Configurations study for the Banzhaf and the Shapley-Shubik indices of power

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Summary How can we count and list all the Banzhaf or Shapley-Shubik index of power configurations for a given number of players? There is no formula in the literature that may give the cardinal of such a set, and moreover, even if this formula had existed, there is no formula which gives the configuration vectors. Even if we do not present such a formula, we present a methodology which enables to determine the set of configurations and its cardinality.

JEL classification: C7, D7

1 Introduction

It is now well known, thanks to the cooperative game theory and more particularly to the theory of the power indices, that the voting power is quite different from the distribution of weights. Hence, having a positive weight does not ensure to have a positive voting power; different weights may lead to the same voting power measure and so on. The literature about power indices is very abundant and the central question is how to measure the power. Unfortunately, there are several measures of power represented by different power indices. All of them admit the importance of a particular player, the decisive player. This player is so that when added to a losing coalition, the latter becomes a winning one. The measure of power depends on the way the coalitions are built. In this paper, we only consider the two most important power indices, the Shapley-Shubik (1954) power index and the Banzhaf

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(1965) power index. A complete description of the power indices is given in Felsenthal and Machover (1998), Laruelle (1998) or in Leech (2002a), among others.

In many problems, it is important to know all the possible distributions of power among the players. For example, if we want to workout the probability for the two power indices to give the same distribution of power, it is necessary to determine all the admissible vectors of power[1]. Our problem in this paper is then to know how to count and list all the possible configurations of power for a given number of players. As far as we know, there is no formula in the literature that may give the cardinal of such a set, and moreover, even if it had existed, there is no formula which gives the different power index configurations. Obviously, to determine all the possible configurations of power, we have to know the different repartitions of weight.

The purpose of this paper is to present some tables with the number of repartitions for a given number of players and give the different configurations for the standard indices of power. Obtaining all the power configurations and listing them is not trivial and but impossible to work without a computer as soon as the number of players is superior to 4.

In order to present some basic elements of cooperative game theory, let $[q; w_1, ..., w_n]$ be a voting game where n is the number of players, q is the quota and w_i is the weight of the player i. We have $\sum_{i=1}^{n} w_i = \bar{w}$ and $q < \bar{w}$ (q and w_i, for all i, are integers). Without loss of generality, we assume that $w_1 \geqslant w_2 \geqslant ... \geqslant w_n$. We say that a coalition S (that is a group of players) is winning if and only if $\sum_{i \in S} w_i \geqslant q$. We denote W the set of all winning coalitions. If $S \in W$, we attribute a value 1 to S, denoted $v(S) = 1$ and if $S \notin W$, we have $v(S) = 0$. We only consider proper voting games as if $S \in W$, then $N \backslash S \notin W$. In other words, $q > \frac{\bar{w}}{2}$.

Several power indices are proposed in the literature and all of them admit the importance of a particular player, the decisive player. This player is so that $v(S) = 1$ when he belongs to the coalition S and $v(S \backslash \{i\}) = 0$ (that is when i leaves the coalition S). The Shapley-Shubik index (1954) is given by the following formula[2]

$$\phi_i = \sum_{S \subseteq N} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S \backslash \{i\})]$$

[1] Such a problem is studied by Barthélémy, Martin, Merlin (2007).

[2] The notation $|A|$ represents the cardinal of the set A.

Since $v(S) = 0$ or $v(S) = 1$, $[v(S) - v(S\backslash\{i\})]$ is non-null only if the player i is decisive in S[3].

The Banzhaf index (1965) is given by the following formula

$$\beta_i = \frac{\sum_{S \subseteq N} [v(S) - v(S\backslash\{i\})]}{\sum_{j \in N} \sum_{S \subseteq N} [v(S) - v(S\backslash\{j\})]}$$

Then each voting game $[q; w_1, ..., w_n] \in \mathbb{N}^{n+1}$ corresponds to a Shapley-Shubik configuration of power (or a vector of power) $(\phi_1, \ldots, \phi_n) \in \mathbb{R}^n$ and a Banzhaf configuration of power $(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$. We insist here on an important point of vocabulary: there is an important distinction between the repartitions of weights and the configurations of power. A repartition of weights (or simply repartition) is just a vector $(w_1, \ldots, w_n) \in \mathbb{N}^n$ and a configuration of power (or simply configuration) is a vector of power. We associate only one configuration to a repartition but there is no boubt that several repartitions may be associated to a configuration.

Various methods are available to compute the Banzhaf and the Shapley-Shubik indices. See Leech (2002a), for instance, for a description of each method and their relative interest. The direct enumeration consists in applying directly the definition of the indices. An important limit of this approach is the number of players, which must remain under 31. Generating functions, suggested by Mann and Shapley (1962) enable to deal with higher number of players (up to 200) and give the exact result. The Monte Carlo simulations presented by Mann and Shapley (1960) are an approximation like the Multilinear Extensions Approximation methods developed by Owen (1972, 1975) and modified by Leech (2003). Here, as we focus on exact power index values, the use of generating functions seems to be more appropriate[4]. In the same spirit, counting and listing all the possible weight repartitions may be linked to a more general problem consisting on counting and listing all the subsets of the set $\{1, 2, \ldots, \bar{w}\}$, the sum of the subsets being always equal to $\bar{w}$. The algorithm of Stojmenivić and Miyakawa (1988) enables to decompose the integer n from 1 to n elements, the sum of the subsets being always equal to n. We are looking for all the vectors $(w_1, w_2, \ldots, w_n)$ with $w_1 \geqslant w_2 \geqslant ... \geqslant w_n$ and $\sum_{i=1}^{n} w_i = \bar{w}$[5].

[3]In the literature, a decisive player is called a pivotal player in the Shapley-Shubik index.

[4]Let us notice that however, the approximation proposed by Leech (2003) is incredibly precise.

[5]Many relative issues are linked to this problem as generating all the r-subsets (subsets containing r elements) of the set $\{1, 2, \ldots, n\}$, for $1 \leqslant r \leqslant n$ or the knapsack problem, where here the seats repartition vector has to be maximised under an inequality constraint, the sum being less or equal to C, the total

The methodology we suggest to list the different Shapley-Shubik and Banzhaf configurations is analyzed in section 2. Next in section 3, we count and list the configurations. Four cases are analyzed: indeed, the total number of weights and the quota may be constrained or not. Clearly, it is different to determine the number of configurations for a given quota and/or a given total number of weights and the number of configurations for any quota and/or any total number of weights. For example, with 3 players, there are 4 configurations if we do not fix the quota and the total number of weights and only 2 if the quota is equal to 5 and the total number of weights is equal to 9. We give a particular attention to the results comparison between the Shapley-Shubik index and the Banzhaf index.

2 Methodology

2.1 Notation

Let $\mathcal{W}_n(\bar{w})$, the set of all the possible repartitions in the case of n players and a total number of weights equals to $\bar{w}$. For instance, in the 3 players case and a total weight equals to 5, there are five different repartitions:

$$\mathcal{W}_3(5) = \Big\{(5,0,0),(4,1,0),(3,2,0),(3,1,1),(2,2,1)\Big\}$$

By adding a quota at each element of $\mathcal{W}_n(\bar{w})$, we define some different voting games. Let $\tilde{q} = q/\bar{w}$, the quota defined in proportion[6]. Let $\mathcal{G}_n(\tilde{q},\bar{w})$ all the n players voting games with a quota of $\tilde{q}$ and a total of weights equals to $\bar{w}$. For instance, for the majority case with 3 players and $\bar{w} = 5$, there are 5 voting games built from $\mathcal{W}_3(5)$:

$$\mathcal{G}_3(\tfrac{1}{2};5) = \Big\{[\tfrac{1}{2};5,0,0],[\tfrac{1}{2};4,1,0],[\tfrac{1}{2};3,2,0],[\tfrac{1}{2};3,1,1],[\tfrac{1}{2};2,2,1]\Big\}$$

Each voting game $[\tilde{q}; w_1,...,w_n]$, corresponds to a configuration (for the Banzhaf and for Shapley-Shubik indices). For each couple $(\tilde{q},\bar{w}) \in [\frac{1}{2},1[\times\mathbb{N}^*$, we define $\mathcal{C}_n(\tilde{q},\bar{w})$ as the

number of seats. Ehrlich (1973) described a loopless procedure for generating subsets of a set of n elements. A procedure based on gray codes is in Nijenhuis and Wilf (1983) where an algorithm for generating all r-subsets in lexicographic order ($1 \leqslant r \leqslant m \leqslant n$) is also proposed. Semba (1984) improved this algorithm. Borgwardt and Tremel (1991) deal with the problem of approximation in the knapsack problem.

[6]The quota q is an integer whose range depends on $\bar{w}$ and n values: $n/2 \leqslant q \leqslant \bar{w}-1$. Then $\tilde{q}$ represents simply a proportion which lies in the $]\frac{1}{2},1[$ interval.

set of all the possible configurations. Hence $\mathcal{C}_n(\tilde{q},\bar{w})$ is the set of all the index of power configurations for the voting games $[\tilde{q}; w_1, ..., w_n]$, where $\sum_{i=1}^{n} w_i = \bar{w}$, the quota being expressed in proportion. For the n players voting games, let us define the notations for the sets of all the possible indices of power configurations according to constraints on $\bar{w}$ and/or on $\tilde{q}$. This leads to four cases as underlined in the table 1.

Table 1: *The different configurations set according to constraints*

$\bar{w}$	$\tilde{q}$	
	constrained	not constrained
constrained	$\mathcal{C}_n(\tilde{q},\bar{w})$ (sections 2.1.1 and 3.1)	$\mathcal{C}_n(.,\bar{w})$ (sections 2.1.3 and 3.3)
not constrained	$\mathcal{C}_n(\tilde{q},.)$ (sections 2.1.2 and 3.2)	$\mathcal{C}_n(.,.)$ (sections 2.1.4 and 3.4)

The four notations will be described in the four next subsections. Let's denote more particularly $\mathcal{C}_n^B()$ and $\mathcal{C}_n^{SS}()$ the set of configurations corresponding respectively to the Banzhaf and to the Shapley-Shubik indices of power.

2.1.1 Set of configurations with $\tilde{q}$ and $\bar{w}$ constrained

To find the set $\mathcal{C}_n(\tilde{q},\bar{w})$, we may list all the possible repartitions of weights and take the union of all the corresponding configurations of power. For instance, in the majority case of 3 players and total weight of 3, there are three different repartitions $(3,0,0)$, $(2,1,0)$ and $(1,1,1)$ which lead respectively to the following configurations $(1,0,0)$, $(1,0,0)$ and $(\frac{1}{2},\frac{1}{2},0)$. Hence, the union of these 3 sets gives the set of all the configurations that exist in the majority case with 3 players and $\bar{w}=3$:

$$\mathcal{C}_3^B(\tfrac{1}{2},3) = \mathcal{C}_3^{SS}(\tfrac{1}{2},3) = \{(1,0,0),(\tfrac{1}{2},\tfrac{1}{2},0)\}$$

with

$$\mathcal{W}_3(3) = \Big\{(3,0,0),(2,1,0),(1,1,1)\Big\} \quad \text{and} \quad \mathcal{G}_3(\tfrac{1}{2};3) = \Big\{[\tfrac{1}{2};3,0,0],[\tfrac{1}{2};2,1,0],[\tfrac{1}{2};1,1,1]\Big\}$$

With $\bar{w}=10$ and $n=3$, the repartitions of weights are the followsing

$$\mathcal{W}_3(10) = \Big\{(10,0,0),(9,1,0),(8,2,0),(8,1,1),(7,3,0),(7,2,1),(6,4,0),\\ (6,3,1),(6,2,2),(5,5,0),(5,4,1),(5,3,2),(4,4,2),(4,3,3)\Big\}$$

and the Banzhaf configurations set becomes:

$$\mathcal{C}_3^B(\tfrac{1}{2},10) = \Big\{(1,0,0),(\tfrac{3}{5},\tfrac{1}{5},\tfrac{1}{5}),(\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})\Big\}$$

Let us notice that the following repartitions $(10,0,0)$, $(9,1,0)$, $(8,2,0)$, $(8,1,1)$, $(7,3,0)$, $(7,2,1)$ and $(6,4,0)$ lead to the same configuration (1,0,0) in the majority case. Moreover, in this example, the set of configurations is the same for a quota of 1/2 or for a quota of 2/3: $\mathcal{C}_3^B(\frac{2}{3},10) = \mathcal{C}_3^B(\frac{1}{2},10)$.

2.1.2 Set of configurations with a constrained $\tilde{q}$ ($\bar{w}$ is not constrained)

For a given quota in proportion $\tilde{q}$, we define the set $\mathcal{C}_n(\tilde{q},.)$ as the set of all the possible configurations for n players, whatever the total number of weights:

$$\mathcal{C}_n(\tilde{q},.) = \bigcup_{\bar{w}\geqslant 1} \mathcal{C}_n(\tilde{q},\bar{w})$$

For instance, with $n=3$ and $\tilde{q}=\frac{1}{2}$, table 2 details for $\bar{w}=1,\dots,4$ the different repartitions and the corresponding configurations[7]. The $\mathcal{C}_3(\frac{1}{2},.)$ sets are obtained by the infinite union of the $\mathcal{C}_3^B(\frac{1}{2},\bar{w})$ for $\bar{w}\geqslant 1$. As we know that there are analytically only 4 different configurations, we may conclude that the union of the four first sets enables to list all the elements of such a set. Hence

$$\mathcal{C}_3^B(\tfrac{1}{2},.) = \mathcal{C}_3^B(\tfrac{1}{2},1)\bigcup\mathcal{C}_3^B(\tfrac{1}{2},2)\bigcup\mathcal{C}_3^B(\tfrac{1}{2},3)\bigcup\mathcal{C}_3^B(\tfrac{1}{2},4)$$
$$\mathcal{C}_3^B(\tfrac{1}{2},.) = \left\{(1,0,0),(\tfrac{3}{5},\tfrac{1}{5},\tfrac{1}{5}),(\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})\right\}$$

In the same way,

$$\mathcal{C}_3^{SS}(\tfrac{1}{2},.) = \left\{(1,0,0),(\tfrac{2}{3},\tfrac{1}{6},\tfrac{1}{6}),(\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})\right\}$$

2.1.3 Set of configurations with a constrained $\bar{w}$ ($\tilde{q}$ is not constrained)

For a given sum of weights $\bar{w}$, we define the set $\mathcal{C}_n(.,\bar{w})$ as the set of all the possible configurations for n players, whatever the quota:

$$\mathcal{C}_n(.,\bar{w}) = \bigcup_{\tilde{q}\geqslant 1/2} \mathcal{C}_n(\tilde{q},\bar{w})$$

Let us notice that as q and $\bar{w}$ are integers, we do not need to analyze all the real values lying in $[\frac{1}{2},1]$. For example, assuming that $\bar{w}=5$, there are only two[8] different quotas, $q=3$ and $q=4$. Remark that for all $q/\bar{w}\in[0.5,0.8[$ the configurations are identical and

[7] We denote $\tilde{q}=\frac{1}{2}$ the well-known majority game where $q=\frac{\bar{w}}{2}+1$ if $\bar{w}$ is even and $q=\frac{\bar{w}+1}{2}$ if $\bar{w}$ is odd.

[8] There is one more available quota which is equal to $\bar{w}$, which corresponds to $\tilde{q}=1$.

Table 2: *Configurations according to $\bar{w}$ with 3 players in the majority case*

$\bar{w}$	Repartition of weights (w_1, w_2, w_3)	Banzhaf configurations	Shapley-Shubik configurations
1	(1,0,0)	(1,0,0)	(1,0,0)
2	(2,0,0)	(1,0,0)	(1,0,0)
	(1,1,0)	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$
3	(3,0,0)	(1,0,0)	(1,0,0)
	(2,1,0)	(1,0,0)	(1,0,0)
	(1,1,1)	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
4	(4,0,0)	(1,0,0)	(1,0,0)
	(3,1,0)	(1,0,0)	(1,0,0)
	(2,2,0)	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$
	(2,1,1)	$(\frac{3}{5},\frac{1}{5},\frac{1}{5})$	$(\frac{2}{3},\frac{1}{6},\frac{1}{6})$

this is same for all $q/\bar{w} \in [0.8, 1[$. More generally, there are: $(\bar{w}/2) - 1$ different values of q if $\bar{w}$ is even and $(\bar{w} - 1)/2$ different values of q if $\bar{w}$ is odd.

For instance, with $n = 3$ and $\bar{w} = 5$, table 3 details for $\tilde{q} = \{3/5, 4/5\}$ the different repartitions and the corresponding configurations.

Table 3: *Configurations according to $\tilde{q}$ with 3 players in the majority case*

Repartitions of weights	Banzhaf configurations according to $\tilde{q}$	
(w_1, w_2, w_3)	$\frac{3}{5}$	$\frac{4}{5}$
(5,0,0)	(1,0,0)	(1,0,0)
(4,1,0)	(1,0,0)	(1,0,0)
(3,2,0)	(1,0,0)	$(\frac{1}{2},\frac{1}{2},0)$
(3,1,1)	(1,0,0)	$(\frac{3}{5},\frac{1}{5},\frac{1}{5})$
(2,2,1)	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{2}, \frac{1}{2}, 0)$

As $\mathcal{C}_3^B(\frac{3}{5}, 5) = \Big\{(1,0,0), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\Big\}$ and $\mathcal{C}_3^B(\frac{4}{5}, 5) = \Big\{(1,0,0), (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}), (\frac{1}{2}, \frac{1}{2}, 0)\Big\}$, we get,

$$\mathcal{C}_3^B(., 5) = \Big\{(1,0,0), (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}), (\tfrac{1}{2}, \tfrac{1}{2}, 0), (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})\Big\}$$

2.1.4 Set of configurations with $\tilde{q}$ and $\bar{w}$ not constrained

We present here the most common case. Given the number of players, our problem is to determine all the configurations without fix $\tilde{q}$ and $\bar{w}$.

Let $\mathcal{C}_n(.,.)$ the set of all the configurations in the case of n players. By construction, these sets are:

$$\mathcal{C}_n(.,.) = \bigcup_{\tilde{q}\geqslant 1/2} \mathcal{C}_n(\tilde{q},.) = \bigcup_{\tilde{q}\geqslant 1/2} \Big(\bigcup_{\bar{w}\geqslant 1} \mathcal{C}_n(\tilde{q},\bar{w}) \Big)$$

2.2 Presentation of the methodology

Our method is based on the fact that all the different configurations for a given number of players (n) in a $\tilde{q}$ majority voting game can be obtained by taking the union of k first $\mathcal{C}_n(\tilde{q},\bar{w})$ sets, as the total number of configurations is finite[9]:

$$\exists k \in N,\ \mathcal{C}_n(\tilde{q},.) = \bigcup_{\bar{w}\leqslant k} \mathcal{C}_n(\tilde{q},\bar{w})$$

For instance, let's take the example presented in table 2. By calculus, we know that there are 4 different configurations with 3 players in the majority case for the Banzhaf and Shapley-Shubik indices of power:

$$\mathcal{C}_3^B(\tfrac{1}{2},.) = \Big\{(1,0,0),(\tfrac{3}{5},\tfrac{1}{5},\tfrac{1}{5}),(\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})\Big\}$$
$$\mathcal{C}_3^{SS}(\tfrac{1}{2},.) = \Big\{(1,0,0),(\tfrac{2}{3},\tfrac{1}{6},\tfrac{1}{6}),(\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})\Big\}$$

Then

$$\Big|\mathcal{C}_3^B(\tfrac{1}{2},.)\Big| = \Big|\mathcal{C}_3^{SS}(\tfrac{1}{2},.)\Big| = 4$$

As shown in table 2, we then have the following results

$$\mathcal{C}_3(\frac{1}{2},.) = \bigcup_{\bar{w}\leqslant 4} \mathcal{C}_3(\frac{1}{2},\bar{w})$$

which implies that the value of k is 4 in this example: that is to say, by taking the union of the first 4 sets of configurations $\mathcal{C}_3^B(\frac{1}{2},1)$, $\mathcal{C}_3^B(\frac{1}{2},2)$, $\mathcal{C}_3^B(\frac{1}{2},3)$ and $\mathcal{C}_3^B(\frac{1}{2},4)$, we get all the configurations for the Banzhaf index, as shown in table 4 (constructed from table 2). Let's notice that k does not correspond to a value of $\bar{w}$ such as all the configurations are obtained: there is no reason for $\mathcal{C}_n(\tilde{q},.) = \mathcal{C}_n(\tilde{q},k)$. Indeed, in the previous example $\Big|\mathcal{C}_3(\frac{1}{2},.)\Big| = 4$, while $\Big|\mathcal{C}_3(\frac{1}{2},4)\Big| = 3$.

Table 4: *Increase of the Banzhaf set of configurations with 3 players according to $\bar{w}$ in the majority case*

$\bar{w}$	New configurations with increments on $\bar{w}$	$\bigcup_{x\leqslant\bar{w}} \mathcal{C}_n^B(\frac{1}{2},x)$
1	(1,0,0)	$\left\{(1,0,0)\right\}$
2	($\frac{1}{2}$,$\frac{1}{2}$,0)	$\left\{(1,0,0),(\frac{1}{2},\frac{1}{2},0)\right\}$
3	($\frac{1}{3}$,$\frac{1}{3}$,$\frac{1}{3}$)	$\left\{(1,0,0),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\right\}$
4	($\frac{3}{5}$,$\frac{1}{5}$,$\frac{1}{5}$)	$\left\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\right\}$

The idea of the method is then to begin with $\bar{w}=1$, and to increment $\bar{w}$ until it is equal to a given value, denoted here k. When $\bar{w}=1$, the configuration is 1 for the first player and 0 for the $(n-1)$ others. Each new value of $\bar{w}$ may lead to a new configuration. One interest of such a method, beginning with low values of $\bar{w}$, is due to the fact that it is much quicker to list weight repartitions for small values of $\bar{w}$. To be convergent, the method has to give the right number of configurations, but it depends on the value of k which is unknown. For instance, in the previous example, if k is set to 2, (instead of the right value which is 4), we only get 2 configurations: $\{(1,0,0),(\frac{1}{2},\frac{1}{2},0)\}$. In order to deal with this problem, we propose to increment $\bar{w}$ as long as we have new configurations. Then, we increment $\bar{w}$ as long as we do not have α consecutive configurations sets not leading to a new configuration. The larger α, the higher the confidence of convergence. Obviously, we have the control on the parameter of convergence α.

For instance, for the example mentioned previously, table 5 illustrates the convergence principle for some values of $\bar{w}\in\{4,5,...,10\}$: $\bigcup_{x\leqslant\bar{w}} \mathcal{C}_3(\frac{1}{2},x)$ is constant. Then 4 different configurations are found: α equals to 6, because six consecutive sets of configurations[10] do not give a new configuration. Even if we know in this example that the estimated set is equal to the real set, we have to mention that it is an estimated set, whose cardinality is estimated as well.

[9]Obviously, k depends on n and $\tilde{q}$ which do not appear to avoid tedious notations.

[10]Those sets are $\mathcal{C}_3^B(\frac{1}{2},5)$ to $\mathcal{C}_3^B(\frac{1}{2},10)$, presented in table 5.

Table 5: *Convergence of the set of configurations with $\tilde{q} = 1/2$ and 3 players*

$\bar{w}$	$\mathcal{C}_3^B(\frac{1}{2}, \bar{w})$	$\bigcup_{x \leqslant \bar{w}} \mathcal{C}_3^B(\frac{1}{2}, x)$
1	$\{(1,0,0)\}$	$\{(1,0,0)\}$
2	$\{(1,0,0),(\frac{1}{2},\frac{1}{2},0)\}$	$\{(1,0,0),(\frac{1}{2},\frac{1}{2},0)\}$
3	$\{(1,0,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$	$\{(1,0,0),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$
4	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0)\}$	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$
5	$\{(1,0,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$
6	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$
7	$\{(1,0,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$
8	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$
9	$\{(1,0,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$
10	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$	$\{(1,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5}),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$

From table A.1 in appendix, we get for the majority case with 4 players:

$$\left|\mathcal{C}_4^B(\tfrac{1}{2},.)\right| = \Big|\Big\{(1,0,0,0),(\tfrac{7}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{1}{10}),(\tfrac{3}{5},\tfrac{1}{5},\tfrac{1}{5},0),(\tfrac{1}{2},\tfrac{1}{2},0,0),(\tfrac{1}{2},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6}),$$
$$(\tfrac{5}{12},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{12}),(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0),(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{6},\tfrac{1}{6}),(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})\Big\}\Big| = 9$$

Generally, the convergence principle may be illustrated graphically by plotting the points $\left(\bar{w}, \left|\bigcup_{x \geqslant \bar{w}} \mathcal{C}_n(\tilde{q}, x)\right|\right)$ in a two dimensional space for successive values of $\bar{w}$.

In the majority case with 3 players, figure 1 represents the convergence principle for the Banzhaf index with $\alpha = 10$ (as $k = 4$, $\bar{w} \leqslant 14$) and figure 2 illustrates the case with 4 players (from table A.2 $k = 8$, then $\bar{w} \leqslant 18$)[11].To study the convergence in this method, we have to list all the configurations for a given $\bar{w}$. Our approach is based on computer calculations which are quiet unavoidable, given the high number of calculus involved in this problematic (except for the well- known cases with 3 or 4 players). In the case of $\mathcal{C}_n(.,.)$, we may analyze all the possible values for the quotas q with a given $\bar{w}$ to make sure we get all the configurations.

[11]In the figures, the notation CN corresponds to "Configurations Number" and CCN corresponds to "Cumulative Configurations Number".

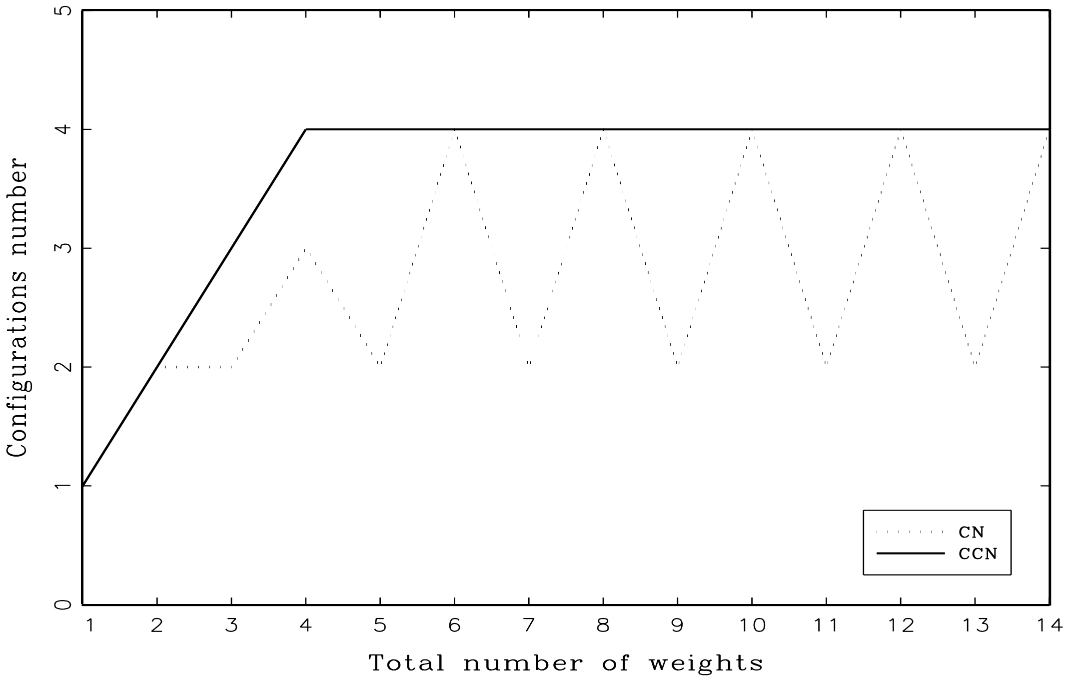

Figure 1: Convergence, $n = 3$, $\tilde{q} = \frac{1}{2}$

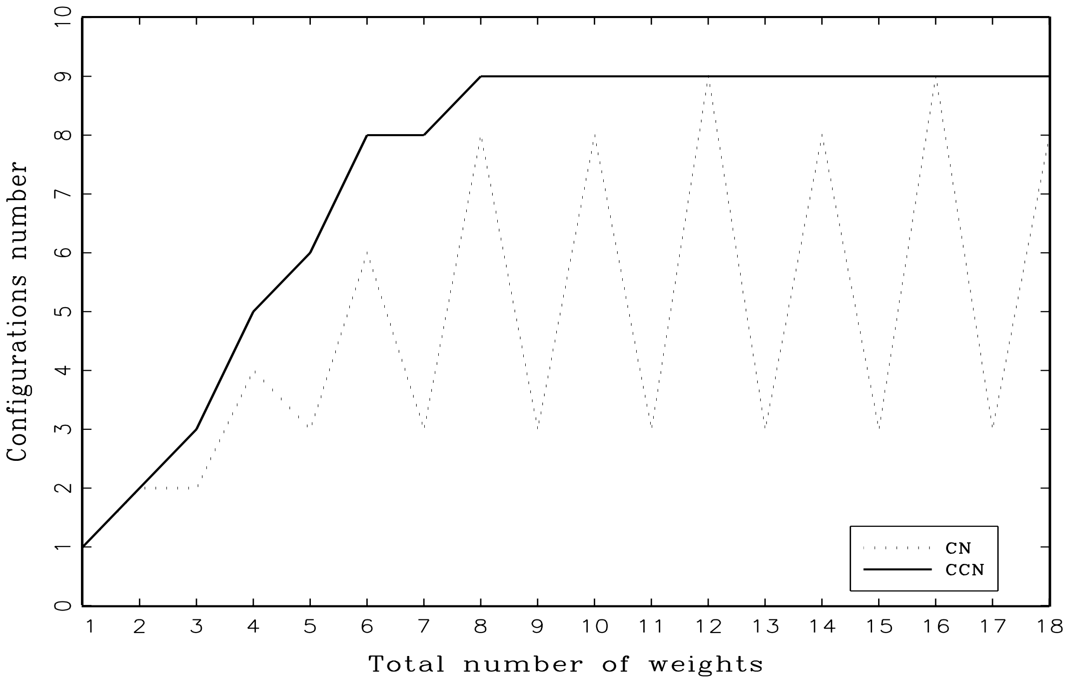

Figure 2: Convergence, $n = 4$, $\tilde{q} = \frac{1}{2}$

Remark 1 *Links between* $\mathcal{C}_n(\tilde{q},.)$ *and* $\mathcal{C}_{n+1}(\tilde{q},.)$

If $\Big\{w_1,\dots,w_n\Big\} \in \mathbb{N}^n$ gives the indices of power $(\phi_1,\dots,\phi_n) \in \mathbb{R}^n$ for Shapley-Shubik and $(\beta_1,\dots,\beta_n) \in \mathbb{R}^n$ for Banzhaf, then the vector $\Big\{w_1,\dots,w_n,0\Big\} \in \mathbb{N}^{n+1}$ gives respectively $(\phi_1,\dots,\phi_n,0) \in \mathbb{R}^{n+1}$ and $(\beta_1,\dots,\beta_n,0) \in \mathbb{R}^{n+1}$. Hence, for a given $\bar{w}$, by taking all the repartitions of weights leading to $\sum_{i=1}^n w_i = \bar{w}$, we get

$$\forall(\phi_1,\dots,\phi_n) \in \mathcal{C}_n^{SS}(\tilde{q},\bar{w}) \Rightarrow (\phi_1,\dots,\phi_n,0) \in \mathcal{C}_{n+1}^{SS}(\tilde{q},\bar{w}) \quad \text{for Shapley-Shubik}$$
$$\forall(\beta_1,\dots,\beta_n) \in \mathcal{C}_n^{B}(\tilde{q},\bar{w}) \Rightarrow (\beta_1,\dots,\beta_n,0) \in \mathcal{C}_{n+1}^{B}(\tilde{q},\bar{w}) \quad \text{for Banzhaf}$$

and we obtain, whatever the index of power

$$\forall n \in N^*, \Big|\mathcal{C}_n(\tilde{q},.)\Big| \leqslant \Big|\mathcal{C}_{n+1}(\tilde{q},.)\Big|$$

3 Our results

In this section, we give the number of configurations in the four cases described in table 1. Obviously, our results depend on the power of the computer and they are more significant when $\tilde{q}$ and $\bar{w}$ are constrained. Indeed, we have seen in the previous section that we obtain the unconstrained cases thanks to the constrained cases.

3.1 Configurations for a given $\bar{w}$ and a given $\tilde{q}$

We want to study the $\mathcal{C}_n(\tilde{q},\bar{w})$ sets and their cardinality given the set of repartitions. In the majority case, table 6 presents the number of configurations for the Banzhaf index or the Shapley-Shubik one in the case where $n = 1,\dots,12$ and $\bar{w} = 1,\dots,25$. The $2/3$ case is presented in table A.3. We observe that even if the sets $\mathcal{C}_n^{B}(\tilde{q},\bar{w})$ and $\mathcal{C}_n^{SS}(\tilde{q},\bar{w})$ differ their cardinality is the same[12]:

$$\forall n, \forall\tilde{q}, \forall\bar{w}, \quad \Big|\mathcal{C}_n^{B}(\tilde{q},\bar{w})\Big| = \Big|\mathcal{C}_n^{SS}(\tilde{q},\bar{w})\Big|$$

Tables 6 and A.3 illustrate that the number of configurations

- is increasing with the number of players (given $\bar{w}$).

[12]This observation holds for all the values of $\tilde{q}$ that we have computed.

- is not increasing monotonously with the sum of weights $\bar{w}$ (given n).

The non monotonicity in $\bar{w}$ when $\tilde{q} = 1/2$ is well-known if we refer to the difference made between odd and even value of $\bar{w}$. As an illustration, with 5 players there are 25 configurations with $\bar{w} = 18$ and 24 configurations with $\bar{w} = 22$. This point is more pronounced in the 2/3 case as shown in table A.3. But, we may notice that this is a bit more complicated issue. For instance, when $\tilde{q} = 1/2$ and for even values of $\bar{w}$ there is still a non monotonicity as shown in figure 3 for the case with 6 players. The case with $\bar{w} = 118$ leads to 126 configurations while there are 135 configurations with $\bar{w} = 108$ or even with $\bar{w} = 84$. With a smaller value, $\bar{w} = 48$, we get 137 configurations ($\bar{w} = 72$ and $\bar{w} = 96$ lead to the same results, with a periodicity of 24). This illustrates the quite complex link between $\bar{w}$ and $\left|\mathcal{C}_n^B(\tilde{q}, \bar{w})\right|$. Hence, we may generate a small configurations set with a higher $\bar{w}$ (remind than previously, we have noticed that taking a high value of $\bar{w}$ does not ensure to get all the configurations ; the smaller value leading to all the configurations was denoted k). This remark reinforces the methodology we propose (see section 2).

Table 6: *Number of configurations according to $\bar{w}$ and n with $\tilde{q} = 1/2$*

$\bar{w}$ \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	2	2	2	2	2	2	2	2	2	2	2	2	2
4	3	4	4	4	4	4	4	4	4	4	4	4	4
5	2	3	4	4	4	4	4	4	4	4	4	4	4
6	4	6	7	8	8	8	8	8	8	8	8	8	8
7	2	3	5	6	7	7	7	7	7	7	7	7	7
8	4	8	11	13	14	15	15	15	15	15	15	15	15
9	2	3	7	10	12	13	14	14	14	14	14	14	14
10	4	8	14	19	22	24	25	26	26	26	26	26	26
11	2	3	7	12	17	20	22	23	24	24	24	24	24
12	4	9	19	29	36	41	44	46	47	48	48	48	48
13	2	3	7	17	27	34	39	42	44	45	46	46	46
14	4	8	21	38	52	63	70	75	78	80	81	82	82
15	2	3	7	19	36	49	60	67	72	75	77	78	79
16	4	9	24	51	76	97	112	123	130	135	138	140	141
17	2	3	7	20	48	73	94	109	120	127	132	135	137
18	4	8	25	63	105	142	171	193	208	219	226	231	234
19	2	3	7	21	60	102	139	167	189	204	215	222	227
20	4	9	26	77	145	208	259	300	330	352	367	378	385
21	2	3	7	21	76	146	210	261	302	332	354	369	380
22	4	8	24	85	183	284	371	443	498	540	570	592	607
23	2	3	7	21	85	186	289	376	448	502	544	574	596
24	4	9	27	102	243	402	545	666	762	838	894	936	966
25	2	3	7	21	100	251	417	563	685	781	857	913	955
26	4	8	24	109	304	539	765	963	1125	1255	1355	1432	1488
27	2	3	7	21	112	324	573	804	1004	1166	1296	1395	1472
28	4	9	26	119	374	715	1062	1375	1639	1856	2027	2161	2262
29	2	3	7	21	119	400	767	1129	1450	1717	1935	2106	2240
30	4	8	25	122	445	924	1437	1921	2340	2692	2975	3201	3376
31	2	3	7	21	125	486	1010	1551	2050	2475	2829	3112	3338
32	4	9	26	129	536	1208	1958	2689	3339	3895	4352	4723	5015
33	2	3	7	21	132	604	1361	2169	2935	3602	4166	4626	4998
34	4	8	24	125	625	1525	2593	3665	4649	5508	6229	6823	7299
35	2	3	7	21	132	713	1732	2891	4022	5036	5911	6638	7234
36	4	9	27	134	732	1934	3434	4987	6451	7755	8868	9799	10558
37	2	3	7	21	133	846	2242	3903	5567	7097	8437	9568	10507
38	4	8	24	126	814	2367	4432	6642	8774	10716	12404	13837	15021
39	2	3	7	21	135	979	2812	5136	7531	9778	11785	13505	14954
40	4	9	26	133	916	2896	5687	8788	11851	14697	17213	19378	21194
41	2	3	7	21	135	1105	3489	6679	10080	13342	16311	18895	21097
42	4	8	25	130	1008	3522	7257	11539	15873	19974	23667	26891	29631
43	2	3	7	21	135	1249	4343	8684	13442	18099	22406	26218	29508
44	4	9	26	131	1120	4306	9279	15152	21224	27068	32417	37149	41221
45	2	3	7	21	135	1419	5424	11323	17965	24581	30784	36354	41215
46	4	8	24	126	1184	5071	11518	19406	27777	35985	43618	50470	56440
47	2	3	7	21	135	1542	6483	14196	23182	32363	41127	49115	56181
48	4	9	27	137	1274	5967	14258	24772	36200	47614	58390	68194	76840
49	2	3	7	21	135	1671	7809	17946	30127	42835	55152	66526	76707
50	4	8	24	127	1322	6975	17592	31489	46944	62648	77696	91560	103936
51	2	3	7	21	135	1817	9408	22637	38988	56388	73491	89481	103955
52	4	9	26	131	1411	8236	21764	40004	60705	82105	102907	122311	139836
53	2	3	7	21	135	1900	11121	28194	49904	73431	96935	119205	139590
54	4	8	25	130	1440	9390	26306	49880	77277	106112	134567	161450	186006
55	2	3	7	21	135	2013	13035	34721	63133	94599	126549	157228	185628

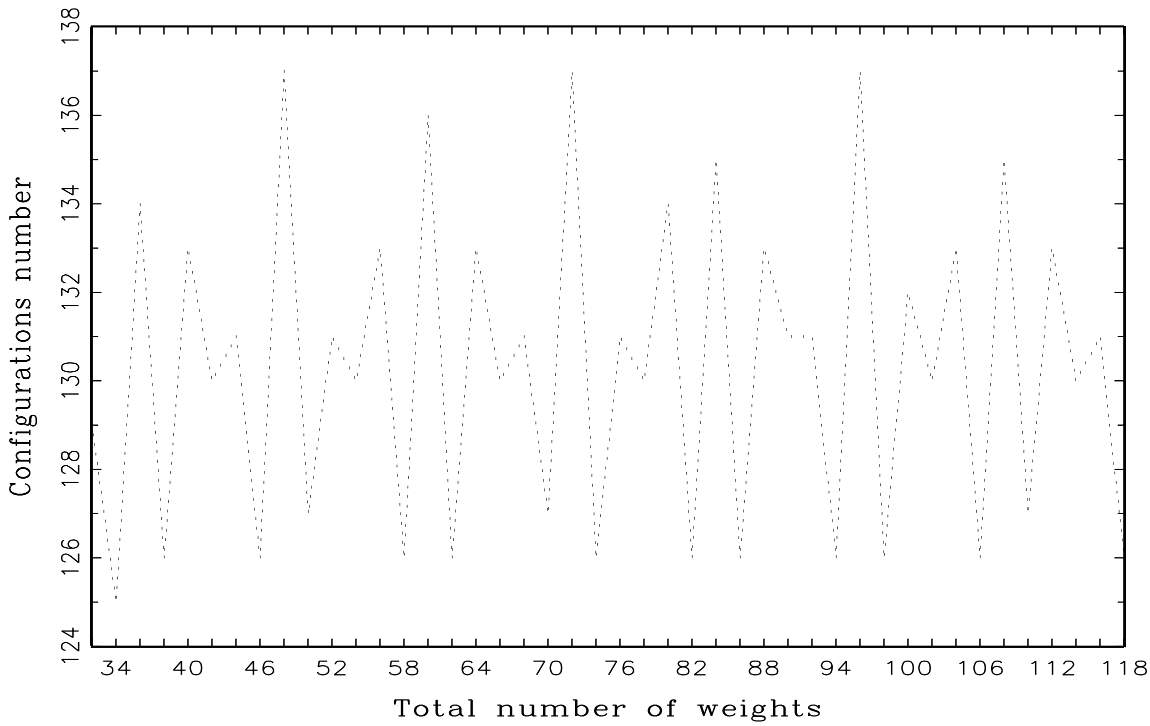

Figure 3: Non monotonicity (even values of $\bar{w}$), $n = 6$, $\tilde{q} = \frac{1}{2}$

3.2 Configurations when $\bar{w}$ is not constrained ($\tilde{q}$ is constrained)

In the previous section, we have seen how to evaluate in exhaustive way the $\mathcal{C}_n(\tilde{q}, \bar{w})$ sets. Here, we are going to study the sets $\mathcal{C}_n(\tilde{q}, .)$: for a given quota in proportion ($\tilde{q}$), a given number of players we list and count all the configurations, whatever the number of weights. The different computations show that, as in the previous section, if the $\mathcal{C}_n^B(\tilde{q}, .)$ and the $\mathcal{C}_n^{SS}(\tilde{q}, .)$ sets are not the same, the cardinalities are equivalent

$$\forall n, \forall \tilde{q}, \quad \left|\mathcal{C}_n^B(\tilde{q}, .)\right| = \left|\mathcal{C}_n^{SS}(\tilde{q}, .)\right|$$

Using the methodology presented in section 2, we increment $\bar{w}$ until we have α new values not leading to a new configuration. The cases with 3 or 4 players are illustrated in section 2 (table 4 and table A.1). For a given n, in the majority case, table 7 gives the number of configurations when $\bar{w} < w_0$. Table 7 corresponds to the cumulative version of table 6, where the elements are:

$$\Big|\bigcup_{1 \leqslant \bar{w} \leqslant w_0} \mathcal{C}_n(\frac{1}{2}, \bar{w})\Big|$$

When this value is constant as w_0 increases, the convergence is obtained. For $n = 3, \ldots, 6$, the convergence principle is illustrated in the figure 4 for the majority case. From table 7, we know that $k = \{4, 8, 16, 32\}$ for the 1/2 case with $n = \{3, 4, 5, 6\}$. We know that $k > 60$ when $n = 7$ (see Figure 5).

For the 2/3 case, table A.4 corresponds to the cumulative version of table A.3, where the elements are:

$$\Big| \bigcup_{1 \leqslant \bar{w} \leqslant w_0} \mathcal{C}_n(\frac{2}{3}, \bar{w}) \Big|$$

for which figure 6 gives an illustration of the convergence principle. Let us notice that the $k = \{4, 7, 13, 27\}$ for the 2/3 case with $n = \{3, 4, 5, 6\}$ which is less than the corresponding value in the majority case.

Finally, table 8 reports the number of configurations for the Banzhaf and Shapley-Shubik indices for these two particular values of $\tilde{q}$.

Table 7: *Cumulative number of configurations cumul according to* $\bar{w}$ *and* n *with* $\tilde{q} = 1/2$

$\bar{w}$ \ n	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3
4	**4**	5	5	5	5	5	5	5	5	5	5	5
5	4	6	7	7	7	7	7	7	7	7	7	7
6	4	8	10	11	11	11	11	11	11	11	11	11
7	4	8	12	14	15	15	15	15	15	15	15	15
8	4	**9**	16	20	22	23	23	23	23	23	23	23
9	4	9	17	24	28	30	31	31	31	31	31	31
10	4	9	20	32	39	43	45	46	46	46	46	46
11	4	9	20	35	47	54	58	60	61	61	61	61
12	4	9	24	45	64	76	83	87	89	90	90	90
13	4	9	24	49	74	93	105	112	116	118	119	119
14	4	9	26	61	96	126	145	157	164	168	170	171
15	4	9	26	63	108	148	178	197	209	216	220	222
16	4	9	**27**	76	139	195	240	270	289	301	308	312
17	4	9	27	77	153	227	288	333	363	382	394	401
18	4	9	27	90	193	296	381	448	493	523	542	554
19	4	9	27	90	207	338	452	543	610	655	685	704
20	4	9	27	105	260	436	592	718	815	882	927	957
21	4	9	27	105	277	493	695	863	995	1092	1159	1204
22	4	9	27	115	336	624	896	1126	1307	1446	1543	1610
23	4	9	27	115	347	688	1035	1336	1579	1767	1906	2003
24	4	9	27	126	422	865	1323	1725	2053	2310	2505	2644
25	4	9	27	126	436	951	1518	2034	2464	2806	3070	3265
26	4	9	27	132	521	1180	1915	2594	3165	3625	3982	4254
27	4	9	27	132	530	1279	2169	3023	3760	4362	4837	5202
28	4	9	27	136	623	1571	2713	3818	4782	5577	6210	6701
29	4	9	27	136	629	1684	3048	4421	5644	6669	7496	8145
30	4	9	27	137	727	2052	3776	5535	7113	8450	9535	10396
31	4	9	27	137	729	2173	4203	6350	8331	10031	11431	12551
32	4	9	27	**138**	840	2634	5175	7883	10407	12591	14404	15867
33	4	9	27	138	843	2782	5734	9003	12121	14865	17170	19049
34	4	9	27	138	949	3332	6997	11086	15018	18510	21462	23888
35	4	9	27	138	950	3476	7668	12553	17363	21704	25422	28503
36	4	9	27	138	1067	4156	9316	15363	21365	26823	31533	35463
37	4	9	27	138	1067	4326	10182	17337	24606	31322	37185	42125
38	4	9	27	138	1169	5106	12262	21060	30052	38429	45786	52030
39	4	9	27	138	1169	5264	13271	23577	34378	44588	53673	61447
40	4	9	27	138	1270	6189	15899	28465	41724	54343	65646	75376
41	4	9	27	138	1270	6349	17138	31730	47530	62802	76637	88654
42	4	9	27	138	1350	7404	20427	38105	57350	76077	93151	108077
43	4	9	27	138	1350	7544	21873	42245	64998	87483	108217	126508
44	4	9	27	138	1433	8790	25997	50515	78041	105404	130795	153327
45	4	9	27	138	1433	8951	27784	55872	88203	120841	151452	178857
46	4	9	27	138	1490	10292	32755	66380	105279	144785	182035	215581
47	4	9	27	138	1490	10413	34690	72870	118249	165090	209754	250296
48	4	9	27	138	1548	11958	40777	86213	140471	196784	250784	300065
49	4	9	27	138	1548	12085	43129	94445	157377	223762	288111	347317
50	4	9	27	138	1578	13757	50470	111272	186114	265484	342828	414377
51	4	9	27	138	1578	13847	53064	121313	207635	300644	392282	477740
52	4	9	27	138	1611	15714	61917	142426	244562	355161	464713	567394
53	4	9	27	138	1611	15801	64919	154851	272100	401067	530221	652257
54	4	9	27	138	1629	17751	75360	180979	319100	471752	625384	771280
55	4	9	27	138	1629	17819	78588	195834	353537	530693	710951	883496

Table 8: *Number of configurations for the Banzhaf and Shapley-Shubik indices of power for a given $\tilde{q}$*

n	$\tilde{q} = 1/2$	$\tilde{q} = 2/3$
3	4	4
4	9	9
5	27	27
6	138	133
7	1 663	1 440
8	63 583	44 934

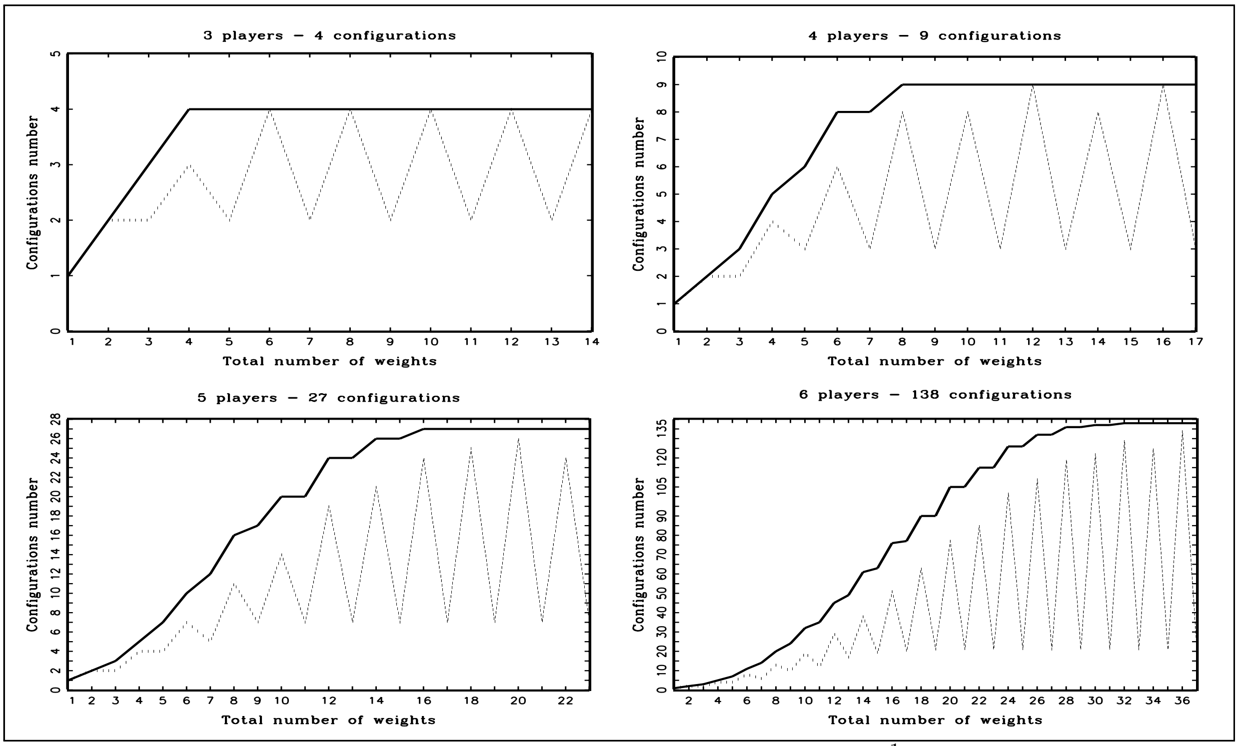

Figure 4: all weight repartitions, $n = 3, ..., 6$, $\tilde{q} = \frac{1}{2}$

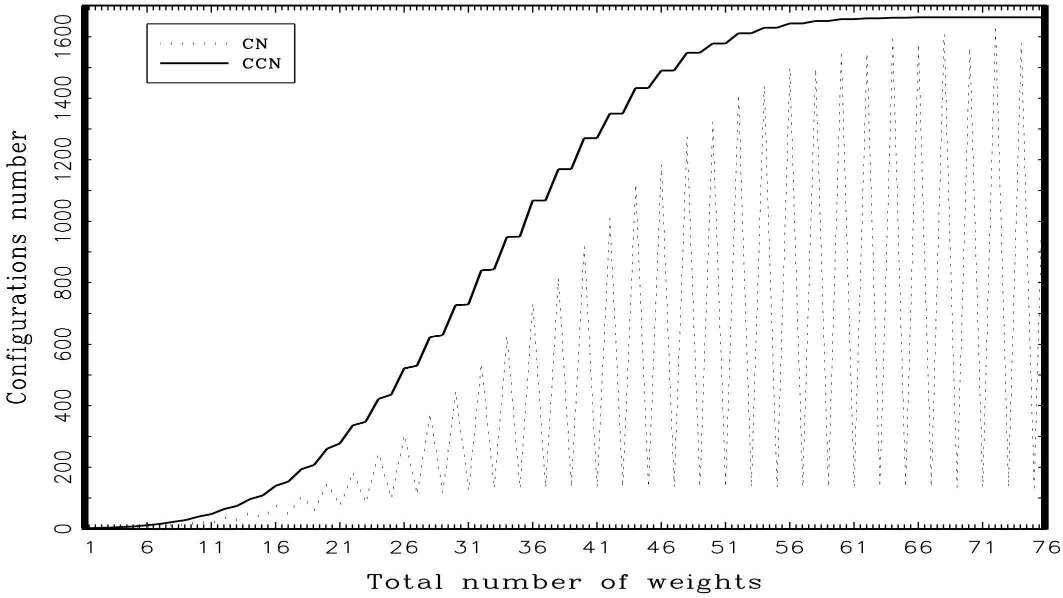

Figure 5: all weight repartitions, $n = 7$, $\tilde{q} = \frac{1}{2}$

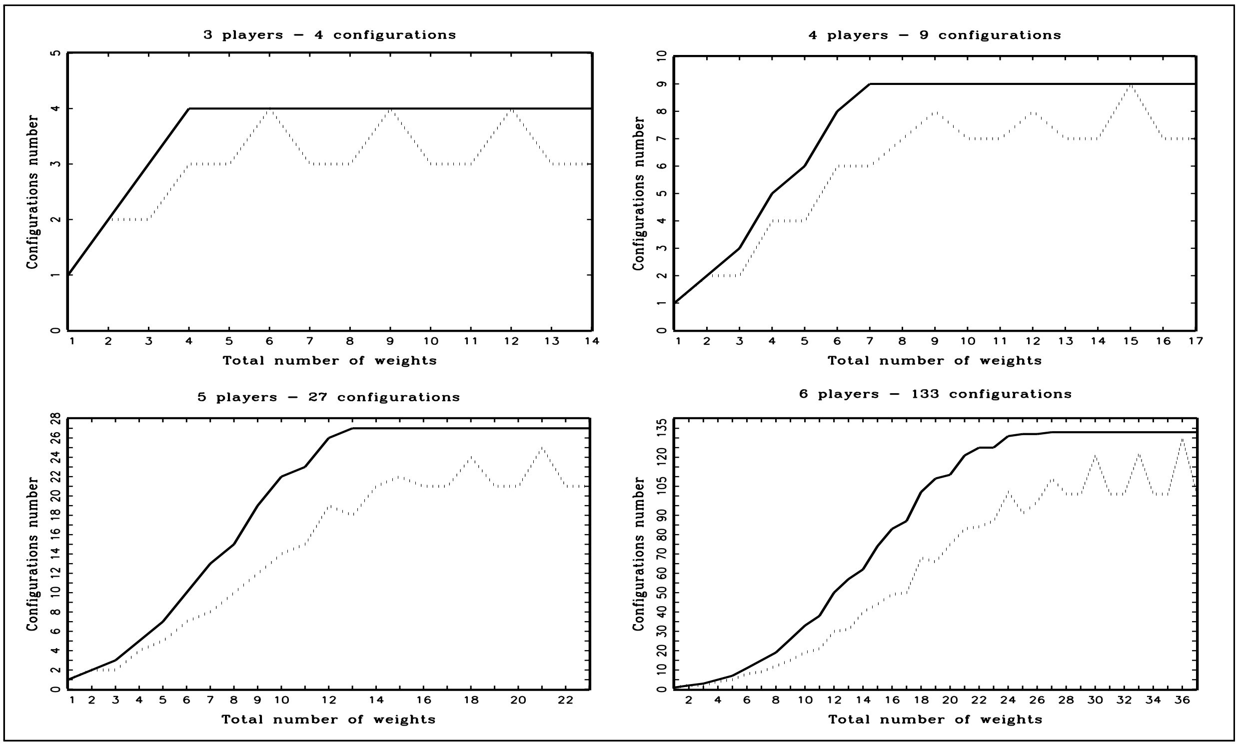

Figure 6: all weight repartitions, $n = 3, ..., 6$, $\tilde{q} = \frac{2}{3}$

3.3 Configurations when $\tilde{q}$ is not constrained ($\bar{w}$ is constrained)

Table 9 reports the configurations number for the Banzhaf and the Shapley-Shubik indices when $\bar{w}$ is fixed, whatever the quota. As $n \geqslant 4$ and $\bar{w} \geqslant 5$, there are more Banzhaf configurations:

$$\left|\mathcal{C}_n^B(\bar{w},.)\right| > \left|\mathcal{C}_n^{SS}(\bar{w},.)\right|$$

Table 10 shows different cardinalities of those two sets in the case of 4 players when $\bar{w} = 10$. The crosses correspond to new configurations and the dots to a configuration associated with a smaller value of q. The last line, "Number of configurations for a given q" illustrates the equivalence of cardinality of the set of configurations. When the quota is 6, there are 8 configurations[13]. With a quota of 7, there are seven configurations for Banzhaf and Shapley-Shubik[14]. Moreover, the line "Number of new configurations" indicates 3 configurations with $q = 7$ not present in $\mathcal{C}_4^B(10, \frac{1}{2})$ and only 2 configurations not present in $\mathcal{C}_4^{SS}(10, \frac{1}{2})$. Note that $q = 9$ does not imply a new configuration, whatever the power index.

[13] $\left|\mathcal{C}_4^B(10, \frac{1}{2}\right| = \left|\mathcal{C}_4^{SS}(10, \frac{1}{2})\right| = 8.$

[14] $\left|\mathcal{C}_4^B(10, \frac{7}{10})\right| = \left|\mathcal{C}_n^{SS}(10, \frac{7}{10})\right| = 7.$

Table 9: *Configurations when $\tilde{q}$ is not constrained ($\bar{w}$ is constrained)*

	$n=3$		$n=4$		$n=5$		$n=6$		$n=7$		$n=8$	
$\bar{w}$	Bz	SS	Bz	SS	Bz	SS	Bz	SS	Bz	SS	Bz	SS
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3
4	3	3	4	4	4	4	4	4	4	4	4	4
5	**4**	**4**	7	6	8	7	8	7	8	7	8	7
6	4	4	8	8	10	9	11	10	11	10	11	10
7	4	4	10	9	16	14	20	16	21	17	21	17
8	4	4	11	11	20	20	26	24	29	25	30	26
9	4	4	11	10	25	23	36	33	45	38	50	40
10	4	4	**12**	**11**	31	30	50	48	61	57	69	61
11	4	4	12	11	35	32	62	58	84	78	100	88
12	4	4	12	11	42	39	82	78	115	110	139	129
13	4	4	12	11	42	38	98	92	151	143	190	177
14	4	4	12	11	50	46	124	118	194	188	253	247
15	4	4	12	11	49	45	139	130	243	234	330	319
16	4	4	12	11	53	49	170	163	314	298	438	420
17	4	4	12	11	54	50	186	177	379	365	555	539
18	4	4	12	11	57	53	230	220	480	462	724	700
19	4	4	12	11	56	52	242	232	554	541	888	872
20	4	4	12	11	57	53	286	273	689	666	1131	1100
21	4	4	12	11	56	52	294	283	792	768	1380	1350
22	4	4	12	11	**57**	**53**	342	330	967	947	1718	1685
23	4	4	12	11	57	53	353	341	1094	1072	2051	2028
24	4	4	12	11	57	53	398	383	1328	1299	2549	2509
25	4	4	12	11	57	53	397	384	1453	1418	2989	2943
26	4	4	12	11	57	53	451	435	1753	1716	3675	3621
27	4	4	12	11	57	53	440	425	1901	1854	4265	4218
28	4	4	12	11	57	53	479	464	2244	2190	5158	5084
29	4	4	12	11	57	53	480	466	2403	2366	5901	5861
30	4	4	12	11	57	53	508	490	2846	2779	7166	7079
31	4	4	12	11	57	53	506	490	2985	2937	8050	7997
32	4	4	12	11	57	53	530	510	3485	3419	9661	9573
33	4	4	12	11	57	53	521	503	3656	3582	10842	10759
34	4	4	12	11	57	53	539	521	4205	4129	12821	12717
35	4	4	12	11	57	53	534	516	4369	4286	14232	14137
36	4	4	12	11	57	53	550	531	5026	4924	16862	16720
37	4	4	12	11	57	53	546	527	5128	5037	18482	18382
38	4	4	12	11	57	53	552	533	5836	5722	21762	21609
39	4	4	12	11	57	53	548	529	5958	5838	23768	23652
40	4	4	12	11	57	53	553	534	6650	6505	27593	27407
41	4	4	12	11	57	53	553	534	6784	6647	29929	29787
42	4	4	12	11	57	53	554	535	7578	7430	34945	34749
43	4	4	12	11	57	53	554	535	7618	7466	37347	37200
44	4	4	12	11	57	53	**555**	**536**	8412	8244	43252	43038
45	4	4	12	11	57	53	554	535	8467	8282	46354	46172
46	4	4	12	11	57	53	555	536	9188	8994	53083	52842
47	4	4	12	11	57	53	555	536	9255	9070	56571	56410
48	4	4	12	11	57	53	555	536	10031	9784	64957	64640
49	4	4	12	11	57	53	555	536	10002	9789	68258	68059
50	4	4	12	11	57	53	555	536	10694	10423	77925	77590

Table 10: *Configurations according to q when $\bar{w} = 10$*

	$q = 6$		$q = 7$		$q = 8$		$q = 9$		for all q	
Configurations	B	SS	B	SS	B	SS	B	SS	B	SS
$(1, 0, 0, 0)$	X	X	.	.	.	.	.	.	X	X
$(\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$		X		.		.				X
$(\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$	X		.		.				X	
$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0)$		X		.		.		.		X
$(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, 0)$	X		.		.		.		X	
$(\frac{7}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12})$				X		.				X
$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	X	X	.	.	.	.	.	.	X	X
$(\frac{1}{2}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10})$			X		.				X	
$(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$	X	X		.		.		.	X	X
$(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$			X		.		.		X	
$(\frac{5}{12}, \frac{5}{12}, \frac{1}{12}, \frac{1}{12})$						X		.		X
$(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12})$	X	X							X	X
$(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8})$					X		.		X	
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	X	X			.	.	.	.	X	X
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$	X	X							X	X
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$			X	X			.	.	X	X
Number of new configurations	8	8	3	2	1	1	0	0	12	11
Number of configurations for a given q	8	8	7	7	8	8	7	7	12	11

3.4 Configurations when $\tilde{q}$ and $\bar{w}$ are not constrained

Our results show that the union of the different sets $\mathcal{C}_n(\tilde{q}, .)$ gives a larger set for Banzhaf as soon as $n \geq 4$:

$$\left|\mathcal{C}_n^B(., .)\right| > \left|\mathcal{C}_n^{SS}(., .)\right|$$

This results is induced by the different values of $\tilde{q}$ as seen in the previous section.

Finally, figures 8 and 9 show the different convergences from 3 to 6 players for Banzhaf and Shapley-Shubik and the different cardinalities are reported in table 11.

Table 11: *Number of configurations for Banzhaf and Shapley-Shubik for all quotas*

Number of players (n)	Banzhaf	Shapley Shubik
3	4	4
4	12	11
5	57	53
6	555	536
7	*14 710*	*14 178*

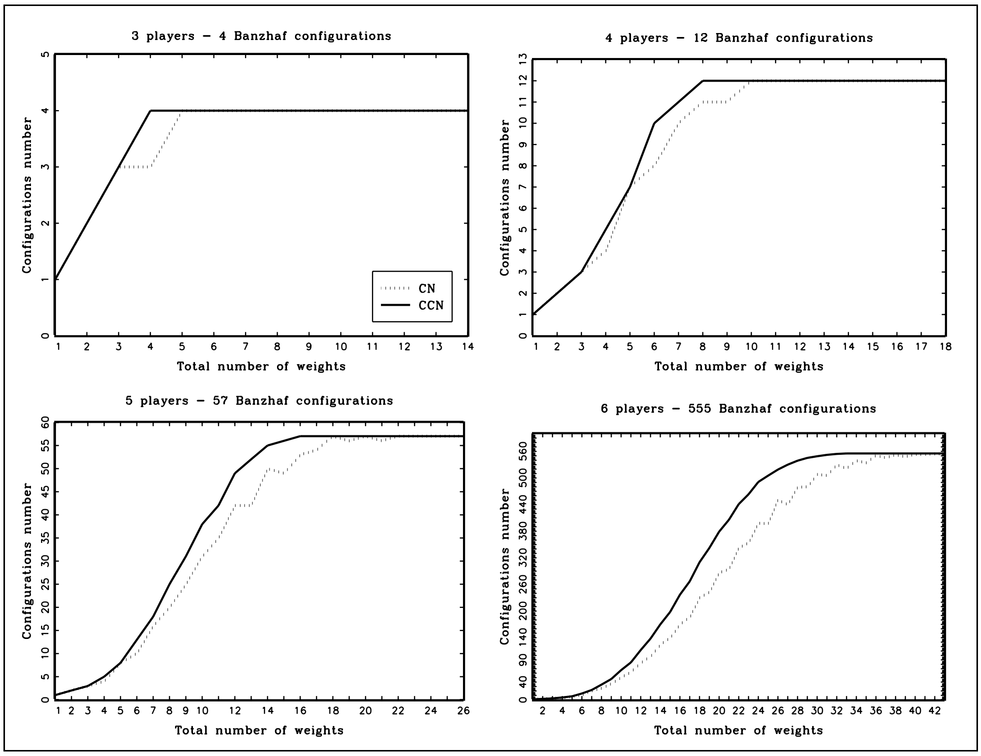

Figure 8: Banzhaf configurations convergence for all quotas, $n = 3, ..., 6$,

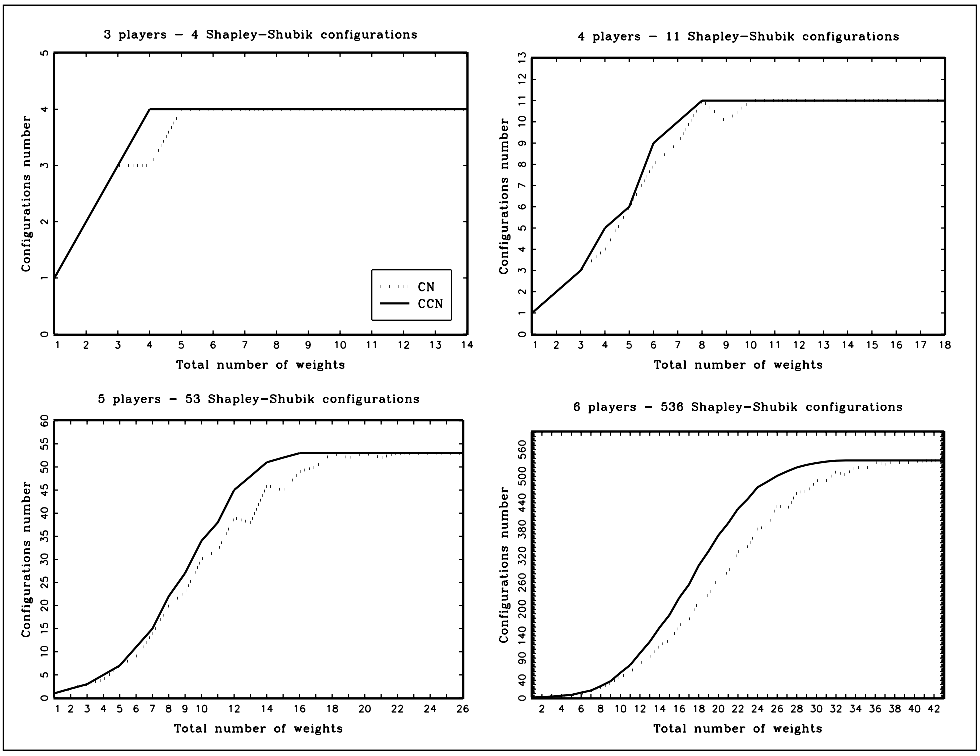

Figure 9: Shapley-Shubik configurations convergence for all quotas, $n = 3, ..., 6$,

4 Conclusion

In this paper, we present a method to enumerate all the possible configurations for the two main important power indices, Shapley-Shubik and Banzhaf. We show that the configurations number is not monotonous with the number of weights which reinforce the use of the proposed methodology. It will be obvious to generate all the configurations with a high number of weights. Hence, when listing all the configurations for a given quota (in proportion) whatever the total number of weights, we propose a convergence principle which leads to give the configurations, beginning the computation with the smallest values of repartitions. A certain number of tables are presented. Obviously, all the tables that we can obtain are not in the text. For example, when $\tilde{q}$ is given, we fix it to 1/2 or to 2/3 in general. The same tables can be obtained for any value of $\tilde{q} \in [1/2, 1[$. In the same way, the entire list of configurations is not given for an obvious lack of space. However, as soon as we give the number of configurations, we can present all of them. Even if we give some results for an important number of players or a high sum of weights, we are limited by the performance of the computer. For instance, when we do not fix $\tilde{q}$ and $\bar{w}$, the exhaustive approach becomes too time consuming for a number of players greater than 7. The purpose of research in the future is to use a simulation approach. The idea is to simulate many repartitions of weights to generate all the possible configurations. In our opinion, this approach is pregnant and will permit to increase noticeably the size of the parameters.

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Table A.1: *Configurations according to* $\bar{w}$ *with* $n = 4$ *and* $\tilde{q} = 1/2$ *;* $||$ *corresponds to 'Cardinality'*

$\bar{w}$	$\mathcal{C}_4^B(\frac{1}{2}, \bar{w})$	$\|\|$
1	$\{(1,0,0,0)\}$	1
2	$\{(1,0,0,0),(\frac{1}{2},\frac{1}{2},0,0)\}$	2
3	$\{(1,0,0,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\}$	2
4	$\{(1,0,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0)\},(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	4
5	$\{(1,0,0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\}$	3
6	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})\}$	6
7	$\{(1,0,0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\}$	3
8	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	8
9	$\{(1,0,0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\}$	3
10	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})\}$	8
11	$\{(1,0,0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\}$	3
12	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	9

Table A.2: *Convergence of the configurations set according to* $\bar{w}$ *with* $n = 4$ *and* $\tilde{q} = 1/2$

$\bar{w}$	$\bigcup_{x \leqslant \bar{w}} \mathcal{C}_4^B(\frac{1}{2}, x)$	$\|\|$
1	$\{(1,0,0,0)\}$	1
2	$\{(1,0,0,0),(\frac{1}{2},\frac{1}{2},0,0)\}$	2
3	$\{(1,0,0,0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\}$	3
4	$\{(1,0,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	5
5	$\{(1,0,0,0),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	6
6	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	8
7	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	8
8	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	9
9	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	9
10	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	9
11	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	9
12	$\{(1,0,0,0),(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}),(\frac{3}{5},\frac{1}{5},\frac{1}{5},0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}),(\frac{1}{3},\frac{1}{3},\frac{1}{3},0),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})\}$	9

Table A.3: Number of configurations according to $\bar{w}$ and n with $\tilde{q} = 2/3$

$\bar{w}$ \ n	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2
3	2	2	2	2	2	2	2	2	2	2	2	2
4	3	4	4	4	4	4	4	4	4	4	4	4
5	3	4	5	5	5	5	5	5	5	5	5	5
6	4	6	7	8	8	8	8	8	8	8	8	8
7	3	6	8	9	10	10	10	10	10	10	10	10
8	3	7	10	12	13	14	14	14	14	14	14	14
9	4	8	12	15	17	18	19	19	19	19	19	19
10	3	7	14	19	22	24	25	26	26	26	26	26
11	3	7	15	21	26	29	31	32	33	33	33	33
12	4	8	19	30	37	42	45	47	48	49	49	49
13	3	7	18	31	42	49	54	57	59	60	61	61
14	3	7	21	40	55	66	73	78	81	83	84	85
15	4	9	22	44	64	78	89	96	101	104	106	107
16	3	7	21	49	77	99	114	125	132	137	140	142
17	3	7	21	50	83	111	132	147	158	165	170	173
18	4	9	24	68	116	156	186	208	223	234	241	246
19	3	7	21	66	121	171	211	241	263	278	289	296
20	3	7	21	75	150	219	273	315	345	367	382	393
21	4	9	25	83	168	250	320	374	415	445	467	482
22	3	7	21	84	194	304	398	473	529	571	601	623
23	3	7	21	87	212	343	460	554	629	684	726	756
24	4	9	27	102	265	440	596	724	823	900	956	998
25	3	7	21	91	265	468	657	817	946	1045	1122	1178
26	3	7	21	97	304	561	802	1012	1181	1314	1415	1492
27	4	9	26	109	348	650	945	1202	1416	1585	1718	1818
28	3	7	21	101	375	747	1118	1451	1729	1953	2127	2262
29	3	7	21	101	397	830	1277	1681	2026	2306	2530	2704
30	4	9	27	121	490	1034	1596	2113	2556	2922	3212	3441
31	3	7	21	101	475	1092	1757	2378	2921	3374	3743	4034
32	3	7	21	101	536	1298	2129	2913	3601	4182	4653	5031
33	4	9	26	122	587	1433	2393	3319	4150	4860	5448	5921
34	3	7	21	101	599	1600	2782	3941	4984	5883	6630	7238
35	3	7	21	101	621	1704	3038	4380	5623	6707	7626	8379
36	4	9	27	130	753	2112	3785	5473	7029	8398	9555	10512
37	3	7	21	101	703	2137	4002	5944	7782	9420	10831	12007
38	3	7	21	101	765	2489	4791	7203	9481	11521	13277	14754
39	4	9	26	127	871	2817	5413	8180	10849	13271	15390	17184
40	3	7	21	101	839	3059	6201	9593	12871	15863	18483	20718
41	3	7	21	101	858	3276	6829	10738	14577	18118	21259	23958
42	4	9	27	132	1005	3854	8072	12775	17428	21757	25609	28943
43	3	7	21	101	903	3864	8547	13932	19352	24447	29037	33036
44	3	7	21	101	949	4339	9776	16121	22591	28734	34299	39188
45	4	9	26	129	1088	4858	11042	18348	25878	33088	39683	45512
46	3	7	21	101	984	5057	12114	20652	29589	38236	46194	53278
47	3	7	21	101	994	5416	13349	23129	33480	43573	52957	61359
48	4	9	27	132	1214	6487	15897	27453	39745	51823	63086	73238
49	3	7	21	101	1032	6356	16624	29704	43822	57824	71025	83000
50	3	7	21	101	1047	7033	19015	34434	51152	67835	83607	97994
51	4	9	26	128	1245	7908	21129	38352	57329	76491	94819	111664
52	3	7	21	101	1054	8016	23033	43140	65491	88209	110018	130163
53	3	7	21	101	1062	8401	24742	47059	72292	98233	123412	146861
54	4	9	27	133	1311	9951	29176	55392	85093	115786	145664	173604
55	3	7	21	101	1067	9535	29892	58777	92164	127091	161469	193875

Table A.4: Cumulative number of configurations according to $\bar{w}$ and n with $\tilde{q} = 2/3$

$\bar{w}$ \ n	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3
4	**4**	5	5	5	5	5	5	5	5	5	5	5
5	4	6	7	7	7	7	7	7	7	7	7	7
6	4	8	10	11	11	11	11	11	11	11	11	11
7	4	**9**	13	15	16	16	16	16	16	16	16	16
8	4	9	15	19	21	22	22	22	22	22	22	22
9	4	9	19	26	30	32	33	33	33	33	33	33
10	4	9	22	33	40	44	46	47	47	47	47	47
11	4	9	23	38	49	56	60	62	63	63	63	63
12	4	9	26	50	68	80	87	91	93	94	94	94
13	4	9	**27**	57	83	101	113	120	124	126	127	127
14	4	9	27	62	98	125	143	155	162	166	168	169
15	4	9	27	74	128	169	198	217	229	236	240	242
16	4	9	27	83	154	212	254	283	302	314	321	325
17	4	9	27	87	173	251	310	352	381	400	412	419
18	4	9	27	102	220	335	423	487	531	561	580	592
19	4	9	27	109	250	396	516	605	669	713	743	762
20	4	9	27	111	279	463	624	747	837	901	945	975
21	4	9	27	121	340	587	813	989	1120	1214	1280	1325
22	4	9	27	125	382	693	987	1225	1404	1536	1630	1696
23	4	9	27	125	411	789	1160	1472	1713	1893	2025	2119
24	4	9	27	131	493	1000	1502	1936	2275	2531	2719	2855
25	4	9	27	132	536	1144	1769	2323	2770	3112	3369	3557
26	4	9	27	132	569	1291	2065	2767	3351	3806	4151	4409
27	4	9	27	**133**	658	1585	2603	3536	4327	4950	5427	5785
28	4	9	27	133	711	1807	3057	4230	5242	6057	6688	7168
29	4	9	27	133	741	2002	3513	4969	6244	7292	8116	8750
30	4	9	27	133	851	2437	4382	6285	7966	9369	10481	11343
31	4	9	27	133	898	2710	5033	7366	9465	11239	12672	13793
32	4	9	27	133	923	2974	5744	8598	11209	13444	15278	16730
33	4	9	27	133	1010	3519	7022	10679	14053	16962	19373	21296
34	4	9	27	133	1054	3908	8070	12524	16690	20325	23369	25830
35	4	9	27	133	1067	4216	9053	14384	19463	23954	27757	30871
36	4	9	27	133	1171	4986	11021	17767	24248	30020	34933	38989
37	4	9	27	133	1201	5426	12398	20398	28230	35298	41384	46455
38	4	9	27	133	1211	5835	13895	23409	32881	41532	49052	55374
39	4	9	27	133	1280	6752	16570	28333	40170	51069	60611	68677
40	4	9	27	133	1301	7326	18642	32552	46774	60031	71741	81723
41	4	9	27	133	1303	7770	20628	36906	53846	69837	84101	96361
42	4	9	27	133	1359	8926	24536	44582	65625	85616	103535	119015
43	4	9	27	133	1368	9552	27253	50596	75525	99482	121168	140047
44	4	9	27	133	1369	10065	30002	57068	86509	115144	141316	164288
45	4	9	27	133	1396	11351	35124	67944	103992	139274	171702	200299
46	4	9	27	133	1400	12105	38908	76872	119231	161201	200120	234718
47	4	9	27	133	1400	12635	42466	86055	135529	185162	231626	273254
48	4	9	27	133	1422	14247	49606	101995	161948	222530	279556	330898
49	4	9	27	133	1424	15013	54328	114186	183846	255083	322790	384222
50	4	9	27	133	1424	15597	59123	127501	208436	292179	372479	445878
51	4	9	27	133	1437	17240	68120	149409	246521	347675	445195	534749
52	4	9	27	133	1437	18091	74557	167240	279737	398208	513392	619921
53	4	9	27	133	1437	18603	80344	185019	314410	452297	587585	713646
54	4	9	27	133	1439	20526	92412	216229	370614	536138	699363	852108
55	4	9	27	133	1439	21331	100106	239529	416352	608136	798992	978902