Universidad de la República
Facultad de Ciencias Sociales
DEPARTAMENTO DE ECONOMIA

Documentos de trabajo

On uniqueness of equilibrium for complete markets with infinitely many goods and in finance models.

Elvio Accinelli

Documento No. 03/98
Diciembre, 1998

On Uniqueness of Equilibrium for Complete Markets with Infinitely Many Goods and in Finance Models.

Elvio Accinelli [*†‡]

Abstract

Our concern in this work is to obtain conditions for the uniqueness of equilibria, with commodity bundles as consumption patterns which depend on the state of the world.

In the first section we consider an economy with complete markets, where consumption spaces are a finite product of measurable function spaces, with separable and proper utility functions and with strictly positive endowments. Using the excess utility function the infinite dimensional problem stated above is reduced to a finite dimensional one. We obtain local uniqueness. The degree theory and specially the Poincaré-Hopf theorem applied to this excess utility function, allow us to characterize the cardinality of the equilibrium set, and we find conditions for the global uniqueness of this set.

On the other hand, we obtain conditions for the uniqueness in economies with incomplete markets and only one good available in each state of the world. When markets are incomplete, equilibrium allocations are typically not Pareto efficient; then the results obtained in section 1, can not be generalized here. Nevertheless we show that for the single consumption good case the first theorem of welfare is satisfied, and then conditions for the uniqueness of equilibrium can be obtained as straightforward extension of our results shown in the first section. This is a particular simple case on incomplete markets but, is a very important one on finance theory.

1 Introduction

The first four sections of the paper concern exchange economies in which each agent's utility depends both upon his consumption vector and the realization of the world. Both trades and prices can be state contingent. A Walrasian equilibrium thus consists of a pair of measurable mappings (p, x) defined on the probability space Ω whose elements are the states of the world, where $p(\cdot)$ specifies the prices and $x(\cdot)$ specifies the net trades as function of the state. The main problem addressed in the first part of the paper concerns the nature of the set of equilibria of

[*]I wish to thank Aloisio Araujo, Paulo K. Monteiro, Flavio Menezes, Daniel Vaz and Ricardo Marchesini for useful comments.

[†]Facultad de Ingeniería (IMERL) CC 30, Montevideo-Uruguay.

[‡]e-mail elvio@fing.edu.uy

such an economy. This is complicated by the fact that the set of states can be infinite, which is the sense in which the paper considers an infinite number of goods. In the more elementary case in which there is not uncertainty (i.e., the state of the world is fixed), a Walrasian equilibrium is given by the solutions of $\epsilon(p) = 0$, where $\epsilon(\cdot)$ is the excess demand function and p is the vector of prices. Here, however, $p(\cdot)$ is a measurable mapping on a probability space, and so to obtain a solution for the equation $\epsilon(p) = 0$ is nontrivial. Moreover, the existence of the excess demand function is not a necessary consequence of a maximization process, its existence is rare in infinite dimensional cases.

In this paper we introduce the excess utility function to characterize the equilibrium set, showing that it is a powerful tool in order to characterize the equilibrium set. In this sense, the excess utility function appears as good substitute in infinite dimensional economies, for the generally inexistent, excess demand function.

On the other hand the excess utility function allows us to obtain a structural relation between the vector of welfare weights, the equilibrium prices and endowments. It follows, as we will show, from the fact that there exists an one-to-one correspondence between the zeros of the excess utility function and the set of Walrasian equilibria.

The excess utility function is definite on the $n-1$ dimensional simplex, and interpreting each element of this simplex as a vector of welfare weights, the weighted sum of the expected utilities of the agents, is maximized subject to the resource constraint at some particular state contingent allocation $x(s)$. The solution to this constrained optimization problem determines implicit prices $p(s)$ (i.e., the Lagrange multipliers at the solution $x(s)$). The excess utility function $e(\cdot)$ calculates, for each vector of welfare weights λ, the budget deficit of each of the n traders at the solution $x(s)$ to the constrained optimization problem. This budget deficit is calculated using the implicit prices $p(s)$ determined by the solution $x(s)$ and λ. The Pareto optimality of a Walrasian equilibrium is then invoked to establish that the set of Walrasian equilibrium is in one-to-one correspondence with the solutions of the equation $e(\lambda) = 0$.

Note that, the aggregate utility function depends on the weighting, λ, and it will be affected when we change their relative magnitudes. Each coordinate of a vector of welfare weights that is a zero of the excess utility function, will be determined by the distribution of initial endowments of individuals. Then the prices in the economy will be affected by the distribution of initial endowments across individuals. In this way we obtain the above mentioned structural relation. On this subject, G. Becker says *"In decentralized economies like our own, families, governments, and other organizations influence what to produce... There is a kind of proportional representation in which the influence of each person is not fixed nor shared equally. but is strictly proportional to* *his command over resources. Influence is exerted by offering to exchange these resources for the goods and services that are desired"* [Becker (71)]

On the other hand, this approach allows us to reduce an infinite dimensional problem to a finite dimensional one. It is shown in lemma 2, that the excess utility function $e(\cdot)$ has some of the useful properties that the excess demand function $\epsilon(\cdot)$ exhibits when there is a single state.

The excess utility function was introduced in [Mas-Colell (85)], (Ch.5, P.174), it is proved that the excess utility function in the finite dimensional case, has the same properties of the excess demand function. In [Mas-Colell (91)], Proposition 1, it is proved that the set of zeros for the excess utility function is generically finite. In this paper, for the infinite dimensional case, from the excess utility function we prove that generically, in the conditions of the model (see section 1) the set of Walrasian equilibria is not empty and that the excess demand function is a vector function in the conditions of the Poincaré-Hopf theorem, then we obtain conditions for uniqueness of equilibrium.

Note that while the not existence of the demand function is not a serious obstacle for the study of the existence of equilibrium, it is a serious one for the knowledge of the cardinality (and uniqueness) of the equilibrium set. Our result generalizes one of Dana, [Dana (93)]. In this work, R. A. Dana obtains a first result on uniqueness for economies with one good in each state of the world, with infinitely many states. Our result concerns a finite number of goods in each state of the world, and allows to use some of well know topological arguments to argue that the set of Walrasian equilibria in infinite dimensional case, with separable utilities, has the local uniqueness property, moreover it is generically finite, and we obtain some sufficient conditions for its uniqueness.

Finally we extend this analysis for the incomplete markets in the special case of one commodity and J assets. In this case the Walrasian equilibrium is Pareto optimal. This is a particular simple case on incomplete markets but, it is a very important one on finance theory. In the general case of incomplete markets, a Walrasian equilibrium need not be a Pareto optimal.

2 The Model

We shall consider a pure exchange economy with uncertainty in the states of the world Ω. We shall treat uncertainty us a probability space $(\Omega, \mathcal{A}, \nu)$, where $\mathcal{A}$ is the $\sigma - algebra$ of subsets of Ω that are events, and ν a probability measure. In each state of the world, there are l commodities available for consumption and n agents.

We assume that each agent has the same consumption space, $\mathcal{M} = \Pi_{j=1}^{l}\mathcal{M}_j$ where $\mathcal{M}_j$ is the space of all positive measurable functions defined on $(\Omega, \mathcal{A}, \nu)$.

Let Λ denote the set of functions $h : R^l_+ \to R$ satisfying:

- h is C^2 on R^l_+, i.e. h has second derivatives on $R^l_{++} = \{x \in R^l$ with all component positive$\}$ and one-sided second derivatives on R^l_+/R^l_{++}, and these are continuous,
- $h(x) = 0$ for all $x \in R^l_+/R^l_{++}$,
- h is differentially monotonic on R^l_+, i.e. $\partial h(x) \geq 0$ (i.e $\frac{\partial h(x)}{\partial x_j} \geq 0$ $j = 1, 2, ...l.$) for all $x \in R^l_+$;
- h is differentially strictly concave on R^l_{++}, i.e. for all $x \in R^l_{++}$, the Hessian matrix of second partial derivatives of h is negative definite and
- h satisfies the ' ' infinite marginal utility condition at zero, i.e. the limit of $|\partial h(x)|$ is infinite, when x approaches to the boundary of R^l_{++}, i.e: the set $B = \{x : x_i = 0$ for some $i = 1, \ldots, n\}$.

Let $\mathcal{U}$ be the set of all measurable functions $U : \Omega \times R^l_{++} \to R$, such that $U(s, \cdot) \in \Lambda$ for each $s \in \Omega$.

For $x, y \in R^l$ we will $x \geq y$ if $x_i \geq y_i$ $i = 1 \ldots$, and we will write $x > y$ if $x_i \geq y_i$ $i = 1 \ldots l$ and $x \neq y$.

Definition 1 *A function u is strictly monotone if $x > y \Rightarrow u(x) > u(y)$.*

We introduce in U the norm

$$\|U\|_K = ess \sup_{s \in \Omega} \max_{z \in K} \left\{ |U(s, z)| + |\partial U(s, z)| + |\partial^2 U(s, z)| \right\}.$$

for any compact $K \subset R^l_{++}$.

Each agent is characterized by his utility function u_i and by his endowment $w_i \in \mathcal{M}$, $i = \{1, 2, \ldots, n\}$, satisfying the following additional conditions:

a) The utility functions $u_i : \mathcal{M} \to R$ are separable. This means that they can be represented by

$$u_i(x) = \int_\Omega U_i(s, x(s)) d\nu(s) \quad i = 1, \ldots, n \tag{1}$$

where for each $i = 1, 2, ..., n$, $U_i : \Omega \times R^l_{++} \to R$ belongs to a fixed compact set of Λ and

b) for each $s \in \Omega$ the functions $U_i(s, \dot{)}$ belong to U.

c) The agent endowments, $w_i \in$ M are bounded away from zero in any component, i.e. there exists, h and H two positive numbers such that, h $< w_{ij}(s) <$ H for each $j = 1 \dots l$, and $s \in \Omega$.

An economy $\mathcal{E}$ is a list $(u_i, w_i), i \in I$, where I is a set of agents or traders (in our case $I = \{1, 2, \dots, n\}$).

The following definitions are standard.

Definition 2 *An allocation of commodities is a list $x = (x_1, \dots, x_n)$ where $x : \Omega \to R_+^{ln}$ and $\sum_{k=1}^n x_k(s) \leq \sum_{k=1}^n w_k(s)$, for a.e.$s \in \Omega$.*

Definition 3 *A commodity price system is a measurable function $p : \Omega \to R_{++}^l$. For any $z : \Omega \to R^l$ we denote by $\langle p, z\rangle$ the real number $\int_\Omega p(s)z(s)d\nu(s)$. (Note that $p(s)z(s)$ is the Euclidean inner product in R^l.)*

The following definition is given in [Mas-Colell (91)]:

Definition 4 *The pair (p, x) is an equilibrium if:*

i) p is a commodity price system and x is an allocation,

ii) $\langle p, x_i\rangle \leq \langle p, w_i\rangle < \infty \quad \forall\, i \in \{1, \dots, n\}$

iii) if $\langle p, z\rangle \leq \langle p, w_i\rangle$ with $z : \Omega \to R_{++}^l$, then

$$\int_\Omega U_i(s, x_i(s))d\nu(s) \geq \int_\Omega U_i(s, z(s))d\nu(s) \quad \forall\ i \in \{1, \dots, n\}.$$

That is there is not an alternative allocation z superior in the sense that is feasible, i.e. $\sum_{i=1}^n z_i(s) \leq \sum_{i=1}^n w_i(s)$ and $u_i(z) \geq u_i(x), i = 1, 2, ..., n$, with strictly inequality for some i.

3 The Excess Utility Function

In order to obtain our results we introduce the excess utility function.

We begin by writing the following well known proposition, see [Kehoe (91)]:

Proposition 1 *For each λ in the $(n-1)$ dimensional open simplex, $\Delta^{n-1} = \{\lambda \in R_{++}^n; \sum \lambda_i = 1\}$ and $U_i \in \Lambda$, there exists $\bar{x}(\lambda) = \{\bar{x}_1(\lambda), \cdots, \bar{x}_n(\lambda)\} \in R_{++}^{ln}$ a Pareto efficient solution of the following problem:*

$$\begin{array}{c} \max_{x \in R^{ln}} \sum_i \lambda_i U_i(x_i) \\ \textit{subject to } \sum_i x_i \leq \sum_i w_i \textit{ and } x_i \geq 0. \end{array} \tag{2}$$

If U_i depend also on $s \in \Omega$, and $U_i(s, \cdot) \in \Lambda$ for each $s \in \Omega$, and $\lambda \in \triangle^{n-1}$, there exists $\bar{x}(s, \lambda) = \bar{x}_1(s, \lambda), ..., \bar{x}_n(s, \lambda)$ solution of the following problem:

$$\begin{array}{c} \max_{x(s) \in R^{ln}} \sum_i \lambda_i U_i(s, x_i(s)) \\ \text{subject to } \sum_i x_i(s) \leq \sum_i w_i(s) \text{ and } x_i(s) \geq 0. \end{array} \tag{3}$$

If $\gamma^j(s, \lambda)$ are the Lagrange multipliers of the problem (3), $j \in \{1, \ldots l\}$, then from the first order conditions we have:

$$\lambda_i \frac{\partial U_i(s, \bar{x}_i(s, \lambda))}{\partial x^j} = \gamma^j(s, \lambda) \text{ with } i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, l\}.$$

Remark 1 *Due to the "infinite marginal utility" condition at zero, the solution of (3) must be strictly positive almost everywhere. Since $U(s, .)$ is a monotone function, we can deduce that $\sum_{i=1}^n \bar{x}_i(s) = \sum_{i=1}^n w_i(s)$.*

Let us now define the excess utility function.

Definition 5 *Let $x_i(s, \lambda); i \in \{1, \ldots, n\}$ be a solution of (3).*
We say that $e : \triangle^{n-1} \to R^n$ $e(\lambda) = (e_1(\lambda), ..., e_n(\lambda))$, with

$$e_i(\lambda) = \frac{1}{\lambda_i} \int_\Omega \gamma(s, \lambda)[x_i(s, \lambda) - w_i(s)] d\nu(s), \ i = 1, \ldots, n. \tag{4}$$

is the excess utility function.

Remark 2 *Since the solution of (3) is homogeneous of degree zero: i.e, $\bar{x}(s, \lambda) = \bar{x}(s, \alpha\lambda)$ for any $\alpha > 0$, then we can consider e_i defined all over R^n_{++} by $e_i(\alpha\lambda) = e_i(\lambda)$ for all $\lambda \in \Delta^{n-1}_{++}, \alpha > 0$.*

4 Equilibrium and the Excess Utility Function.

For $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\} \in R^n$, let us consider the following problem:

$$\begin{array}{c} \max_{x \in \mathcal{M}} \sum_i \lambda_i \int_\Omega U_i(s, x_i(s)) d\nu(s) \\ \text{subject to } \sum_i x_i(s) \leq \sum_i w_i(s) \text{ and } x_i(s) \geq 0. \end{array} \tag{5}$$

It is a well known proposition, [Mas-Colell (91)] that an allocation $\bar{x}$, is Pareto optimal if and only if we can choose a $\bar{\lambda}$, such that $\bar{x}$ solves the above problem with $\lambda = \bar{\lambda}$. Moreover, since a consumer with zero social weight receive nothing of value at a solution of this problem, we have that if $\bar{x}$ is a strictly positive allocation, that is $\{\bar{x} \in R^l_{++}\}$, all consumption has a positive social weight. See for instance [Kehoe (91)]. Reciprocally if $\bar{\lambda}$ is in the interior of the simplex, then from remark (1) the solution $x(., \lambda)$ of (6) is a strictly positive Pareto optimal allocation, [Kehoe (91)])

The first theorem of welfare establishes that every equilibrium allocation is Pareto optimal. In our setting this theorem hold. To see this, suppose that there exists a feasible allocation z that Pareto dominates the equilibrium allocation x. As the equilibrium price p, is nonnegative for all $s \in \Omega$ it follows that $< p, z_i > \geq < p, w_i >$, for all $i = 1, 2, ..., n$, and strictly form some of them. It follows that $< p, \sum_{i=1}^n z_i > > < p, \sum_{i=1}^n w_i >$ $(*)$. As z is a feasible allocation, $< p, \sum_{i=1}^n z_i > \leq < p, \sum_{i=1}^n w_i >$, holds, contrary to (*).

Let $\bar{x}$ be an equilibrium allocation, then there exists a $\bar{\lambda}$ such that $\bar{x} = \{\bar{x_1}, \ldots, \bar{x_n}\} : \Omega \to R^n$, is a solution for the problem (6).

In the conditions of our model, the first order conditions either for problem (6) or for (3) are the same. Then if a pair $(\bar{p}, \bar{x})$ is an price-allocation equilibrium, there exists a $\bar{\lambda}$ such that $\bar{x}(s) = \bar{x}(s, \bar{\lambda})$; solves (6) and $\bar{p}(s) = \gamma(s, \bar{\lambda})$, solves (4) for a.e.$s \in \Omega$.

Moreover we have the following proposition:

Proposition 2 *A pair $(\bar{p}, \bar{x})$ is an equilibrium, if and only if there exists $\bar{\lambda} \in \triangle^{n-1}$ such that $\bar{x}(s) = \bar{x}(s, \bar{\lambda})$ solves (6), and $\bar{p}(s) = \gamma(s, \bar{\lambda})$, solves (4) for a.e.s and $e(\bar{\lambda}) = 0$.*

Proof: Suppose that $\bar{x}(\cdot, \bar{\lambda})$ solves (6) and $\gamma(s, \bar{\lambda})$ solves (4), then $e(\bar{\lambda}) = 0$. Because $\bar{x}$ is Pareto optimal and from the strictly concavity of each $u_i, i = 1, 2, ..., n$ there is not a feasible allocation z such that $u_i(z) \geq u_i(\bar{x})$ and $< p, z > = < p, w_i >$ for all consumer. Then the pair $(\bar{p}, \bar{x})$, with $\bar{p} = \gamma(\cdot, \bar{\lambda})$ and $\bar{x} = x(\cdot, \bar{\lambda})$, is an equilibrium. Reciprocally, if $(\bar{p}, \bar{x})$ is an equilibrium, then from the first welfare theorem, there exists $\bar{\lambda} \in \triangle^{n-1}$, such that $\bar{x}$ is a solution for (6). Since p is an equilibrium price, it is a support for $\bar{x}$, i.e. if for some x we have that $u_i(x) \geq u_i(\bar{x}), i = \{1, ..., n\}$, strictly for some i, then $\langle \bar{p}, x_i \rangle > \langle \bar{p}, w_i \rangle$ and from the first order conditions we have that: $\bar{p}(s) = \gamma(\bar{\lambda}, s)$. Then $e(\bar{\lambda}) = 0$. 2

Let be $S^n_{++} = \{\lambda \in R^n : \|\lambda\|^2 = \sum_{i=1}^n \lambda_i^2 = 1, \lambda_i > 0\}$.

From remark 2, with $\alpha = \frac{1}{\|\lambda\|}$, we can consider e defined on S^n_{++}.

We give now the definition of the equilibrium set.

Definition 6 *We say that λ is an equilibrium for the economy if $\lambda \in E$, where $E = \{\lambda \in S^n_{++} : e(\lambda) = 0\}$. The set E will be called, the equilibrium set of the economy.*

A pair formed by a utility function and an endowment will be called a characteristic.

We will endow the set of characteristics $\mathcal{C} = \mathcal{U} \times \mathcal{M}$ with the topology generated by the norm :

$$\|(U, w)\|_K = \|(U)\|_K + \|w\| = ess \sup_{s \in \Omega} \max_K (|U| + |\partial U| + |\partial^2 U| + \|w(s)\|).$$

Let Γ be the set of economies with characteristics in $\mathcal{C}$ such that zero is a regular value of its excess utility function. That is, for any λ such that $e(\lambda) = 0$ we have that rank of the Jacobian of $e(\lambda)$, is $n-1$, i.e: $rankJ[e(\lambda)] = n-1$. from [Mas-Colell (91)], we know that Γ is open and dense in the set of economies. From now on we will work with economies in Γ.

Let be $T_\lambda S^n_{++} = \{\bar{\lambda} \in R^n : \bar{\lambda}\lambda = 0, \lambda \in S^n_{++}\}$, and Π_λ the orthogonal projection from R^n onto $T_\lambda S^n_{++}$. Since whenever $e(\bar{\lambda}) = 0$, $J[e(\bar{\lambda})]$ maps $T_{\bar{\lambda}} S^n_{++}$ into $T_{\bar{\lambda}} S^n_{++}$ (to verify it differentiate $\lambda e(\lambda) = 0$), if $\bar{\lambda}$ is a regular value, $J[e(\bar{\lambda})]$ maps $T_{\bar{\lambda}} S^{n-1}_{++}$. onto $T_{\bar{\lambda}} S^{n-1}_{++}$. Its determinant is equal to the determinant of the following matrix, (see [Mas-Colell (85)] B.5.2):

$$[\Pi_{\bar{\lambda}} J[e(\bar{\lambda})]] = \begin{bmatrix} J(e(\bar{\lambda})) & \bar{\lambda} \\ -\bar{\lambda}^t & 0 \end{bmatrix}.$$

Since $\Pi_{\bar{\lambda}} J[e(\bar{\lambda})]$ is an isomorphism from $T_{\bar{\lambda}} S^{n-1}_{++}$ onto $T_{\bar{\lambda}} S^{n-1}_{++}$, its determinant is not zero.

We will put $\text{sign}J(e(\lambda)) = +1(-1)$ according to whether $det[\Pi_T J(e(\lambda))] > 0 (< 0)$

We may now state our main result:

Theorem 1 *Consider an economy in Γ with infinitely dimensional consumption set, differentiable strictly convex proper and separable utilities functions and satisfying the conditions a), b), c) in section (1,1), then:*

(1) The cardinality of E is finite and odd,

(2) If $signJ(e(\lambda))$ is constant in E, there exists an unique equilibrium, where $J(e(\lambda))$ denotes the Jacobian of the excess utility function.

The main tool that will be used to prove theorem 1, is the Poincaré Hopf theorem.

Let us recall it.

Poincaré Hopf theorem. Let N be a compact n-dimensional C^1 manifold with boundary and f a continuous vector field on N. Suppose that:

(i) f points outward at δN [this means that $f(x)g(x) > 0$ for all $x \in \delta N$, where g is the Gauss map [1]] and

(ii) f has a finite number of zeros.

then the sum of the indices of f at the different zeros equals the Euler characteristic of N.

[1]Recall that if N is a closed C^2 n-dimensional manifold with boundary, then we can define a C^1 function g from the boundary of M into S^{n-1}, called the Gauss map, see [Mas-Colell (85)].

For the definition of index of f at x (zero of f) and the Euler characteristic of N, see [Mas-Colell (85)].

We need, also, the following lemmas:

Lemma 1 *The excess utility function is C^1.*

Proof: The lemma follows immediately from the following assertion: *The Lagrange multiplier $\gamma(s,\lambda)$ and the Pareto optimal allocation $x_i(s,\lambda)$ are C^1 with respect to λ.* Let us consider the following system of equations:

$$\begin{array}{c} \lambda_i \partial U_i(s, x(s,\lambda)) = \gamma(s,\lambda) \\ \sum_{i=1}^n x_i(s,\lambda) = \sum_{i=1}^n w_i(s). \end{array} \tag{6}$$

From the implicit function theorem, taking derivatives in the above system, with respect to x and γ, we obtain a matrix with the following form:

$$M = \begin{bmatrix} A & B \\ B^t & 0 \end{bmatrix}$$

where A is a $(nl) \times (nl)$ matrix; and B is a $(nl) \times l$ matrix and $U_h^{jk} = \frac{\partial^2 U_h}{\partial x^j \partial x^k}$,

$$A = \begin{bmatrix} U_1^{11} & \cdots & U_1^{l1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & \cdots & \cdots & 0 \\ U_1^{1l} & \cdots & U_1^{ll} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & U_n^{11} & \cdots & U_n^{l1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_n^{l1} & \cdots & U_n^{1l} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

That is B is a $(nl) \times l$ matrix

We claim that there is no vector $z = (v, w) \neq 0$ with $v \in R^{nl}$ and $w \in R^l$ such that $Mz = 0$.

Indeed, if v is such that $Mz = 0$, then

$$B^t v = 0 \tag{7}$$

and

$$Av + Bw = 0. \tag{8}$$

Then from (8) and (9), we have that

$$v^t A v = 0 \tag{9}$$

If v is in the kernel of B^t then

$$\begin{array}{cccc} v_1 + & v_{l+1} + & \cdots + & v_{(n-1)l+1} = 0 \\ v_2 + & v_{l+2} + & \cdots + & v_{(n-1)l+2} = 0 \\ \vdots & \vdots & \vdots & \vdots \\ v_l + & v_{2l} + & \cdots + & v_{nl} \ = \ 0. \end{array}$$

Observe that

$$\partial \sum_{i=1}^{n} \lambda_i U_i =$$

$$\{\lambda_1 \frac{\partial U_1}{\partial x_1}, \lambda_1 \frac{\partial U_1}{\partial x_2}, \ldots, \lambda_1 \frac{\partial U_1}{\partial x_l}, \cdots, \lambda_n \frac{\partial U_n}{\partial x_1}, \lambda_n \frac{\partial U_n}{\partial x_2}, \ldots, \lambda_n \frac{\partial U_n}{\partial x_l}\} =$$

$$= \{\gamma_1, \gamma_2, \ldots, \gamma_l, \cdots, \gamma_1, \gamma_2, \ldots, \gamma_l, \cdots, \gamma_1, \gamma_2, \ldots, \gamma_l\}.$$

Then

$$\partial\{\sum_{i=1}^{n} \lambda^i U^i\}.v = \gamma_1(v_1 + v_{l+1} + \ldots + v_{(n-1)l+1}) + \cdots + \gamma_l(v_l + v_{2l} + \ldots + v_{nl}) = 0. \tag{10}$$

From (8), (9) and the strictly differentiable convexity of $\sum_{i=1}^{n} \lambda^i U^i$ we deduce that $v = 0$. Then since B is a injective matrix, from (6) $w = 0$. We have that $z = 0$, proving our claim. From the claim and the fact that $(U_i(s, \cdot)$, is in a compact set of Λ, the lemma follows. 2

Lemma 2 *The excess utility function has the following properties:*

1) $e(\lambda)$ *is homogeneous of degree zero;*

2) $\lambda e(\lambda) = 0$, $\forall\, \lambda \in R^n_{++}$*;*

3) there exists $k \in R$ *such that* $e(\lambda) \ll k\mathbf{1}$*, where* $\mathbf{1} = (1, 1, 1..., 1) \in R^n$.

4) $||e(\lambda)|| \to \infty$ *as* $\lambda_j \to 0$ *for any* $j \in \{1, ..., n\}$ *and* $\lambda \in \triangle^{n-1}$*;*

5) Let $e : S^n \to R^n$ *for economies in* Γ*, and* $\bar{\lambda} \in E$ *rank*$J[e(\bar{\lambda})] = n - 1$ *i.e. maps* $T_{\bar{\lambda}} S^n_{++}$ *onto* $T_{\bar{\lambda}} S^n_{++}$.

Proof: Property 1) follows from remark 2. Property 2) follows from remark 2), and definition 7). Property 5) That, whenever $e(\bar{\lambda}) = 0, J[e(\bar{\lambda})]$ maps into $T_{\bar{\lambda}} S^n$ is a general property of the vector field, [Mas-Colell (85)]. An economy is regular if and only if $T_{\bar{\lambda}} S^n$ maps onto $T_{\bar{\lambda}} S^n$. This property identifies de concept of regularity with the non nullity of determinants.

To prove property 3), note that from equation (2) we can write

$$e_i(\lambda) = \int_\Omega \partial U_i(s, x_i(\lambda))[x_i(s, \lambda) - w_i(s)] d\nu(s).$$

From the concavity of U_i it follows that:

$$U_i(s, x_i(s, \lambda)) - U_i(s, w_i(s)) \geq \partial U_i(s, x_i(s, \lambda))(x_i(s, \lambda) - w_i(s)).$$

Therefore,

$$e_i(\lambda) \leq \int_\Omega U_i(s, x_i(s, \lambda)) - U_i(w_i(s))\, d\nu(s) \leq \int_\Omega U_i(\sum_{j=1}^n w_j(s))\, d\nu(s), \ \forall\, \lambda.$$

If we let

$$k_i = \int_\Omega U_i(\sum_{i=1}^n w_i(s))\, d\nu(s) \ \text{ and } \ k = \sup_{1 \leq i \leq n} k_i.$$

Hence, property 3) follows.

To see property 4) recall that $\gamma(s, \lambda) > 0$ and that a consumer with zero social weight receive nothing of value. Then, since endowments are strictly positive and $x_i(s, \cdot)$ is a continuous function (see lemma 1), the property follows. 2

We can now prove the following lemma:

Lemma 3 *: The excess utility function is an outward pointing vector field at the boundary of* S^n_{++}.

Proof: From property 2 of lemma 2 it follows that $e(\lambda) \in T_\lambda S^n_{++}$.

To prove that $e(\lambda)$ is an outward pointing vector field, let us now define z_i

$$z_i = \lim_{\lambda^m \to \lambda \in \delta S^n_{++}} \frac{e_i(\lambda^m)}{\|e(\lambda^m)\|}.$$

By Property 3 of Lemma 2, we know that $e_i(\lambda^m)$ is uniformly bounded above, and because $\|e(\lambda)\| \to \infty$ (Property 4 of Lemma 2), the limit of $e_i(\lambda^m)/|e(\lambda^m)|$ must be non-positive. Then we conclude that $z_i \leq 0$.

Furthermore, z_i could be different from zero only if λ_i were zero. This follows from the fact that if λ_i is different from zero, then we can write

$$e_i(\lambda^m) = \frac{-1}{\lambda_i^m} \sum_{j \neq i} e_j(\lambda^m) \;\geq\; -\frac{kn}{\lambda_i^m}.$$

Letting $k' = -kn/\lambda_i^m$, we have that $k' \leq e_i(\lambda^m) \leq k$. Hence $z_i = 0$. 2

Strictly speaking, we have proved that we have a continuous outward pointing vector field for *almost any point* in the boundary of S_{++}^{n-1}. The excess utility function has properties similar to those of the excess demand function. Mas-Colell (1985) proves that for excess demand functions there is an homotopic inward vector field for *all points* of the boundary S_{++}^{n-1}. In our case, with an analogous proof, we can obtain an homotopic outward vector field for excess utility functions.

Proof of Theorem 1

Since S_{++}^{n-1} is homeomorphic to the $(n-1)$-dimensional disk, its Euler characteristic is one.

The equilibrium set E, is a compact set. Moreover, from the fact that zero is a regular value of e, we have that E is a finite set. On the other hand, $e(\lambda)$ is a C^1 vector field on the tangent space pointing outward at the boundary of S_{++}^{n-1}. Then we can apply the Poincaré Hopf theorem.

In our case the index of the vector field e at $\lambda \in E$ is the sign of determinant of $J[e(\lambda)]$.

So, we obtain that:

$$1 = \sum_{\{\lambda : e(\lambda) = 0\}} signdet J(e(\lambda)).$$

The theorem follows by simple cardinality arguments. 2

5 The Case of Incomplete Markets

The exchange economy has n traders and two periods, $t = 0, 1$. There is a state space $(\Omega, \mathcal{A}, \nu)$, which is a probability space. There is one commodity available in each $s \in \Omega$.

Utility functions and endowments are the same as in section 1. At $t = 0$ there are $J < \infty$ assets. Each asset is specified by a measurable bounded *return function* $f_j : \Omega \to R_{++}^l$. Assets have real returns. There are not initial endowments for assets.

Following [Mas-Colell, Monteiro (96)] we define an equilibrium for the economy as a set $(q, \bar{\theta}, p, \bar{x})$, where:

(a) $q \in R^j, q \neq 0$ is an asset price.

(b) $\bar{\theta} = (\bar{\theta}^1, \cdots, \bar{\theta}^n) \in R^{Jn}$ is a vector of assets portfolios such that $\sum_{i=1}^n \bar{\theta}^i = 0$, and $q\bar{\theta}^i \leq 0 \; \forall i = 1, \ldots, n$.

(c) $p : \Omega \to R_+$ is a non zero, measurable spot commodity price function.

(d) $\bar{x} = (\bar{x}^1, \ldots, x^n)$ is an allocation. Each $x^i : \Omega \to R$ is a (measurable) function such that: $\sum_{i=1}^n x^i(s) = \sum_{i=1}^n w^i(s)$, for a.e. s and for every i.

(e) $(\bar{x}^i, \bar{\theta}^i)$ solves:

$$\max_x \int_\Omega U^i(s, x^i(s)) d\nu(s) \tag{11}$$

$$\text{subject to } \; p(s)x^i(s) \leq p(s)(w^i(s) + \sum_{i=1}^J \theta_j^i f_j(s), \text{ for a.e. } s \text{ and } \; q\bar{\theta}^i \leq 0$$

It is proved that the equilibrium exists. [Mas-Colell, Monteiro (96)]

As we have a single commodity we can suppose that the spot price $p(s) = 1 \; \forall s \in \Omega$. Then the decision problem (3.1) can be reduced to the pure portfolio choice problem:

$$\max_\theta u^i(\theta) = \int_\Omega U^i(s, \sum_{j=1}^J w^i(s) \; + \; \theta_j^i f_j(s)) d\nu(s) \tag{12}$$

$$\text{subject to } \; w^i(s) + \sum_{j=1}^J \theta_j^i f_j(s) \geq 0 \; \text{ and } \; q\theta^i \leq 0$$

Definition 7 *An equilibrium in incomplete markets (GEI equilibrium) with one consumption good is a vector of asset prices $\bar{q} \in R^J$ and an allocation of assets $\bar{\theta} \in R^{nJ}$, such that :*

1) $\bar{\theta}^i$ solves (13).

2) $\bar{\theta}$ is a feasible allocation; i.e. $\sum_{i=1}^n \bar{\theta}^i = 0$.

3) If $\theta \succeq \bar{\theta}$ i.e.: $u_i(\bar{\theta}) \geq u_i(\theta)$, for all $i = 1, 2, ..., n$ and with strictly inequality for at least one of them, θ is not in the budget set.

Since $U^i(\cdot)$ is a strictly concave function, and f_j is a positive one, then $U^i(w^i(s) + \sum_{j=1}^J f_j(s)(\cdot)) : R^j \to R$ is a strictly concave function. The set $\Theta^i = \{\theta \in R^J; w^i(s) + \sum_{j=1}^J f_j(s)\theta_j \geq \; 0 \;$ for a.e.$s \in \Omega\}$ is a lower bounded set. Then an economy with incomplete markets and only one good available in each state of the world is an Arrow-Debreu model.

The following proposition proves this statement.

Proposition 3 *Every equilibrium assets allocation is Pareto optimal.*

Proof. Suppose that $\bar{\theta} \in R^{nj}$ is an equilibrium allocation, and there is a feasible θ with $\theta \succ \bar{\theta}$. That is, $U^i(w^i(s) + \sum_{i=1}^J f_j(s)\theta_j^i) \geq U^i(w^i(s) + \sum_{i=1}^J f_j(s)\bar{\theta}_j^i)$ strictly for at least one $j \in \{1, 2, \ldots, n\}$, and $\sum_{i=1}^n \theta^i = 0, \forall\ s$ in a non null subset of Ω. Then $q\theta^j > q\bar{\theta}^i$. From $0 = \sum_{i=1}^n \theta^i = \sum_{i=1}^n \bar{\theta}^i$ we have $0 = \sum_{i=1}^n q\theta^i = \sum_{i=1}^n q\bar{\theta}^i$. Therefore, $q\theta^k < q\bar{\theta}^k$ must hold for at least one k, and thus $\theta^k \neq \bar{\theta}^k$. Then, it follows from the strict convexity of the preferences that $\frac{1}{2}\theta^k + \frac{1}{2}\bar{\theta}^k \succ_k \bar{\theta}^k$ holds. Therefore we have $\frac{1}{2}q\theta^k + \frac{1}{2}q\bar{\theta}^k > 0 > q\bar{\theta}^k$. From this we have $q\theta^k > q\bar{\theta}^k$, which is a contradiction. Then $\bar{\theta}$, is a Pareto optimal allocation. 2

So, it is possible to derive the equilibrium set using the excess utility function.

5.1 Equilibrium in The Portfolio Choice Problem

In the above conditions, for the portfolio choice problem, if $\bar{\theta}$ is a Pareto optimal portfolio, we know that there exists a positive "social weight" vector $\lambda \in R^n$ such that $\bar{\theta}$ solves the problem:

$$\sup_{\theta} \sum_{i=1}^n \lambda_i \int_\Omega U_i(w(s) + \sum_{j=1}^J \theta_j^i f_j(s)) d\nu(s) \text{ subject to } \sum_i^n \theta_i = 0. \tag{13}$$

First order equations for this maximization problem are:

$$\lambda_i \frac{\partial}{\partial \theta} \int_\Omega U_i(w(s) + \sum_{j=1}^J f_j(s)\theta_i(\lambda) d\nu(s) = \gamma(\lambda). \tag{14}$$

As in section 1, we construct the excess utility function:

$$e_i(\lambda) = \frac{1}{\lambda_i}\gamma(\lambda)\theta_i(\lambda) \tag{15}$$
$$e(\lambda) = \{e_1(\lambda), \cdots, e_n(\lambda)\}.$$

As in this case,

$$\frac{\partial}{\partial \theta} \int_\Omega U_i(w(s) + \sum_{j=1}^J f_j(s)\theta_{ij}(\lambda) d\nu(s) = \int_\Omega \frac{\partial}{\partial \theta} U_i(w(s) + \sum_{j=1}^J f_j(s)\theta_{ij}(\lambda) d\nu(s).$$

From the first order conditions we obtain the following equation:

$$e_i(\lambda) = \int_\Omega \partial \left[U_i(w_i(s) + \sum_i^J \theta_{ij}(\lambda) f_j(s)) \right] \theta_i(\lambda) d\nu(s). \tag{16}$$

We say that the "social weight" vector $\bar{\lambda}$ is an equilibrium if and only if $e(\bar{\lambda}) = 0$.

As in section 1, in order to obtain conditions for the uniqueness of equilibrium we consider the Jacobian of the excess utility function.

The term in the i row and j column in the Jacobian of the excess utility function, is given by

$$J(e(\lambda))_i(\lambda_j) \;=\; \int_\Omega \frac{\partial \theta_i}{\partial \lambda_j}\left[\partial^2 U_i.\theta_i + \partial U_i\right] d\nu(s). \tag{17}$$

An Example with Uniqueness

In order to obtain an example with a unique of equilibrium, consider an economy with two agents and two assets.

From the above equations we obtain that the Jacobian of the excess utility function has the following form:

$$\frac{\partial e_i(\lambda)}{\partial \lambda_j} = \int_\Omega \left[f_1 \frac{\partial \theta_{i1}}{\partial \lambda_j} + f_2 \frac{\partial \theta_{i2}}{\partial \lambda_j}\right]\left[\partial^2 U_i(w_i + f_1\theta_1 + f_2\theta_2)\,(f_1\theta_1 + f_2\theta_2) + \partial U_i(w_i + f_1\theta_1 \;+\; f_2\theta_2)\right] d\nu(s). \tag{18}$$

Suppose that there are two agents with endowments $w_i > 0$, two assets and only one good available in each state.

Let us consider the following utility functions:

$$u_i(x) = \int_\Omega x(s)^{\frac{1}{2}} d\nu(s), \quad i = \{1, 2\}.$$

We have the following portfolio choice problem:

$$\sup_\theta \sum_{i=1}^{2} \lambda_i \int_\Omega \left(w_i(s) + \sum_{j=1}^{2} \theta_{ij} f_j(s)\right)^{\frac{1}{2}} d\nu(s)$$

$$\text{s.t.} \sum_{i=1}^{2} \theta_i \;=\; 0.$$

The excess utility function is:

$$e_i(\lambda) = \int_\Omega \frac{1}{2}\left(w_i(s) + \sum_{j=1}^{2} f_j(s)\theta_{ij}(\lambda)\right)^{-\frac{1}{2}}\left(\sum_{j=1}^{2} f_j(s)\theta_{ij}(\lambda)\right) d\nu(s).$$

From this equation we obtain:

$$\frac{\partial e_i}{\partial \lambda_i} = \frac{1}{2}\int_\Omega \left[\sum_{j=1}^{2} f_j \frac{\partial \theta_{ij}}{\partial \lambda_i}\right]\left[\partial^2 U_i[\sum_{j=1}^{2} f_j\theta_{ij}] + \partial U_i)\right] d\nu(s).$$

Since $\theta_{1j} + \theta_{2j} = 0, \ j = \{1,2\}$, we obtain that: $\frac{\partial \theta_{ij}}{\partial \lambda_i} = -\frac{\partial \theta_{kj}}{\partial \lambda_i}; \ \ i \neq k$ and $i = \{1,2\}; \ \ k = \{1,2\}$.
Since $\partial^2 U_i \left[\sum_{i=1}^{2} f_j(s)\theta_{ij}(\lambda)\right] + \partial U_i \geq 0$ we obtain:

$$\text{if } \frac{\partial e_i}{\partial \lambda_i} > (<) 0 \ \text{ then } \ \frac{\partial e_i}{\partial \lambda_j} < (>) 0,$$

hence $e(\lambda)$ has the "Gross Substitute" property [Dana (93)] and uniqueness of equilibrium follows.
2

References

[Araujo, Monteiro (89)] (1989) Araujo, A., Monteiro, P.,K. "Equilibrium Without Uniform Conditions". *Journal of Economic Theory* **48,** 416-427.

[Becker (71)] Becker G.S. (1971) "Economic Theory" *Alfred A. Knopf. N.Y.*

[Dana (93)] Dana R.A., (1993), "Existence, Uniqueness and Determinacy of Arrow-Debreu Equilibria in Finance Models". *Journal of Mathematical Economics* **22** 563-580

[Kehoe (91)] Kehoe, T.,J. (1991) "Computation and Multiplicity of Equilibria". *Handbook of Mathematical Economics,* **Vol IV.** Edited by Hildenbrand, W. and Sonnenschein, H.

[Mas-Colell (85)] Mas-Colell, A. (1985) "The Theory of General Equilibrium, A Differentiable Approach". Cambridge University Press.

[Mas-Colell (91)] Mas-Colell, A., (1991), " Indeterminacy in Incomplete Market Economies". *Economic Theory* **1** 45-62.

[Mas-Colell, Monteiro (96)] Mas-Colell, A., Monteiro, P.K. (1996). "Self-Fulfilling Equilibria: An Existence Theorem for a General Space". *Journal of Mathematical Economics* **26,** No.1.

[Mas-Colell, Zame (91)] Mas-Colell, A. Zame, W.R. (1991) "Equilibrium Theory in Infinite Dimensional Spaces", *Handbook of Mathematical Economics* **Vol IV**.