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THE SIGNIFICANCE OF THE MARKET PORTFOLIO

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Abstract

The market portfolio (world portfolio) is in one sense the least important portfolio to provide to investors; there is always a better portfolio for social planners to make available to them. In a J-agent one-period stochastic endowment economy, where preferences are quadratic, the market portfolio is never spanned by the optimal markets a social planner would create. With identical preferences, the market portfolio is orthogonal to all $J-1$ portfolios which achieve a first best solution. These conclusions rely on the assumption that the contract planner has perfect information about agents' utilities. We also show that as the contract designer's information about agents' utilities becomes more imperfect, the optimal contracts approach contracts that weight individual endowments in proportion to elements of eigenvectors of the variance matrix of the endowments. If there is a substantial market component to endowments then a social planner, for reasons of robustness and simplicity, may conclude that creating a contract to allow trading the market portfolio would be a significant innovation.

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1 Introduction

The "market portfolio," the portfolio of all endowments in the world, has great significance in the capital asset pricing model (CAPM) in finance. The Sharpe–Lintner CAPM characterization of optimal risk sharing implies that in equilibrium no one will be subject to a random shock that is not shared by everyone else.[1] Thus, the CAPM gives us the "mutual fund theorem," which asserts that only one risky portfolio need be available to individual investors, the mutual fund that holds the market portfolio. In this paper we seek further clarification of the significance of the market portfolio beyond the bounds of the restrictive assumptions of the CAPM.

The original version of the CAPM was designed to describe how agents should invest in existing financial assets. Thus each agent has some stock of wealth and she must choose how much of her wealth to invest in each asset. There is some zero cost intermediary that allows the agents to purchase the assets. One of the key insights of the CAPM is that each agent needs only the market portfolio (the portfolio of all the financial assets in the world) and the risk free bond to be available to them to trade in so that they obtain their optimal allocation of risk. It is in this sense that the market portfolio is so important in the CAPM.

In our analysis, we will drop the (highly unrealistic) assumption of the CAPM that all risks are tradable; we include in our model non-financial endowments such as labor income. Thus in general no one will be able to hold the market portfolio (the portfolio of all the endowments of the world) unless unprecedented new institutional arrangements are made to permit it to be traded.[2] We instead develop a CAPM-type model in which each individual has a random endowment that is initially not marketable, and we will consider adding one, two, or more contracts that make it possible to buy or sell portfolios of claims on the endowments. We assume that these contracts are to be traded in markets open to everyone, and a market price will be generated such that total excess demand by all agents is zero. Thus, by creating these contracts, we are creating new markets for portfolios of endowments, making a risk tradable that had not been so before.

We confine our attention to designing N contracts, where N is small,

[1] CAPM will refer to the Sharpe–Lintner version unless specified otherwise.

[2] It is an historical accident that the portfolio of all risky assets in the world is called the "market portfolio," a term that sounds odd when we consider the fact that one cannot in practice buy a share in the entire world which includes claims to all countries' future income streams. We persist in this old terminology.

in order to prescribe in simple terms the most important risk management actions that should be taken by groups of people and to ask if the "market portfolio" (the world portfolio) is the most important contract, or is even in the span of the most important contracts. Most people take no more than simple prescriptions from existing models. Practitioners usually do not use the CAPM to arrive at precise definitions of optimal portfolios, but merely refer to the conclusion of the CAPM that investors should hold the market portfolio of investable assets. The indexed funds that are now commonplace were designed with this objective. But this common prescription disregards the correlation of portfolio returns with other endowments.

It is very important, at the time financial innovation takes place, to consider what are conceptually the most important markets. We cannot have liquid markets for everything, and history shows that markets that are not sufficiently valuable to participants will not succeed, and markets will sometimes disappear when better markets are created.

It seems intuitive that it might be a good idea to create a market for all the endowments of the world, not just the financial assets, and that such a market portfolio might be the most important portfolio of all. It is possible some day that the market portfolio and other major aggregates will be traded. New derivative contracts cash settled on income or price indexes can achieve this goal. Methods of creating cash-settled futures contracts for long-term claims on indexes of national income or of occupational income are discussed in Shiller and Athanasoulis [1995]. This paper examines the theoretical arguments for setting up a market for the portfolio of world endowments.

In our model it is immediate that, regardless of the number or kind of markets created, whether or not a market for the market portfolio (world portfolio) is created, risk premia, represented by prices of our contracts here, are as in the CAPM determined exclusively by covariances with the market portfolio. The market portfolio is also in another sense the most important market: with a normalization rule that we define in the paper, the market portfolio is the portfolio which would carry the highest risk premium (highest absolute value of price), Theorem 3 below. In fact, all other portfolios uncorrelated with the market portfolio will have a zero risk premium.

And yet we find, curiously, that of all possible markets to create, a market for claims on the market portfolio would be, by a social welfare criterion, a least important market to create, not a most important market, Theorem 2 below. If we are in the business of creating markets for endowments that are not tradable, then there is a natural order to creating such markets. There is a most important market to create, and then, after this, a market that would be the next best market to create, and so on. The market for the market portfolio turns out to be a completely unimportant market in this ordering, and then the welfare gain to creating it is zero. This is not to say that a market for the market portfolio would not be useful to people if it were created first, or if it were created second or third, only that there would always be something better to do instead. This result may be regarded as, in a sense, the very *antithesis of the mutual fund theorem*.

Neither will we ever want to create markets for individual endowments or for portfolios weighting all endowments with the same sign. Optimal contracts will always involve portfolios of risky endowments with both positive and negative quantities and their weighted sum is zero. The optimal contracts are thus always essentially swaps, i.e., one side trades the negatives for the positives. This result may be regarded as, in a sense, the *apotheosis of swaps*.

The results that there is no need for a market for the market portfolio and that only swaps will be created rest on the assumption that the contract designer who is creating the new markets knows everything about utilities. We show one representation of lack of knowledge on the part of the market designer that brings the market portfolio back to some potential significance, Theorem 7 below. If lack of knowledge is high and if there is a strong market component to endowments, then something approximating the market portfolio may well be of first importance. Creating the market portfolio makes possible more robust definitions of subsequent markets.

Theorem 1 below is part of a framework developed in Athanasoulis [1995] and Shiller and Athanasoulis [1995]; it was developed independently by Demange and Laroque [1995b]. A related analysis is found in Duffie and Jackson [1989] and Willen [1997]. See Geanakoplos [1990] for an introduction to General Equilibrium with incomplete markets. Cass, Chichilnisky and Wu [1996] show how the number of assets needed to obtain a complete markets solution can be greatly reduced by constructing a set of mutual insurance contracts and a smaller set of Arrow securities when compared to an Arrow–Debreu world. This is related to our results as we only need assets far less than the number of states of the world to obtain a first best solution. We however consider which assets are best to construct if we cannot complete asset markets. Demange and Laroque [1995a] show that in an economy with general utilities (not necessarily quadratic), when all residual risk is hedged, then the only important assets remaining to construct in the economy are nonlinear assets, such as options, whose realizations depend exclusively on the realization of the market portfolio. Our results are complementary to this Demange and Laroque result rather than being a competing result. Our analysis here starts from no markets at all, and studies a sequence of markets to allow linear spanning of the original endowments; Demange and Laroque [1995a] are considering moving yet beyond the linear spanning, and it is in the subsequent nonlinear markets alone that the market portfolio has (under their assumptions) such importance.

The paper is organized as follows. We first lay out the assumptions of the general equilibrium model and then solve the agent's problem for a given set of available contracts. The resulting expressions for equilibrium prices and quantities will be used in all subsequent parts of this paper. We then go through several variations on the maximization problem faced by the contract designer, differing in assumptions about pre-existing markets and about the information available to the designer. We then conclude with some practical advice for contract designers.

2 The Model

There are J agents in this economy indexed by $j = 1, ..., J$, each representing an individual except in Section 11 where each agent represents a large number of individuals. All random variables are defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is the set of states of the world and $\omega \in \Omega$ is the state of the world. $\mathcal{F}$ is a σ-algebra of subsets of Ω known as events and $\mathcal{P} : \mathcal{F} \to [0, 1]$ satisfying $\mathcal{P}(\emptyset) = 0$ and $\mathcal{P}(\Omega) = 1$ is a probability measure on $(\Omega, \mathcal{F})$ held commonly by all agents in the economy.

There is a single good in the economy which is consumed. Each agent j has an endowment $x_j \in L^2(\Omega, \mathcal{F}, \mathcal{P})$ where $L^2(\Omega, \mathcal{F}, \mathcal{P})$ is the set of random variables which are square integrable, i.e., have finite mean and variance. We will denote the demeaned stochastic endowment as $\tilde{x}_j = x_j - E(x_j)$. Define x to be the $1 \times J$ vector of random endowments in the economy and similarly let $\tilde{x}$ be the $1 \times J$ vector of demeaned stochastic endowments. Then $E(\tilde{x}'\tilde{x}) = \Sigma$ is the $J \times J$ covariance matrix of the endowments in the economy. Define $E(\tilde{x}'\tilde{x}_j) = \Sigma_j$ and $E(\tilde{x}_j\tilde{x}_j) = \Sigma_{jj}$.

The $N \leq J$ contracts indexed by $n = 1, ..., N$ designed in this paper are futures contracts. Let $f_n \in L^2(\Omega, \mathcal{F}, \mathcal{P})$ be the risky transfer made in the n^{th} futures contract resulting in $f_n(\omega)$ units of consumption contingent on state $\omega \in \Omega$. To purchase the contract n, the agent must promise to pay a riskless price $p_n \in \mathcal{R}$ in the period where the state of the world is resolved. Thus if the state $\omega \in \Omega$ is realized, agents who take a long positive position in contract n receive $f_n(\omega) - p_n$, those who take a short position pay this amount. Define f to be the $N \times 1$ vector whose n^{th} element is f_n and P to be the $N \times 1$ vector whose n^{th} element is p_n. Without loss of generality we construct the futures contracts such that $E(f) = 0$ and $E(ff') = I_N$ where I_N is the $N \times N$ identity matrix. These are two normalizations that have no effect on the economy. For example, if $E(f_n) = 1$, then we need only increase the price p_n by one. So the equilibrium is invariant to these linear transformations. If $\text{var}(f_n) = 2$ then we need only increase the price of contract n by the square root of 2.

Given that we restrict our attention to the set of linear equilibria, i.e., we use quadratic utility, it must be that the optimally chosen risky transfers, f, are in the space spanned by the initial endowment risks, $\tilde{x}$. This is a point shown by Demange and Laroque [1995]. Furthermore, it must be that since f is in the space spanned by $\tilde{x}$ and we are studying linear equilibria, the optimal contracts will be linear combinations of the elements of $\tilde{x}$. Consequently we define $f = A'\tilde{x}'$ where A is a $J \times N$ matrix and $A_n'\tilde{x}' \in L^2(\Omega, \mathcal{F}, \mathcal{P})$, $n = 1, ..., N$ and A_n is the n^{th} column of A. Therefore, according to our notation, $E(ff') = A'\Sigma A = I_N$.

3 Agents

Each representative agent has a utility function $U_j : L^2(\Omega, \mathcal{F}, \mathcal{P}) \to \mathcal{R}$. We make the simplifying assumption that each agent has mean-variance utility as follows,

$$U_j = \text{E}(c_j) - \frac{\gamma_j}{2}\text{Var}(c_j) \tag{1}$$

where c_j is the consumption of agent j, the same as the endowment plus proceeds from hedging. Each agent j takes the futures contracts f which is a vector $L^2(\Omega, \mathcal{F}, \mathcal{P})$ process and the futures prices $P \in \mathcal{R}^N$ as given and solves for her optimal futures positions, q_j, as

$$q_j = \underset{q_j \in \mathcal{R}^N}{\arg\max} \{U_j | c_j = x_j + q_j'(f - P)\}. \tag{2}$$

We can rewrite this in a simpler form as

$$q_j = \underset{q_j \in \mathcal{R}^N}{\arg\max} \left\{ E(x_j) - q_j'p - \frac{\gamma_j}{2}(\Sigma_{jj} + q_j'A'\Sigma A q_j + 2q_j'A'\Sigma_j) \right\}. \tag{3}$$

Remembering that $A'\Sigma A = I_N$, the optimal demand for this agent is

$$q_j = -\frac{1}{\gamma_j}P - \mathrm{Cov}(f, \tilde{x}_j) = -\frac{1}{\gamma_j}P - A'\Sigma_j. \quad (4)$$

This demand curve tells us that agent j will purchase more of a security as its price declines. She will purchase less of the security the more it covaries with her endowment since it provides less hedging services. To help the exposition of this paper it is convenient to form the $N \times J$ matrix Q whose j^{th} column is q_j and rewrite (4) as

$$Q = -P\iota'\Gamma^{-1} - \mathrm{Cov}(f, \tilde{x}) = -P\iota'\Gamma^{-1} - A'\Sigma \quad (5)$$

where Γ is the $J \times J$ diagonal matrix with the j^{th} diagonal element equal to γ_j and ι the $J \times 1$ unit vector.

4 Equilibrium

The equilibrium condition in this economy is simply that the futures contracts are in zero net supply. We can represent equilibrium in this economy as

$$Q\iota = 0 = -P\iota'\Gamma^{-1}\iota - A'\Sigma\iota. \quad (6)$$

From equilibrium condition (6) we can derive the equilibrium pricing equation

$$P = -A'\Sigma\iota\left(\iota'\Gamma^{-1}\iota\right)^{-1}. \quad (7)$$

Definition: The market portfolio is defined by its dividend

$$f_m = \tilde{x}m \quad (8)$$

where m is a scaled unit vector

$$m \equiv \frac{\iota}{(\iota'\Sigma\iota)^{.5}}. \quad (9)$$

If we multiply and divide the right hand side of (7) by $(\iota'\Sigma\iota)^{.5}$ then the price of a contract n depends on $A'_n\Sigma\iota(\iota'\Sigma\iota)^{-.5} \equiv A'_n\Sigma m$, the covariance of contract n with the market. Thus we can derive the CAPM pricing equation from equation (7). If the covariance of a contract with the market is zero, as for example with a risk free asset, then the price of this asset is $p_f = 0$. The price of the market portfolio asset is $p_m = -\frac{(\iota'\Sigma\iota)^{.5}}{\iota'\Gamma^{-1}\iota}$. It follows that:

$$p_n - p_f = \frac{\text{cov}(f_n, f_m)}{\text{var}(f_m)}(p_m - p_f) \tag{10}$$

which is the familiar CAPM pricing equation and $\frac{\text{cov}(f_n,f_m)}{\text{var}(f_m)}$ is the familiar beta of the CAPM model. Similar results are obtained by Magill and Quinzii [1996], Duffie and Jackson [1989], Oh [1996] and Mayers [1972]. Substituting (7) into (5) we also obtain

$$Q = -A'\Sigma M \tag{11}$$

and we define $M \equiv I_J - \iota\iota'\Gamma^{-1}\left(\iota'\Gamma^{-1}\iota\right)^{-1}$ and $A \equiv [A_1 : A_2 : \cdots : A_N]$, where A_n is the n^{th} column of A. These are the equilibrium demands in matrix form. Looking closely at the above expressions we see that ΣM is the $J \times J$ matrix whose j^{th} column is the amount of each risk agent j wants to sell off at market-clearing prices. In the end, if markets are complete, each agent will hold the inverse of her own risk aversion times the harmonic mean of all individuals' risk aversion, of the market. This result will be recognizable to those familiar with the CAPM economy, see Huang and Litzenberger [1988].

5 Contract Design

The contract designer's problem is to maximize welfare, total utility, in the economy given she is constrained to choose $N \leq J$ contracts. The contract designer will choose the $J \times N$ matrix A to maximize the sum of utilities in the economy. From (3) we know that each agent's utility is given by

$$E(x_j) - q_j'p - \frac{\gamma_j}{2}(\Sigma_{jj} + q_j'A'\Sigma A q_j + 2q_j'A'\Sigma_j). \tag{12}$$

If we sum over all J agents, drop $E(x_j)$, and put this in matrix form we obtain

$$\text{tr}\left(-Q'P\iota' - \tfrac{1}{2}\Gamma\left(\Sigma + Q'Q + 2Q'A'\Sigma\right)\right) \tag{13}$$

where tr denotes the trace. If we substitute (11) and (7) into (13) we obtain

$$\text{tr}\left(\tfrac{1}{2}A'\Sigma M\Gamma M'\Sigma A - \tfrac{1}{2}\Gamma\Sigma\right) \tag{14}$$

where the term $\frac{1}{2}\Gamma\Sigma$ has no effect on the contract designer's decision. Thus, using $\text{tr}(AB) = \text{tr}(BA)$, the contract designer's problem simplifies to

$$A \in \underset{A_n \in \mathcal{R}^J, n=1,\ldots,N}{\arg\max} \left\{\text{tr}\left(A'\Sigma M\Gamma M'\Sigma A\right) | A'\Sigma A = I_N\right\}. \tag{15}$$

This leads to a fundamental theorem shown separately by Demange and Laroque [1995b] and by Shiller and Athanasoulis [1995]:

Theorem 1: *The A matrix that solves* (15) *has columns corresponding to the N eigenvectors with highest eigenvalues of:*

$$M\Gamma M'\Sigma. \tag{16}$$

Proof: We may write the Lagrangian as

$$\begin{aligned} \mathcal{L} &= A_1'\Sigma M\Gamma M'\Sigma A_1 + \cdots + A_N'\Sigma M\Gamma M'\Sigma A_N \\ &\quad -\lambda_1(A_1'\Sigma A_1 - 1) + \cdots + \lambda_N(A_N'\Sigma A_N - 1). \end{aligned} \tag{17}$$

We are requiring in this problem that the diagonal of the matrix $A'\Sigma A$ is equal to ι. The first order conditions can be written as

$$\Sigma M\Gamma M'\Sigma A_i = \lambda_i \Sigma A_i, \qquad \forall i = 1, \cdots, N \tag{18}$$

and

$$A_i'\Sigma A_i = 1, \qquad \forall i = 1, \cdots, N \tag{19}$$

If we define Λ to be the $N \times N$ diagonal matrix with the n^{th} element to be λ_n we can combine the first order conditions to obtain

$$\Sigma M\Gamma M'\Sigma A = \Sigma A\Lambda \tag{20}$$

and

$$\text{diag}(A'\Sigma A) = \iota. \tag{21}$$

Thus, taking the inverse of Σ through equation (20) gives us the result. Finally if one premultiplies equation (20) by A, one obtains $A'\Sigma M\Gamma M'\Sigma A = \Lambda$. The trace of the left hand side of this is the objective function the planner is trying to maximize.

Since this equals Λ, it is diagonal and as such the planner will choose the N eigenvectors corresponding to the N largest eigenvalues.[3] □

Note that if we take a Cholesky decomposition of the variance matrix Σ, $\Sigma = C'C$, and premultiply through equation (20) by C'^{-1}, then $CM\Gamma M'C'$ is positive semidefinite and symmetric with eigenvectors CA. The eigenvalues of a positive semidefinite symmetric matrix are all real and nonnegative,

[3] The second order conditions that we have a maximum are satisfied.

and these are the same as the eigenvalues of $M\Gamma M'\Sigma$. Since the rank of M is $J-1$, there are only $J-1$ nonzero eigenvalues, and hence only $J-1$ contracts are of any value. Thus, there is no point in creating all J possible contracts, at most $J-1$ are needed and A need have no more than $J-1$ columns. If there is a fixed cost to creating markets, then N, the number of markets created, can be chosen optimally. We create all markets whose eigenvalues are greater than this cost.

One will notice in the above problem that we did not constrain the off diagonal elements of $A'\Sigma A$ to be zero. Notice, however, that Λ is diagonal and since $C'M\Gamma M'C$ is positive semidefinite and symmetric with eigenvectors CA, it follows that $A'\Sigma M\Gamma M'\Sigma A$ is diagonal. Since $A'\Sigma M\Gamma M'\Sigma A = A'\Sigma A\Lambda$ it must be that $A'\Sigma A$ is diagonal. Thus the constraint that the off diagonal elements are zero are satisfied in the unconstrained problem. This was shown by Darroch [1965] and by Okamoto and Kanazawa [1968].

6 The Market Portfolio Is Least Important

It is now very easy to prove our featured result, that the market portfolio is in a sense the least important portfolio to allow trading in:

Theorem 2: *The A matrix that solves problem* (15) *is orthogonal to* $g \equiv \iota'\Gamma^{-1}(\iota'\Gamma^{-1}\Sigma\Gamma^{-1}\iota)^{-.5}$, *and all $N \leq J-1$ markets together do not span the market portfolio.*

Proof: By (20) it follows that $A = M\Gamma M'\Sigma A\Lambda^{-1}$. Since $gM = 0$, it follows that $gA = 0$, i.e., the A matrix is orthogonal to g. We can then show by contradiction that all $N \leq J-1$ assets do not span the market portfolio: if there exists a vector v such that $Av = m$, then $gAv = gm = \iota'\Gamma^{-1}\iota(\iota\Sigma\iota)^{-.5}(\iota'\Gamma^{-1}\Sigma\Gamma^{-1}\iota)^{-.5} \neq 0$ which is a contradiction. □

Lemma 1: *In the case where agents have the same risk aversion, γ_j, the market portfolio is orthogonal to all optimal contracts.*

Proof: By Theorem 2, g is orthogonal to all optimal contracts and $g = m$ here. □

For any J, the result that $gA = 0$ means that non linear combination of the N optimal contracts can be constructed with all positive elements.

Only “swaps” between endowments can be constructed by portfolios of the optimal contracts. No matter how many markets we choose to create (regardless of N), it will be impossible to renormalize these markets, define different markets as linear combinations of them, so that any market is not a swap. All possible portfolios constructed from the optimal portfolios represent exchanges of endowments for other endowments. Since the market portfolio holds positive quantities of all endowments, it is an example of a market that cannot be constructed from the optimal contracts constructed from the above method. Consider the case where all agents have the same risk aversion, Lemma 1, so that Γ is proportional to the identity matrix. It then follows from this theorem and lemma that in all possible portfolios constructed from the optimal contracts defined by A, the sum of the portfolio weights in terms of endowments are zero and the portfolios are orthogonal to the market portfolio. Furthermore, if $N = J - 1$, then the resulting equilibrium is Pareto optimal. See Magill and Quinzii [1996, p. 181] for Pareto optimality of the CAPM equilibrium. This follows here since the case with $J - 1$ contracts results in the CAPM equilibrium, i.e., CAPM allocation of risk.

To understand these results better let us consider a two-agent example: A two-agent example ignores some of the complexity that the optimal market solution method is supposed to handle, but it will make some basic concepts more transparent. We can then illustrate the solution to the contract designer's problem on a simple two-dimensional graph, Figures 1 and 2, with the first element of A_1, a_1, on the horizontal axis and the second element of A_1, a_2, on the vertical axis. On these figures the constraint $A_1' \Sigma A_1 = 1$ is that the A_1 vector must end somewhere on the ellipse shown. The ellipse shown illustrates a case of positive correlation between the two endowments, where both endowments have the same variance and a correlation coefficient of one half. On each figure, iso-welfare curves are parallel straight lines (one pair of which is shown). The further from the origin the higher the welfare.

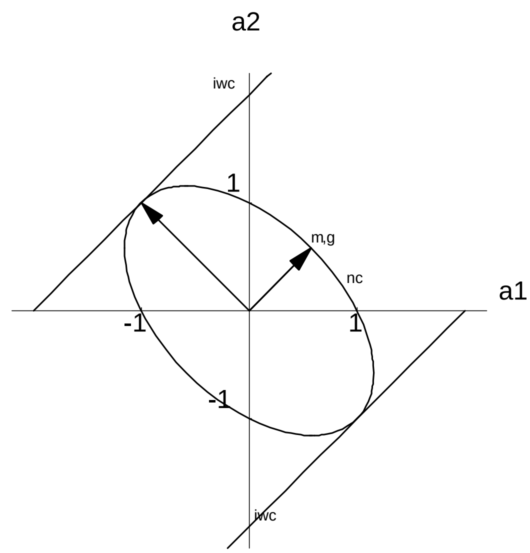

FIGURE 1: Illustration of optimal portfolio weights when both agents have same risk aversion, iwc is an iso-welfare curve, nc is the normalization constraint.

The optimal vector A_1 must be orthogonal to g, which means that the vector is in the upper left quadrant (or lower right), and is not the in the same quadrant where the market portfolio vector m is. In Figure 1, the case is shown where all the γ's are one, and so g equals the market portfolio vector. Each agent will use the optimal contract to swap half of her endowment risk for half of the other's, and both agents will end up holding the market portfolio. In this case, the optimal contract is orthogonal to the market portfolio, and the market portfolio contract would be utterly useless to the agents if it were created instead of the optimal contract. The optimal contract is found on the graph by finding the highest iso-welfare curve, iwc, that satisfies the constraint, tangent to the ellipse. Clearly in this symmetric situation there is no value to being able to trade the market portfolio for these agents, as they would both like to take the same position.

In Figure 2, the case is shown where γ_1 equals 3 and γ_2 equals 1. Now, the g vector no longer coincides with the market portfolio vector, m, and now the optimal A_1 vector results in an unequal swap. In the swap, the more risk averse agent gives up three times as much of the risky component of her endowment to the other agent, and pays a price to the other agent for doing so. After the swap, the more risk averse agent is bearing only one quarter of world income risk, the less risk averse agent is bearing three quarters. This is the Pareto optimal outcome: there are no more risk sharing opportunities, and each agent is bearing world income risk in accordance with his or her own risk preferences. Note that in this case had we instead created the market portfolio first, it would have been of some use though it would touch an iso-welfare curve that is closer to the origin. In both figures, the isoquants for the objective function in (15) are parallel straight lines with just such a slope that the tangency between them and the ellipse $A_1' \Sigma A_1 - 1 = 0$ occurs at a point defining a vector perpendicular to g.

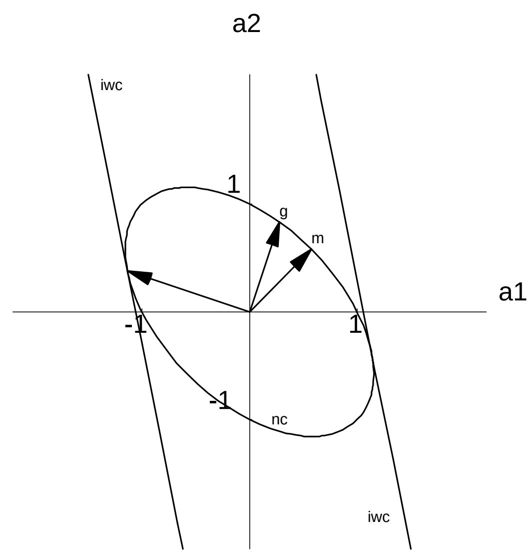

FIGURE 2: Illustration of optimal portfolio weights when agent 1 is three times more risk averse than agent 2, *iwc* is an iso-welfare curve, *nc* is the normalization constraint.

7 The Market Portfolio Has the Highest Absolute Value of Price

Even though, as we have concluded, the market portfolio is in this model an unimportant contract to trade, it remains true that the market portfolio is the most important market by a different measure: it is the contract (subject to our normalization) that has the highest possible price. Let us change the objective of the contract designer in designing the first market to maximize the price of the contract, in designing the second market to maximize price subject to zero covariance with the first market, and so on. Using equation

(7) the problem becomes

$$A_1 = \underset{A_1 \in \mathcal{R}^J}{\arg\max} \left\{ \left(A_1' \Sigma \iota \left[\iota' \Gamma^{-1} \iota \right]^{-1} \right) | A_1' \Sigma A_1 = 1 \right\} \tag{22}$$

Theorem 3: *The contract that satisfies* (22) *is the market portfolio, i.e.,* $A_1 = m$.

Proof: The first order conditions can be written as

$$\Sigma \iota \left[\iota' \Gamma^{-1} \iota \right]^{-1} = \lambda \Sigma A_1. \tag{23}$$

and

$$A_1' \Sigma A_1 = 1. \tag{24}$$

Solving these we obtain the result. □

Thus as in the CAPM, the only insurance which costs anything is to insure oneself against the market.

It may seem puzzling that the market is completely unimportant to construct by a social welfare criterion and yet has the highest absolute value of price. But note that the expression to be maximized by the social welfare criterion (15) is the trace of $A' \Sigma M \Gamma M \Sigma A$, and

$$A' \Sigma M \Gamma M' \Sigma A \equiv A' \Sigma \Gamma \Sigma A - A \Sigma u' \Sigma A \left(\iota' \Gamma^{-1} \iota \right)^{-1} \equiv A' \Sigma \Gamma \Sigma A - P P' \left(\iota' \Gamma^{-1} \iota \right). \tag{25}$$

The trace of the first term in the above expression is the expression to be maximized by a contract designer choosing A to minimize a γ-weighted sum of variance of agents' endowments subject to the normalization constraint and subject to the constraint that all prices are zero, see Shiller and Athanasoulis [1995], and the trace of the second term is proportional to minus the sum of squared prices. Higher absolute value of price counts against welfare gain because higher price means more of the risk is redistributed (market risk) rather than pooled (residual risk).

8 Pre-existing Markets

The above theorems take no account of pre-existing markets, markets for some endowments or linear combinations of endowments that already exist before the contract designer begins to define new markets (contracts).

Suppose that we modify problem (15) to represent that there is a single pre-existing contract, where the coefficients of the endowments in the linear combination that defines this pre-existing contract are given by the $J \times 1$ vector A_1, the first column of A, which, without loss of generality, we normalize so that $A_1' \Sigma A_1 = 1$. (It is trivial to extend our results to more than one pre-existing contract.) The contract designer will then design $N^* = N - 1$ markets, choose $A^* = [A_1^* \ A_2^* \cdots A_{N^*}^*]$, the remaining columns of A, $(A = [A_1 : A^*])$ subject to the normalization rule $A'\Sigma A = I$. Then A^* is defined by: s

$$A^* \in \underset{A_n^* \in \mathcal{R}^J, n=1,\ldots,N^*}{\arg\max} \left\{ \operatorname{tr}\left(A^{*\prime} \Sigma M \Gamma M' \Sigma A^* \right) | A^{*\prime} \Sigma A^* = I_{N^*}, A^{*\prime} \Sigma A_1 = 0 \right\} \tag{26}$$

Theorem 5: *The A^* matrix that solves* (26) *has columns corresponding to the N^* eigenvectors with highest eigenvalues of:*

$$\Phi M \Gamma M' \Phi' \Sigma \tag{27}$$

where $\Phi \equiv I_J - A_1 A_1' \Sigma$.

Proof: We can write the Lagrangian as

$$\begin{aligned} \mathcal{L} = & \ A_1^{*\prime} \Sigma M \Gamma M' \Sigma A_1^* + \cdots + A_{N^*}^{*\prime} \Sigma M \Gamma M' \Sigma A_{N^*}^* - \lambda_1 \left(A_1^{*\prime} \Sigma A_1^* - 1 \right) \\ & + \cdots - \lambda_{N^*} \left(A_{N^*}^{*\prime} \Sigma A_{N^*}^* - 1 \right) - \delta_1 A_1^{*\prime} \Sigma A_1 + \cdots - \delta_N^* A_{N^*}^{*\prime} \Sigma A_1 \end{aligned} \tag{28}$$

The first order conditions are

$$2\Sigma M \Gamma M' \Sigma A_{n*} - 2\lambda_{n^*} \Sigma A_{n^*} - \delta_{n^*} \Sigma A_1 = 0, \ \forall n^* = 2, \cdots, N^* \tag{29}$$

and

$$A_{n^*}^{*\prime} \Sigma A_{n^*}^* - 1 = 0, \quad \forall n^* = 2, \cdots, N^* \tag{30}$$

$$A_{n^*}^{*\prime} \Sigma A_1 = 0, \quad \forall n^* = 2, \cdots, N^* \tag{31}$$

If we premultiply equation (29) by A_1', then we obtain $\delta_{n^*} = 2A_1' \Sigma M \Gamma M' \Sigma A_{n^*}$. If we substitute δ_{n^*} into equation (29), form the N^* equations $n^* = 1, \cdots, N^*$ into a matrix and rearrange, we arrive at an equation in terms of eigenvectors of (27). If we premultiply equation (29) by $A_{n^*}^{*\prime}$ then we obtain $A_{n^*}^{*\prime} \Sigma M \Gamma M' \Sigma A_{n^*}^{*\prime} = \lambda_{n^*}$ which is the elements of the expression of the function the planner is trying to maximize. Thus the planner chooses the columns of A^* as the N^* eigenvectors with the highest eigenvalues of (27). $\square$

An example can be constructed that illustrates that with one pre-existing market, if we are to create only one more market optimally as we have defined, then the resulting two markets may span the market portfolio. Suppose that A_1 is a column of zeros except for the first element, which is strictly positive. The pre-existing market is just a market for the endowment of the first agent. Suppose, for simplicity, that Σ equals the identity matrix and that Γ also equals the identity matrix except that the upper left element is not one but a "very" large number; the first agent is very risk averse. With these assumptions, if there had been no pre-existing market, the first market to create would have been a swap between the first agent and the rest of the world, with all other agents receiving equal weight in the contract. With the pre-existing market, the optimal A^* will be proportional to a column of ones with the first element replaced with zero; creating this market will enable the first agent to swap her endowment risk for the rest of the world's, by shorting the first market and going long the second. In this example A_1 and A^* together also span the market portfolio.

The result that pre-existing markets may cause the contract designer optimally to create contracts that allow spanning of the market portfolio does not mean that the market portfolio is in any real sense important. In the above example, the agents use the two markets to construct a swap between the first agent's endowment and world endowment, not to take a position in world endowment. Had the contract designer, in constructing the contract represented by A^*, ignored orthogonality with the pre-existing market and just created the contract defined as the solution to (15), thereby directly creating the swap between the first agent's endowment and the rest-of-the-world endowment, then almost all the welfare improvement available to hedgers would be available just by using the second market. One may suppose that if the welfare gain available through the pre-existing market is small enough then it might well disappear after the second market is created.

9 The Market Portfolio as a Pre-existing Market

It is instructive to consider the problem for the contract designer with the constraint that the first market must be the market for the market portfolio that is, assuming that $A_1 = m$. While a market for the world portfolio of endowments does not now exist, we shall see that there may be reasons to construct it. At the very least, as we shall see in this section, these markets are conceptually relatively simple to understand, and such simplicity might promote more effective use of the markets.

Lemma 2: *If the market portfolio exists (i.e., if $A_1 = m$) then all other assets (constructed so that our normalization $A'\Sigma A = I_N$ holds) will necessarily have a zero price.*

Proof: If the first contract is the market then it must be the case that the rest of the contracts $A^*_{n^*}$, $n^* = 1, \cdots, N^*$ are constructed such that $A^{*\prime}_{n^*}\Sigma A_1 = \frac{A^{*\prime}_{n^*}\Sigma\iota}{\sqrt{\iota'\Sigma\iota}} = 0$. If this is the case, then $A'_{n^*}\Sigma\iota = 0$ and from equation (7), the result follows. □

Let us define the $N^* \times J$ matrix Q^* such that its j^{th} column is the demand vector for agent j of the N^* contracts. We then have:

Theorem 5: *When $A_1 = m$ the A^* matrix that solves* (26) *has the property that $Q^* = -A^{*\prime}\Sigma M$ has columns corresponding to the N^* eigenvectors with highest eigenvalues of:*

$$\Phi'\Sigma\Phi\Gamma \tag{32}$$

Proof: Using equation (25) and Lemma 2, the problem the social planner solves is

$$A^* \in \underset{A^*_n \in \mathcal{R}^J, n^*=1,\ldots,N^*}{\arg\max} \{\text{tr}(A^{*\prime}\Sigma\Gamma\Sigma A^*)|A^{*\prime}\Sigma A^* = I_{N^*},\ A^{*\prime}\Sigma A_1 = 0,\ A_1 = m\} \tag{33}$$

Proceed as with Theorem 4. □

$\Phi'\Sigma\Phi$ is the variance matrix of residuals when the endowments are regressed on the world endowment. If $\Gamma = I_J$, i.e., that is if everyone has the same risk aversion, then the optimal markets are defined in terms of eigenvectors of this simple variance matrix. Moreover, since $Q^{*\prime} = -\Sigma A^*$, the position that agent j holds of the n^{th} contract is just the regression coefficient corresponding to the n^{th} contract when the endowment of that agent is regressed on the vector of contract payoffs xA^*. These results, coupled with the above-noted zero prices for all contracts other than the market contract, make this equilibrium a simple one to understand. Once the market portfolio exists, the problem facing the contract designer is a weighted variance minimization problem weighted by risk aversion.

10 Uncertainty About Preferences

The preceding analysis assumed great knowledge on the part of the contract designer: the designer was assumed to know perfectly all utility functions. The unrealism of this assumption would appear to be an issue if we try to apply this analysis to the design of actual markets. We show that the relaxation of this assumption may restore the importance of the market portfolio.

Uncertainty about preferences poses a real problem to the contract designer since we cannot assume that agents have the same uncertainty about their own preference parameters that the contract designer does. Agents have perfect knowledge about their own preference parameters and maximize their expected utility knowing their γ_j. The above analysis of market equilibrium, equations (6)–(11), must be done for the agents' true risk preferences. When we arrive at the contract designer's problem, (15), we face the problem that the contract designer does not know the true M and Γ matrices. The only reason the contract designer does not know agent's demands is because she does not know the coefficients of risk aversions. Supposing now that the true elements of Γ are unknown to the market designer, we will suppose that the market designer chooses $N \leq J$ contracts to solve a maximization problem which is the same as (15) but replacing the unknown value to be maximized in (15) with its expected value:

$$A \in \underset{A_n \in \mathcal{R}^J, n=1,\ldots,N}{\arg\max} \left\{ \operatorname{tr}\left[E\left(A'\Sigma M \Gamma M' \Sigma A \right) \right] | A'\Sigma A = I_N \right\} \tag{34}$$

Note that since M is a function of Γ, the expression involves expectations of a nonlinear function of Γ. In order to deal with (34), we rewrite the matrix $A'\Sigma M \Gamma M' \Sigma A$ as

$$A'\Sigma M \Gamma M' \Sigma A \equiv A'\Sigma \Gamma \Sigma A - A'\Sigma \iota \iota' \Sigma A \left(\iota' \Gamma^{-1} \iota \right)^{-1}. \tag{35}$$

One obtains equation (35) by substituting in for M.

Theorem 6: *The A matrix that solves* (34) *has columns corresponding to the N eigenvectors with highest eigenvalues of:*

$$E(\Gamma)\Sigma - \iota\iota'\Sigma E(\iota'\Gamma^{-1}\iota)^{-1} \tag{36}$$

Proof: Substitute (35) into (34) and proceed as in Theorem 1. □

Note that, unless $E\left[\Gamma - \iota\iota'(\iota'\Gamma^{-1}\iota)^{-1}\right]$ is singular, the matrix (36) is generally non-singular, and so our conclusion above that only $J-1$ markets are needed no longer holds. If there is no constraint on the number of markets constructed, the contract designer will create all J contracts, and then the contracts will span the market portfolio. Let us assume the γ's for all $j = 1, ..., J$ are iid. This assumption represents a symmetric state of knowledge of all individuals' risk aversion parameters. With this assumption we can rescale (36) as

$$\Sigma - c\iota\iota'\Sigma \tag{37}$$

where $c = \frac{E(\iota'\Gamma^{-1}\iota)^{-1}}{E(\gamma)}$.

With (37) we can easily take account of specific distributional assumptions about Γ. We need only derive the expected value and expected value of the harmonic mean of the elements of Γ, to define the scalar c.

The limiting case of this problem, when the standard deviation of γ increases to infinity, is particularly interesting. This is the case where the contract designer's information is becoming more diffuse.

Theorem 7: *If γ_j, $j = 1, \cdots, J$ are* iid *lognormal variates then as the variance, σ^2, of* $\ln(\gamma_j)$ *goes to infinity, the A matrix that solves* (34) *approaches a matrix whose columns are N eigenvectors of Σ with the corresponding highest eigenvalues.*

Proof: Define the geometric mean of the risk aversion to be $G = \left(\prod_{j=1}^{J}\gamma_j\right)^{\frac{1}{J}}$ and the harmonic mean as $H = \left(\frac{1}{J}\sum_{j=1}^{J}\gamma_j^{-1}\right)^{-1}$. Under the lognormal assumption $\frac{E(G)}{E(\gamma)} = \frac{\exp(\mu+(\sigma^2/2J))}{\exp(\mu+(\sigma^2/2))} = \exp\left(-\sigma^2\left(\frac{(J-1)}{2J}\right)\right)$. Therefore $\lim_{\sigma^2\to\infty}\frac{E(G)}{E(\gamma)} = 0$. Since $H < G$ everywhere, (see for example Hardy et al. [1964], p. 26) then $\lim_{\sigma^2\to\infty}\frac{E(H)}{E(\gamma)} = \lim_{\sigma^2\to\infty} c = 0$. Thus, the limit of the matrix (37) as σ^2 goes to infinity is Σ. Since the solution of problem (34) is a continuous function of the elements of the matrix (37), and since the limit of a continuous function is the function of the limit, the theorem follows. □

If one is going to construct some contract given she knows nothing about the utilities in the economy, what should the contract be? One wants to somehow maximize the probability that their contract will have the highest welfare improvement in the economy. As such the contract designer should construct the contract that markets the largest component of risk in the economy. This is exactly the result of Theorem 7. Given we know nothing about risk aversion, we have the best chance of welfare improvement in the economy by allowing agents to hedge the most risk possible. The first principal component of Σ is unrestricted by our theory. It could have all positive elements and could approximate the market portfolio.

If the first principal component of Σ is approximately the market portfolio and its eigenvalue is large, then people have a substantial covariance with the market. Among those agents with similar market exposures, those who are more risk averse can sell a share of the market portfolio to less risk averse agents, thereby reducing their risk. We do not need to know who is more risk averse in setting up markets to make this possible.

Let us return to the two-by-two examples that were plotted in Figures 1 and 2. If we do not know which agent is the more risk averse, then this maximization problem facing the contract designer is not as simple as it appeared from that figure. We do not know the position of the vector g, that is whether Figure 1, Figure 2 or some other figure is relevant. Thus the position of the optimal A_1 vector cannot be determined.

We plot instead in Figures 3 and 4 the expected iso-welfare-curve to the maximization problem (34). These are not parallel straight lines but ellipses. If we have only a little uncertainty about risk aversion, see for example Figure 3 where $c = .49$, the expected iso-welfare curves are elongated and near the origin resemble the parallel straight lines of Figure 1. But if our uncertainty about risk aversion is large, see Figure 4, where $c = 0$, the expected iso-welfare curves are elongated in the perpendicular direction. In the extreme case, where the uncertainty about agents' risk aversion makes it very probable that one is much more risk averse than the other, then, not knowing which is the more risk averse, the best contract we can design in this example is simply a market for the market portfolio.

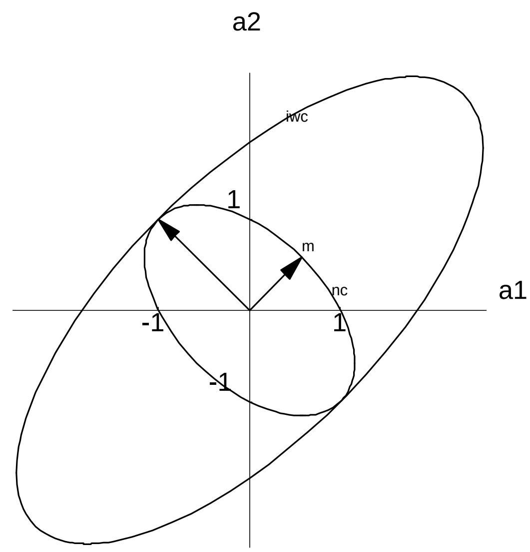

FIGURE 3: Illustration of optimal portfolio weights when risk aversions are iid and $c = .49$. *eiwc* is an expected iso-welfare curve and *nc* is the normalization constraint.

With very little uncertainty in these terms about the γ's, the optimal A_1 for our two-agent example with iid γ's will still be a vector perpendicular to the market portfolio, a vector with a slope of minus one. Note that Figure 3, where $c = 0.49$, resembles Figure 1 in the vicinity of the origin. Figure 1 corresponds to $c = 0.5$. However, even a small amount of uncertainty means that there will still be a reason to create a second market, and A_2 will be the market portfolio vector, in the first quadrant, with slope of plus one. As the uncertainty about the γ's increases, the eigenvalue corresponding to A_1 shrinks relative to the eigenvalue corresponding to A_2, and at some point becomes the lower; at this point we must switch the order of the columns of A, and the market portfolio becomes the best portfolio to create. What has happened finally is that uncertainty about the γ's has become so great that we can no longer predict what kinds of swaps will be useful to agents. The market portfolio may still be useful if either agent is more risk averse than the other; that agent can sell part of the market component of her endowment to the other.

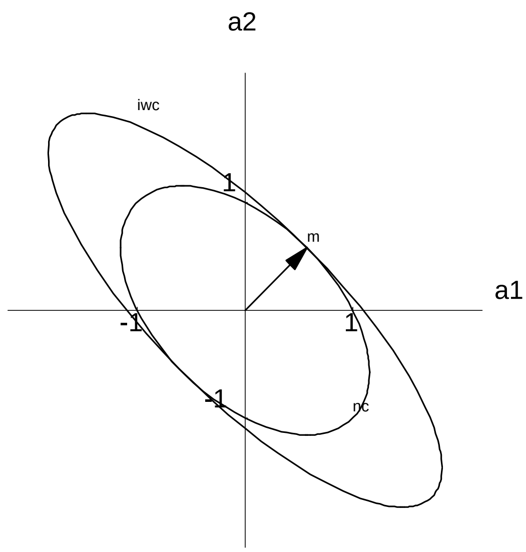

FIGURE 4: Illustration of optimal portfolio weights when risk aversions are iid and $c = 0$. *eiwc* is an expected iso-welfare curve and *nc* is the normalization constraint.

Note that this conclusion using the lognormal assumption might be generalized to other distributions but it is not true of all distributions of $\gamma_i > 0$ with finite means. The important point of the theorem is that the contract designer's information about agents' utilities becomes more diffuse. If for some reason, as the variance approaches infinity, the contract designer's information becomes less diffuse, then the contract designer can better construct assets since she has more information which result in more welfare improvement.

Consider, for example, a case where γ_j can only take on two values, γ_{j1} and γ_{j2}. γ_{j1} is fixed, the mean $\bar{\gamma}$ is fixed and we vary γ_{j2}. The probability we observe γ_{j1} or γ_{j2} are pr_1 and pr_2 respectively. Thus we have

$$pr_1\gamma_{j1} + pr_2\gamma_{j2} = \bar{\gamma} \tag{38}$$

and

$$\mathrm{Var}(\gamma) = pr_1(\gamma_{j1} - \bar{\gamma})^2 + pr_2(\gamma_{j2} - \bar{\gamma})^2. \tag{39}$$

We increase the variance of γ_j by moving the higher value γ_{j2} towards infinity. As we do this we reduce the probability pr_2 that risk aversion for person j equals γ_{j2}. It is easy to show that in the limit, as the variance is increased to infinity, i.e., as $\gamma_{j2} \to \infty$ the expected value of the harmonic mean of J values approaches γ_{j1}. In the limit, the probability approaches one that all J values are the same so that the probability approaches one that the expected value equals the harmonic mean of the J values. This example shows that all peoples risk aversion approaches γ_{j1} in the limit and thus as the variance goes to infinity, the contract designer becomes more informed.

11 Each Agent Represents K People

We now suppose that each of the J "agents" is a group of K people who share the same endowment, but may differ from each other in terms of risk tolerances as measured by γ. Each "agent" may represent a country or an occupational group.

Allowing multiple individuals per "agent" is important, since in practice we are likely to want to apply the methods for contract design not to data on individual endowments but to data on endowments of groupings of individuals. It is also important to consider multiple individuals per "agent" since our uncertainty about risk aversion may be better thought of as recognition of diversity of risk aversions within each group, rather than as uncertainty about the average risk aversion of all people in each group.

Assuming that all individuals in a group share the same endowment, the variance matrix of individual endowments is now $\bar{\Sigma} = \Sigma \otimes (\kappa\kappa')$ where $\otimes$ denotes the kronecker product and κ is a K-element column vector of ones. Assuming that all individuals risk parameters γ are iid regardless of the "agent" group to which the individual belongs, we suppose that the contract designer desires to maximize total utility of all individuals, i.e., to find the matrix $\bar{A}$ that solves:

$$\bar{A} \in \underset{\bar{A}_n \in R^J, n=1,\ldots N}{\arg\max} \left\{ \text{tr}\left(\bar{A}'\bar{\Sigma}\bar{\Sigma}\bar{A} - \bar{c}\bar{A}'\bar{\Sigma}\bar{\iota}\bar{\iota}'\bar{\Sigma}\bar{A}\right) | \bar{A}'\bar{\Sigma}\bar{A} = I_N \right\} \tag{40}$$

where $\bar{c} \equiv \frac{E(\bar{\iota}'\bar{\Gamma}^{-1}\bar{\iota})^{-1}}{E(\gamma)}$ and where $\bar{\iota} = \iota \otimes \kappa$ and $\bar{\Gamma}$ is $\Gamma \otimes I_K$.

Theorem 8: *The $\bar{A}$ that solves* (40) *equals $A \otimes \kappa$ where A solves* (34).[4]

Proof: To prove this we use the multiplication rule for kronecker products, $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$. Note that $\bar{c} = \frac{c}{K}$, where $c \equiv$

[4] There are an infinite number of $\bar{A}$'s which will solve (40) of which $A \otimes \kappa$ is one. All result in the same equilibrium.

$\frac{E(\iota'\Gamma^{-1}\iota)^{-1}}{E(\gamma)}$. We have $\bar{H} \equiv \bar{\Sigma} - \overline{c\iota\iota'}\bar{\Sigma} = \Sigma \otimes (\kappa\kappa') - \frac{c}{K}((\iota\iota') \otimes (\kappa\kappa'))(\Sigma \otimes \kappa\kappa')$ $= \Sigma \otimes (\kappa\kappa') - \frac{c}{K}((\iota\iota'\Sigma) \otimes (\kappa\kappa')(\kappa\kappa')) = \Sigma \otimes (\kappa\kappa') - c(\iota\iota'\Sigma \otimes (\kappa\kappa')) = H \otimes (\kappa\kappa')$, where $H \equiv \Sigma - c(\iota\iota')\Sigma$. Now, from above we know that $HA = A\Lambda$. $\bar{H}(A \otimes \kappa) = (H \otimes (\kappa\kappa'))(A \otimes \kappa) = (HA) \otimes (\kappa\kappa'\kappa) = KHA \otimes \kappa$. We can also show that $(A\Lambda) \otimes \kappa = (A \otimes \kappa)\Lambda$. Hence $\bar{H}(A \otimes \kappa) = (A \otimes \kappa)(K\Lambda)$. Thus, the same set of eigenvectors that solve (34) solve (40), with eigenvalues multiplied by K. □

Thus, the bigger problem of designing optimal markets for all NK people collapses to the simpler problem discussed in the preceding section. Note that since $\bar{H}$ is the same rank as H, there are no more nonzero eigenvalues, the presence of K individuals per "agent" does not introduce the need for any more than J markets.

12 Uncertainty About Preferences with a Pre-existing Market Portfolio

We have seen in an earlier section that if the pre-existing market is the market portfolio, then all remaining contracts constructed, such that $A'\Sigma A = I$, have a zero price, Lemma 2. It is interesting to ask what the optimal contracts are if there is uncertainty about γ's and the market portfolio already exists. The contract designer chooses A^* to

$$A^* \in \underset{A_n^* \in \mathcal{R}^J, n^*=1,\ldots,N^*}{\arg\max} \{E\text{tr}(A^{*\prime}\Sigma M \Gamma M'\Sigma A^*) | A^{*\prime}\Sigma A^* = I_{N^*}, A^{*\prime}\Sigma A_1 = 0, A_1 = m\}. \tag{41}$$

Using (35) and noting from the constraints that $A^{*\prime}\Sigma\iota = 0$, we may rewrite the contract designer's problem as

$$A^* \in \underset{A_n^* \in \mathcal{R}^J, n^*=1,\ldots,N^*}{\arg\max} \{\text{tr}(A^{*\prime}\Sigma E(\Gamma)\Sigma A^*) | A^{*\prime}\Sigma A^* = I_{N^*}, A^{*\prime}\Sigma A_1 = 0, A_1 = m\}. \tag{42}$$

Theorem : *The A^* matrix that solves (42) has the property that $Q^* = -A^{*\prime}\Sigma$ has columns corresponding to the N^* eigenvectors with highest eigenvalues of:*

$$\Phi'\Sigma\Phi E(\Gamma). \tag{43}$$

Proof: Proceed as with Theorem 5. □

This theorem shows that given the expectations of Γ, uncertainty about the γ's does not affect the optimal markets when the market portfolio is a pre-existing market. We know that the amount of uncertainty, Theorem 6, or diversity, Theorem 8, of the γ's affects the optimal contracts if the market portfolio is not pre-existing. As such one reason to construct the market portfolio first is that the remaining markets' definitions are robust to misspecification of the uncertainty or diversity of γ's.

13 Conclusion and Practical Implications for Contract Design

We have presented several alternative maximization problems for contract designers to define optimal risk management contracts. Thus, we have several alternative definitions of the optimal markets to create.

The simplest maximization problem, (15), is the most restrictive: it assumes no preexisting markets and no uncertainty about preferences. It yielded the striking conclusion that the contracts created would never allow trading the market portfolio, and no linear combination of the portfolios defined in the contracts could even have non-negative quantities of all endowments. The question is, how restrictive are the assumptions in (15)?

Of course, we are not in a situation where there are no pre-existing markets, and so one might conclude that the alternative maximization problem that accounts for these, (26), is the more relevant. We are, however, somewhat inclined against this view. We should not automatically assume that we are constrained by pre-existing markets. History shows that pre-existing derivative markets actually do sometimes wither away when another derivative market appears that serves hedgers better.[5]

A more important issue is uncertainty about preferences which leads us to problem (34), or if there are K individuals per agent, problem (40). These lead to the same solution and so our maximization problem (34) may be the most relevant. As a matter of historical fact, market designers have found it very difficult to predict in advance of creating a new market who will want to take positions in the new market. Our representation of uncertainty about preference parameters can be regarded as a metaphor for our difficulty in predicting investor behavior.

[5] An example of this is the demise of the GNMA CDR futures resulting from the formation of the Treasury-Bond futures, see Johnston and McConnell (1989).

Thus, taking account of this uncertainty as in (34) would be of great practical importance for contract designers. If contract designers assumed enormous uncertainty about preferences, so that the limiting case described in Theorem 7 applies, then, if there is a substantial market component in the economy one might think that something approximating the market portfolio would be the most important market.

There are reasons to suspect that endowment (income) shocks have a substantial world component; technology shocks proliferate around the world and economies are linked through trade. If we do not know Σ accurately but believe there is a strong world component to endowments, we may want to impose on our estimate of Σ a prior that this component is important. This might lead a contract designer to construct the market portfolio. It may be noted that a possible outcome of estimating Σ, using (34) and specifying moderate prior uncertainty about risk parameters would be a conclusion that something approximating the market portfolio is not the most important new market to create, but still one of the more important markets.

Actual markets we create should be easy to describe and understand to ensure their success. Despite fundamental uncertainty about future variances and risk aversions, we think it is safe to advice that a market for the market portfolio should be constructed. Even though it may not be precisely on the list of most important markets with estimated Σ, it allows for more robust contract definition and enhances the simplicity and understandability of equilibrium.

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Symbol List

A. Latin Symbols

A : $J \times N$ matrix whose jn^{th} element is the share of the unforecastable component of agent j's endowment that is paid from a short in one of the n^{th} contracts to a long in the contract

A^* : $J \times N^*$ matrix whose jn^{th} element is the share of the unforecastable component of agent j's endowment that is paid from a short in one of the n^{th} new (not pre-existing) contracts to a long in the contract

A_1 : The $J \times 1$ vector whose j^{th} element is the share of the unforecastable component of agent j's endowment that is paid from a short in one of the n^{th} (pre-existing) contract to a long in the contract

f : $N \times 1$ vector of dividends of the contracts from the short to the long

$g = \iota'\Gamma^{-1}\left(\iota'\Gamma^{-1}\Sigma\Gamma^{-1}\iota\right)^{-1}$

$\bar{H} \equiv \bar{\Sigma} - \overline{c\iota\iota'}\bar{\Sigma}$

$H \equiv \Sigma - c(\iota\iota')\Sigma$

j : Index for representative agents

J : Number of representative agents in the economy, $K = 1$ unless otherwise specified

M : A $J \times J$ matrix such that xM is the $1 \times J$ vector whose j^{th} element is the different between the unforecastable component of agent j's endowment and agent j's risk-parameter-adjusted share of the unforecastable component of world total endowments

$m = \iota(\iota'\Sigma\iota)^{-1}$

N : Number of contracts available to investors, $\leq J$

N^* : Number of contracts in addition to pre-existing contracts that are available to investors, $N = N^* + 1$

P : $N \times 1$ vector whose j^{th} element is the price of contract n, the amount paid from the long in the contract to the short before the uncertain endowments are realized

Q : $N \times J$ matrix whose nj^{th} element is the number of the j^{th} contracts demanded by agent j

Q^* : $N^* \times J$ vector whose n^{th} element is the number of the j^{th} new contracted demanded by agent j who also has one pre-existing contracts available to trade in

x : The J element row vector whose j^{th} element is the endowment of agent j

$\tilde{x}$: The j element row vector whose j^{th} element is the endowment of agent j minus its expected value

B. Greek Symbols

γ_j : Risk aversion parameter of agent j

Γ : The $J \times J$ diagonal matrix whose j^{th} diagonal element is the risk aversion parameter of agent j

δ_n : Lagrangian multiplier for the constraint that the n^{th} new contract is uncorrelated with the pre-existing contract

ι : The $J \times 1$ vector all of whose elements are one

κ : The $K \times 1$ vector all of whose elements are one

λ_n : The Lagrangian multiplier for the contract designer's problem, corresponding to the constraint that $A_n' \Sigma A_n = 1$

Λ : The $N \times N$ diagonal matrix whose n^{th} diagonal element is λ_n

Φ : The $J \times J$ matrix $\Phi = I - A_1 A_1' \Sigma$

Σ : The $J \times J$ variance matrix for agent's endowments

Summary of Basic Relations

A. Pertaining to All contract designers problems:

$f = a' \tilde{x}'$

$P = -A' \Sigma \iota (\iota' \Gamma^{-1} \iota)^{-1}$

$Q = -(A'\Sigma + P\iota'\Gamma^{-1})$

$Q\iota = A'\Gamma^{-1}\iota = 0$

$QA = -I$

$Q = QM$

$M = I - \iota(\iota'\Gamma^{-1}\iota)^{-1}\iota'\Gamma^{-1}$

$A = -M\Gamma Q'\Lambda^{-1}$

$A = MA$

$M\iota = 0$

$\iota\Gamma^{-1}M = 0$

$M'\Gamma^{-1}M = \Gamma^{-1}M$

$Q = -A'\Sigma M \ xA = xMA$

$xA = (xM)A$

B. If $N = J - 1$:

$M'\Sigma M = Q'$

$AQ = -M$

$I + AQ = \iota\iota'\Gamma^{-1}(\iota'\Gamma^{-1}\iota)^{-1}$

The ij^{th} element of $I + AQ$ is the exposure of agent i to agent j's endowment.

C. If there is a pre-existing contract for market portfolio, 26, given $A_1 = m$:

$\Phi = I - \iota\iota'\Sigma(\iota'\Sigma\iota)^{-1}$

$\Phi'\Sigma\Phi = \Sigma\Phi$

$\Phi M = \Phi$

$M\Phi = M$

$$\Sigma A^* = M'\Sigma A^*$$

$$\Sigma A^* = \Phi'\Sigma A^*$$

$$\Sigma A^* = -Q^*$$