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TESTABLE RESTRICTIONS ON THE EQUILIBRIUM MANIFOLD

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by

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Abstract: We present a finite system of polynomial inequalities in unobservable variables and market data that observations on market prices, individual incomes and aggregate endowments must satisfy to be consistent with the equilibrium behavior of some pure trade economy. Quantifier elimination is used to derive testable propositions on finite data sets for the pure trade model.

Keywords: General equilibrium, nonparametric restrictions, quantifier elimination
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1. Introduction

The core of the general equilibrium research agenda has centered around questions of existence and uniqueness of competitive equilibria and stability of the price adjustment mechanism. Despite the resolution of these concerns, i.e., the existence theorem of Arrow and Debreu; Debreu's results on local uniqueness; Scarf's example of global instability of the tâtonnement price adjustment mechanism; and the Sonnenschein–Debreu–Mantel theorem, general equilibrium theory continues to suffer the criticism that it lacks falsifiable implications or, in Samuelson's terms, "meaningful theorems."

Comparative statics analysis is the primary source of testable restrictions in economic theory. This mode of analysis is most highly developed within the theory of the household and theory of the firm, e.g., Slutsky's equation, Shepard's lemma, etc. As is well known from the Sonnenschein–Debreu–Mantel theorem, the Slutsky restrictions on individual excess demand functions do not extend to market excess demand functions. In particular, as shown by Mas-Colell [12], utility maximization subject to a budget constraint imposes no testable restrictions on the set of equilibrium prices. The disappointing attempts of Walras, Hicks and Samuelson to derive comparative statics for the general equilibrium model are chronicled in Inagro and Israel [11], Moreover, there has been no substantive progress in this field since Arrow and Hahn's discussion of monotone comparative statics for the Walrasian model [5].

If we denote the market excess demand function as $F_{\widehat{w}}(p)$ where the profile of individual endowments $\widehat{w}$ is fixed but market prices p may vary, then $F_{\widehat{w}}(p)$ is the primary construct in the research on existence and uniqueness of competitive equilibria, the stability of the price adjustment mechanism, and comparative statics of the Walrasian model. A noteworthy exception is the monograph of Balasko [6] who addressed these questions in terms of properties of the equilibrium manifold. To define the equilibrium manifold we denote the market excess demand function as $F(\widehat{w}, p)$, where both $\widehat{w}$ and p may vary. The equilibrium manifold is defined as the set $\{(\widehat{w}, p) | F(\widehat{w}, p) = 0\}$. Contrary to the result of Mas-Colell, cited above, we shall show that utility maximization subject to a budget constraint does impose testable restrictions on the equilibrium manifold. Hence, general equilibrium theory does have implications that are falsifiable.

To this end we consider an alternative source of testable restrictions within economic theory: the nonparametric analysis of revealed preference theory as developed by Samuelson, Houthakker, Afriat, Richter, Diewert, Varian and others for the theory of the household and the theory of the firm. For us, the seminal proposition in this field is Afriat's theorem [1] for data on prices and consumption bundles. Recall that Afriat, using the Theorem of the Alternative, proved the equivalence of a finite family of linear inequalities — now called the Afriat inequalities – that contain unobservable utility levels and marginal utilities of income; his axiom of revealed preference, "cyclical consistency" — finite families of linear inequalities that contain only observables (i.e., prices and consumption bundles); and the existence of a concave, continuous monotonic utility function rationalizing the observed data. The equivalence of the Afriat inequalities and cyclical consistency is an instance of a deep theorem in model theory, the Tarski–Seidenberg theorem on quantifier elimination.

The Tarski-Seidenberg theorem — see Van Den Dries [15] for an extended discussion — proves that any finite system of polynomial inequalities can be reduced to an equivalent finite family of polynomial inequalities in the coefficients of the given system. They are equivalent in the sense that the original system of polynomial inequalities has a solution if and only if the parameter values of its coefficients satisfy the derived family of polynomial inequalities. In addition, the Tarski–Seidenberg theorem provides an algorithm which, in principle, can be used to carry out the elimination of the unobservable — the quantified — variables, in a finite number of steps. Each time a variable is eliminated, an equivalent system of polynomial inequalities is obtained, which contains all the variables except those that have been eliminated up to that point. The algorithm terminates in one of three mutually exclusive and exhaustive states: (i) $1 \equiv 0$, i.e., the original system of polynomial inequalities is never satisfied; (ii) $1 \equiv 1$, i.e., the original system is always satisfied; (iii) an equivalent finite family of polynomial inequalities in the coefficients of the original system that is satisfied only by some parameter values of the coefficients.

To apply the Tarski–Seidenberg theorem, we must first express the structural equilibrium conditions of the pure trade model as a finite family of polynomial inequalities. Moreover, to derive equivalent conditions on the data, the coefficients in this family of polynomial inequalities must be the market observables — in this case, individual endowments and market prices — and the unknowns must be the unobservables in the theory — in this case, individual utility levels, marginal utilities of income, and consumption bundles. A family of equilibrium conditions having these properties consists of the Afriat inequalities for each agent; the budget constraint of each agent; and the market clearing equations of each observation. Using the Tarski–Seidenberg procedure to eliminate the unknowns must therefore terminate in one of the following states: (i) $1 \equiv 0$ — the given equilibrium conditions are inconsistent, (ii) $1 \equiv 1$ — there is no finite data set that refutes the model, or (iii) the equilibrium conditions are testable.

Unlike Gaussian elimination — the analogous procedure for linear systems of equations — the running time of the Tarski-Seidenberg algorithm is in general not polynomial and in the worst case can be doubly exponential. See the volume edited by Arnon and Buchberger [4] for more discussion of the complexity of the Tarski–Seidenberg algorithm. Fortunately, it is often unnecessary to apply the Tarski–Seidenberg algorithm to determine if the given equilibrium theory has testable restrictions on finite data sets. It suffices to show that the algorithm cannot terminate with $1 \equiv 0$ or with $1 \equiv 1$. In fact, as we shall show, this is the case for the pure trade model.

It follows from the Arrow–Debreu existence theorem that the Tarski–Seidenberg algorithm applied to this system will not terminate with $1 \equiv 0$. In the next section, we construct an example of a pure trade model where no values of the unobservables are consistent with the values of the observables. Hence the algorithm will not terminate with $1 \equiv 1$. Therefore the Tarski–Seidenberg theorem implies for any finite family of profiles of individual endowments $\widehat{w}$ and market prices p that these observations lie on the equilibrium manifold of a pure trade economy, for some family of concave, continuous and monotonic utility functions, if and only if they satisfy the derived family of polynomial inequalities in $\widehat{w}$ and p. This family of polynomial inequalities in the data constitutes the set of testable restrictions of the Walrasian model of pure trade.

It may be difficult, using the Tarski–Seidenberg algorithm, to derive these testable restrictions on the equilibrium manifold in a computationally efficient manner for every finite data set, although we are able to derive restrictions for two observations. If there are more than two observations our restrictions are necessary but not sufficient. That is, if there are at least three observations, then even if our conditions hold for every pair of observations, the data need not lie on any equilibrium manifold. Consequently, we call our conditions the weak axiom of revealed equilibrium or WARE. Of course, if our conditions are violated for any pair of observations then the Walrasian model of pure trade is refuted.

An important distinction between our model and Afriat's model is that we do not assume individual consumptions are observed as did Afriat. As a consequence the Afriat inequalities in our model are nonlinear in the unknowns.

This paper is organized as follows. Section 2 presents necessary and sufficient conditions for observations on market prices, individual incomes and total endowments to lie on the equilibrium manifold of some pure trade economy. Section 3 specializes the results to equilibrium manifolds corresponding to economies whose consumers have homothetic utility functions. In the final section of the paper we discuss extensions and empirical applications of our methodology.

2. Restrictions in the Pure Trade Model

We consider an economy with K commodities and T traders, where the intended interpretation is the pure trade model. The commodity space is $\mathbb{R}^K$ and each agent has $\mathbb{R}^K_+$ as her consumption set. Each trader is characterized by an endowment vector $w_t \in \mathbb{R}^K_{++}$ and a utility function $V_t : \mathbb{R}^K_+ \to K$. Utility functions are assumed to be continuous, monotone and concave.

An *allocation* is a consumption vector x_t for each trader such that $x_t \in \mathbb{R}^K_+$ and $\Sigma_{t=1}^T x_t = \Sigma_{t=1}^T w_t$. The *price simplex* $\Delta = \{p \in \mathbb{R}^K_+ | \Sigma_{i=1}^K p_i = 1\}$. We shall restrict attention to strictly positive prices $S = \{p \in \Delta | p_i > 0 \text{ for all } i\}$. A *competitive equilibrium* consists of an allocation $\{x_t\}_{t=1}^T$ and prices p such that each x_t is utility maximizing for agent t subject to her budget constraint. The prices p are called *equilibrium prices.*

Suppose we observe a finite number N of profiles of individual endowment vectors $\{w_t^r\}_{t=1}^T$ and market prices p^r, where $r = 1, ..., N$, but we do not observe the utility functions or consumption vectors of individual agents. For each family of utility functions $\{V_t\}_{t=1}^T$ there is an equilibrium manifold, which is simply the graph of the Walras correspondence, i.e., the map from profiles of individual endowments to equilibrium prices.

We say that the pure trade model is *testable* if for every N there exists a finite family of polynomial inequalities in w_t^r and p^r for $t = 1, ..., T$ and $r = 1, ..., N$ such that observed pairs of profiles of individual endowments and market prices satisfy the given system of polynomial inequalities iff they lie on some equilibrium manifold.

To prove that the pure trade model is testable, we first recall Afriat's Theorem [1]:

AFRIAT'S THEOREM: *The following conditions are equivalent:*

(A.1) *There exists a nonsatiated utility function that "rationalizes" the data* $(p^i, x^i)_{i=1,...,N}$*; i.e., there exists a nonsatiated function* $u(x)$ *such that for all* $i = 1, ..., N$*, and all* x *such that* $p^i x^i \geq p^i x$, $u(x^i) \geq u(x)$.

(A.2) *The data satisfy "Cyclical Consistency* (CC)*"; i.e., for all* $\{r, s, t, ..., q\}$ $p^r x^r \geq p^r x^s$, $p^s x^s \geq p^s x^t$, ..., $p^q x^q \geq p^q x^r$ *implies* $p^r x^r = p^r x^s$, $p^s x^s = p^s x^t$, ..., $p^q x^q = p^q x^r$.

(A.3) *There exist numbers* U^i, $\lambda^i > 0$, $i = 1, ..., n$ *such that* $U^i \leq U^j + \lambda^j p^j (x^j - x^j)$ *for* $i, j = 1, ..., N$.

(A.4) *There exists a nonsatiated, continuous, concave, monotonic utility function that rationalizes the data.*

Versions of Afriat's theorem for SARP (the Strong Axiom of Revealed Preference, due to Houthakker [10]) and SSARP (the Strong SARP, due to Chiappori and Rochet [9]) can be found in Matzkin and Richter [13] and in Chiappori and Rochet [9], respectively.[1]

We consider the structural equilibrium conditions for N observations on pairs of profiles of individual endowment vectors $\{w_t^r\}_{t=1}^T$ and market prices p^r for $r = 1, ..., N$, which are:

[1] Chiappori and Rochet [9] show that SSARP characterizes demand data that can be rationalized by strictly monotone, strictly concave, C^∞ utility functions. Define the binary relationship R^0 by: $x^i R^0 x$ if $p^i x^i \geq p^i x$. Let R be the transitive closure of R^0. Then, SARP is satisfied iff for all t, s: $[(x^t R x^s \ \& \ x^t \neq x^s) \Rightarrow (\text{not } x^s R x^t)]$; SSARP is SARP together with $[(p^s \neq \alpha p^r \text{ for all } \alpha > 0) \Rightarrow (x^s \neq x^r)]$.

$\exists\{\overline{V}_t^r\}_{r=1,...,N;t=1,...,T}$, $\{\lambda_t^r\}_{r=1,...,N;t=1,...,N}$, $\{x_t^r\}_{r=1,...,N;t=1,...,T}$ such that

$$\overline{V}_t^r - \overline{V}_t^s - \lambda_t^s p^s(x_t^s - x_t^s) \leq 0 \quad r,\, s = 1,\, ...,\, N;\; t = 1,\, ...,\, T \tag{1.1}$$

$$\lambda_t^r > 0,\; x_t^r \geq 0 \quad r = 1,\, ...,\, N;\; t = 1,\, ...,\, T \tag{1.2}$$

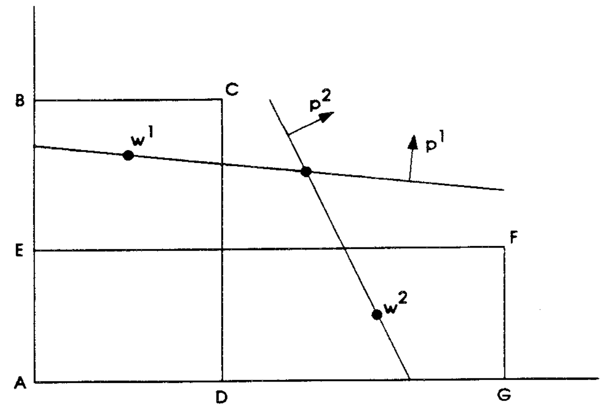
$$p^r x_t^r = p^r w_t^r \quad r = 1,\, ...,\, N;\; t = 1,\, ...,\, T \tag{1.3}$$

$$\Sigma_{t=1}^T x_t^r = \Sigma_{t=1}^T w_t^r \quad r = 1,\, ...,\, N \tag{1.4}$$

this family of conditions will be called the *equilibrium inequalities.* The observable variables in this system are the w_t^r and p^r; hence this is a nonlinear family of polynomial inequalities in unobservable utility levels, $\overline{V}_t^r$; marginal utilities of income, λ_t^r; and consumption vectors x_t^r. Choose T concave, continuous and monotonic utility functions and N profiles of individual endowment vectors. Then by the Arrow–Debreu existence theorem there exist equilibrium prices and competitive allocations, together with the competitive prices and allocations and profiles of endowment vectors, satisfy the equilibrium inequalities. Therefore, the Tarski–Seidenberg algorithm applied to the equilibrium inequalities will not terminate with $1 \equiv 0$.

The following example of a pure trade economy with two goods and two traders proves that the algorithm will not terminate with $1 \equiv 1$. In Figure 1, we superimpose two Edgeworth boxes, which are defined by the aggregate endowment vectors w^1 and w^2. The first box, (1), is ABCD and the second box, (2), is AEFG. The first agent lives at the A vertex in both boxes and the second agent lives at vertex C in box (1) and at vertex F in box (2). The individual endowments w_1^1, w_2^1; s_1^2, w_2^2 and the two price vectors p^1 and p^2 define the budget sets of each consumer. The sections of the budget hyperplanes that intersect with each Edgeworth box constitute the set of potential equilibrium allocations. All pairs of allocations in box (1) and box (2) that lie on the given budget lines violate Cyclical Consistency, i.e., the Weak Axiom of Revealed Preference (WARP) in this case, for the first agent (the agent living at vertex A). By Afriat's theorem there is no solution to the equilibrium inequalities. This example is easily extended to pure trade models with any finite number of goods or traders.

Figure 1



THEOREM 1: *The pure trade model is testable.*

PROOF: The system of equilibrium inequalities is a finite family of polynomial inequalities, hence we can apply the Tarski–Seidenberg algorithm. We have shown above that the algorithm cannot terminate with $1 \equiv 0$ or with $1 \equiv 1$.

Since it is often difficult to observe individual endowment vectors, in the next theorem we restate the equilibrium inequalities where the observables are the market prices, incomes of consumers and aggregate endowments. Let I_t^r denote the income of consumer t in observation r and w^r the aggregate endowment in observation r.

THEOREM 2: *Let* $\langle p^r, \{I_t^r\}_{t=1}^T, w^r\rangle$ *for* $r = 1, ..., N$ *be given. Then there exists a set of continuous, concave and monotone utility functions* $\{V_t\}_{t=1}^T$ *such that for each* $r = 1, ..., N$; p^r *is an equilibrium price vector for the exchange economy* $\langle\{V_t\}_{t=1}^T, \{I_t^r\}_{t=1}^T, w^r\rangle$ *iff there exist numbers* $\{\overline{V}_t^r\}_{t=1,...,T;r=1,...,N}$ *and* $\{\lambda_t^r\}_{t=1,...,T;r=1,...,N}$ *and vectors* $\{x_t^r\}_{t=1,...,T;r=1,...,N}$ *satisfying*

$$\overline{V}_t^r - \overline{V}_t^s - \lambda_t^s p^s(x_t^r - x_t^s) \le 0 \quad r, s = 1, ..., N; \ t = 1, ..., T \tag{2.1}$$

$$\lambda_t^r > 0, \ x_t^r \ge 0 \quad r = 1, ..., N; \ t = 1, ..., T \tag{2.2}$$

$$p^r x_t^r = I_t^r \quad r = 1, ..., N; \ t = 1, ..., T \tag{2.3}$$

$$\Sigma_{t=1}^T x_t^r = w^r \quad r = 1, ..., N \tag{2.4}$$

PROOF: Suppose that there exist $\{\overline{V}_t^r\}$, $\{\lambda_t^r\}$ and $\{x_t^r\}$ satisfying (2.1)–(2.4). Then, by Afriat's Theorem, (2.1)–(2.3) imply that for each t, there exists a continuous, concave, and monotone utility function $V_t : \mathbb{R}_+^K \Rightarrow \mathbb{R}$ such that for each r, x_t^r is one of the maximizers of V_t subject to the budget constraint $p^r y \leq I_t^r$. Hence, since $\{x_t^r\}_{t=1}^T$ define an allocation, i.e., satisfy (2.4), p^r is an equilibrium price vector for the exchange economy $\langle\{V_t\}_{t=1}^T, \{w_t^r\}_{t=1}^T\rangle$ for each $r = 1, ..., N$.

The converse is immediate, since given continuous, concave and monotone utility functions, V_t, the equilibrium price vectors p^r and allocations $\{x_t^r\}_{t=1}^T$ satisfy (2.3) and (2.4) by definition. The existence of $\{\lambda_t^r\}_{t=1}^T$ such that (2.1) and (2.2) hold follows from the Kuhn–Tucker Theorem, where $\overline{V}_t^r = V_t(x_t^r)$.

For two observations ($r = 1, 2$) and the Chiappori–Rochet version of Afriat's theorem we use quantifier elimination, in the Appendix, to derive from the equilibrium inequalities the testable restrictions for the pure trade model with two consumers ($t = a, b$). We call the family of polynomial inequalities obtained from this process the *Weak Axiom of Revealed Equilibrium* (WARE). Define the vector $\overline{z}_t^r$ ($r = 1, 2; t = a, b$) by $\overline{z}_t^r = \arg\max_x\{p^s x | p^r x = I_t^r, 0 \leq x \leq w^r\}$ where $r \neq s$. Hence, among all the bundles that are feasible in observation r and are on the budget hyperplane of consumer t in observation r, $\overline{z}_t^r$ is the bundle that costs the most under prices p^s ($s \neq r$) ($\overline{z}_t^r$ is unique since prices are strictly positive).

We will say that observations $\{p^r\}_{r=1,2}$, $\{I_t^r\}_{r=1,2;t=a,b}$, $\{w^r\}_{r=1,2}$ *satisfy WARE* if

(I) $\forall r = 1, 2\ ,\ \ I_a^r + I_b^r = p^r w^r$

(II) $(\forall r, s = 1, 2\ (r \neq s))\ (\forall t = a, b)\ ,\ \ [(p^s \overline{z}_t^r \leq I_t^s) \Rightarrow (p^r \overline{z}_t^s > I_t^r)]$

(III) $(\forall r, s = 1, 2\ (r \neq s))\ ,\ \ [(p^s \overline{z}_a^r \leq I_a^s)\ ,\ \ (p^s \overline{z}_b^r \leq I_b^s)] \Rightarrow (p^r w^s > p^r w^r)$

In the next theorem we show that WARE characterizes data that lie on some equilibrium manifold. Condition (I) says that the sum of the individuals' incomes equals the value of the aggregate endowment. Condition (II) applies when all the bundles in the budget hyperplane of consumer t in observation r that are feasible in observation r can be purchased with the income and prices faced by consumer t in observation s ($s \neq r$) (i.e., $p^s \overline{z}_t^r \leq I_t^s$). The condition then requires that some of the bundles that are feasible in observation s and are in the budget hyperplane of consumer t in observation s cannot be purchased with the income and prices faced by consumer t in observation r (i.e., $p^r \bar{z}_t^s > I_t^r$). Clearly, unless this condition is satisfied, it will not be possible to find consumption bundles consistent with equilibrium and satisfying SSARP. Note that this condition is not satisfied by the observations in Figure 1. Condition (III) says that when for each of the agents all the bundles that are feasible and affordable under observation r can be purchased with the agent's income and the price of observation s, then the aggregate endowment in observation s must cost more than the aggregate endowment in observation r when evaluated at the prices of observation r. This guarantees that at least one of the pairs of consumption bundles in observation s that contain for each agent feasible and affordable bundles that could not be purchased with the income and price of observation r are such that they add up to the aggregate endowment.

THEOREM 3: *Let* $\{p^r\}_{r=1,2}$, $\{I_t^r\}_{r=1,2;t=a,b}$, $\{w^r\}_{r=1,2}$ *be given such that* p^1 *is not a scalar multiple of* p^2. *Then the equilibrium inequalities for strictly monotone, strictly concave,* C^∞ *utility functions have a solution, i.e., the data lies on the equilibrium manifold of some economy whose consumers have strictly monotone, strictly concave,* C^∞ *utility functions, iff the data satisfy* WARE.

This result follows from the Tarski–Seidenberg theorem, because WARE can be derived by quantifier elimination. In the Appendix we provide a direct proof of Theorem 3, which helps to interpret WARE.

3. Restrictions When Utility Functions Are Homothetic

In applied general equilibrium analysis — see Shoven and Whalley [14] — utility functions are often assumed to be homothetic. We next derive testable restrictions on the pure trade model under this assumption. These restrictions can be used as a specification test for computable general equilibrium models, say in international trade, where agents have homothetic utility functions.

Afriat [2, 3] and Varian [17] developed the Homothetic Axiom of Revealed Preference (HARP), which is equivalent to the Afriat inequalities for homothetic utility functions. For two observations $\{p^r, x^r\}_{r=1,2}$ HARP reduced to: $(p^r x^r)(p^s x^r) \geq (p^r x^r)(p^s x^s)$ for $r, s = 1, 2$ $(r \neq s)$. If we substitute these for the Afriat inequalities in the equilibrium inequalities (1.1)–(1.4), we obtain a nonlinear system of polynomial inequalities were the unknowns (or unobservables) are the consumption vectors x_t^r for $r = 1, 2$ and $t = a, b$. In the Appendix, we use quantifier elimination to derive the testable restrictions of this model on the observable variables, which we call the *Homothetic-Weak Axiom of Revealed Preferences* (H-WARE).

Given observations $\{p^r\}_{r=1,2}$, $\{I_t^r\}_{r=1,2;t=a,b}$, $\{w^r\}_{r=1,2}$, we define the following terms:

$$\gamma_a = I_a^1 I_a^2,\ \gamma_b = I_b^1 I_b^2,\ \gamma_w = (p^1 w^2)(p^2 w^1)$$

$$\psi_1 = \gamma_b - \gamma_a - \gamma_w,\ \psi_2 = (\gamma_b - \gamma_a - \gamma_w)^2 - 4\gamma_a \gamma_w$$

$$r_1 = \frac{\gamma_a}{p^1 \overline{z}_a^2},\ r_2 = p^2 w^1 - \frac{\gamma_b}{p^1 \overline{z}_b^2}$$

$$t_1 = \frac{-\psi_1 - (\psi_2)^{1/2}}{2p^1 w^2},\ t_2 = \frac{-\psi_1 + (\psi_2)^{1/2}}{2p^1 w^2}$$

$$s_1 = \max\{0, r_1, t_1\},\ s_2 = \min\{r_2, t_2\}$$

$$\underline{z}_t^r = \arg\min_x \{p^s x | p^r x = I_t^r,\ 0 \leq x \leq w^r\} \quad \text{where} \quad r \neq s\ (r = 1, 2;\ t = a, b).$$

Our *Homothetic-Weak Axiom of Revealed Equilibrium* (H-WARE) is

(H.I) $\Psi_2 \geq 0$,

(H.II) $s_1 \leq s_2$

(H.III) $s_1 \leq p^2 \overline{z}_a^1$

(H.IV) $p^2 \underline{z}_a^1 \leq s_2$

(H.V) $I_a^1 + I_b^1 = p^1 w^1$ and $I_a^2 + I_b^2 = p^2 w^2$

Condition (H.I) guarantees that t_1 and t_2 are real numbers. Conditions (H.II)–(H.IV) guarantee the existence of a vector x_a^1 whose cost under prices p^2 is between s_1 and s_2. The values of s_1

and s_2 guarantee that equilibrium allocations can be found. Condition (H.V) says that the sum of the individuals' incomes equals the value of the aggregate endowment.

THEOREM 4: *Let* $\{p^2\}_{r=1,2}$, $\{I_t^r\}_{r=1,2;t=a,b}$, $\{w^r\}_{r=1,2}$ *be given. Then the equilibrium inequalities for homothetic utility functions have a solution, i.e., the data lies on the equilibrium manifold of some economy whose consumers have homothetic utility functions, iff the data satisfy* H-WARE.

This result follows by the Tarski–Seidenberg theorem, because H-WARE can be derived by quantifier elimination. The necessity and sufficiency of H-WARE may, however, be better understood by reading the direct proof of Theorem 4, which we provide in the Appendix.

4. Empirical Applications and Extensions

For an empirical test of the pure exchange model, one might use cross-sectional data to obtain the necessary variation in market prices and individual incomes. Assuming that sampled cities or states have the same distribution of tastes but different income distributions and consequently different market prices, the observations can serve as market data for our model. In the stylized economies in our examples one should think of each "trader" as an agent type that represents numerous small consumers each having the same tastes and incomes.

There is a large variety of situations that fall into the structure of a general equilibrium exchange model and for which data are available. For example, our methods can be used in a multiperiod capital market model, where agents have additively separable (time invariant) utility functions, to test whether spot prices are equilibrium prices, by using only observations on the spot prices and the individual endowments in each period. They can be used to test the equilibrium hypothesis in an assets market model where agents maximize indirect utility functions over feasible portfolios of assets, using observations on the outstanding shares of the assets, each trader's initial asset holdings, and the asset prices. Or, they can be used in a household labor supply model of the type considered in Chiappori [6], to test whether the unobserved allocation of consumption within the household is determined by a competitive equilibrium, using data on the labor supply, wages, and the aggregate consumption of the household.

To apply the methodology to large data sets, it is necessary to devise a computationally efficient algorithm for solving large families of equilibrium inequalities. A promising approach is to restrict attention to special classes of utility functions. As an example, if traders are assumed to have quasilinear utility functions — all linear in the same commodity (say the kth) — then the equilibrium inequalities can be reduced to a family of linear inequalities by choosing the kth commodity as numeraire. We can then use the simplex algorithm or the interior point algorithm of Karmarkar — which runs in polynomial time — to test for or compute solutions of the equilibrium inequalities.

The more challenging problem in economic theory is to recast the equilibrium inequalities to allow random variation in tastes. Some recent progress has been made in this area by Brown and Matzkin [7]. They consider a random utility model, which gives rise to a stochastic family of Afriat inequalities, that can be identified and consistently estimated. If their approach can be extended to random exchange models then this is a significant step in empirically testing the Walrasian hypothesis.

The methodology can also be extended to find testable restrictions on the equilibrium manifold of economies with production technologies. Only observations on the market prices, individuals' endowments, and individuals' profit shares are necessary to test the equilibrium model in production economies. Inn particular, for a Robinson Crusoe economy, where the consumer has a nonsatiated utility function we have derived the following restrictions on the observable variables, for any number of observations. A direct proof of the result is given in the Appendix.

THEOREM 5: *The data* $\langle p^r, w^r \rangle$ *for* $r = 1, \ldots, N$ *lies in the equilibrium manifold of a Robinson Crusoe economy iff* $\langle p^r, w^r \rangle$ *for* $r = 1, \ldots, N$ *satisfy Cyclical Consistency* (CC).

Testable restrictions for other economic models can also be derived using the methodology that we have presented in this paper.

APPENDIX

Derivation of WARE Using Quantifier Elimination:

The polynomial inequalities characterizing the pure trade model for two observations and two consumers with strictly monotone, strictly concave, C^∞ utility functions are: $\exists\{\overline{V}_t^r\}_{r=1,2;t=a,b}$, $\{\lambda_t^r\}_{r=1,2;t=a,b}$, $\{x_t^r\}_{r=1,2;t=a,b}$ such that

(C.1) $\overline{V}_t^2 - \overline{V}_t^1 - \lambda_t^1 p^1(x_t^2 - x_t^1) < 0\ ,\ \ t = a, b$

(C.2) $\overline{V}_t^1 - \overline{V}_t^2 - \lambda_t^2 p^2(x_t^1 - x_t^2) < 0\ ,\ \ t = a, b$

(C.3) $\lambda_t^r > 0\ ,\ \ r = 1, 2;\ t = a, b$

(C.4) $p^r x_t^r = I_t^r\ ,\ \ r = 1, 2;\ t = a, b$

(C.5) $p^1 \neq p^2 \Rightarrow x_t^1 \neq x_t^2\ ,\ \ t = a, b$

(C.6) $x_t^r \geq 0\ ,\ \ r = 1, 2;\ t = a, b$

(C.7) $x_a^r + x_b^r = w^r\ ,\ \ r = 1, 2$

((C.1)–(C.5) follow from Chiappori and Rochet [9].) The equivalent expression, after eliminating $\{\lambda_t^r\}_{r=1,2;t=a,b}$, is: $\exists\{\overline{V}_t^r\}_{r=1,2;t=a,b}$, $\{x_t^r\}_{r=1,2;t=a,b}$ such that

(C.1′) $p^1(x_t^2 - x_t^1) \leq 0 \Rightarrow \overline{V}_t^2 < \overline{V}_t^1\ ,\ \ t = a, b$

(C.2′) $p^2(x_t^1 - x_t^2) \leq 0 \Rightarrow \overline{V}_t^1 < \overline{V}_t^2\ ,\ \ t = a, b$

(C.4) $p^r x_t^r = I_t^r\ ,\ \ r = 1, 2;\ t = a, b$

(C.5) $p^1 \neq p^2 \Rightarrow x_t^1 \neq x_t^2\ ,\ \ t = a, b$

(C.6) $x_t^r \geq 0\ ,\ \ r = 1, 2;\ t = a, b$

(C.7) $x_a^r + x_b^r = w^r\ ,\ \ r = 1, 2$

(C.1′) and (C.2′) are the necessary and sufficient conditions on $\{\overline{V}_t^r\}_{r=1,2;t=a,b}$, $\{x_t^r\}_{r=1,2;t=a,b}$, and $\{I_t^r\}_{r=1,2;t=a,b}$ guaranteeing that there exist $\{\lambda_t^r\}_{r=1,2;t=a,b}$ satisfying (C.1)–(C.3). Elimination of $\{\overline{V}_t^r\}_{r=1,2;t=a,b}$ yields the equivalent expression: $\exists\{x_t^r\}_{r=1,2;t=a,b}$ such that

(C.1″) $p^1(x_t^2 - x_t^1) \leq 0 \Rightarrow p^2(x_t^1 - x_t^2) > 0\ ,\quad t = a,\, b$

(C.4) $p^r x_t^r = I_t^r\ ,\quad r = 1,\, 2;\ t = a,\, b$

(C.5) $p^1 \neq p^2 \Rightarrow x_t^1 \neq x_t^2\ ,\quad t = a,\, b$

(C.6) $x_t^r \geq 0\ ,\quad r = 1,\, 2;\ t = a,\, b$

(C.7) $x_a^r + x_b^r = w^r\ ,\quad r = 1,\, 2$

(C.1″) is the necessary and sufficient condition that guarantees there exist $\{\overline{V}_t^r\}_{r=1,2;t=a,b}$ satisfying (C.1′)–(C.2′). Note that we have just shown how, for two observations, SSARP can be derived by quantifier elimination. Next, elimination of $\{x_b^r\}_{r=1,2}$ and usage of (C.4) yields the expression: $\exists x_a^1,\, x_a^2$ such that

(C.1‴.1) $p^s x_a^r \leq I_a^s \Rightarrow p^r x_a^s > I_a^r\ ,\quad s \neq r$

(C.1‴.2) $p^s(w^r - x_a^r) \leq I_b^s \Rightarrow p^r(w^s - x_a^s) > I_a^r\ ,\quad s \neq r$

(C.4′) $p^r x_a^r = I_a^r\ ,\quad r = 1,\, 2$

(C.5′) $p^1 \neq p^2 \Rightarrow [(x_a^1 \neq x_a^2)\ \&\ (w^1 - x_a^1 \neq w^2 - x_a^2)]$

(C.6′) $0 \leq x_a^r \leq w^r\ ,\quad r = 1,\, 2$

(C.7′) $I_a^r + I_b^r = p^r w^r\ ,\quad r = 1,\, 2$

Let $\underline{z}_t^r = \arg\max_x\{p^s x | p^r x = I_t^r,\ 0 \leq x \leq w^r\}$, where $r \neq s$. Note that $I_a^r + I_b^r = p^r w^r$ implies that $\overline{z}_a^r + \underline{z}_b^r = w^r$. Then, elimination of x_a^2 gives: $\exists x_a^1$ such that

(C.1⁗.1) $p^2 x_a^1 \leq I_a^2 \Rightarrow p^1 \overline{z}_a^2 > I_a^1$

(C.1⁗.2) $p^2(w^1 - x_a^1) \leq I_b^2 \Rightarrow p^1(w^2 - \underline{z}_a^2) > I_b^1$

(C.1''''.3) $[(p^2x_a^1 \le I_a^2)\ \&\ (p^2(w^1 - x_a^2) \le I_b^2)] \Rightarrow p^1w^2 > p^1w^1$

(C.4'') $p^1x_a^1 = I_a^1$

(C.6') $0 \le x_a^1 \le w^1$

(C.7') $I_a^r + I_b^r = p^r w^r \ , \quad r = 1, 2$

((C.5') does not impose restrictions.) Finally, elimination of x_a^1 yields

(C.1*.1) $p^1\overline{z}_a^2 \le I_a^1 \Rightarrow p^2\overline{z}_a^1 > I_a^2$

(C.1*.2) $p^1(w^2 - \underline{z}_a^2) \le I_b^1 \Rightarrow p^2(w^1 - \underline{z}_a^1) > I_b^2$

(C.1*.3) $[(p^1\overline{z}_a^2 \le I_a^1)\ \&\ (p^1(w^2 - \underline{z}_a^2) \le I_b^1)] \Rightarrow p^2w^1 > p^2w^2$

(C.1*.4) $[(p^2\overline{z}_a^1 \le I_a^2)\ \&\ (p^2(w^1 - \underline{z}_a^1) \le I_b^2)] \Rightarrow p^1w^2 > p^1w^1$

(C.7') $I_a^r + I_b^r = p^r w^r \ , \quad r = 1, 2$

This family of polynomial inequalities can be written as:

(I) $\forall r = 1, 2 \ , \quad I_a^r + I_b^r = p^r w^r$

(II) $(\forall r, s = 1, 2\ (r \ne s))\ (\forall t = a, b) \ , \quad [(p^s\overline{z}_t^r \le I_t^s) \Rightarrow (p^r\overline{z}_t^s > I_t^r)]$

(III) $(\forall r, s = 1, 2\ (r \ne s)) \ , \quad [(p^s\overline{z}_a^r \le I_a^s)\ \&\ (p^s\overline{z}_b^r \le I_b^s)] \Rightarrow (p^r w^s > p^r w^r)$

which is our *Weak Axiom of Revealed Equilibrium* (WARE).

Direct Proof of Theorem 3: Suppose that there exist strictly monotone, strictly concave C^∞ utility functions V_a and V_b such that p^r is an equilibrium price vector for (I_a^r, I_b^r, w^r) for $r = 1, 2$. Let x_t^r be the unique maximizer of V_t subject to the budget constraint determined by (p^r, I_t^r) for $r = 1, 2$ and $t = a, b$. Then

$$0 \le x_t^r \ , \quad r = 1, 2;\ t = a, b \tag{3.1}$$

$$x_a^r + x_b^r = w^r \ , \quad r = 1, 2 \tag{3.2}$$

Let us show that WARE is satisfied. First note that

$$I_a^r + I_b^r = p^r x_a^r + p^r x_b^r = p^r w^r \quad \text{for} \quad r = 1, 2$$

Next suppose that $p^r \overline{z}_t^s \leq I_t^r$. Then for all x such that $0 \leq x \leq w^s$ and $p^s x = I_t^s$, $p^r x \leq I_t^r$. In particular, $p^r x_t^s \leq I_t^r$. Since x_t^s and x_t^r satisfy SSARP, $p^s x_t^r > I_t^s$. Hence $p^s \overline{z}_c^r > I_c^s$ and we have shown that conditions (I) and (II) of WARE are satisfied.

Next suppose that when $s \neq r$, $p^r \overline{z}_a^s \leq I_a^r$ and $p^r \overline{z}_b^s \leq I_b^r$. Then $p^r x_a^s \leq I_a^r$ and $p^r x_b^s \leq I_b^r$. By SSARP $p^s x_a^r > I_a^s$ and $p^s x_b^r > I_b^s$. Since $x_a^r + x_b^r = w^r$, it follows that $p^s w^r = p^s(x_a^r + x_b^r)$ $> I_a^s + I_b^s p^s w^s$. So $p^s w^r > p^s w^s$ and we have shown that condition (III) of WARE is satisfied. This completes the proof of necessity.

To prove sufficiency, we show that WARE implies that there exists $\{x_t^r\}_{r=1,2;t=a,b}$ satisfying SSARP, the budget equations, and the market clearing equations. The result will then follow from Chiappori and Rochet [9].

We first note that WARE implies that there exists $s = 1$ or $s = 2$ such that either

CASE 1: $p^r \overline{z}_a^s > I_a^r$ and $p^s \overline{z}_b^r > I_b^s$ for $r \neq s$, or

CASE 2: $p^r \overline{z}_a^s > I_a^r$, $p^r \overline{z}_b^s > I_b^s$, and $p^r w^s > p^r w^r$ for $r \neq s$.

To see this, note that when

$$(p^s \overline{z}_a^r \leq I_a^s) \ \& \ (p^s \overline{z}_b^r \leq I_b^s) \tag{3.3}$$

Conditions (II) and (III) in WARE imply that Case 2 occurs. When (3.3) is not satisfied, one of the following situations must occur: either

$$(p^s \overline{z}_a^r \leq I_a^s) \ \& \ (p^s \overline{z}_b^r > I_b^s) \ , \quad \text{or} \tag{3.4}$$

$$(p^s \overline{z}_a^r > I_a^s) \ \& \ (p^s \overline{z}_b^r \leq I_b^s) \ , \quad \text{or} \tag{3.5}$$

$$(p^s \overline{z}_a^r > I_a^s) \ \& \ (p^s \overline{z}_b^r > I_b^s) \ . \tag{3.6}$$

If (3.4) or (3.5) occurs, Condition (II) implies that Case 1 occurs. We can divide the case where (3.4) occurs into $[(3.4) \ \& \ (p^s w^r > p^s w^s)]$ and $[(3.4) \ \& \ (p^s w^r \leq p^s w^s)]$. The first situation falls in Case 2. In the second situation, Condition (III) in WARE implies that either $(p^r \overline{z}_a^s > I_a^r)$ or $(p^r \overline{z}_b^s > I_b^r)$, which imply that Case 1 occurs.

When Case 1 is true, let $x_a^s = \overline{z}_a^s$ and $x_b^r = \overline{z}_a^s$. Then,

$$p^s x_a^s = I_a^s \ , \quad p^r x_b^r = I_b^r \ , \tag{3.7}$$

$$p^r x_a^s > I_a^r \ , \quad p^s x_b^r > I_b^s \ . \tag{3.8}$$

Let $x_b^s = w^s - x_a^s$ and $x_a^r = w^r - x_b^r$. Then $\{x_t^r\}_{r=1,2;t=a,b}$ satisfy the equilibrium equalities by definition and SSARP by (3.8). By (3.7) and Condition (I) in WARE, x_b^s and x_a^r satisfy the budget equations.

When Case 2 is true,[2] it follows from Condition (I) in WARE that

$$I_a^r = p^r w^r - I_b^r < p^r w^s - I_b^r \ . \tag{3.9}$$

Since $p^r \overline{z}_b^s > I_b^r$ and $\overline{z}_b^s = w^s - \underline{z}_a^s$, it follows that $p^r \underline{z}_a^s < p^r w^s - I_b^r$. So, since $p^r \overline{z}_a^s > I_a^r$ and $I_a^r < p^r w^s - I_b^r$, there must exist x_a^s satisfying

$$0 \leq x_a^s \leq w^s \ , \tag{3.10}$$

$$p^s x_a^s = I_a^s \ , \quad \text{and} \tag{3.11}$$

$$I_a^r < p^r x_a^s < p^r w^s - I_b^r \ . \tag{3.12}$$

Let $x_b^s = w^s - x_a^s$. Then x_b^s and x_a^s satisfy the equilibrium equality. By (3.11) and Condition (I) in WARE, $p^s x_b^s = I_b^s$. By (3.12) $p^s x_b^s = p^r(w^s - x_a^s) = p^r w^s - p^r x_a^s > p^r w^s - p^r w^s + I_b^r = I_b^r$. So,

$$p^r x_b^s > I_b^r \ . \tag{3.13}$$

Let x_a^r be any vector such that $0 \leq x_a^r \leq w^r$ and $p^r x_a^r = I_a^r$. Let $x_b^r = w^r - x_a^r$. It then follows that $p^r x_b^r = p^r w^r - p^r x_a^r = I_b^r$ and $0 \leq x_b^r \leq w^r$. Finally, by (3.12) and (3.13), $\{x_t^r\}_{r=1,2;t=a,b}$ satisfy SSARP.

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[2] The original proof of this case contained an error. We thank Susan Snyder for pointing it out to us.

Derivation of H-WARE Using Quantifier Elimination: We have to eliminate the quantifiers in the following expression: $\exists x_a^1,\, x_a^2,\, x_b^1,\, x_b^2$ such that

(H.1) $(p^1 x_a^2)(p^2 x_a^1) \geq \gamma_a$

(H.2) $(p^1 x_b^2)(p^2 x_b^1) \geq \gamma_b$

(H.3) $p^r x_t^r = I_t^r \ , \quad r = 1, 2;\ t = a, b$

(H.4) $x_t^r \geq 0 \ , \quad r = 1, 2;\ t = a, b$

(H.5) $x_a^r + x_b^r = w^r \ , \quad r = 1, 2$

This is equivalent to: $\exists x_a^1,\, x_a^2$ such that

(H.1) $(p^1 x_a^2)(p^2 x_a^1) \geq \gamma_a$

(H.2′) $(p^1(w^2 - x_a^2))(p^2(w^1 - x_a^1)) \geq \gamma_b$

(H.3′) $p^r x_a^r = I_a^r \ , \quad r = 1, 2$

(H.4′) $w^r \geq x_a^r \geq 0 \ , \quad r = 1, 2$

(H.5′) $I_a^r + I_b^r = p^r w^r \ , \quad r = 1, 2$

which is equivalent to: $\exists x_a^1$ such that

(H.1.1) $(p^1 \overline{z}_a^2) \geq \dfrac{\gamma_a}{p^2 x_a^1}$

(H.1.2) $p^1 w^2 - \dfrac{\gamma_b}{p^2(w^1 - x_a^1)} \geq p^1 \underline{z}_a^2$

(H.1.3) $p^1 w^2 - \dfrac{\gamma_b}{p^2(w^1 - x_a^1)} \geq \dfrac{\gamma_a}{p^2 x_a^1}$

(H.3″) $p^1 x_a^1 = I_a^1$

(H.4″) $w^1 \geq x_a^r \geq 0$

(H.5″) $I_a^r + I_b^r = p^r w^r \ , \quad r = 1, 2$

Or, equivalently, $\exists x_a^1$ such that

(H.1.1′) $\dfrac{\gamma_w - (p^1 \underline{z}_a^2)(p^2 w^1) - \gamma_b}{(p^1 w^2 - p^1 \underline{z}_a^2} \geq p^2 x_a^1 \geq \dfrac{\gamma_a}{p^1 \overline{z}_a^2}$

(H.1.2′) $(p^1 w^2)(p^2 x_a^1)^2 + (\gamma_b - \gamma_w - \gamma_a)(p^2 x_a^1) + \gamma_a p^2 w^1 \leq 0$

(H.3″) $p^1 x_a^1 = I_a^1 \ , \quad r = 1, 2$

(H.4″) $w^1 \geq x_a^1 \geq 0 \ , \quad r = 1, 2$

(H.5″) $I_a^r + I_b^r = p^r w^r \ , \quad r = 1, 2$

Using the fact that $w^2 - \underline{z}_a^2 = \overline{z}_b^2$, (H.1.1′) can be written as

(H.1.1″) $p^2 w^1 - \dfrac{\gamma_b}{p \overline{z}_b^2} \geq p^2 x_a^1 \geq \dfrac{\gamma_a}{p^1 \overline{z}_a^2}$

The necessary and sufficient condition for the existence of x_a^1 satisfying (H.1.1″), (H.1.2′), (H.3″)–(H.5″) are:

(H.1*) $\dfrac{\gamma_a}{p^1 \overline{z}_a^2} \leq p^1 \overline{z}_a^1 \ , \quad p^2 \underline{z}_a^1 \leq p^2 w^1 - \dfrac{\gamma_b}{p \overline{z}_b^2} \ , \quad \text{and} \dfrac{\gamma_a}{p^1 \overline{z}_a^2} \leq p^2 w^1 - \dfrac{\gamma_b}{p \overline{z}_b^2}$

(H.2*) $\Psi_2 = (\Psi_1)^2 - 4\gamma_a \gamma_w \geq 0$

(H.3*) $\dfrac{-\Psi_1 - \sqrt{\Psi_2}}{2p^1 w^2} \leq p^1 \overline{z}_a^1 \ , \quad p^2 \underline{z}_a^1 \leq \dfrac{-\Psi_1 + \sqrt{\Psi_2}}{2p^1 w^2}$

(H.4*) $I_a^r + I_b^r = p^r w^r \ , \quad r = 1, 2$

Conditions (H.1*)–(H.4*) are our Homothetic Weak Axiom of Revealed Equilibrium.

Direct Proof of Theorem 4: We first prove the necessity of H-WARE. Suppose that there exist monotone, concave, and homothetic utility functions rationalizing the data. Let $\{x_t^r\}_{r=1,2;t=a,b}$ be obtained by maximizing such utility functions over the budget constraints determined by $\{p^r\}_{r=1,2}$ and $\{I_t^r\}_{r=1,2}$. Then,

$$x_t^r \geq 0 \ , \quad r = 1, 2; \ t = a, b \tag{4.1}$$

$$p^r x_t^r = I_t^r \ , \quad r = 1, 2; \ t = a, b \tag{4.2}$$

$$x_a^r + x_b^r = w^r \ , \quad r = 1, 2 \tag{4.3}$$

$$(p^1x_a^2)(p^2x_a^1) \geq \gamma_a \text{ and} \tag{4.4}$$

$$(p^1x_b^2)(p^2x_b^1) \geq \gamma_b \ . \tag{4.5}$$

By (4.3) and (4.5), $\gamma_b \leq p^1(w^2 - x_a^2)p^2(w^1 - x_a^1)$. So that

$$\begin{aligned} \Psi_1 &= \gamma_b - \gamma_a - \gamma_w \\ &\leq (p^1w^2)(p^2w^1) - (p^1w^2)(p^2x_a^1) - (p^1x_a^2)(p^2w^1) + (p^1x_a^2)(p^2x_a^1) - \gamma_a - \gamma_w \\ &= -(p^1x_a^2)p^2(w^1 - x_a^1) - (p^1w^2)(p^2x_a^1) - \gamma_a < 0 \ . \end{aligned}$$

Using (4.4), (4.5), the definitions of $\overline{z}_a^2$ and $\overline{z}_b^2$, and (4.3), it follows that

$$p^2x_a^1 = (p^2x_a^1)\frac{(p^1x_a^2)}{(p^1x_a^2)} \geq \frac{\gamma_a}{(p^1x_a^2)} \geq \frac{\gamma_a}{(p^1\overline{z}_a^2)} \ r_1 \tag{4.6}$$

and

$$p^2(w^1 - x_a^1) = p^2x_b^1 = \frac{(p^2x_b^1)(p^1x_b^2)}{p^1x_b^2} \geq \frac{\gamma_b}{p^1x_b^2} \geq \frac{\gamma_b}{p^1\overline{z}_b^2}$$

so that

$$p^2x_a^1 \leq p^2w^1 - \frac{\gamma_b}{p^1\overline{z}_b^2} = r_2 \ . \tag{4.7}$$

Using again (4.4) and (4.5) we get that

$$p^1x_a^2 \geq \frac{\gamma_a}{p^2x_a^1} \quad \text{and} \quad p^1x_b^2 \geq \frac{\gamma_b}{p^2(w^1 - x_a^1)}$$

so that adding up and using (4.3) we get

$$p^1w^2 \geq \frac{\gamma_a}{p^2x_a^1} + \frac{\gamma_b}{p^2(w^1 - x_a^1)} \quad \text{or} \quad (p^1w^2)(p^2x_a^1)^2 + (\gamma_b - \gamma_a - \gamma_w)(p^2x_a^1) + \gamma_a p^2w^1 \leq 0 \ .$$

Consider the quadratic function $f(t) = (p^1w^2)t^2 + (\gamma_b - \gamma_a - \gamma_w)t + \gamma_a p^2w^1$. Notice that $f(0) = \gamma_a p^2w^1 > 0$, $f'(0) = (\gamma_b - \gamma_a - \gamma_w) = \Psi_1 < 0$, $f(p^2x_a^1) \leq 0$, and $f''(t) = 2p^1w^2 > 0$ for all t. Hence the equation $f(t) = 0$ must have real roots t_1 and t_2, which implies that $\Psi_2 = (\gamma_b - \gamma_a - \gamma_w)^2 - 4\gamma_a\gamma_w \geq 0$, proving (H.1), and

$$0 < t_1 \leq p^2x_a^1 \leq t_2 \ . \tag{4.8}$$

Since by (4.6) and (4.7), $r_1 \leq p^2x_a^1 \leq r_2$, it follows that $s_1 = \max\{0, r_1, t_1\} \leq p^2x_a^1 \leq \min\{r_2, t_2\} = s_2$, which proves (H.II). Moveover, since $s_1 \leq p^2x_a^1 \leq s_2$, it follows by the definitions of $\overline{z}_a^1$ and $\underline{z}_a^1$ that $p^2\underline{z}_a^1 \leq s_2$ and $p^2\overline{z}_a^1 \geq s_1$, which prove (H.III) and (H.IV). Finally, (H.V) is satisfied by (4.2) and (4.3). This completes the proof of necessity. To show the sufficiency of H-WARE, we first note that by (H.1) t_1 and t_2 are real numbers and by (H.II) and the definition of s_1, $0 \leq t_2$. By (H.II)–(H.IV), there exists x_a^1 such that $p^1x_a^1 = I_a^1$, $0 \leq x_a^1 \leq w^1$, and $s_1 \leq p^2x_a^1 \leq s_2$. Take one such x_a^1. It follows that

$$t_1 \leq p^2x_a^1 \leq t_2 \text{ and} \tag{4.9}$$

$$r_1 \leq p^2x_a^1 \leq r_2 \ . \tag{4.10}$$

The latter expression implies that

$$p^2x_a^1 \leq p^2w^1 - \frac{\gamma_b}{p^2\overline{z}_b^2} = p^2w^1 - \frac{\gamma_b}{p^1(w^2 - \underline{z}_a^2)} \text{ and } p^2x_a^1 \geq \frac{\gamma_a}{p^1\overline{z}_a^2} \text{ so that}$$

$$p^1\underline{z}_a^2 \leq p^1w^2 - \frac{\gamma_b}{p^1(w^1 - x_a^1)} \text{ and} \tag{4.11}$$

$$p^1\overline{z}_a^2 \geq \frac{\gamma_a}{p^2x_a^1} \ . \tag{4.12}$$

Let $f(t) = (t - t_1)(t - t_2)$. For all $t \in [t_1, t_2]$ $f(t) \leq 0$. Hence, (4.9) implies that $(p^2x_a^1 - t_1)(p^2x_a^1 - t_2) \leq 0$, or, for

$$t_1 = \frac{-\Psi_1 - (\Psi_2)^{1/2}}{2p^1w^2} \text{ and } t_2 = \frac{-\Psi_1 + (\Psi_2)^{1/2}}{2p^1w^2} \text{ we have}$$

$$(p^2x_a^1)^2 + (p^2x_a^1)\frac{\Psi_1}{p^1w^2} + \gamma_a\frac{p^2w^1}{p^1w^2} \leq 0 \ .$$

Rearranging terms, we get that

$$\frac{\gamma_a}{p^2x_a^1} \leq p^1w^2 - \frac{\gamma_b}{p^2(w^1 - x_a^1)} \ .$$

Hence, by (4.11) and (4.12) it follows that there exists x_a^2 such that $0 \leq x_a^2 \leq w^2$, $p^2x_a^2 = I_a^2$, and

$$\frac{\gamma_a}{p^2x_a^1} \leq p^1x_a^2 \leq p^1w^2 - \frac{\gamma_b}{p^2(w^1 - x_a^1)} \ .$$

Hence,

$$\gamma_a \leq (p^1x_a^2)(p^2x_a^1) \text{ and } \gamma_b \leq p^1(w^2 - x_a^2)p^2(w^1 - x_a^1) \ . \tag{4.13}$$

With (4.13) we have completed the proof that $\{p^r\}_{r=1,2}$, $\{I_a^r\}_{r=1,2}$, and $\{x_a^r\}_{r=1,2}$ satisfy HARP. Let $x_b^1 = w^1 - x_a^1$ and $x_b^2 = w^2 - x_a^2$. Since $0 \leq x_a^1 \leq w^1$, $0 \leq x_a^2 \leq w^2$, $p^1x_a^1 = I_a^1$, and $p^2x_a^2 = I_a^2$, it follows, using (H.V), that $0 \leq x_b^1 \leq w^1$, $0 \leq x_b^2 \leq w^2$, $p^1x_b^1 = I_b^1$, $p^2x_b^2 = I_b^2$, and by above $\gamma_b \leq (p^1x_b^2)(p^2x_b^1)$. Hence, also $\{p^r\}_{r=1,2}$, $\{I_b^r\}_{r=1,2}$, and $\{x_b^r\}_{r=1,2}$ satisfy HARP.

Proof of Theorem 5: Let x^r and y^r denote, respectively, a consumption and production plan in observation r. If $\langle p^r, w^r\rangle_{r=1}^N$ satisfy CC, then $\langle p^r, x^r = w^r, y^r = 0\rangle_{r=1,\ldots,N}$ satisfy the Afriat inequalities for utility maximization and profit maximization (see Varian [18]), and markets clear. Suppose that $\langle p^r, w^r\rangle_{r=1}^N$ does not satisfy CC but lies in the equilibrium manifold. Let x^r and y^r denote, respectively, any equilibrium consumption and equilibrium production plan in observation r. Since CC is violated, there exists $\{s, v, f, ..., e\}$ such that

$$p^sw^v \leq p^sw^s,\ p^vw^f \leq p^vw^v,\ ...,\ p^ew^s \leq p^ew^e \tag{5.1}$$

where at least one of the inequalities is strict. Profit maximization ($p^sy^v \leq p^sy^s$, $p^vy^f \leq p^vy^v$, ..., $p^ey^s \leq p^ey^e$) and markets clearing ($x^v = w^v + y^v$, $x^s = w^s + y^s$, $x^f = w^f + y^f$, ..., $x^e = w^e + y^e$) imply with (5.1) that

$$p^sx^v \leq p^sx^s,\ p^vx^f \leq p^vx^v,\ ...,\ p^ex^s \leq p^ex^e \tag{5.2}$$

where at least one of the inequalities is strict. Since (5.2) is inconsistent with utility maximization, a contradiction has been found.

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