of period t is the inner product $q_t'\, p_t$. It would be unnecessarily tedious to repeat all of the steps which led to Theorem 1 except to point out that in the multi-product case Lemma 1 would require restatement as follows:

$$(18) \qquad \Sigma\, \beta^t\, q_t'\, A(L)\, d_t = \Sigma\, \beta^t\, d_t'\, A'(\beta E)\, q_t$$

and it should be noted that the operator $A(L)$ becomes transposed into $A'(\beta E)$. For the multi-product case the counterpart to Theorem 1 is

<u>Theorem 11</u>: In order that the bounded sequence of vectors $\{q_t\}$ maximize P.V. uniquely, it is necessary and sufficient that q_t be the bounded solution of

$$(19) \qquad [A(L)+ A'(\beta E)]\, q_t = f_t$$

and that

$$(20) \qquad \Sigma\, \beta^t\, d_t'\, A(L)\, d_t > 0$$

for any bounded sequence $\{d_t\}$ not identically zero.

$A(L)$ is the matrix operator defined in (4).

The conditions on A_u , $u = 0, 1, \ldots, n$ which guarantee that (20) is satisfied is a matrix generalization of the Herglotz Lemma and Bochner's Theorem. The results are stated here for polynomials although they can be extended to cover matrix power series. As in Section II we give our results for a complex version of (10). If p_t and q_t were permitted to be complex, the maximand (17) would be

$$\Sigma \beta^t q_t^* p_t$$

and (20) would become

$$\Sigma \beta^t d_t^* A(L) d_t > 0 \tag{20*}$$

where the * denotes the complex conjugate so that $d_t^* = \bar{d}_t'$. Clearly, if (20^*) is positive for complex d_t then (20) is positive for real d_t .

It is interesting to note that the choice of $d_t = 0$ for $t \geq 1$ in (20^*) implies that A_o must be positive definite <u>and</u> symmetric as is easily verified. However, condition (20) implies only that A_o must be positive definite, not that it must be symmetric as well. Thus the extension to complex sequences narrows the class of admissible matrix operators at least with respect to A_o .

By analogy with equation II.18, define

$$A_{-u} = A_u' \tag{21}$$

The elements of the matrices A_u are always assumed to be real. If the A's were allowed to be complex then A_{-u} would be the complex conjugate of A_u . It is readily verified that

$$\Sigma\, \beta^t\, d_t^*\, A(L)\, d_t = (1/2)\left(\Sigma\, \beta^{t/2}\, d_t^*\, [A(\beta^{1/2}\, L) + A'(\beta^{1/2}\, E)]\, \beta^{t/2}\, d_t\right) \tag{22}$$

(compare with Lemma 2), and the latter can be written out as follows:

$$(1/2)\left[\Sigma\, \beta^{u/2}\, d_u^*\, \beta^{|u-v|/2}\, A_{u-v}\, \beta^{v/2}\, d_v + \Sigma\, \beta^{u/2}\, d_u^*\, A_o\, \beta^{u/2}\, d_u\right] \tag{23}$$

(Cf. II.14'). We used here the fact that A_o is symmetric. The band matrix of the form given by (22) is illustrated for $n = 2$.

$$\begin{bmatrix} 2\,A_o & \beta^{1/2}\,A_1 & \beta\,A_2 & 0 & \cdots \\ \beta^{1/2}\,A_{-1} & 2\,A_o & \beta^{1/2}\,A_1 & \beta\,A_2 & \cdots \\ \beta\,A_{-2} & \beta^{1/2}\,A_{-1} & 2\,A_o & \beta^{1/2}\,A_1 & \cdots \\ & \cdots & & & \end{bmatrix} \tag{24}$$

The theorem which follows refers to a finite sum of the form (23).

<u>Theorem 12</u>: The form

$$(1/2)\left[\sum_{u,v=1}^{S} d_u^*\, A_{u-v}\, d_v + \sum_{u}^{S} d_u^* + A_o\, d_u\right] \tag{25}$$

is positive for all choices of finite integers $S > 0$ and finite sequences $\{d_u\}$ not identically zero, if and only if for $|z| = 1$, the form

$$\eta^*\, [A'(\bar{z}) + A(z)]\, \eta > 0 \tag{26}$$

for any vector $\eta \neq 0$.

Proof:

Define

(27) $$W(x) = A(e^{ix}) + A^*(e^{ix})$$

where $z = e^{ix}$.[17] Taking the Fourier transform of $W(x)$ gives

(28) $$A_u = (1/2\pi) \int W(x)\, e^{-iux}\, dx , \qquad u = 1, \ldots, n$$

and

(29) $$A_o = (1/4\pi) \int W(x)\, dx .$$

The integration of a matrix is understood to be done by individual elements, over the interval $[-\pi, \pi]$.

Hence (25) can be written as follows:

(30) $$(1/4\pi) \left[\sum_{u,v=0}^{S} d_u^* \left(\int W(x)\, e^{-i(u-v)x} dx \right) d_v \right]$$
$$= (1/4\pi) \int \left(\sum_{u=0}^{S} d_u^*\, e^{-iux} \right) W(x) \left(\sum_{v=0}^{S} d_v\, e^{ivx} \right) dx$$

Hence if (25) is positive then (30) must be positive which proves necessity.

Conversely, assume that (30) is positive for all finite integers $S > 0$ and sequences $\{d_u\}$. For arbitrary x and nonzero η we wish to prove that

(31) $$\eta^* W(x)\, \eta > 0 .$$

It is sufficient to show that

(32) $$(1/N\pi) \int \; [(\sin N(s - x))/(s - x)]^2\, \eta^* W(x)\, \eta\, dx$$
$$= \int F(s - x)\, \eta^* W(x)\, \eta\, dx > 0$$

since for large N , (32) approximates (31). Let the Fourier series of $F(s)$ be

(33) $$F(s) = \Sigma_{-\infty}^{\infty} c_h e^{ihs}$$

where the c_h are scalars. Since F is real and even, $c_h = c_{-h}$ and c_h is real. The series can be approximated uniformly by

(34) $$\Sigma_{-S}^{S} c_h e^{ihs}$$

and the latter in turn can be written as

(35) $$\left(\Sigma_{o}^{S} g_u e^{-ius}\right)\left(\Sigma_{o}^{S} g_u e^{ius}\right)$$

To obtain the representation given by (35) in which the g_u are complex scalars, it is necessary to factor (34) as a reciprocal equation on the unit circle as shown by (35). In more detail write (34) as follows:

(36) $$\Sigma_{-S}^{S} c_h z^h$$

If r is real and $z-r$ is a factor of (36), then $z^{-1} - r$ is also a factor. If r is complex and $z - r$ is a factor then so are $z - \bar{r}$, $z^{-1} - r$, and $z^{-1} - \bar{r}$. This verifies the representation (35). Now choose

$$d_u = g_u \eta .$$

This gives the desired result that (32) is arbitrarily close to a positive quantity.

The theorem gives necessary and sufficient conditions for the finite sum (25) to be positive. However, the maximization is taken over an infinite sequence. Hence it is necessary to obtain conditions that ensure the positivity of

(37) $$\Sigma\, \beta^{u/2}\, d_u^*\, \beta^{|u-v|/2}\, A_{u-v}\, \beta^{r/2}\, d_v + \Sigma\, d_u^*\, A_o\, d_u \;.$$

For $\beta = 1$, (37) is the same as (23). The argument used to prove Lemma 3 applies to the multi-product case and we state

Corollary: The form (37) is positive for all bounded nonzero sequences $\{d_u\}$ if and only if (26) is positive for all nonzero vectors η .

It is possible to extend Theorem 2 to the multi-product case.

Theorem 13: The form (23) is positive for all bounded sequences $\{d_u\}$ and $0 < \beta \leq 1$ if and only if $W(x)$ defined in (27) is positive definite for $-\pi \leq x \leq \pi$.

Proof:

First, notice that for any vector η , the scalar inner product

(38) $$\eta^* \,[A(z) + A^*(z)]\, \eta = 2\mathrm{Re}\, \eta^*\, A(z)\, \eta \;.$$

Therefore, the inner product is a harmonic function of x .

To prove sufficiency we assume that $W(x)$ is positive definite for all x . Therefore, (38) is positive on the unit circle and being a harmonic function attains its extrema on the boundary of the closed unit disk. Hence (28) is positive for all $|z| \leq 1$. Since any point inside the unit circle can be represented by $\beta^{1/2}\, e^{ix}$, this completes the proof of necessity. Conversely, if (23) is positive for all $0 < \beta \leq 1$, then Theorem 12 implies that (26) must hold.

Part of Theorem 3 also generalizes to vectors.

<u>Theorem 14</u>: $W(x) = A(e^{ix}) + A^*(e^{ix})$ is positive definite for $-\pi \leq x \leq \pi$ if and only if

$$A(z) + A'(z^{-1})$$

is nonsingular for all $|z| = 1$ and

(39) $$\eta^* [A(0) + A'(0)] \eta > 0$$

for all vectors $\eta \neq 0$.

Proof:

Evidently, $A(z) + A'(z^{-1})$ and $W(x)$ are the same for all $|z| = 1$. If $W(x)$ is positive definite then it must be nonsingular so that the reciprocal equation

(40) $$\det [A(z) + A'(z^{-1})] = 0$$

cannot have any roots on the unit circle. Since $\eta^*[A(z) + A^*(z)] \eta$ is a harmonic function in the closed unit disk and positive on the boundary, it is also positive in the interior. Thus (39) is true.

Conversely, assume that (40) has no roots on the unit circle and that (39) holds. We must show that for all x and $\eta \neq 0$

$$\eta^* W(x) \eta > 0$$

Suppose the contrary; that is for some z_o with $|z_o| = 1$

(41) $$\eta^* [A(z_o) + A'(z_o^{-1})] \eta < 0$$

Now for $|z| = 1$, (41) has the same values as (38). Since (38) is harmonic it must take a value for $|z| = 1$ at least as large as its value for $z = 0$. Thus it must be positive for some z_1 with $|z_1| = 1$. Thus (41) is positive for $z = z_1$ and negative for $z = z_0$ and must vanish for some z with $|z| = 1$. This gives a contradiction since (40) has no roots on the unit circle.

Corollary: If $W(x) = A(e^{ix}) + A^*(e^{ix})$ is positive definite for all $-\pi \leq x \leq \pi$ then the matrix operator $A(L)$ is stable.

Proof:

The matrix operator $A(L)$ is stable if and only if the matrix polynomial $A(z)$ is nonsingular for all $|z| \leq 1$. Since the positive definiteness of $W(x)$ implies

$$\text{Re}\ \eta^* A(z)\, \eta > 0$$

for all $|z| \leq 1$, it follows that the only solution of

$$A(z)\, \eta = 0 \quad \text{for all} \quad |z| \leq 1$$

is the trivial one $\eta = 0$.

In Theorem 3 it was shown that the reciprocal polynomial could be factored into the product of two polynomials such that one of the factors has no roots inside the unit circle and the other factor's roots are the reciprocals of the roots of the first factor. It is considerably more difficult to factor the matrix $A'(z) + A(z^{-1})$. Moreover, the problem of factoring $W(x)$ is of considerable importance in function theory and is the subject of considerable current research by mathematicians. The existence of factorizations for positive semi-definite $W(x)$ has been shown in two papers, the first by Wiener and Masani and the second by Helson and Lowdenslager.[18] However, if $W(x)$ is not of full rank, so that $\det W(x) = 0$ identically, then the existence of factors has been shown only for the special case of $k = 2$ by Masani and Wiener.[19] For general $W(x)$ not of full rank it is not known whether factors exist. Even if $W(x)$ is positive semi-definite so that factors exist, algorithms for calculating the factors are not known. As far as we know it is possible to calculate the factors of $W(x)$ only when $A(e^{ix})$ is a polynomial matrix and $W(x)$ is positive definite and not merely positive semi-definite. In our factorization we follow a suggestion of Whittle.[20]

<u>Lemma 8</u>: Let $W(x) = A(e^{ix}) + A^*(e^{ix})$ where

$$A(e^{ix}) = A_o + A_1\, e^{ix} + \ . \ . \ . + A_n\, e^{inx}$$

and assume that $W(x)$ is positive definite for all $-\pi \leq x \leq \pi$.

If $W(x)$ can be factored such that

$$W(x) = B(e^{ix})\, B^*(e^{ix})$$

where $B(z)$ is a matrix polynomial then (i) $B(z)$ is nonsingular for all $|z| \leq 1$ (ii) $A(z)$ and $B(z)$ must be of the same degree and the matrix coefficients of $B(z)$ must be real.[21]

Proof:

Again we exploit the fact that

$$\eta^*[A(z) + A^*(z)]\, \eta$$

is a harmonic function which must be positive on the closed unit disk because the positive definiteness of $W(x)$ implies it is positive on the boundary. Therefore, if a factorization by matrix polynomials exists then

$$\eta^* B(z) B^*(z) \eta > 0$$

is also a harmonic function for all $|z| \leq 1$ and $B(z)$ must be nonsingular for all $|z| \leq 1$. In particular,

$$\det B(0) = \det B_o \neq 0 .$$

To prove the second assertion suppose that the degree of $B(z)$ is m and that $m > n$. Since

$$A(e^{ix}) + A^*(e^{ix}) = B(e^{ix}) B^*(e^{ix})$$

we can multiply out the right side and equate the matrix coefficients of like powers of e^{ix}. Specifically,

$$B_o B_m^* = 0$$

since $A(e^{ix})$ is of degree n. Hence $B_m^* = 0$ since B_o is nonsingular. Similarly,

$$B_m^* = B_{m-1}^* = . . . = B_{n+1}^* = 0$$

which proves that $B(z)$ must be of degree n as asserted. That the B's are real follows from the maintained hypothesis that the matrix coefficients of $A(z)$ are real.

<u>Lemma 9</u>: If $W(x)$ is positive definite then

(42) $$[A(z) + A'(z^{-1})]^{-1} = \Sigma_1^\infty G_{-u} z^{-u} + G_o' + G_o + \Sigma_1^\infty G_u z^u$$

is the uniquely defined Laurent expansion for the annulus

$$|r| < |z| < |r^{-1}|, |r| < 1$$

where $|r|$ is the largest modulus of the roots of the reciprocal equation

(43) $$\det [A(z) + A'(z^{-1})] = 0$$

that are inside the unit circle, and the G_u's are certain linear combinations of $r_j^{|u|}$ which denote the distinct roots of (43) inside the unit circle. In addition G_u is real and $G_{-u} = G_u'$.

Proof:

The positive definiteness of $W(x)$ and Theorem 14 imply that the reciprocal equation (43) has no roots on the unit circle. Hence the inverse of $A(z) + A'(z^{-1})$ has a representation in an annulus including the unit circle as follows:

(44) $$[A(z) + A'(z^{-1})]^{-1} = \text{adj}[A(z) + A'(z^{-1})]/ \det[A(z) + A'(z^{-1})]$$
$$= \text{adj}[A(z) + A'(z^{-1})]/ \Pi (1 - r_j z) (1 - r_j z^{-1})$$

and $|r_j| < 1$. The denominator of (34) has a partial fraction expansion in terms of $(1 - r_j z)$ and $(1 - r_j z^{-1})$ (and their powers in case of repeated roots) while the numerator is a matrix polynomial in powers of z and z^{-1}. The representation of (42) is an immediate consequence of the expansion of (44) by means of partial fractions. The matrix coefficients are uniquely determined for the annulus

(45) $$|r_1| < |z| < |r_1^{-1}| \quad \text{where} \quad |r_j| \leq |r_1| < 1 .$$

It is clear from the construction of the Laurent series that G_u is a certain linear combination of all $r_j^{|u|}$. A_u is real implies G_u is real. To prove that

(46) $$G_u' = G_{-u}$$

take transposes in (42) and substitute z^{-1} for z . Since the result is an identity in z and z^{-1} by the uniqueness of the Laurent series representation, the matrix coefficients of like powers of z must be identical and this yields (46).

By the method with which the G_u can be constructed it follows that the matrix coefficients of the Laurent series obey a certain recurrence relation (see definition 5). Thus for $t \geq 0$

(47) $$[A(L) + A'(E)]\, G_t = \Delta_{ot}$$

where Δ_{ot} is a matrix generalization of the Kronecker delta so that

$$\Delta_{ot} = \begin{cases} I & t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In brief, G_t is the bounded solution of

(48) $$[A(L) + A'(E)]\, G_t = 0 \qquad t \geq n$$

subject to the initial conditions

(49) $$[A(L) + A'(E)]\, G_t = \Delta_{ot} \qquad 0 \leq t < n$$

where

$$L^t\, G_{t-u} = G_{-u} = G'_u \qquad u \geq 0$$

The problem of factoring the optimal operator reduces to that of calculating the difference equation of least degree which will generate the matrix coefficients of the Laurent expansion. Recalling that $E = L^{-1}$, (48) is a difference equation of degree equal to $2n$ and G_t is its bounded solution so that it is a certain linear combination of the roots of the characteristic polynomial which are inside the unit circle. Since there are n initial conditions as shown in (49) we seek the nth degree difference equation which meets these given initial conditions. Provided $W(x)$ is positive definite it is possible to give a constructive proof of the factorability of $A(z) + A'(z^{-1})$.

<u>Theorem 15</u>: Assume that $A(z) + A'(z^{-1})$ is nonsingular for $|z| = 1$. First, there is a polynomial matrix $B(z)$ with real matrix coefficients

(50) $$B(z) = B_o + B_1 z + \ldots + B_n z^n, \qquad \det B_o \neq 0$$

such that

(51) $$[A(z) + A'(z^{-1})]^{-1} B(z)$$

contains no positive powers of z if and only if $C_1, \ldots, C_n$ satisfies the equations as follows:

$$
(52) \quad
\begin{aligned}
G_1 + (G_o + G_o')\, C_1 &+ \quad . \quad . \quad . \quad + G_{-n+1} \quad C_n = 0 \\
G_2 + \quad G_1\, C_1 &+ (G_o + G_o')C_2 + \ldots + G_{-n+2} \quad C_n = 0 \\
&\quad . \quad . \quad . \\
G_n + \quad G_{n-1}C_1 &+ \ . \ . \ . \qquad + (G_o + G_o')\, C_n = 0
\end{aligned}
$$

$$
(53) \qquad G_{t+n} + G_{t+n-1}C_1 + \ . \ . \ . \ + G_t\, C_n = 0 \ , \ t > 0
$$

where

$$
(54) \qquad C_j = B_j\, B_o^{-1}
$$

Second, B_o is determined up to multiplication by an arbitrary orthogonal matrix, $C_1, \ldots, C_n$ is the unique solution of (52), and

$$
(55) \qquad A(z) + A'(z^{-1}) = B(z)\ B'(z^{-1})
$$

where $B(z)$ is nonsingular for all $|z| \leq 1$.

Proof:

By the preceding lemma, the hypothesis that $[A(z) + A'(z^{-1})]$ is nonsingular for $|z| = 1$, assures the absolute convergence of the uniquely determined Laurent series in the annulus defined in (45). Clearly, (52) through (54) hold if and only if there is a n/th degree polynomial matrix such that

$$
(56) \qquad [A(z) + A'(z^{-1})]^{-1}\, B(z) = D(z^{-1})
$$

where $D(z^{-1})$ is a power series in nonpositive powers of z . Therefore, it is only required to show there is a non-trivial set of $C_1, \ldots, C_n$ that will satisfy (52) and (53).

By the method of constructing the matrix coefficients of the Laurent series (42), it is clear that the power series

(57) $$G(z) = G_o' + G_o + \Sigma_1^\infty G_u z^u$$

is a recurrent series which is absolutely convergent in the disk

$$|z| < |r_1^{-1}|$$

Therefore, by Lemma 7, $G(z)$ is a rational matrix function of the scalar z and there are polynomial matrixes

(58) $$C(z) = I + C_1 z + \ldots + C_n z^n$$

(59) $$P(z) = P_o + P_1 z + \ldots + P_{n-1} z^{n-1}$$

such that

(60) $$G(z)\ C(z) = P(z)\ .$$

It only remains to choose the matrix coefficients of $P(z)$ to satisfy (52) and (53). The following choice of P_u accomplishes the purpose:

$$P_o = -[G_{-1} C_1 + G_{-2} C_2 + \ldots + G_{-n} C_n]$$

$$P_1 = -[G_{-1} C_2 + \ldots + G_{-n+1} C_n]$$

$$\ldots$$

$$P_{n-1} = -[G_{-1} C_n]$$

As remarked above, the degrees of the polynomials $P(z)$ and $C(z)$ are fixed by the requirement that n initial conditions must be satisfied.

The hypothesis that $[A(z) + A'(z^{-1})]$ is nonsingular on the unit circle permits the application of Theorem 14 and bringing to bear Theorem 12

$$[G_{u-v}] , \quad u,v = 0, 1, \ldots, n-1$$

is positive definite so that there is a unique set of non-trivial real $C_1, \ldots, C_n$ which satisfies (52).

We have shown that $D(z^{-1})$ defined in (56) contains no positive powers of z . Take transposes in (56) and substitute z^{-1} for z . This yields

$$B'(z^{-1}) [A'(z^{-1}) + A(z)] = D'(z) . \tag{61}$$

Multiply (56) on the left by $B'(z^{-1})$ and (61) on the right by $B(z)$ and obtain

$$B'(z^{-1}) D(z^{-1}) = D'(z) B(z) \tag{62}$$

which is an identity in z . Since the right side has no negative powers of z and the left side no positive powers of z ,

$$D'(z) B(z) = K \tag{63}$$

where K is an arbitrary nonsingular matrix of constants. Therefore,

$$D(z^{-1}) = B'(z^{-1})^{-1} K \tag{64}$$

and substituting into (56) this yields,

(65) $$A(z) + A'(z^{-1}) = B(z)\, K^{-1}\, B'(z^{-1})$$

Clearly, K must be positive definite but is otherwise arbitrary. It follows that K can be factored by the classical algebraic tools into the product of a matrix and its transpose. Therefore, (65) is equivalent to the desired representation (55).

Since $B(z)$ is a polynomial factor, Lemma 8 applies and shows that $B(z)$ must be nonsingular on the closed unit disk. This is evident directly from the fact that (53) is a stable difference equation. Write

$$[A(z) + A'(z^{-1})]^{-1}\, B(z) = B'(z^{-1})^{-1}$$

and equate the coefficients of the constant term. This gives

$$(G_o + G_o')\, B_o + G_{-1}B_1 + G_{-2}B_2 + \dots + G_{-n}B_n = B_o'^{-1}$$

which reduces to

(66) $$G_o + G_o' + G_{-1}C_1 + \dots + G_{-n}\, C_n = (B_oB_o')^{-1}$$

Since the G's are the uniquely determined coefficients of the Laurent expansion and the C's are the unique solutions of (52), the matrix product B_oB_o' is uniquely determined. Let M be any orthogonal matrix.

$$B_oB_o' = B_o\, MM'\, B_o \quad \text{since} \quad MM' = I\,.$$

Therefore, B_o can be any nonsingular matrix and is fixed only up to multiplication by an arbitrary orthogonal matrix.

We now reintroduce β and establish the factorization pertinent to the problem:

<u>Corollary 1:</u> If $A(z) + A'(z^{-1})$ is nonsingular for all $|z| = 1$, then the optimal matrix operator $A(L) + A'(\beta E)$ can be factored so that

(67) $$A(L) + A'(\beta E) = B'(\beta E)\ B(L)$$

where $B(z)$ is the factor with all of the properties established in Theorem 15.

Proof:

Consider the matrix polynomial

(68) $$A'(\beta z) + A(z)$$

which corresponds to the optimal matrix operator $A'(\beta E) + A(L)$ after the substitution of z for E and z^{-1} for L. Let

$$z = w/\beta^{1/2}$$

which permits the rewriting of (68) as a reciprocal polynomial matrix

(69) $$A'(\beta^{1/2}\, w) + A(\beta^{1/2}\, w^{-1})\ .$$ [22]

The hypothesis allows application of Theorem 14 so that

$$A(z) + A'(z^{-1})$$

is positive definite for all $|z| = 1$. Since

$$\eta^*[A(z) + A^*(z)]\eta$$

is harmonic for $|z| \leq 1$ and is positive on the boundary, (69) is positive definite for $|w| = 1$ and Theorem 15 applies to yield the representation

(70) $$A'(\beta^{1/2} w) + A(\beta^{1/2} w^{-1}) = B'(\beta^{1/2} w)\, B(\beta^{1/2} w^{-1})$$

$B'(\beta^{1/2} w)$ is of degree n and is nonsingular for all $|w| \leq 1$ and all $0 < \beta \leq 1$. Change back from w to z via $z = w/\beta^{1/2}$ to obtain

(71) $$A'(\beta z) + A(z^{-1}) = B'(\beta z)\, B(z^{-1}) .$$

The preceding remark shows that $B'(z)$ is necessarily nonsingular in the closed unit disk. Finally using the correspondence of z for E and z^{-1} for L, (71) implies (67).

The next corollary corresponds to Theorem 4.

Corollary 2: If $A(z) + A'(z^{-1})$ is nonsingular on the unit circle, then

(72) $$\det\,[A(z^{-1}) + A'(\beta z)] = 0$$

has no roots in the annulus

$$\beta^{1/2} < z < 1/\beta^{1/2}$$

Proof:

Let $z = w/\beta^{1/2}$ and transform (72) into the following reciprocal polynomial;

(73) $$\det\,[A'(\beta^{1/2} w) + A(\beta^{1/2} w^{-1})] = 0$$

Clearly, if (72) has a root in the annulus, then (73) has a root r such that

(74) $$\beta^{1/2} < |r| < 1$$

Therefore, there would be a nonzero vector η such that

$$[A'(\beta^{1/2} r) + A(\beta^{1/2} r^{-1})] \eta = 0$$

However, by hypothesis

$$A(\beta^{1/2} w) + A^*(\beta^{1/2} w)$$

is positive definite for all $|w| \leq 1$ and $0 < \beta \leq 1$. Since $|r \beta^{1/2}| < 1$ and $|\beta^{1/2} r^{-1}| < 1$, the assumption of the existence of a root r which satisfies (74) would contradict the hypothesis of positive definiteness. Hence there can be no roots in the annulus

$$\beta^{1/2} < |z| < 1/\beta^{1/2}$$

Summing up, if the demand operator is such that there is a finite maximum present value, then the optimal operator is factorable. Hence the optimal output path must satisfy

$$B(L) q_t = B'(\beta E)^{-1} f_t . \quad (75)$$

In addition $B(L)$ is stable and it is possible to calculate the optimal output path recursively by means of (75). The characteristic polynomial of $B(L)$ is

$$\det B(z^{-1}) = 0 .$$

Consider

$$z^{kn} \det B(z-1) = 0 \quad (76)$$

Since B_o is nonsingular, Theorem 10 applies and the degree of (76) is kn . However, some of the roots of (76) can be zero. There are zero roots of (76) if and only if B_n is singular. Moreover, B_n is singular if and only if A_n

is singular. Therefore, singularity of A_n implies that

(77) $$\det B(z) = 0$$

is of lower degree than (76) and has fewer roots. This possibility was mentioned above in the analysis of matrix difference equations. In general, A_n can be expected to be singular which would mean that the components of q_t obey difference equations of diverse degrees. For example, it is possible for some components of q_t to satisfy trivial difference equations of zero degree, that is, to respond to price without any lag whatever.

Finally, there is the problem of nonnegativity of q_t. Although it is more complicated for the multi-product case, given the hypothesis of unconditional nonnegativity of the matrix operator, many of the results for scalar difference equations apply to vectors. In particular, the definition of unconditional nonnegativity becomes

<u>Definition 7</u>: The matrix operator $A(L)$ is unconditionally nonnegative provided for any bounded nonnegative sequence $\{\eta_t\}$

$$A(L)\ \xi_t = \eta_t$$

has a bounded nonnegative solution ξ_t.

Lemma 5 generalizes to matrix operators. Thus $A(L)$ is u.n.n. if and only if the recurrent matrix power series

(78) $$A(z)^{-1} = \Sigma\ D_t\ z^t$$

has nonnegative matrix coefficients. Therefore, every element of every D_t

is nonnegative. Since

$$A(z)^{-1} = \text{adj } A(z) \; / \; \det A(z)$$

$A(L)$ is u.n.n. if and only if every element of $A(z)^{-1}$ is a characteristic function, or, more precisely, a positive constant times a characteristic function. Since the denominator of every element of $A(z)^{-1}$ is the same, namely $\det A(z)$, Theorem 5 applies to the rational functions of $A(z)^{-1}$ which are the elements of $A(z)^{-1}$. Thus every one of the k^2 elements of $A(z)^{-1}$ has the properties of a characteristic function. In particular, the smallest root of $\det A(z) = 0$ must be positive and $A_1 < 0$. There are two sufficient conditions for u.n.n. matrix operators. If $A_1, \ldots, A_n$ have nonpositive elements or if all of the roots of $\det A(z) = 0$ are positive then $A(L)$ is u.n.n.

A variant of Theorem 6 carries over for vectors. Thus if $A(L)$ is u.n.n. and $A(z) + A'(z^{-1})$ is nonsingular on the unit circle, then every element of

$$\begin{aligned} A(e^{ix})^{-1} + A^*(e^{ix})^{-1} &= A(e^{ix})^{-1} \, [A^*(e^{ix}) + A(e^{ix})] \, A^*(e^{ix})^{-1} \\ &= A(e^{ix})^{-1} \, B^*(e^{ix})B(e^{ix}) \, A^*(e^{ix})^{-1} \end{aligned}$$

is a characteristic function and is subject to certain bounds in modulus similar to those described in Theorem 6.

It is remarkable that even the stabilization property of Theorem 8 carries over to vectors. This fact is worth a formal statement.

<u>Theorem 16</u>: Assume that $A(z) + A'(z^{-1})$ is nonsingular on the unit circle so that

$$A(z) + A'(z^{-1}) = B'(z)\; B(z^{-1})$$

Let s_1 denote the largest root of

$$\det B(z^{-1}) = 0 .$$

If s_1 is real then there is a real root of

$$\det A(z^{-1}) = 0$$

such that

$$|s_1| < |r_1| < 1$$

Proof:

Since s_1 is real, there is a real vector $\eta \neq 0$ such that

$$[A(s_1) + A'(s_1^{-1})]\; \eta = 0 .$$

Therefore,

$$\eta'[A(s_1) + A'(s_1^{-1})]\; \eta = 0 .$$

By the positive definiteness of $A(z) + A^*(z)$ for all $|z| \leq 1$,

$$\eta'[A(s_1) + A'(s_1)]\; \eta > 0$$

so that

$$\eta'\; A(s_1)\; \eta > 0 .$$

Therefore,

$$\eta' A'(s_1^{-1}) \eta < 0 \quad \text{and} \quad \eta' A(s_1^{-1}) \eta < 0 .$$

Thus there is a point r_1^{-1} on the line joining s_1 and s_1^{-1} with

$$\eta' A(r_1^{-1}) \eta = 0 .$$

Now $|r_1^{-1}| > 1$ since $\eta' A(r) \eta > 0$ for $|r| \leq 1$.

Thus r_1^{-1} is a singularity of $A(z)$ and

$$1 < |r_1^{-1}| < |s_1^{-1}|$$

This proves the theorem. Therefore, if the optimal operator is u.n.n. so that its largest root inside the unit circle must be positive, then it is asymptotically stabilizing relative to the demand operator.

IV. Conclusions.

We have presented necessary and sufficient conditions for the solution of the following problem:

$$\max \ \Sigma \beta^t p_t' q_t \quad \text{subject to} \quad A(L) q_t = f_t - p_t .$$

These conditions must be satisfied by the optimal sequence $\{q_t\}$ and by the matrix operator $A(L)$ in order that the problem have a finite maximum.
In addition it is possible to give sufficient conditions for $A(L)$ in order that the optimal sequence q_t be nonnegative for any nonnegative sequence $\{f_t\}$.

An interesting generalization of this problem would allow the operator $A(L)$ to possess an infinite number of terms. Thus

$$A(L) = \sum_{o}^{\infty} A_u L^u$$

For example, the operator might be

$$A(L) = I - G\, e^{-C(L)}$$

where $C(L)$ is a polynomial in L . Many of our results could be expected to carry over to such analytic operators although the proofs would require the use of more powerful theorems of function theory. Factoring, however, could not be carried out except possibly for special cases since only algorithms for rational functions are known.

FOOTNOTES

1. This research is part of a project analyzing competition which is being supported by a National Science Foundation grant, GS-365, to the University of Chicago.

2. For one of the earliest formal statements of a closely related problem see Evans [3].

3. The standard references on linear decision rules are Holt, Modigliani, Muth and Simon [10] and Theil [13].

4. In the subsequent development we shall occasionally find it helpful to display b explicitly and this will always be clear from the context.

5. Hence revenue (in money terms) may grow at a rate which is less than the discount rate. See also p. 34.

6. For a statement and proof of the Herglotz Lemma and Bochner's Theorem, see Loève [11] pp. 207-210. See also Bochner [2] p. 326.

7. We use herein a well known result on characteristic functions stated in a Corollary, Loève [11] p. 188.

8. A harmonic function is the real part of an analytic function. See Ahlfors [1] for a fine exposition of the properties of harmonic functions. The unit disk is the interior of the unit circle; the closed unit disk is the closure of the unit disk.

9. See Helson and Lowdenslager [7] I. and Hoffman [9] especially Chapter 4.

10. For the basic properties of characteristic function of discrete valued random variables, see Gnedenko and Kolmogorov [5] pp. 55-61.

11. Bochner [2] pp. 325-328.

12. A span is the difference between two successive values of a discrete valued random variable. In the theorem the span is one. See Gnedenko and Kolmogorov [5] for an extensive discussion.

13. We use Poisson's formula. See Ahlfors [12] pp. 179-181 and 184.

14. See Frazer, Duncan and Collar [4] for a discussion of some properties of polynomial matrices.

15. The definition of a recurrent matrix power series and Lemma 7 generalize Hardy's results for a scalar power series, Hardy [6] pp. 392-393.

16. A partial statement and proof of what amounts to Theorem 10 is given in Frazer, Duncan and Collar [4] pp. 162-163.

FOOTNOTES (Cont.)

17. It simplifies notation to define $A^*(e^{ix})$ as the conjugate transpose of $A(e^{ix})$. The conjugate transpose of $A(z)$ is obtained by transposing the polynomial elements of $A(z)$ and substituting $\bar{z}$ for z .

18. See Wiener and Masani [15] and Helson and Lowdenslager [7]. There are also pertinent articles in Russian which we have not examined. A non-technical summary of the current status of the problem by Lowdenslager is given in the Appendix to Yaglom [15]. For a technical exposition of the most recent developments see Helson [8].

19. Masani and Wiener [12].

20. Whittle suggests the use of the Yule-Walker relations to calculate the factors, Whittle [14] p. 101. However, he gives no precise statement of the conditions which would allow the use of these relations and does not prove the validity of the algorithm. On a charitable interpretation his statement at the bottom of p. 100 is obscure. The Yule-Walker relations are our equations (52) and are well-known in regression theory.

21. Although our Lemma 8 sufficies for our purposes, a stronger result is given in Theorem 20, Helson [8].

22. In our problem we wish to factor $A'(z) + A(z^{-1})$ whereas Theorem 14 refers to $A(z) + A'(z^{-1})$. Clearly, the first is factorable if and only if the second is.

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