No unbounded arbitrage, weak no market arbitrage and no arbitrage price system conditions: The circular results.*

Manh-Hung Nguyen[a] and Thai Ha-Huy[b]

[a]THEMA, Cergy-Pontoise University

E-mail: *mnguyen@u-cergy.fr*

[b]CES, Paris-1 Pantheon-Sorbonne University

E-mail: *thai.ha-huy@m4x.org*

March 1, 2008

Abstract

Page and Wooders (1996) prove that the no-unbounded-arbitrage (NUBA) condition introduced by Page (1987) is equivalent to the existence of a no arbitrage price (NAPS) when no agent has non-null useless vectors. Allouch, Le Van and Page (2002) show that their generalized NAPS condition is actually equivalent to the weak-no-market-arbitrage (WNMA) condition introduced by Hart (1974). They mention that this result implies the one given by Page and Wooders (1996). In this note, we show that these results are actually circular.

Keywords: Arbitrage, Equilibrium, Recession cone.
JEL Classification: C62, D50.

*We would like to thank to Cuong Le Van for having suggested the result obtained in this note. Corresponding author: M.H. Nguyen.

1 Introduction

Allouch, Le Van and Page (2002) consider the problem of competitive equilibrium in an unbounded exchange economy by exploiting the geometry of arbitrage. They generalize the definition of no-arbitrage-price-systems (NAPS) introduced by Werner (1987) to the case where some agent in the economy has only useless vectors. They show that this generalized NAPS condition is actually equivalent to the weak-no-market-arbitrage (WNMA) condition introduced by Hart (1974). They mention that this result implies the one given by Page and Wooders (1996) who prove that the no-unbounded-arbitrage (NUBA) introduced by Page (1987) is equivalent to NAPS when no agent has non-null useless vectors. The proof of the claims consist of two parts. One is very easy (NAPS implies WNMA or NAPS implies NUBA). The converse part is more difficult. The purpose of this note is to show that if the statement NUBA implies NAPS (Page and Wooders (1996)) is true then we have WNMA implies NAPS (Allouch, Le Van and Page (2002)). But it is obvious that if the second statement holds then the first one also holds. The novelty of the result of this note is that the results are self-contained. While Allouch, Le Van and Page (2002) prove WNMA implies NAPS by using a difficult Lemma in Rockafellar (1970) then deduce that NUBA implies NAPS when no agent has non-null useless vectors, we show that these results are actually circular. In some mathematical senses, these results let us to think of the circular tours of Brouwer and Kakutani fixed-point theorems. Moreover, the proofs which we provide are simple and elementary.

We consider an unbounded exchange economy $\mathcal{E}$ with m agents indexed by $i = 1, \ldots, m$. Each agent has an endowment $e^i \in \mathbb{R}^l$, a consumption set X_i which is a closed, convex non-empty subset of $\mathbb{R}^l$ and a upper semi-continuous, quasi-concave utility function u^i from X_i to $\mathbb{R}$.

For a subset $X \subset \mathbb{R}^l$, let denote $\text{int}X$ the interior of X, X^o is the polar of X where $X^o = \{p \in \mathbb{R}^l \mid p.x \leq 0, \ \forall\ x \in X\}$ and $X^{oo} = (X^o)^o$. If X is closed, convex and contains the origin then $X^{oo} = X$. Denote also $\overline{X}$ the closure of X.

For $x \in X_i$, let

$$\widehat{P}^i(x) = \{y \in X_i \mid u^i(y) \geq u^i(x)\}$$

be the weak preferred set a x by agent i and let $R_i(x)$ be its recession cone (see Rockaffellar (1970)). It is called the set of *useful vectors* for u^i and is defined as

$$R_i(x) = \{w \in \mathbb{R}^l \mid u^i(x + \lambda w) \geq u^i(x), \text{ for all } \lambda \geq 0\}$$

The lineality space of i is defined by

$$L_i(x) = \{w \in \mathbb{R}^l \ \mid \ u^i(x + \lambda w) \geq u^i(x), \text{ for all } \lambda \in \mathbb{R}\} = R_i(x) \cap -R_i(x)$$

Elements in L_i will be called *useless vectors* . Let denote $R_i = R_i(e^i), \ L_i = L_i(e^i)$ and $L_i^{\perp}$ is the orthogonal space of L_i. It is easy to check that $R_i(x)$ is a closed convex cone.

Let us first recall the *no-unbounded-arbitrage* condition denoted now on by NUBA introduced by Page (1987) which requires non-existence of an unbounded set of mutually compatible net trades which are utility non decreasing.

Definition 1 *The economy satisfies the NUBA condition if* $\sum_{i=1}^m w^i = 0$ *and* $w^i \in R_i$ *for all* i *implies* $w^i = 0$ *for all* i*.*

There exists a weaker condition, called the *weak-no-market-arbitrage* condition (WNMA), introduced by Hart[1974] which requires that all mutually compatible net trades which are utility non-decreasing be useless.

Definition 2 *The economy satisfies the WNMA condition if* $\sum_{i=1}^m w^i = 0$ *and* $w^i \in R_i$ *for all* i *implies* $w^i \in L_i$ *for all* i*.*

If $L_i = \{0\}$, $\forall \ i$, then WNMA is NUBA.

We shall use the concepts of *no-arbitrage-price system* condition (NAPS) as in Allouch, Le Van, Page (2002). Define the notion of *no-arbitrage price*:

Definition 3 $S_i = \begin{Bmatrix} \{p \in L_i^{\perp} \mid p.w > 0, \forall \ w \in (R_i \cap L_i^{\perp})\backslash\{0\} \ \textit{if} \ R_i \backslash L_i \neq \emptyset\} \\ L_i^{\perp} \ \textit{if} \ R_i = L_i \end{Bmatrix}$

Observe that, when $L_i = \{0\}$, then we can write

$$S_i = \{p \in \mathbb{R}^l \mid p.w > 0, \forall \ w \in R_i \backslash \{0\}\}.$$

Definition 4 *The economy* $\mathcal{E}$ *satisfies the NAPS condition if* $\cap_i S_i \neq \emptyset$*.*

2 The circular results

As we mentioned above, the proofs of the implications NAPS$\Longrightarrow$NUBA and NAPS $\Longrightarrow$WNMA are easy. We now give elementary proofs for NUBA$\Longrightarrow$NAPS and WNMA$\Longrightarrow$NAPS.

The following lemma is useful in our proof:

Lemma 1 *WNMA*$\Longrightarrow \sum_i (R_i \cap L_i^{\perp})$ *is closed.*

In particular, if $L_i = \{0\}$ *for all* i*, then NUBA* $\Longrightarrow \sum_i R_i$ *is closed.*

Proof: Assume that there exists a sequence $\sum_i w_n^i \longrightarrow w$, with $w_n^i \in R_i \cap L_i^{\perp}$ for all i and n. We shall prove that the sum $\sum_i \| w_n^i \|$ is bounded, and then the vector w is in $\sum_i (R_i \cap L_i^{\perp})$. Suppose that

$$\lim_{n\to\infty} \sum_i \| w_n^i \| = +\infty$$

Then we have

$$\begin{aligned} \lim_{n\to+\infty} \sum_{i=1}^{m} \frac{w_n^i}{\sum_i \| w_n^i \|} &= 0, \\ \lim_{n\to+\infty} \sum_{i=1}^{m} \frac{\| w_n^i \|}{\sum_i \| w_n^i \|} &= 1. \end{aligned}$$

Therefore we can suppose that $\frac{w_n^i}{\sum_i \| w_n^i \|} \to w^i$ when $n \to +\infty$. Note that R_i is a closed convex cone, we have $w^i \in R_i$ and $\sum_i w^i = 0$, $\sum_i \| w^i \| = 1$. But WNMA condition implies that $w^i \in L_i$, we also have $w^i \in L_i^{\perp}$. Hence, for all i, $w^i = 0$ that leads to a contradiction. ∎

The following result has been proven by Page and Wooders (1996) where they used Dubovitskii-Milyutin (1965) theorem. We give here an elementary proof to make the note self-contained.

Proposition 1 *Assume* $L_i = \{0\}$, $\forall\, i$*, then NUBA* $\Longrightarrow$ *NAPS.*

Proof: Since $L_i = \{0\}$, then $S_i \neq \emptyset \ \forall\, i$. Assume now that $\cap_i S_i = \emptyset$. Then $\cap_i \overline{S}^i$ is contained in a linear subspace $H \subset \mathbb{R}^l$ since $\text{int}\cap_i S_i = \text{int}\cap_i \overline{S}^i = \emptyset$.

It follows from $\overline{S}^i = -(R_i)^o$ that $\cap_i \overline{S}^i = -(\sum_i R_i)^o \subset H$.

This implies

$$H^{\perp} \subset (\sum_i R_i)^{oo}.$$

The sum $\sum_i R_i$ is closed by lemma 1, then $\sum_i R_i = (\sum_i R_i)^{oo}$ since it is closed convex set and contains the origin. Hence, $H^{\perp} \subset \sum_i R_i$ and $\sum_i R_i$ contains a line.

Thus there exist $r \in H^{\perp}$, $r \neq 0$, $-r \in H^{\perp}$ and $(r^1, \ldots, r^m) \neq 0$, $r^i \in R_i$ such that

$$r = \sum_{i=1}^{m} r^i$$

Since $-r \in \sum_i R_i$, there exit $(r'^1, \ldots, r'^m) \neq 0$, $r'^i \in R_i$ such that

$$\sum_i r'^i = -r.$$

Therefore $\sum_i (r^i + r'^i) = 0$ and $r^i + r'^i \in R_i$ since R_i is the convex cone. By the NUBA condition, we have $r^i = -r'^i$. This means that, for some i, R_i contains a line and $S_i = \emptyset$: a contradiction. ■

Allouch, Le Van and Page (2002) prove the equivalence of NAPS and WNMA by using a lemma which is based on the support function (Corollary 16.2.2 in Rockafellar (1970)). Actually, from Proposition 1, we get the following proposition, the proof of which is elementary.

Proposition 2 *WNMA* $\Longrightarrow \cap_i S_i \neq \emptyset$

Proof: Consider a new economy $\widetilde{\mathcal{E}} = (\widetilde{X}_i, \widetilde{u}^i, \widetilde{e}^i)$ defined as

$$\widetilde{X}_i = X_i \cap L_i^\perp,\ \widetilde{u}^i = u^i \mid_{\widetilde{X}_i},\ \widetilde{e}^i = (e^i)^\perp$$
$$\widetilde{R}_i = R_i \cap L_i^\perp$$

We have $\widetilde{L}_i = (R_i \cap L_i^\perp) \cap -(R_i \cap L_i^\perp) = \{0\}$. Hence, in the economy $\widetilde{\mathcal{E}}$, WNMA is NUBA. Proposition 1 implies that $\cap_i \widetilde{S}_i \neq \emptyset$ where

$$\widetilde{S}_i = \{p \in \mathbb{R}^l \ \mid \ p.w > 0,\ \forall\ w \in (R_i \cap L_i^\perp) \backslash \{0\}\}.$$

It is easy to see that $\widetilde{S}_i = S_i + L_i$. Thus, if $(\cap_i \widetilde{S}_i) \cap (\cap_i L_i^\perp) \neq \emptyset$, then $\cap_i S_i \neq \emptyset$.

We will show that $(\cap_i \widetilde{S}_i) \cap (\cap_i L_i^\perp) \neq \emptyset$.

On the contrary, assume that $(\cap_i \widetilde{S}_i) \cap (\cap_i L_i^\perp) = \emptyset$. By using a separation theorem, note that $\cap_i \widetilde{S}_i$ is open and $\cap_i L_i^\perp$ is a subspace, there exists a vector $w \neq 0$ such that:

$$w.p > 0 = w.l,\ \forall\ p \in \cap_i \widetilde{S}_i,\ \forall\ l \in \cap_i L_i^\perp.$$

Therefore, we get

$$w \in \sum_{i=1}^m L_i.$$

Moreover, we have

$$w.p \geq 0\ \forall p \in \overline{\cap_i \widetilde{S}_i}.$$

Since for every i, $\widetilde{S}_i$ is open, and $\cap_i \widetilde{S}_i \neq \emptyset$ we have $\overline{\cap_i \widetilde{S}_i} = \cap_i \overline{\widetilde{S}_i}$. From the lemma 1, $\sum_i \widetilde{R}_i$ is closed. We then have:

$$w \in -(\cap_i \widetilde{S}_i)^o = (\sum_i \widetilde{R}_i)^{oo} = \sum_i \widetilde{R}_i$$

Therefore, there exist,$\forall i$, $l_i \in L_i$, $\widetilde{w}^i \in \widetilde{R}_i$ such that $w = \sum_i l_i = \sum_i \widetilde{w}^i$ or $\sum_i (l_i - \widetilde{w}^i) = 0$. The WNMA implies that $l_i - \widetilde{w}^i \in L_i$. Since $\widetilde{R}_i = R_i \cap L_i^{\perp}$, it implies that $\widetilde{w}^i \in L_i$ and $\widetilde{w}^i \in L_i^{\perp}$. Thus $\widetilde{w}^i = 0$ for all i and $w = 0$: we obtain a contradiction. The proof is complete. ■

The following result is trivial:

Proposition 3 *If Proposition 2 holds then Proposition 1 holds.*

References

[1] Allouch, N., C.Le Van and F.H.Page:. The geometry of arbitrage and the existence of competitive equilibrium. *Journal of Mathematical Economics* 38, 373-391.

[2] Dana, R.A, C.Le Van and F.Magnien (1999): On the different notions of arbitrage and existence of equilibrium. *Journal of Economic Theory* 86, 169-193.

[3] Hart, O. (1974): On the Existence of an Equilibrium in a Securities Model. *Journal of Economic Theory* 9, 293-311.

[4] Page, F.H. Jr, Wooders M.H, (1996): A necessary and sufficient condition for compactness of individually rational and feasible outcomes and existence of an equilibrium. *Economics Letters* 52, 153-162.

[5] Page, F.H (1987): On equilibrium in Hart's securities exchange model. *Journal of Economic Theory* 41, 392-404.

[6] Rockafellar, R.T (1970): *Convex Analysis*, Princeton University Press, Princetion, New-Jersey.

[7] Werner, J.(1987): Arbitrage and the Existence of Competitive Equilibrium. *Econometrica* 55, 1403-1418.