Expected Utility Theory Without the Completeness Axiom

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Expected Utility Theory without the Completeness Axiom*

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Abstract

We study axiomatically the problem of obtaining an expected utility representation for a potentially incomplete preference relation over lotteries by means of a *set* of von Neumann-Morgenstern utility functions. It is shown that, when the prize space is a compact metric space, a preference relation admits such a multi-utility representation provided that it satisfies the standard axioms of expected utility theory. Moreover, the representing set of utilities is unique in a well-defined sense.

Keywords: Expected utility, incomplete preferences.

JEL Classification numbers: D11, D81.

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1 Introduction

The von Neumann-Morgenstern expected utility theorem is one of the most fundamental results of the theory of individual decision making. It shows that a preference relation defined on a lottery space has an expected utility representation, provided that it is a complete and transitive binary relation that satisfies the standard independence and continuity axioms. Given the importance of this result, it is not surprising that there is a large number of studies that investigate its alterations which arise due to the relaxation of its various postulates. However, only few of these studies focus on the completeness assumption; it is presently not known if there is a reasonable way of modifying the expected utility theorem to include incomplete preferences within its coverage. Our objective is to offer a remedy for this situation.

Before stating more carefully our goal and the contribution thereof, let us note that there are several economic reasons why one would like to study incomplete preference relations. First of all, as advanced by several authors in the literature, it is not evident if completeness is a fundamental rationality tenet the way the transitivity property is. Aumann (1962), Bewley (1986) and Mandler (1999), among others, defend this position very strongly from both the normative and positive viewpoints. Indeed, if one takes the psychological preference approach (which derives choices from preferences), and not the revealed preference approach, it seems natural to define a preference relation as a potentially incomplete preorder, thereby allowing for the occasional "indecisiveness" of the agents. Secondly, there are economic instances in which a decision maker is in fact composed of several agents each with a possibly distinct objective function. For instance, in coalitional bargaining games, it is in the nature of things to specify the preferences of each coalition by means of a vector of utility functions (one for each member of the coalition), and this requires one to view the preference relation of each coalition as an incomplete preference relation. The same reasoning applies to social choice problems; after all, the most commonly used social welfare ordering in economics, the Pareto dominance, is an incomplete preorder. Finally, we note that incomplete preferences allow one to enrich the decision making process of the agents by providing room for introducing to the model important behavioral traits like status quo bias, loss aversion, procedural decision making, etc.

Since these issues are discussed at length in the literature, we shall not discuss the potential importance of incomplete preferences for economic modeling at large, but rather proceed to discuss how one may handle the problem of actually representing such preferences.[1] Curiously, the basic idea has already been suggested, albeit elusively, by von Neumann and Morgenstern (1944, pp. 19-20):

[1] A closely related issue was studied by Aumann (1962) and Kannai (1963). These authors were interested in finding an extension of an incomplete preference relation defined over lotteries that admits an expected utility representation. Unfortunately, as also noted by Majumdar and Sen (1976), this approach falls short of yielding a representation theorem, for it does not *characterize* the preference relations under consideration. Put differently, the Aumann-Kannai approach fails to capture the indecisiveness region of an individual, and hence provides only partial information about the associated choice behavior. More on this in Remark 2 below.

> "... We have conceded that one may doubt whether a person can always decide which of two alternatives ... he prefers. If the general comparability assumption is not made, a mathematical theory ... is still possible. It leads to what may be described as a many-dimensional vector concept of utility. This is a more complicated and less satisfactory set-up, but we do not propose to treat it systematically at this time."[2]

In evaluation of this statement, Aumann (1962, p. 449) notes that "... Details were never published. What they probably had in mind was some kind of mapping from the space of lotteries to a canonical partially ordered euclidean space, ... but it is not clear to me how this approach can be worked out." Our objective here is actually nothing other than formalizing Aumann's interpretation of the von Neumann-Morgenstern suggestion.

To make things a bit more precise, let us denote by X the set of certain prizes, and consider a preference relation $\succsim$ which is defined as a (potentially incomplete) preorder on the set of all lotteries on X. It is obvious that one cannot represent $\succsim$ in the standard way by using a single von Neumann-Morgenstern utility function, if $\succsim$ is actually incomplete. But one may do so by means of a *set* of utility functions defined on X. Thus the representation notion we suggest requires one to come up with a set $\mathcal{U}$ of real functions on X such that, for all lotteries p and q,

$$p \succsim q \quad \text{if and only if} \quad \mathbf{E}_p(u) \geq \mathbf{E}_q(u) \quad \text{for all } u \in \mathcal{U}$$

where $\mathbf{E}_r(u)$ stands for the expectation of u with respect to the lottery $r = p, q$. We are, then, interested in obtaining an *expected multi-utility representation* for incomplete preference relations. This seems to correspond well to the intuition indicated in the von Neumann-Morgenstern and Aumann quotations given above.

A close relative of the above representation concept is actually suggested also by Shapley and Baucells (1998) (see Remark 3 below), and is studied in the context of utility theory under certainty by Ok (1999). This concept clearly carries a stochastic dominance flavor, and hence brings the expected utility theory one step closer to the theory of stochastic orders.[3] More generally, this particular formulation of utility representation ties the expected utility theory to the theory of multi-objective decision making. While this link is often suggested to motivate the study of incomplete preferences (as in the coalitional bargaining example), an axiomatization of the representation we suggest here will clearly make the connection a concrete one. What is more, such an axiomatization sheds light into the role of the completeness assumption in the classical expected utility theorem. For all practical purposes, our approach shows precisely how this theorem modifies in the absence of the completeness axiom.

Put concretely, we focus in this paper on the case in which X is a compact metric space, and prove that the standard independence axiom and a mild strengthening of the standard continuity

[2] Also quoted in Aumann (1962) and Vind (2000).

[3] In fact, the preorders that admit such a vector-valued representation are called *integral stochastic orders* (Whitt, 1986), and have been studied extensively in the literature on applied probability; see, *inter alia*, Mosler and Scarsini (1994) - which is an annotated bibliography -, Shaked and Shanthikumar (1994), and Müller (1997). To the best of our knowledge, however, the integral stochastic orders are so far not investigated axiomatically.

property suffice to yield an expected multi-utility representation in terms of continuous utility functions. In the sequel, we shall also determine in what sense such a representation may be regarded as unique, show how it can be strengthened in the case of monetary lotteries, and discuss the potential difficulties in extending the present approach to a more general class of prize spaces.

2 Expected Multi-Utility Representation

We take an arbitrary metric space X as the set of all certain prizes (degenerate lotteries), and let $C_b(X)$ stand for the set of all continuous and bounded real maps on X, respectively. The set of all Borel probability measures (lotteries) over X is denoted by $\mathcal{P}(X)$. In turn, we let $ca(X)$ stand for the set of all finite Borel signed measures on X, that is,

$$ca(X) := \text{span } (\mathcal{P}(X)).$$

It is well known that $(C_b(X), ca(X))$ is a dual pair under the duality map $(f, \mu) \mapsto \int_X f d\mu$. We thus endow $C_b(X)$ with the weak topology and $ca(X)$ with the weak*-topology induced by this dual pair structure.[4] It is worth noting that this weak*-topology on $ca(X)$ induces on $\mathcal{P}(X)$ the topology of weak convergence for probability measures.

The main representation theorem of this paper will be obtained under the assumption that X is compact. When this assumption is posited, we shall write $C(X)$ for $C_b(X)$. Since $ca(X)$ (normed by the total variation norm) is isometrically isomorphic to the topological dual of $C(X)$ (normed by the sup-norm), the weak and weak*-topologies on $C(X)$ and $ca(X)$ induced by the above mentioned dual pair structure are simply the standard weak and weak*-topologies.

We define a *preference relation* as any reflexive and transitive binary relation on $\mathcal{P}(X)$. This should be contrasted with the standard theory in which a preference relation is assumed also to be complete. To stress this point, we note that the first order stochastic dominance ordering (defined on $\mathbf{R}$) is a preference relation in the general sense of the term adopted here, while this is not the case in the standard theory.

The two fundamental postulates of the expected utility theory are the independence and the continuity axioms which we state formally next.

Independence Axiom. For any $p, q, r \in \mathcal{P}(X)$ and any $\lambda \in (0, 1)$,

$$p \succsim q \quad \text{implies} \quad \lambda p + (1 - \lambda)r \succsim \lambda q + (1 - \lambda)r.$$

[4] For concreteness, we recall that, under these topologies, a net (μ_α) in $ca(X)$ converges to $\mu \in ca(X)$ (denoted $\mu_\alpha \overset{w^*}{\to} \mu$) iff $\int_X f d\mu_\alpha \to \int_X f d\mu$ for all $f \in C_b(X)$. On the other hand, a net (f_α) in $C_b(X)$ converges to $f \in C_b(X)$ (denoted $f_\alpha \overset{w}{\to} f$) iff $\int_X f_\alpha d\mu \to \int_X f d\mu$ for all $\mu \in ca(X)$. (See Aliprantis and Border (1999), pp. 206-7.)

Continuity Axiom.[5] For any convergent sequences (p_n) and (q_n) in $\mathcal{P}(X)$,

$$p_n \succsim q_n \text{ for all } n \quad \text{imply} \quad \lim p_n \succsim \lim q_n.$$

While, as noted in the Introduction, the significance of incomplete preference relations is noted in the literature, a definitive expected utility representation for such preorders is yet to be found. The first issue on this regard is to agree on a "representation" notion for a preference relation $\succsim$ which need not be complete. Given the well-known characterization of the stochastic dominance orderings in terms of linear functionals that possess an expected utility form, we would like to propose here a *multi-utility* representation for such a preorder. Put more precisely, we seek here a *set $\mathcal{U}$ of utility functions on X* such that

$$p \succsim q \quad \text{if and only if} \quad \int_X u dp \geq \int_X u dq \quad \text{for all } u \in \mathcal{U} \tag{1}$$

for all $p, q \in \mathcal{P}(X)$. As discussed above, this is a somewhat natural notion of an integral multi-utility representation, and it is indeed suitable for applications. Its appealing nature is also noted recently by Shapley and Baucells (1998), Dubra and Ok (1999), and Mitra and Ok (2000).

The main result of this paper states that any preference relation that satisfies the independence and continuity axioms admits an expected multi-utility representation, provided that the prize space X is compact. This result is proved next.

Expected Multi-Utility Theorem. *Let X be a compact metric space, and let $\succsim$ be a preference relation on $\mathcal{P}(X)$. $\succsim$ satisfies the independence and continuity axioms if, and only if, there exists a set $\mathcal{U} \subseteq C(X)$ such that* (1) *holds for each $p, q \in \mathcal{P}(X)$.*

Proof. The necessity of the axioms for the representation is easy to verify; we shall rather focus here on their sufficiency. Let $\succsim$ satisfy the independence and continuity axioms. The idea of the proof stems from the following two elementary observations.[6]

Claim 1. For any $p, q \in \mathcal{P}(X)$ and any $\lambda \in (0, 1]$, $\lambda p + (1-\lambda)r \succsim \lambda q + (1-\lambda)r$ implies $p \succsim q$.

Proof of Claim 1. Let $p, q \in \mathcal{P}(X)$ and $\lambda \in (0, 1]$ be such that $\lambda p + (1-\lambda)r \succsim \lambda q + (1-\lambda)r$. Let

$$\bar{\alpha} := \sup \{\alpha \in [0, 1] : \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r\}.$$

Clearly $\bar{\alpha} \geq \lambda > 0$. Using the continuity of $\succsim$ it is easily verified that $\bar{\alpha}p + (1-\bar{\alpha})r \succsim \bar{\alpha}q + (1-\bar{\alpha})r$. Now set $\beta := \dfrac{1}{1+\bar{\alpha}}$ and observe that the independence axiom yields

$$\begin{aligned} \beta(\bar{\alpha}p + (1-\bar{\alpha})r) + (1-\beta)p &\succsim \beta(\bar{\alpha}q + (1-\bar{\alpha})r) + (1-\beta)p = \beta(\bar{\alpha}p + (1-\bar{\alpha})r) + (1-\beta)q \\ &\succsim \beta(\bar{\alpha}q + (1-\bar{\alpha})r) + (1-\beta)q \end{aligned}$$

[5] In the literature the following weaker property is sometimes used (Grandmont (1972)): For all $q \in \mathcal{P}(X)$, the sets $\{p : p \succsim q\}$ and $\{p : q \succsim p\}$ are closed in $\mathcal{P}(X)$. We do not know if the main theorem of this paper can be proved with this weaker continuity condition, except in the case where X is a finite set. Conceptually speaking, however, there is evidently little difference between the two continuity conditions. In fact, some textbooks (such as Mas-Colell, Whinston and Green (1995)) "define" the continuity axiom precisely as we do here.

[6] Both of these observations were noted first in an unpublished paper by Shapley and Baucells (1998). We include their proofs here for completeness.

so that $\frac{2\bar{\alpha}}{1+\bar{\alpha}}p+\frac{1-\bar{\alpha}}{1+\bar{\alpha}}r \succsim \frac{2\bar{\alpha}}{1+\bar{\alpha}}q+\frac{1-\bar{\alpha}}{1+\bar{\alpha}}r$. But by definition of $\bar{\alpha}$, $\frac{2\bar{\alpha}}{1+\bar{\alpha}} \leq \bar{\alpha}$, that is, $\bar{\alpha}^2-\bar{\alpha} \geq 0$. Since $\bar{\alpha} > 0$, therefore, we have $\bar{\alpha} = 1$, and hence the previous observation gives $p \succsim q$. ‖

Claim 2. For any $p, q \in \mathcal{P}(X)$, we have $p \succsim q$ if, and only if, there exist a $\lambda > 0$ and $r, s \in \mathcal{P}(X)$ with $r \succsim s$ and $p - q = \lambda(r-s)$.

Proof of Claim 2. Take any $\lambda > 0$ and $r, s \in \mathcal{P}(X)$ such that $r \succsim s$ and $p - q = \lambda(r-s)$. Observe that the independence axiom gives

$$\frac{1}{1+\lambda}p + \frac{\lambda}{1+\lambda}s = \frac{1}{1+\lambda}q + \frac{\lambda}{1+\lambda}r \succsim \frac{1}{1+\lambda}q + \frac{\lambda}{1+\lambda}s.$$

Applying Claim 1, $p \succsim q$ obtains. The converse claim is trivial. ‖

Now define

$$\mathcal{C}(\succsim) := \{\lambda(p-q) : \lambda > 0 \text{ and } p \succsim q\}.$$

The importance of this set stems from the following observation.[7]

Claim 3. $\mathcal{C}(\succsim)$ is a convex cone in $ca(X)$ such that $p \succsim q$ iff $p - q \in \mathcal{C}(\succsim)$.[8]

Proof of Claim 3. While that $\mathcal{C}(\succsim)$ is a cone is trivial, its convexity follows from the independence axiom; we omit the routine details. The second claim is, on the other hand, an immediate consequence of Claim 2. ‖

Claim 4. $\mathcal{C}(\succsim)$ is weak*-closed.

Proof of Claim 4. We shall first show that $\mathcal{C}(\succsim)$ is sequentially weak*-closed. Take then a sequence $(\lambda_n(p_n - q_n))$ in $\mathcal{C}(\succsim)$, and assume that $(\lambda_n(p_n - q_n))$ converges in $ca(X)$ in the weak*-topology. Then, by definition, $\int_X f d(\lambda_n(p_n - q_n))$ must be a convergent real sequence for all $f \in C(X)$, which implies that $\sup\{\int_X f d(\lambda_n(p_n - q_n)) : n = 1, 2, ...\}$ is finite. By the Banach-Steinhaus theorem, therefore, there exists a real number K such that

$$\|\lambda_n(p_n - q_n)\| \leq K, \qquad n = 1, 2, ... \tag{2}$$

Now, by using the Jordan decomposition theorem, we can write $p_n - q_n = \gamma_n(r_n - w_n)$ for two mutually singular $r_n, w_n \in \mathcal{P}(X)$ such that $r_n \succsim w_n$ and $\gamma_n \geq 0$. By mutual singularity, $\|r_n - w_n\| = 2$. But then

$$\|\lambda_n(p_n - q_n)\| = \|\lambda_n\gamma_n(r_n - w_n)\| = \lambda_n\gamma_n \|(r_n - w_n)\| = 2\lambda_n\gamma_n$$

so that, by (2), we may conclude that $(\lambda_n\gamma_n)$ is a real sequence that lies in the closed interval $[0, K/2]$. This sequence must then have a convergent subsequence $(\lambda_{n_k}\gamma_{n_k})$. But since X is

[7] We note that the significance of the set $\mathcal{C}(\succsim)$ for expected utility theory without the completeness axiom was observed first by Aumann (1962). Like that of Aumann, the primary element of the approach we adopt here is the investigation of the geometry of $\mathcal{C}(\succsim)$. This approach is also adopted by a number of authors in the literature, among which are Kannai (1963), Fishburn (1975), Bewley (1986), Shapley and Baucells (1998), and Vind (2000).

[8] In this paper by a *convex cone* (in any vector space) we mean a nonempty convex set that is closed under nonnegative scalar multiplication. For any set A, cone(A) stands for the smallest convex cone that contains A.

compact, $\mathcal{P}(X)$ is a weak*-compact set in $ca(X)$, and hence both (r_{n_k}) and (w_{n_k}) must have (weak*-)convergent subsequences.[9] Passing to these subsequences consecutively, we end up with convergent subsequences $(\lambda_{n_{k_t}}\gamma_{n_{k_t}})$, $(r_{n_{k_t}})$, and $(w_{n_{k_t}})$. Let us write $\lambda_{n_{k_t}}\gamma_{n_{k_t}} \to \lambda$, $r_{n_{k_t}} \to p$ and $w_{n_{k_t}} \to q$ as $t \to \infty$. By continuity of $\succsim$, we have $p \succsim q$. Moreover,

$$\lambda_{n_{k_t}}(p_{n_{k_t}} - q_{n_{k_t}}) = (\lambda_{n_{k_t}}\gamma_{n_{k_t}})(r_{n_{k_t}} - w_{n_{k_t}}) \to \lambda(p-q)$$

as $t \to \infty$. Since every subsequence of a convergent sequence converges to the limit of the mother sequence, we must then have $\lim \lambda_n(p_n - q_n) = \lambda(p-q) \in \mathcal{C}(\succsim)$, and hence we may conclude that $\mathcal{C}(\succsim)$ is sequentially weak*-closed.

Since X is compact, $C(X)$ is separable, and $ca\,(X)$ is equal (i.e., isometrically isomorphic) to the topological dual of $C(X)$. But by the Krein-Šmulian theorem every sequentially weak*-closed convex set in the dual of a separable normed space is weak*-closed.[10] Consequently, the previous observation implies that $\mathcal{C}(\succsim)$ is weak*-closed in $ca(X)$. ‖

We are now prepared to prove the theorem. Define

$$\mathcal{U} := \left\{ u \in C(X) : \int_X u d\mu \geq 0 \text{ for all } \mu \in \mathcal{C}(\succsim) \right\}.$$

If $p \succsim q$, then $p - q \in \mathcal{C}(\succsim)$ so that $\int_X udp \geq \int_X udq$ for all $u \in \mathcal{U}$. To establish the converse, take any p' and q' in $\mathcal{P}(X)$ with

$$\int_X udp' \geq \int_X udq' \quad \text{for all } u \in \mathcal{U},$$

and assume that $p' \succsim q'$ does not hold. This means that the sets $\{p' - q'\}$ and $\mathcal{C}(\succsim)$ are disjoint. Since $\mathcal{C}(\succsim)$ is a weak*-closed convex cone, then, by the Hahn-Banach separation theorem, there exists a continuous linear T on $ca(X)$ and a real α such that $T(\mu) \geq \alpha > T(p' - q')$ for all $\mu \in \mathcal{C}(\succsim)$.[11] Since $0 \in \mathcal{C}(\succsim)$, we have $0 = T(0) \geq \alpha$ so that $0 > T(p' - q')$. Moreover, since $\mathcal{C}(\succsim)$ is a cone, we have $mT(\mu) = T(m\mu) \geq \alpha$ for any $\mu \in \mathcal{C}(\succsim)$ and $m \in \mathbf{N}$. This implies that $T(\mu) \geq 0$ for all $\mu \in \mathcal{C}(\succsim)$.[12] That is, $T(\mu) \geq 0 > T(p' - q')$ for all $\mu \in \mathcal{C}(\succsim)$. Since T is linear and continuous in the weak*-topology, there exists a $v \in C(X)$ such that $T(\mu) = \int_X vd\mu$ for all $\mu \in ca(X)$.[13] Thus, we have

$$\int_X vd\mu \geq 0 > \int_X vd\left(p' - q'\right) \qquad \text{for all } \mu \in \mathcal{C}(\succsim).$$

This means that $v \in \mathcal{U}$ and $\int_X vdp' < \int_X vdq'$, which is a contradiction. *Q.E.D.*

[9] Since the weak*-topology on $\mathcal{P}(X)$ is identical to the standard topology of weak convergence on $\mathcal{P}(X)$, this is an immediate consequence of the Prohorov theorem. Alternatively, one can supply a nonprobabilistic proof by using Alaoglu's theorem.

[10] See Megginson (1998, p. 242.), Corollary 2.7.13.

[11] See Aliprantis and Border (1999), Theorem 5.58.

[12] The last three sentences and the geometric form of the Hahn-Banach theorem show that a closed convex cone can be strictly separated from a point in its exterior by a closed hyperplane which passes through the origin.

[13] See Aliprantis and Border (1999), Theorem 5.83, p. 208.

We now determine in what sense the set $\mathcal{U}$ found in the above representation theorem can be considered as unique. It turns out that there is actually quite a tractable way of generalizing the uniqueness part of the classic von Neumann-Morgenstern theorem in our multi-utility context (even when X is not necessarily compact). Indeed, if the sets $\mathcal{U}$ and $\mathcal{V}$ in $C_b(X)$ represent a preference relation $\succsim$ as in (1), then $\mathcal{V}$ must belong to the closed convex cone generated by $\mathcal{U}$ and all constant functions; this is the content of the forthcoming uniqueness theorem.[14] This is an important observation a special case of which is the standard uniqueness result of expected utility theory: if a single utility function u in $C_b(X)$ represents $\succsim$, then another such function is necessarily a positive affine transformation of u.

To state formally our general uniqueness result on the set-valued expected utility representations, we define the operator $\langle\cdot\rangle : 2^{C_b(X)} \to 2^{C_b(X)}$ as

$$\langle\mathcal{U}\rangle := \mathrm{cl}\left(\mathrm{cone}(\mathcal{U}) + \{\theta\mathbf{1}_X\}_{\theta\in\mathbf{R}}\right),$$

where the closure operator is applied with respect to the weak topology (or equivalently, with respect to the sup-norm topology when X is compact). It is easy to verify that if $\mathcal{U}$ represents $\succsim$, so does $\langle\mathcal{U}\rangle$. The following result tells us further that $\langle\mathcal{U}\rangle$ is in fact the largest set of utility functions in $C_b(X)$ that represents $\succsim$ as in (1). This observation can be viewed as a general uniqueness theorem for expected multi-utility representations.

Uniqueness Theorem. *Two nonempty sets $\mathcal{U}$ and $\mathcal{V}$ in $C_b(X)$ satisfy, for each $p, q \in \mathcal{P}(X)$,*

$$\int_X udp \geq \int_X udq \quad \text{for all } u \in \mathcal{U} \quad \text{if and only if} \quad \int_X vdp \geq \int_X vdq \quad \text{for all } v \in \mathcal{V},$$

if, and only if, $\langle\mathcal{U}\rangle = \langle\mathcal{V}\rangle$.

Proof. Since the "if" part is trivial, we shall prove here only the "only if" part. Suppose that we can find a $v \in C_b(X)$ such that $v \in \langle\mathcal{V}\rangle \setminus \langle\mathcal{U}\rangle$. Given that $C_b(X)$ is endowed with the weak topology, we may apply the separating hyperplane theorem to find a nonzero signed measure $\mu \in ca(X)$ such that

$$\int_X vd\mu > 0 \geq \int_X ud\mu \qquad \text{for all } u \in \langle\mathcal{U}\rangle. \tag{3}$$

The latter inequalities imply that $0 \geq \int_X \theta\mathbf{1}_X d\mu = \theta\mu(X)$ for all real θ, and hence we have $\mu(X) = 0$. Of course, we have $\mu = \mu^+ - \mu^-$ for some finite Borel measures μ^+ and μ^- on X. By the previous observation, $\mu^+(X) = \mu^-(X) = c \geq 0$. Since $c = 0$ would imply that $\mu = 0$, we must actually have $c > 0$. Thus $p := \mu^+/c$ and $q := \mu^-/c$ belong to $\mathcal{P}(X)$. So, by (3), we get $\int_X vdp > \int_X vdq$ and $\int_X udp \leq \int_X udq$ for all $u \in \langle\mathcal{U}\rangle$, which is a contradiction. *Q.E.D.*

We conclude the present discussion with a number of complementary comments.

Remark 1. (***Preferences over Monetary Lotteries***) An important special case of the present setup which is widely used in applications is the case of monetary lotteries where X is a

[14] A number of versions and special cases of this result have actually been noted elsewhere in the literature; see, for instance, Müller (1997), Castagnoli and Maccheroni (1998), and Dubra and Ok (1999).

closed interval in the real line, say, $X = [0,1]$. Since in this case it is natural to incorporate the idea that "more money is preferred to less," one should examine the structure of the preference relations $\succsim$ on $\mathcal{P}[0,1]$ such that $p \succ_{\text{FSD}} q$ implies $p \succ q$ for all $p, q \in \mathcal{P}(X)$, where $\succ_{\text{FSD}}$ is the irreflexive part of the first order stochastic dominance relation $\succsim_{\text{FSD}}$ on $\mathcal{P}[0,1]$. The question is then to determine the structure of preferences that satisfy not only the axioms of independence and continuity, but also this *monotonicity* condition. To answer this question let us agree to call a set $\mathcal{U}$ in $\mathbf{R}^X$ *strictly increasing*, if each $u \in \mathcal{U}$ is increasing, and if $0 \leq a < b \leq 1$ implies $u(a) < u(b)$ for some $u \in \mathcal{U}$. The following is an important corollary of our main representation theorem.

Expected Multi-Utility Theorem on $\mathcal{P}[0,1]$. *Let $\succsim$ be a preference relation on $\mathcal{P}[0,1]$. $\succsim$ satisfies the independence, continuity and monotonicity axioms if, and only if, there exists a strictly increasing set $\mathcal{U} \subseteq C[0,1]$ such that* (1) *holds for each $p, q \in \mathcal{P}[0,1]$.*

Given the general expected multi-utility theorem we have proved above, all we need to do here is to verify the monotonicity of a preference relation $\succsim$ for which there exists a strictly increasing $\mathcal{U}$ in $C[0,1]$ such that (1) holds for each $p, q \in \mathcal{P}[0,1]$. Take any $p, q \in \mathcal{P}[0,1]$ with $p \succ_{\text{FSD}} q$. Then $F_p^{-1} > F_q^{-1}$, that is, $F_p^{-1}(s) \geq F_q^{-1}(s)$ for all $s \in (0,1)$ and $F_p^{-1}(s^*) > F_q^{-1}(s^*)$ for some $s^* \in (0,1)$.[15] Since $\mathcal{U}$ is strictly increasing, there exists a $u^* \in \mathcal{U}$ such that $u^*\left(F_p^{-1}(s^*)\right) > u^*\left(F_q^{-1}(s^*)\right)$, so that $u^* \circ F_p^{-1} > u^* \circ F_q^{-1}$. But $u^* \circ F_p^{-1}$ and $u^* \circ F_q^{-1}$ are left continuous, and hence

$$\int_{[0,1]} u^* dp = \int_0^1 u^*\left(F_p^{-1}(s)\right) ds > \int_0^1 u^*\left(F_q^{-1}(s)\right) ds = \int_{[0,1]} u^* dq.$$

Moreover (since $\mathcal{U}$ consists of increasing functions), $\int_{[0,1]} u dp \geq \int_{[0,1]} u dq$ for all $u \in \mathcal{U}$. Hence we may conclude that $p \succ q$. $\|$

Remark 2. (***The Extension Approach***) As noted in the Introduction, earlier studies on relaxing the completeness axiom within the paradigm of expected utility have focused on the problem of *extending* a preference relation that satisfies the independence and (various forms of) the continuity axioms in such a way that the extended relation admits a von Neumann-Morgenstern representation. The important work of Aumann (1962), in particular, is geared towards finding a function $u : X \to \mathbf{R}$, referred to as an *Aumann utility* below, such that

$$p \left\{ \begin{array}{c} \succ \\ \sim \end{array} \right\} q \quad \text{implies} \quad \int_X u dp \left\{ \begin{array}{c} > \\ = \end{array} \right\} \int_X u dq$$

for all $p, q \in \mathcal{P}(X)$. A major disadvantage of this approach is that one cannot recover the preference relation $\succsim$ from its Aumann utility. So, in contrast to $\mathcal{U}$ in (1), the information contained in an Aumann utility for $\succsim$ is strictly less than $\succsim$. Maximization of an expected Aumann utility on a

[15] The *pseudoinverse distribution function* of a probability measure $p \in \mathcal{P}[0,1]$ is defined by $F_p^{-1}(s) := \min\{t \in [0,1] : p([0,t]) \geq s\}$ for all $s \in (0,1)$. It is easily checked to be increasing and left continuous. Moreover, pseudounverses display these two useful features: (i) $\int_{[0,1]} u dp = \int_0^1 u\left(F_p^{-1}(s)\right) ds$ for all $u \in C[0,1]$, and (ii) $p \succsim_{\text{FSD}} q$ iff $F_p^{-1} \geq F_q^{-1}$.

given constraint set S leads to a $\succsim$-maximal element in S, whereas the vector-maximization of all expected members of $\mathcal{U}$ leads to the set of all $\succsim$-maximal elements in S.

It is, however, still worth knowing if an Aumann utility exists in the present context. Fortunately, mostly because we work with a continuity condition stronger than that adopted by Aumann, the answer is yes.[16]

Theorem. *Let X be a compact metric space, and let $\succsim$ be a preference relation on $\mathcal{P}(X)$.* If *$\succsim$ satisfies the independence and continuity axioms, then it must possess a continuous Aumann utility.*

To prove this, we apply the expected multi-utility theorem to find a set $\mathcal{U}$ in $C(X)$ such that (1) holds for all $p, q \in \mathcal{P}(X)$. Thanks to the Weierstrass theorem, it is without loss of generality to assume that $u \geq 0$ for all $u \in \mathcal{U}$. Since X is compact, $C(X)$ is separable, and hence $\mathcal{U}$ is itself a separable metric space. Let $\{v_1, v_2, ...\}$ be a dense set in $\mathcal{U}$. It is readily verified that

$$p \succsim q \quad \text{if and only if} \quad \int_X v_n dp \geq \int_X v_n dq \text{ for all } n = 1, 2, ...$$

Let $u_n := 2^{-n} \frac{v_n}{\|v_n\| + 1}$ for each n, and observe that

$$p \succsim q \quad \text{if and only if} \quad \int_X u_n dp \geq \int_X u_n dq \text{ for all } n = 1, 2, ...$$

Define $w := \sum^{\infty} u_n \in C(X)$, and take any $p, q \in (X)$. It is obvious that $p \sim q$ implies $\int_X w dp = \int_X w dq$. On the other hand, by denseness of $\{v_1, v_2, ...\}$ in $\mathcal{U}$ and (1), $p \succ q$ implies that there exists a positive integer N such that $\int_X u_N dp > \int_X u_N dq$, where, of course, we have $\int_X u_n dp \geq \int_X u_n dq$ for all n. Therefore, by the monotone convergence theorem,

$$\int_X \left(\sum u_n\right) dp = \sum \left(\int_X u_n dp\right) > \sum \left(\int_X u_n dq\right) = \int_X \left(\sum u_n\right) dq,$$

that is, $\int_X w dp > \int_X w dq$. Thus, w is a continuous Aumann utility for $\succsim$. $\|$

Remark 3. (***The Algebraic Approach***) Despite the quotation by von Neumann and Morgenstern (1944) mentioned in the Introduction, a notion of expected utility representation by means of a set of utility functions has, to the best of our knowledge, not studied in the literature so far. However, we should note that Shapley and Baucells (1998) advance a representation notion which actually admits the corresponding notion we introduced here as a special case. These authors identify conditions for a preference relation $\succsim$ on $\mathcal{P}(X)$ (actually on an arbitrary mixture space) to have a representation of the form

$$p \succsim q \quad \text{if and only if} \quad T(p) \geq T(q) \quad \text{for all } T \in \Omega$$

[16] This is perhaps somewhat surprising, because one major message of Aumann (1962) is that an expected utility theory without the completeness axiom *cannot* be pursued along the extension approach, *when X is infinite.* However, since Aumann's related example does not work for a space of lotteries (it is proved in the mixture space $\mathbf{R}^{\infty}$), there is reason to believe that the said message is in fact overly pessimistic. What is more, with a slight strengthening of the continuity axiom (as we adopted here), both the extension and the multi-utility approaches stand strong, at least in the case of lotteries defined over an arbitrary compact metric space.

where Ω is a nonempty set of affine functionals on $\mathcal{P}(X)$. The approach of Shapley and Baucells contrasts with the present one in that it is algebraic as opposed to topological. While penetrating, it is as such not immediately useful in dealing with the problem of expected multi-utility representation of incomplete preferences, for it says little about when the functionals T in Ω can be chosen to be continuous in the weak*-topology (or put differently, when $\mathcal{C}(\succsim)$ can be expressed as an intersection of half spaces the corresponding hyperplanes of which are not dense in $ca(X)$). Moreover, Shapley-Baucells approach is based on a crucial "properness" assumption which, by definition, ensures that the cone $\mathcal{C}(\succsim)$ has a nonempty algebraic interior. Unfortunately, it is not at all easy to see what sort of a primitive axiom on a preference relation would support such a technical requirement. ‖

Remark 4. (***Larger Classes of Prize Spaces***) While our main representation theorem is strong enough to cover almost all cases of interest, it does not function in the general domain that the classical expected utility theorem functions, namely, for preferences defined over lotteries on an arbitrary Polish space. Whether our result can be extended to this general domain is presently an open technical problem. It may be worth noting that the main difficulty on this regard is that, when X is not compact, the "natural" topologies on $C_b(X)$ and $ca(X)$ (induced by the dual pair structure $(C_b(X), ca(X))$ under the duality map $(f,\mu) \mapsto \int_X f d\mu$) differs from the standard weak and weak*-topologies (induced by the sup-norm). This, in turn, invalidates the arguments given in Claim 4 of the proof of our main theorem; in particular, the use of the Krein-Šmulian theorem is not warranted anymore. ‖

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