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ON BEHAVIORAL COMPLEMENTARITY AND ITS IMPLICATIONS

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Abstract. We study the behavioral definition of complementary goods: if the price of one good increases, demand for a complementary good must decrease. We obtain its full implications for observable demand behavior (its testable implications), and for the consumer's underlying preferences. We characterize those data sets which can be generated by rational preferences exhibiting complementarities. In a model in which income results from selling an endowment (as in general equilibrium models of exchange economies), the notion is surprisingly strong and is essentially equivalent to Leontief preferences. In the model of nominal income, the notion describes a class of preferences whose extreme cases are Leontief and Cobb-Douglas respectively.

Resumen. Estudiamos la definición conductual de bienes complementarios: si el precio de un bien aumenta cae la demanda de un bien complementario. Obtenemos las consecuencias empíricas sobre el comportamiento del consumidor y sobre sus preferencias. Caracterizamos las observaciones que son compatibles con la complementariedad en demanda. En un modelo en que el ingreso resulta de la venta de una dotación, la noción de bienes complementarios resulta sorprendentemente fuerte y es esencialmente equivalente a preferencias Leontief. En un modelo de ingreso nominal, es equivalente a preferencias que de algún modo estan entre las Leontief y las Cobb-Douglas.

Keywords. Afriat's Theorem, Weak Axiom of Revealed Preference, Complementary goods.

JEL Classification. D11, D12.

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1. Introduction

We study the behavioral notion of complementarity in demand (which we refer to throughout simply as *complementarity*): when the price of one good decreases, demand for a complementary good increases. We deal both with the case where consumers' nominal income is fixed and where income is derived from selling a fixed endowment at prevailing prices.

Our work is specifically aimed at the study of two goods. While the definition is quite natural for two goods, for three or more goods, things become more complex. In fact, when there are more than two goods, the obvious extension of the definition (that two goods are complements if the reduction in price of one good implies an increase in consumption of the other) is not adequate as an intuitive definition of complementarity.[1] There are other ways of extending the definition. Discussion of these ideas is left to future research.

We obtain the full implications of complementarity both for observable demand behavior (its testable implications) and for the underlying preferences. In the former exercise, we characterize all finite sets of price-demand pairs consistent with complementarity. The latter exercise characterizes the class of preferences generating complementarity.

The complementarity property, which we call "behavioral" to emphasize that demand, not preference, is primitive, is a classical notion. It is the notion taught in Principles of Economics textbooks (e.g. McAfee (2006), Stiglitz and Walsh (2003) and Krugman and Wells (2006)) and Intermediate Microeconomics textbooks (e.g. Nicholson and Snyder (2006), Jehle and Reny (2000), and Varian (2005)). It is a crucial property in applied work: marketing researchers test for complementarities among products they plan to market; managers' pricing strategy takes a special

[1]Samuelson (1974) gives the example of coffee, cream, and sugar. Both cream and sugar are intuitively complementary to coffee. However, sugar may be "more complementary" with coffee than cream. Hence, a reduction in the price of cream may lead to a decrease in the consumption of sugar, and a corresponding decrease in the consumption of coffee.

form when they market complementary goods; regulatory agencies are interested in complements for their potential impact on competitive practices; complementarity is relevant for decisions on environmental policies; complementary goods receive a special treatment in the construction of price indexes; complementary export goods are important in standard models of international trade, etc. etc. The literature on applications of complementarity is too large to review here.

Yet, the notion discussed here has received surprisingly little theoretical attention. The general testable implications of complementarity were, until now, unknown. In many applications, one needs to decide empirically whether two goods are complements. Hence, a test which can falsify complementarity is both useful and important. Empirical researchers' tests typically estimate cross-partial elasticities in highly parametric models. However, such an exercise actually jointly tests several hypotheses. In contrast, we elicit the complete testable implications of complementarity in a general framework.

We consider two models: a model in which consumers carry endowments and form their demand as a function of prices and the income derived from selling endowment, and a model in which consumers are simply endowed with a nominal income. In the nominal-income model, we provide a necessary and sufficient condition for expenditure data to be consistent with the rational maximization of a preference which exhibits complementarity in demand. In the income-from-endowment model, complementarity is equivalent to all observed demands lying on a continuous monotone path.

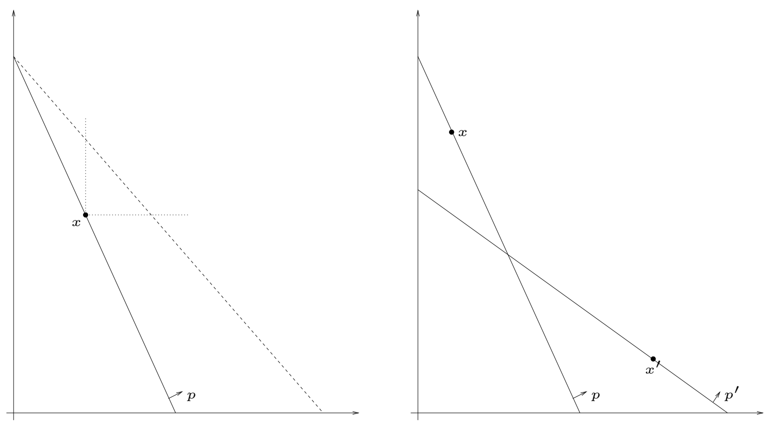
We also characterize the class of preferences that generate complementarity. Again the results depend on the model under consideration. In the nominal-income model, complementarity effectively requires that demand be monotonic with respect to set inclusion of budgets (and hence normal). In addition, complementarity in this model automatically implies rationalizability by an upper semicontinuous utility function–a consequence of the continuity of demand (which is also an implication of complementarity). Within the class of smooth rationalizations, complementarity is characterized by a bound on the percentage change in the marginal rate of substitution with respect to a change in either commodity. Cobb-Douglas preferences are exactly those preferences meeting this bound.

However, when income obtains from selling an endowment, we obtain a sharper result. If for each endowment and price, there exists some other price for which consumption of the two goods move weakly in the same direction, then the preferences are "generalized" Leontief. Hence, the only preferences exhibiting global demand complementarity for all endowments are preferences featuring "perfect complements." This result may speak to the fact that most canonical examples of complementary goods (coffee and sugar, gin and tonic, etc.) are chosen in fixed proportions.

Our results are for demand for two goods. Complementarity is always a statement about pairs of goods, so ours is in a sense the canonical setup, and the most relevant from the viewpoint of conceptually sorting out the right notion of complementarity. We imagine our results being used after aggregating appropriately (as in Blundell, Browning, and Crawford (2003)), or conditional on demand for other goods being constant. In any case, testing for all goods being complementary makes little sense, so some degree of aggregation or conditioning seems unavoidable.

1.1. **Discussion of results.** We illustrate and discuss graphically some of our results. See Section 2 for the formal statements.

Consider Figure 1(a), which depicts a hypothetical observation of demand $x = (x_1, x_2)$ at prices $p = (p_1, p_2)$. Figure 1(a) illustrates the notion of complementarity: goods 1 and 2 are complements if, when we decrease the price of one good, demand for the other good increases. In the figure, complementarity require that demand at the dotted budget line involves more of both goods. Note that we are assuming no Giffen goods, which is implied by normal demand. Symmetrically, a decrease in the price of good 2 would also imply a larger demanded bundle.



(a) Complementarities.

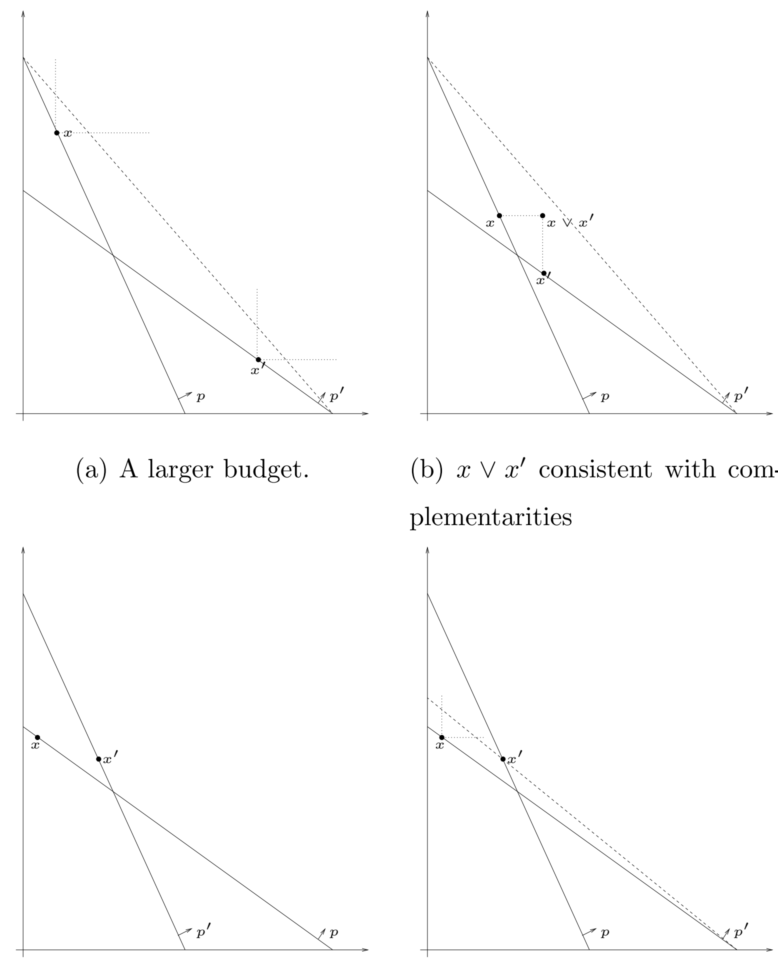
(b) Demands x and x' at prices p and p'.

FIGURE 1. When is observed demand consistent with complementarity?.

Given Figure 1(a), one may think that the testable implications of complementarity amount to verifying that, whenever one finds two budgets like the ones in the figure, one demand is always higher than the other. Consider then Figure 1(b), where one budget is not larger than another. Are the observed demands of x at prices p, and x' at p', consistent with demand complementarity? The answer is negative, as can be seen from Figure 2(a): the larger budget drawn with a dotted line is obtained from either of the p or p' budgets by making exactly one good cheaper. So it would need to generate a demand larger than both x and x', which is not possible.

Figure 2(b) shows a condition on x and x' which is necessary for complementarity: the pointwise maximum of demands, $x \vee x'$, must be affordable for any budget larger than the p and p' budgets. Since there is a smallest larger budget, the least upper bound on the space of budgets (the dotted-line budget), we need $x \vee x'$ to be affordable at the least upper bound of the p and p' budgets.

Since demand is homogeneous of degree zero, we can normalize prices and incomes so that income is 1. Then the least upper bound of the p and p' budgets is the budget obtained with income 1 and prices $p \wedge p'$, the component-wise minimum price. The necessary condition in Figure 2(b) is that $(x \vee x') \cdot (p \wedge p') \leq 1$.



(a) A larger budget.

(b) $x \vee x'$ consistent with complementarities

(c) Demands x and x' at prices p and p'.

(d) Violation of WARP.

Figure 2. Observed demands.

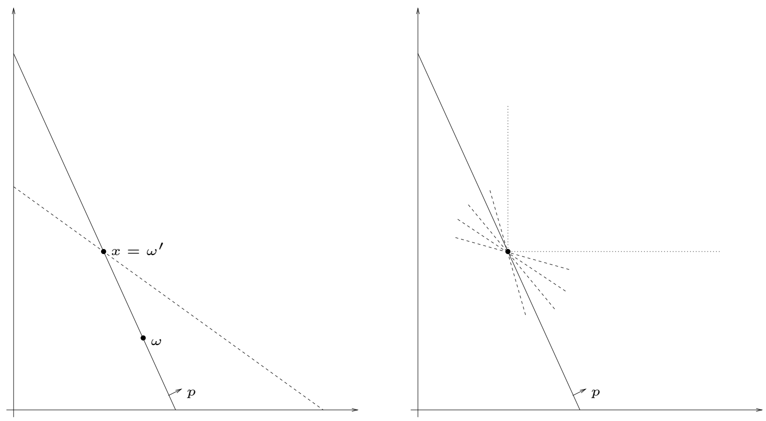
There is a second necessary condition. Consider the observed demands in Figure 2(c). This a situation where, when we go from p to p', demand for the good that gets cheaper decreases while demand for the good that gets more expensive increases. This is not in itself a violation of either complementarity or the absence of Giffen goods. However, consider Figure 2(d): were we to increase the budget from p to the dotted prices, complementarity would imply a demand at the dotted prices that is larger than x. But no point in the dotted budget line is both larger than x and satisfies the weak axiom of revealed preference (WARP) with respect to the choice of x'.

So a simultaneous increase in one price and decrease in another cannot yield opposite changes in demand. This property is a strengthening of WARP: Fix p, p' and x as in Figure 2(c). Then WARP requires that x' not lie below the point where the p and p' budget lines cross. Our property requires that x' not lie below the point where the horizontal dotted line crosses the p' budget, or that demand for good 1 not be smaller in x' than in x. In fact, this property is implied by either of the two following sets of conditions: *i*) rationalizability and the absence of Giffen goods or *ii*) rationalizability and normal demand.

We show (Theorem 1 of Section 2) that the two necessary properties, the $(x \vee x') \cdot (p \wedge p')$ property in Figure 2(b) and the strengthening of WARP, are also sufficient for a complementary demand. That is: given a finite collection of observed demands x at prices p, these could come from a demand function for complementary goods if and only if any pair of observations satisfies the two properties. Thus, the two properties constitute a non-parametric test for complementary goods, in the spirit of the revealed-preference tests of Samuelson (1947) and Afriat (1967).[2]

We now turn to a geometric intuition for one of our results on preferences. Suppose that prices affect incomes—a consumer obtains her income from selling an endowment ω of goods at the prevailing prices. Consider Figure 3(a), which shows demand x at prices p and endowment ω, i.e. income is $p \cdot \omega$. We shall describe the consumer's indifference curve at x. Note that demand does not change if we set the endowment to be $\omega' = x$. Consider the dotted prices in Figure 3(a). Demand at these prices cannot be to the left of x because it would violate WARP, and demand to the right of x would violate complementarity, as it would demand less of the good complementary to the good whose price decreased. But then demand has to be x at the dotted prices. By repeating this argument for all prices, Figure 3(b),

[2]See Varian (1982) for an exposition and further results. Matzkin (1991) and Forges and Minelli (2006) discuss more general sets of data. Brown and Calsamiglia (2007) present a test for quasi-linear utility.



(a) (b)

FIGURE 3. Complementarity implies Leontief preferences.

we conclude that the only indifference curve supported by all prices at x is the one obtained from Leontief preferences.

Our result is in fact stronger than the previous argument suggests. We show (Theorem 4 of Section 2) that if, for every endowment, there is *one* price change for which demand of both goods move in the same direction, then preferences must have a Leontief form.

1.2. **Historical Notes.** Before proceeding, we discuss briefly the history of the theory of complementary goods. Much of this discussion is borrowed from Samuelson (1974), which serves as an excellent introduction to the topic.

Perhaps the first notion of complementary goods is that formulated by Edgeworth and Pareto on introspective grounds (Samuelson, 1974). They believed that if two goods were complementary, then the marginal utility of an extra unit of each should be greater than the sum of the marginal utilities of an extra unit of either. In other words, the marginal utility of the consumption of either good should be increasing in the consumption of the other good; the utility function should have nonnegative cross-derivatives. This is an intuitively appealing definition based on preferences, not behavior; however, it clearly depends on cardinal utility comparisons. Hicks and Allen (1934), Hicks (1939) and Samuelson (1947) recognized this, and suggested that as a local measure of complementarity, it was useless. At any given consumption bundle, any utility function can be transformed to have nonnegative cross-derivatives. Milgrom and Shannon (1994) established that, despite not being an ordinal notion, the Edgeworth Pareto definition does in fact have ordinal implications. Chambers and Echenique (2007) on the other hand, showed that if the notion has any implications for observable demand data, they can only be tested with an infinite set of data.

Most of the modern notions build on the increasing marginal utility notion, using some cardinal function. For example, the notion discussed by Hicks and Allen notion for three goods works as follows. Consider some bundle $\left(\overline{x}, \overline{y}, \overline{t}\right)$. Now, define a function $T(x, y) = \left\{t : U(x, y, t) = U\left(\overline{x}, \overline{y}, \overline{t}\right)\right\}$. Then the first two goods are complements if and only if

$$\frac{\partial^2}{\partial x \partial y}\left(-T\left(\overline{x}, \overline{y}\right)\right) \geq 0.$$

In particular, if $u(x, y, t) = U(x, y) + t$, then goods one and two are complementary if and only if U has nonnegative cross derivatives (Samuelson, 1974, p. 1270). Samuelson goes a bit further, suggesting that complementarity be defined with respect to a particular cardinalization of preference. His proposal is to use either McKenzie's money-metric utility function, or a von Neumann-Morgenstern utility index for expected utility maximizers.

We now discuss the main objection to the behavioral notion of complementarity we have studied, as well as the Hicks-Allen theory proposed to rectify it. While our notion, sometimes called "gross complementarities," is both natural and commonly understood, there are other such notions. The primary criticism of our definition is that it can be "asymmetric" in a sense. It is possible that raising the price of good one leads to an increase in consumption of good two, while raising the price of good two leads to a decrease in consumption of good one. This asymmetry led Hicks (1939) and other early researchers to take interest in other notions (although they never claimed the notion we discuss was incorrect). Hicks and Allen (1934) developed a theory of complementarity of demand based on compensated price changes.

The type of price change considered by Hicks is the following. The price of good one is increased *and* the income of the agent is simultaneously increased just enough to leave the consumer on the same indifference curve. Good one is complementary to good two if a compensated increase in the price of good two leads to a lower consumption of good one. Indeed, Hicksian demand is usually understood as an expenditure minimizing bundle attaining a certain level of utility. It is well-known that with such a definition, good one is complementary to good two if and only if good two is complementary to good one.

Samuelson suggests that Hicks' notion only real defense is the fact that it is not susceptible to the same "criticisms" that earlier definitions are (Samuelson, 1974, p. 1284). Ostensibly, the reason for studying compensated price changes is to provide a single-dimensional measure of complementarity of any pair of goods. Implicit in this approach is the notion that all goods *must be either complements or substitutes.* While a single-dimensional measure of complementarity is certainly interesting, we believe there is also room for the study of other concepts (perhaps leading to other, less decisive, measures of complementarity).

Further, the Hicks definition can also be "criticized": in the case of two commodities (which is the case under consideration here), *all goods are economic substitutes by necessity.* This is a consequence of downward sloping indifference curves–requiring both goods to be complements essentially results in generalized Leontief preferences. Thus, the definition does not allow for a meaningful study of complementarity in what is arguably the most natural framework for discussing the concept. In contrast, with our definition (in the nominal income model), goods are both complements and substitutes if and only if preferences are Cobb-Douglas.

Finally, compensated price changes present a challenge from the empirical perspective we adopt in this paper: compensated demand changes are unlikely to be observed in real data. In other words, it is unclear what observable phenomena in the real world correspond to compensated price changes. The notion of complementarity we adopt is the only purely behavioral notion.

To sum up, we study the standard textbook-notion of complementarity of demand. We avoid the criticism of asymmetry simply by specifying from the outset that two goods are complementary if a change in price in either good leads to consumption changing in the same direction for both goods.

2. Statement of Results

We discuss complementarity in two different contexts: first we study changes in price when nominal income is fixed, and second, when an endowment of goods is fixed. In the latter environment, price changes affect income, as income results from selling the endowment at prevailing prices. Theorems 1, 2, and 3 are for the nominal income model, $D(p, I)$. Theorem 4 is for the endowment model. The proof of Theorem 1 is in Section 6; the proof of Theorem 4 is in Section 4; the proof of Theorem 3 is in Section 5. Theorem 2 follows from lemmas 6.15 and 6.16 in Section 6.

2.1. **Preliminaries.** Let $\mathbb{R}^2_+$ be the domain of consumption bundles, and $\mathbb{R}^2_{++}$ the domain of possible prices. We use standard notational conventions: $x \leq y$ if $x_i \leq y_i$ in $\mathbb{R}$, for $i = 1, 2$; $x < y$ if $x \leq y$ and $x \neq y$; and $x \ll y$ if $x_i < y_i$ in $\mathbb{R}$, for $i = 1, 2$. We write $x \cdot y$ for the inner product $x_1 y_1 + x_2 y_2$.

A function $u : \mathbb{R}^2_+ \Rightarrow \mathbb{R}$ is *monotone increasing* if $x \leq y$ implies $u(x) \leq u(y)$. It is *monotone decreasing* if $(-u)$ is monotone increasing.

A function $D : \mathbb{R}^2_{++} \times \mathbb{R}_+ \to \mathbb{R}^2_+$ is a *demand function* if it is homogeneous of degree 0 and satisfies $p \cdot D(p, I) = I$, for all $p \in \mathbb{R}^2_{++}$ and $I \in \mathbb{R}_+$.

Say that a demand function satisfies *complementarities* if, for fixed p_2 and I, $p_1 \mapsto D((p_1, p_2), I)$ is monotone decreasing, and for fixed p_1 and I, $p_2 \mapsto D((p_1, p_2), I)$ is monotone decreasing.[3]

For all $(p, I) \in \mathbb{R}^2_{++} \times \mathbb{R}_+$, define the *budget* $B(p, I)$ by $B(p, I) = \left\{x \in \mathbb{R}^2_+ : p \cdot x \leq I\right\}$. Note that $B(p, I)$ is compact, by the assumption that prices are strictly positive.

A demand function D is *rational* if there is a monotone increasing function $u : \mathbb{R}^2_+ \to \mathbb{R}$ such that

$$D(p, I) = \operatorname{argmax}_{x \in B(p,I)} u(x). \tag{1}$$

[3]This is equivalent to the notion that if $p' \leq p$, then $D(p) \leq D(p')$. Formally, we may require the weaker statement that $D_2((p_1, p_2), I)$ is weakly monotone decreasing in p_1 and that $D_1((p_1, p_2), I)$ is weakly monotone decreasing in p_2. That is, none of our results would change if we allowed for the theoretical possibility of Giffen goods (they will be ruled out anyhow).

In that case, we say that u is a *rationalization of* (or that it *rationalizes*) D. Note that $D(p, I)$ is the unique maximizer of u in $B(p, I)$.

Say that a partial demand function satisfies the *the weak axiom of revealed preference* if $p \cdot D(p', I') > I$ whenever $p' \cdot D(p, I) < I'$ (with two goods, the weak axiom is equivalent to the strong axiom of revealed preference).

2.2. **Nominal Income.** We shall use homogeneity to regard demand as only a function of prices: $D(p, I) = D((1/I)p, 1)$, so we can normalize income to 1. In this case, we regard demand as a function $D : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ with $p \cdot D(p) = 1$ for all $p \in \mathbb{R}^2_{++}$.

A *partial demand function* is a function $D : P \to \mathbb{R}^2_+$ where $P \subseteq \mathbb{R}^2_{++}$ and $p \cdot D(p) = 1$ for every $p \in P$; P is called *the domain* of D. So a demand function is a partial demand function whose domain is $\mathbb{R}^2_{++}$. The concept of the partial demand function allows us to study finite demand observations. We imagine that we have observed demand at all prices in P (see e.g. Afriat (1967), Diewert and Parkan (1983) or Varian (1982)).

Theorem 1 (Observable Demand)**.** *Let P be a finite subset of $\mathbb{R}^2_{++}$ and let $D : P \to \mathbb{R}^2_+$ be a partial demand function. Then D is the restriction to P of a rational demand that satisfies complementarity if and only if for every $p, p' \in P$ the following conditions are satisfied*

(1) $(p \wedge p') \cdot (D(p) \vee D(p')) \leq 1$.
(2) *If $p' \cdot D(p) \leq 1$ and $p'_i > p_i$ for some product $i \in \{1, 2\}$ then $D(p')_j \geq D(p)_j$ for $j \neq i$.*

The following theorem gives several topological implications of rationalizability.

Theorem 2 (Continuity)**.** *Let $D : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be a rationalizable demand function which satisfies complementarity. Then D is continuous. Furthermore, D is rationalized by an upper semicontinuous, weakly monotonic utility function.*

Theorem 3 requires demand to be rationalized by a twice continuously differentiable (C^2) function u. We write

$$m(x) = \frac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2}$$

to denote the *marginal rate of substitution* of u at an interior point x.

Theorem 3 (Smooth Utility)**.** *Let D be a rational demand function with interior range and a monotone increasing, C^2, and strictly quasiconvex rationalization u. Then D satisfies complementarity if and only if the marginal rate of substitution m associated to u satisfies that*

$$\frac{\partial m(x)/\partial x_1}{m(x)} \leq \frac{-1}{x_1} \quad \textit{and} \quad \frac{\partial m(x)/\partial x_2}{m(x)} \geq \frac{1}{x_2}$$

2.3. **Endowment Model.** We also study what happens when income results from selling an endowment $\omega \in \mathbb{R}^2_+$ at prices p. In this case, $I = p \cdot \omega$ and demand is therefore given by $D(p, p{\cdot}\omega)$. Importantly, a change in prices implies a corresponding change in income.

In this model, D satisfies *complementarity* if, for all (p, ω) and all p',

$$[D_1(p', p' \cdot \omega) - D_1(p, p \cdot \omega)]\,[D_2(p', p' \cdot \omega') - D_2(p, p \cdot \omega)] \geq 0. \tag{2}$$

A substantially weaker property will be of interest: A demand function D satisfies *weak complementarity* if, for every (p, ω) there is at least one price $p' \neq p$ satisfying (2). Note that Fisher (1972) has a characterization of preferences generating gross substitutes in the endowment model; like us, he assumes that substitutes holds at all endowments (but he assumes differentiability).

Theorem 4 (Endowment Model)**.** *Let D be a rational demand function with a continuous and monotone increasing rationalization. Then, in the endowment model, the following are equivalent:*

1. *D satisfies complementarity.*
2. *D satisfies weak complementarity.*
3. *There exist continuous strictly monotone functions $f_i : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$, $i = 1, 2$, at least one of which is everywhere real valued ($f_i(\mathbb{R}_+) \subseteq \mathbb{R}$), so that*

$$u(x) = \min \{f_1(x_1), f_2(x_2)\}$$

   is a rationalization of D.

2.4. **Discussion and remarks.** The following observations are of interest:

1. Theorem 1 derives the testable implications of complementarity in the nominal income model, $D(p, I)$. With expenditure data (as in, e.g., Afriat (1967)), it should be straightforward to verify Conditions 1 and 2 in the theorem.
2. The testable implications of complementarity in the endowment model are, by Theorem 4, trivial: with Leontief preferences all observed consumption bundles would lie on a continuous monotone path in consumption space.
3. Property 2 of Theorem 1 follows from the weak axiom of revealed preference and the monotonicity in own price (absence of Giffen goods, see the discussion in the introduction).
4. In Theorem 4, we may without loss of generality normalize the real-valued f_i to be the identity function; this good then acts as a kind of endogenous "numeraire."
5. A version of Theorem 4 holds for any number of goods. We present a proof for the general n good case in Section 4. For the other results, we present a discussion of the two-good assumption in the introduction.

Theorem 1 implies that a partial demand satisfying (1) and (2) is rationalizable by a monotone increasing, upper semicontinuous, utility. One may want the rationalizing utility to be in addition continuous, Example 1 shows that complementarity does not impli rationalization by a continuous utility. It is interesting to note here that Richter (1971) and Hurwicz and Richter (1971) present results on the existence of monotone increasing and continuous rationalizations, but require the range of demand to be convex. Demand in Example 1 has non-convex range (see also the remark after Lemma 6.16).

Example 1. Consider the following utility

$$u(x_1,x_2) = \begin{cases} \min(x_1,x_2), & \text{if } \min(x_1,x_2) < 1 \\ x_1 \cdot x_2, & \text{if } x_1,x_2 \geq 1. \end{cases}$$

So u behaves like a Leontief preference when $\min(x,y) < 1$ and Cobb-Douglas otherwise. In other words, if the consumer cannot afford to buy at least 1 from both products then she buys the same amount from each product. Otherwise, she spends half of your money on each product, making sure to buy at least 1 from each. The demand generated by this preference relation is given by

$$D(p_x,p_y) = \begin{cases} \big(1/(p_x+p_y), 1/(p_x+p_y)\big), & \text{if } p_x+p_y \geq 1 \\ \big(1/(2p_x), 1/(2p_y)\big), & \text{if } p_x,p_y \leq 1/2 \\ \big(1,(1-p_x)/p_y\big), & \text{if } p_y \leq 1/2 \text{ and } 1/2 \leq p_x \leq 1-p_y \\ \big((1-p_y)/p_x, 1\big), & \text{if } p_x \leq 1/2 \text{ and } 1/2 \leq p_y \leq 1-p_x \end{cases}$$

and let D be the corresponding demand function. It is easy to verify that D is monotone. So D is continuous by Lemma 6.15.

However D cannot be rationalized by a continuous utility function. Indeed, assume that v is a utility that rationalizes D. Then for every $\epsilon > 0$ we have $v(1-\epsilon,3) < v(1,1)$, Since $(1,1)$ is revealed prefer to $(1-\epsilon,3)$: If $p = (1-\eta,\eta)$ for small enough η then $D(p) = (1,1)$ and $(1-\epsilon,3) \in L(p)$. On the other hand

$v(1,3) > v(1,1)$ since $(1,3)$ is revealed preferred to $(1,1)$: If $p = (1/2, 1/6)$ then $D(p) = (1,3)$ and $(1,1) \in L(p)$. Therefore v cannot be continuous.

Finally, discussions of complementarity are often centered around the elasticity of substitution (Samuelson, 1974). In addition, Fisher (1972) presents a characterization of the gross substitutes property in terms of elasticities. The following corollary to Theorem 3 may be of interest.

Let D be differentiable, in addition to the hypotheses of Theorem 3. Let $\eta_i(p, I)$ be the own-price elasticity, and $\theta_i(p, I)$ be the cross elasticity, of demand for good i; i.e.

$$\eta_1(p,I) = \frac{\partial D_1(p,I)}{\partial p_1}\frac{p_1}{D_1(p,I)} \quad \theta_1(p,I) = \frac{\partial D_1(p,I)}{\partial p_2}\frac{p_2}{D_1(p,I)}.$$

Corollary 1. *If D satisfies complementarity, then, for $i = 1, 2$,*

$$\frac{\eta_i(p,I) + \theta_i(p,I)}{\eta_1(p,I)\eta_2(p,I) - \theta_1(p,i)\theta_2(p,i)} \leq -1.$$

Now, we consider the case of additive separability.

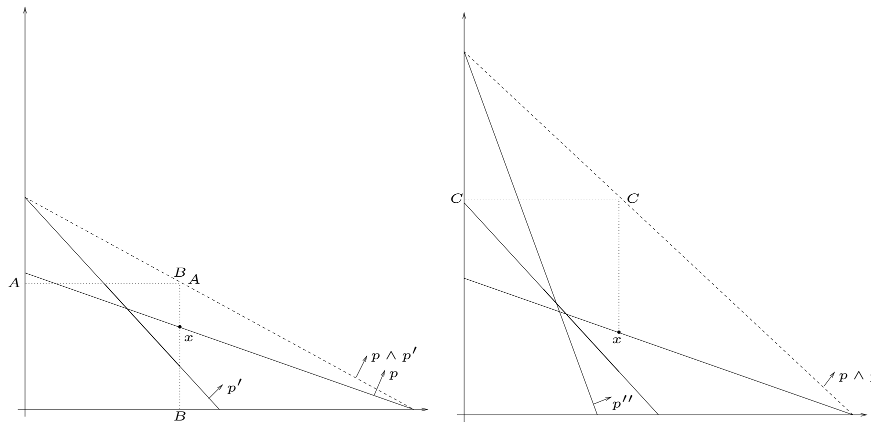
Corollary 2. *Suppose the hypotheses of Theorem 3 are satisfied,and in addition, suppose that $u(x,y) = f(x) + g(y)$. Then complementarities is satisfied if and only if*

$$\frac{f''(x)}{f'(x)} \leq -\frac{1}{x}, \frac{g''(x)}{g'(x)} \leq -\frac{1}{x}.$$

Therefore, an additively separable utility satisfies complementarity if and only if each of its components are more concave than the natural logarithm. This result is essentially in Wald (1951), for the case of gross substitutes (Varian (1985) clarifies this issue and presents a different proof; the appendix to Quah (2007) has a proof for the the non-differentiable case). For a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, the number

$$-\frac{f''(x)}{f'(x)}$$

is often understood as a local measure of curvature at the point x. In particular, one can demonstrate that for subjective expected utility, when $u(x,y) = \pi_1 U(x) + \pi_2 U(y)$, complementarity is satisfied if and only if the rate of relative risk aversion



(a) Budget p: between A–A and B–B. (b) Budget p'': below C–C.

FIGURE 4. Implications of (x, p).

is greater than one. It may be of interest to compare this with Quah's (2003) result that the "law of demand" is, in this case, equivalent to the rate of risk aversion never varying by more than four.

We proceed in Section 3 with a heuristic argument for Theorem 1. We then proceed with proofs of all three results.

3. A geometric intuition for Theorem 1.

The proof of Theorem 1 is based on extending D, one price at a time, to a countable dense subset of $\mathbb{R}^2_{++}$. It turns out that the crucial step is to extend D from two prices to a third price. Here we present a geometric version of the argument, for one of the special cases we need to cover in the proof.

Fix two prices, p and p'', with corresponding demands $x = D(p)$ and $x'' = D(p'')$. Let p' be a third price. We want to show that we can extend D to p' while respecting properties 1 and 2. We fix x as shown in Figure 4(a)

In Figure 4(a) we present the implications of x for demand $x' = D(p')$, if x' is to satisfy the conditions in the theorem. Compliance with Property 1 requires demand to be below the line A–A, as the intersection of A–A with the p'-budget line gives equality in Property 1. Compliance with Property 2 requires demand to be to the left of B–B. Hence, the possible x' are in the bold interval on the p' budget line.

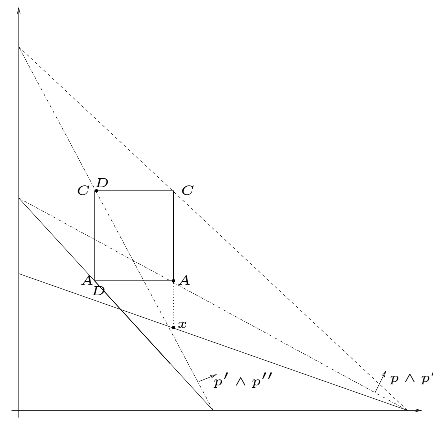


FIGURE 5. Compliance with x and any x'' that complies with x.

Consider Figure 4(b), where we introduce prices p''. Since x'' and x satisfy properties 1 and 2, x'' must lie below the line C–C on the p'' budget line. We want to show that we can choose an x' that agrees with the implications of both p and p''. In particular, that any x'' below C–C is compatible with a choice of x' on the bold segment of the p'-budget line.

In Figure 5, we represent the implications of x on x'', and its indirect implications on the demand at p'. To make the figure clearer, we do not represent the p'' budget, but we keep the C–C line: Note that the highest possible position of x'' determines a point on the $p' \wedge p''$-budget line, the point where C–C intersects the $p' \wedge p''$-budget line. This point, in turn, determines a point on the p'-budget line, the point where the D–D line intersects the p' budget line; note that, were x' to lie to the left of D–D, it would violate Properties 1 with respect to x''.

So, Property 1 applied to (x, p) and (x'', p'') requires that x'' lies below the intersection of C–C with the p''-budget line. This implies that the position of demand on the p'-budget line must lie to the right of the intersection with D–D. But this requirement is *the same* as the compliance with Property 1 with respect to x: note that A–A and D–D intersect p' at the same point. So demand for p' lies below A–A, as dictated by x if and only if it lies to the right of D–D, as dictated by any x'' that complies with Property 1 with respect to x.

That D–D and A–A should coincide on the p'-budget line may seem curious at this stage, but it is a result of the special case we are considering. Here, the budget set of p' is the meet of the budget sets corresponding to prices $p \wedge p'$ and $p' \wedge p''$; that is $p' = (p \wedge p') \vee (p' \wedge p'')$. Let y and z be, respectively, the intersection of B–B with the $p \wedge p'$ budget line, and of C–C with the $p' \wedge p''$ line. Then, *in the case we show in Figure 5*, $y \vee z$ coincides with y in the good that is cheaper for p', and with z in the good that is cheaper in p''. As a result, $(p' \wedge p'') \cdot (y \vee z) = 1$ says that expenditure on the two cheapest goods adds to 1. But at the same time $y \wedge z$ coincides with y in the good that is more expensive for p, and similarly for z and p''. So $(p \wedge p'') \cdot (y \vee z) = 1$ also says also that the sum of expenditures on the two most expensive goods, when evaluated at prices $p \vee p''$, must equal 1. Hence $(p \vee p'') \cdot (y \wedge z) = 1$.

4. Proof of Theorem 4

We present the proof for n goods. For the purpose of this section, then, let $\mathbb{R}^n_+$ be consumption space and $\mathbb{R}^n_{++}$ be the set of possible prices. A *demand function* is a function $D : \mathbb{R}^n_{++} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that, for all $(p, w) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_+$, $p \cdot D(p, w) \leq p \cdot w$.

We generalize the notion of complementarity and weak complementarity to the case of n goods. We will say that demand D satisfies *complementary* if, for all i, j, for all $(p, w), (p', w')$ for which $w' = w$,

$$[D_i(p', w') - D_i(p, w)][D_j(p', w') - D_j(p, w)] \geq 0.$$

We say that D satisfies *weak complementary* if, or every (p, w) the set of prices p' such that

$$[D_i(p', w) - D_i(p, w)][D_j(p', w) - D_j(p, w)] \geq 0 \text{ for every } i, j \tag{3}$$

has a nonempty interior

We shall prove the following:

Theorem 5. *A rational demand function D satisfies weak complementarity if and only if there exist continuous strictly monotone functions $f_i : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ for* *which for some i, $f_i(\mathbb{R}_+) \subseteq \mathbb{R}$ (is always real-valued), so that*

$$D(p,w) = \arg\max_{B(p,w)} \min_{i=1,\ldots,n} \{f_i(x_i)\}.$$

Let us first discuss a simple argument for the case in which complementarity (as opposed to weak complementarity) is assumed. The argument holds for n goods. The argument here is simpler than the argument for weak complementarity, and proceeds by establishing that all price vectors have the same (strictly increasing and continuous) Engel curves. The proof of the weak complementarity argument is more difficult and works by characterizing weak upper contour sets of certain commodity bundles.

Define a function $g : \mathbb{R}_+ \to \mathbb{R}^n_+$, and then prove that it is weakly increasing. Let $\mathbf{1}$ indicate a vector of ones, and define

$$g(\alpha) = D((1/n)\mathbf{1}, \alpha\mathbf{1}).$$

Then g is a function specifying demand when total wealth is α, and prices are equal. Moreover, it is clear by rationalizability (from Walras' law) that $\sum_i g_i(\alpha) = \alpha$. We establish that g is weakly increasing, so that for all i, and all $\alpha < \beta$, $g_i(\alpha) \le g_i(\beta)$. To this end, suppose by means of contradiction that there exist $\alpha < \beta$ and i^* for which $g_{i^*}(\beta) < g_{i^*}(\alpha)$. As $\sum_i g_i(\alpha) = \alpha$, there exists some j^* for which $g_{j^*}(\alpha) < g_{j^*}(\beta)$.

We will establish the existence of $p^* \in \mathbb{R}^n_{++}$ such that $p^* \cdot (g(\beta) - g(\alpha)) = 0$. Suppose there does not exist such a p^*. Define

$$P = \{p : p \cdot (g(\beta) - g(\alpha)) = 0\};$$

the space of vectors orthogonal to $(g(\beta) - g(\alpha))$. We know that $P \cap \mathbb{R}^n_{++} = \varnothing$. Hence, there exists a hyperplane $q \in \mathbb{R}^n \backslash \{0\}$ for which for all $p \in P$, $q \cdot p \ge 0$ and for all $p \in \mathbb{R}^n_{++}$, $q \cdot p \le 0$. As P is a vector space, we may conclude that for all $p \in P$, $q \cdot p = 0$. Moreover, as $\mathbb{R}^n_{++}$ is open, we may conclude that for all $p \in \mathbb{R}^n_{++}$, $q \cdot p < 0$. In particular, as q is orthogonal to the orthogonal subspace of $(g(\beta) - g(\alpha))$, there exists some $\gamma \neq 0$ for which $q = \gamma(g(\beta) - g(\alpha))$. Hence, we may conclude that either $(g(\beta) - g(\alpha)) \cdot p < 0$ for all $p \in \mathbb{R}^n_{++}$ or $(g(\beta) - g(\alpha)) \cdot p > 0$ for all $p \in \mathbb{R}^n_{++}$. But this is clearly false, as there exist i^* for which $g_{i^*}(\beta) - g_{i^*}(\alpha) < 0$ and j^* for which $g_{j^*}(\beta) - g_{j^*}(\alpha) > 0$. Hence, there exists such a p^*.

Now, $B((1/n)\mathbf{1}, \alpha\mathbf{1}) = B((1/n)\mathbf{1}, g(\alpha))$, so that by rationalizability, $D((1/n)\mathbf{1}, g(\alpha)) = g(\alpha)$. Similarly, $D((1/n)\mathbf{1}, g(\beta)) = \beta$. Now, there does not exist $x \in B(p^*, g(\alpha))$ for which $x \geq g(\alpha)$ and $x \neq g(\alpha)$. Hence, by rationality and complementarity, it follows that $D(p^*, g(\alpha)) = g(\alpha)$. Similarly, we may establish that $D(p^*, g(\beta)) = g(\beta)$. But note that $B(p^*, g(\alpha)) = B(p^*, g(\beta))$. Hence $g(\alpha) = D(p^*, g(\alpha)) = D(p^*, g(\beta)) = g(\beta)$, a contradiction. Hence, $\alpha \leq \beta$ implies $g(\alpha) \leq g(\beta)$.

This latter fact in particular, along with the fact that $\sum_i g_i(\alpha) = \alpha$, implies that g is continuous (we establish that any monotonic demand function is continuous, see Lemma 6.15). We establish that for any (p, w), $D(p, w) = \max_\alpha \{g(\alpha) : g(\alpha) \in D(p, w)\}$. Let

$$x = \max_\alpha \{g(\alpha) : g(\alpha) \in D(p, w)\} = g(\alpha^*).$$

This follows in a similar way to the preceding argument: For all p and all α one easily establishes that $D(p, g(\alpha)) = g(\alpha)$ by complementarity and the preceding argument. As the maximum is attained (by continuity and the fact that g is unbounded), as $B(p, w) = B(p, g(\alpha^*))$, we conclude that $D(p, w) = g(\alpha^*)$, so that $D(p, w) = \max_\alpha \{g(\alpha) : g(\alpha) \in D(p, w)\}$.

Recall that rationalizability implies the existence of a monotonic, continuous R rationalizing D.[4] Define $U(g(\alpha)) \equiv \{x : xRg(\alpha)\}$. By monotonicity, $\mathbb{R}^n_+ + \{g(\alpha)\} \subseteq U(g(\alpha))$. Moreover, for any $x \notin \mathbb{R}^n_+ + \{g(\alpha)\}$, there exists some $p \in \mathbb{R}^n_{++}$ for which $p \cdot x \leq p \cdot g(\alpha)$ (by a simple separating hyperplane argument) as $x \notin D(p, g(\alpha))$, we may conclude that $g(\alpha) Px$. Hence, $x \notin U(g(\alpha))$. Therefore, $U(g(\alpha)) = \mathbb{R}^n_+ + \{g(\alpha)\}$. Therefore, by continuity, for all $x \geq g(\alpha)$ for which $x \gg g(\alpha)$ is false, we conclude that $xIg(\alpha)$. Therefore, for all $\alpha < \beta$, it follows that $g(\alpha) \ll g(\beta)$; as otherwise, for all p, $g(\alpha)$ maximizes R in $B(p, g(\beta))$, contradicting rationalizability.

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[4]For R, we denote the asymmetric part by P and the symmetric part by I.

We conclude that g is strictly increasing and continuous. It remains to define f_i. But g_i is strictly increasing and continuous for all i, so simply define $f_i(x) = g_i^{-1}(x)$ on the range of g_i, and ∞ otherwise. Note that as $\sum_i g_i(\alpha) = \alpha$, it follows that some g_i must be unbounded above; hence for some i, g_i^{-1} is well-defined and real-valued on all of $\mathbb{R}_+$. It is now straightforward to verify that $u(x) = \min_i \{f_i(x_i)\}$ generates $D(p, w)$.

Now, we proceed with the argument for the weak complementarity case. The intuition for the proof is quite simple. Consider a price-endowment pair, and the demand at that pair. Without loss of generality, by rationality, we may assume that in fact, the demand coincides with the endowment. Let us imagine the upper-contour set of utility at the demand. We claim that there is a "kink" in this upper contour set at the demand point. This follows because there exists *some* other price (at the same endowment) for which demand of both commodities moves weakly in the same direction. But, since endowment and demand coincide for the original price, in order for both commodities to move weakly in the same direction, demand must remain unchanged. Hence there exist two distinct prices for which demand coincides: there must therefore be a kink in the indifference curve at this point. However, for *any support point* of the upper contour set, the same property is satisfied. This tells us that every support point is actually a kink, and therefore the upper contour set must coincide with a translation of the nonnegative orthant: establishing that the rationalizing preference is generalized Leontief.

Say a set U is *upper comprehensive* if for all $x \in U$, $\{x\} + \mathbb{R}^n_+ \subseteq U$. Recall that the *support function* $h : \mathbb{R}^n_+ \to \mathbb{R}$ of a closed upper comprehensive set $U \subseteq \mathbb{R}^n$ is given by

$$h(p) = \inf\{p \cdot x | x \in U\}. \tag{4}$$

The infimum is achieved for $p \in \mathbb{R}^n_{++}$, and the support function is continuous and concave on this domain. A simple modification of Theorem 1.7.4 of Schneider (2003) establishes that that the subgradient of h at p is given by

$$\partial h(p) = \operatorname{conv}\left(\arg\min_{x\in U} p\cdot x\right) \tag{5}$$

for every $p \in \mathbb{R}^n_{++}$.

Before proving the theorem we prove the following lemmas (many are common knowledge and appear in standard economics textbooks–we obviously make no claim to priority). The set $U_\alpha = \{x : u(x) \geq \alpha\}$ for some $\alpha \in \mathbb{R}$ is called an *upper contour set of* u *at* α.

Lemma 4.1. *Let* u *be a continuous function on* $\mathbb{R}^n_+$*,* U_α *an upper contour set of* u *and* $p \in \mathbb{R}^n_{++}$*, and let* $w_0 \in$ *argmin*$_{x\in U_\alpha} p\cdot x$*. Then*

$$u(w_0) = \alpha = \max_{B(p,w_0)} u, \text{ and} \tag{6}$$

$$\mathit{argmin}_{x\in U_\alpha} p\cdot x = U_\alpha \cap B(p,w_0) = \mathit{argmax}_{x\in B(p,w_0)} u(x). \tag{7}$$

Proof. First, if $x \in B(p,w_0)$ and $z < x$ then $p\cdot z < p\cdot x \leq p\cdot w_0$ and therefore, since $w_0 \in \operatorname{argmin}_{x\in U_\alpha} p\cdot x$, it follows that $z \notin U_\alpha$, i.e. $u(z) < \alpha$. Since u is continuous and this is true for every $z < x$ it follows that $u(x) \leq \alpha$ for every $x \in B(p,w_0)$. Thus $\max_{B(p,w_0)} u \leq \alpha$. Since $w_0 \in B(p,w_0)$ and $u(w_0) \geq \alpha$ (as $w_0 \in U_\alpha$) we get (6).

The left equality of (7) follows directly from the fact that $w_0 \in \operatorname{argmin}_{x\in U_\alpha} p\cdot x$, and the right equality follows from the fact that $\max_{B(p,w_0)} u = \alpha$. ∎

When the utility function rationalizes a single-valued demand function we know that $\operatorname{argmax}_{B(p,w)} u = D(p,w)$ is a singleton for every p, w. For an upper comprehensive set U, we define the support function as $h : \mathbb{R}^2_{++} \to \mathbb{R}$ given by

$$h(p) = \inf_{x\in U} p\cdot x.$$

We get the following Corollary.

Corollary 3. *Let* D *be a (single-valued) demand function which is rationalized by the continuous and monotone utility function* u *and let* $U = U_\alpha$ *be an upper contour set of* u*, and* h *the support function of* U_α*. Then:*

(1) *$\partial h(p)$ is a singleton for every $p \in \mathbb{R}^n_{++}$ (that is, h is differentiable at p).*

(2) *$\partial h(p)$ is the unique element $w \in \mathbb{R}^n$ such that $u(w) = \alpha$ and $w = D(p, w)$.*

Proof. Let $w_0 \in \text{argmin}_{x \in U_\alpha} p \cdot x$. Then

$$\text{argmin}_{x \in U_\alpha} p \cdot x = \text{argmax}_{B(p,w_0)} u = D(p, w_0), \tag{8}$$

where the first equality follows from (7) and the second from the definition of D. In particular, $\text{argmin}_{x \in U_\alpha} p \cdot x$ is a is a singleton, that is $w_0 = \text{argmin}_{x \in U_\alpha} p \cdot x$. By (5) $\partial h(p)$ is also a singleton and $\partial h(p) = \text{argmin}_{x \in U_\alpha} p \cdot x = w_0$. Moreover, it also follows from (8) that $w_0 = D(p, w_0)$ and from (6) that $u(w_0) = \alpha$.

Assume now that $u(w) = \alpha$ and $w = D(p, w)$ for some $w \in \mathbb{R}^n_{++}$. We claim that $w = w_0$. Indeed, since $w \in U_\alpha$ and $w_0 = \text{argmin}_{x \in U_\alpha} p \cdot x$ it follows that $p \cdot w_0 \leq p \cdot w$. In particular $w_0 \in B(p, w)$. But $u(w_0) = \alpha = u(w)$ and by assumption $w = D(p, w) = \text{argmax}_{B(p,w)} u$. Therefore $w_0 = w$. ∎

Lemma 4.2. *Assume that D is rationalized by u. Then $p \cdot D(p, w) = p \cdot w$ for every $p \in \mathbb{R}^n_{++}$ and $w \in \mathbb{R}^n_+$.*

Proof. By monotonicity of u the maximum in (1) is achieved on the boundary of $B(p, w)$. Since the the maximizer is by assumption unique the result follows. ∎

Lemma 4.3. *Assume that D is rationalized by the relation u. Then*

$$D(p, D(p, w)) = D(p, w)$$

for every $p \in \mathbb{R}^n_{++}$ and $w \in \mathbb{R}^n_+$.

Proof. The lemma follows from (1) and the fact that by the previous lemma $B(p, D(p, w)) = B(p, w)$. ∎

Consequences of Weak Complementarity. The following lemma shows that weak complementarity, when applied to demand, is not as innocuous as it appears.

Lemma 4.4. *Assume that D satisfies rationalizability. Let $w^* = D(p, w)$ for some $p \in \mathbb{R}^n_{++}$ and $w \in \mathbb{R}^n_+$. Assume that, for some $p' \in \mathbb{R}^n_{++}$ (3), is satisfied with $w = w^*$. Then*

$$D(p, w^*) = w^* = D(p', w^*).$$

Proof. Condition (3) means that either $D(p', w^*) \geq D(p, w^*)$ or $D(p', w^*) \leq D(p, w^*)$ coordinate-wise. Assume, without loss of generality, the former. By Lemma 4.3 $D(p, w^*) = w^*$, therefore

$$D(p', w^*) \geq w^*.$$

By Lemma 4.2

$$p' \cdot D(p', w^*) = p' \cdot w^*.$$

Since $p' \in \mathbb{R}^n_{++}$ it follows from the last two equations that

$$D(p', w^*) = w^* = D(p, w^*).$$

∎

Proof of Theorem 4. Let U be an upper contour set of R and let h be its support function. Let $p \in \mathbb{R}^n_{++}$. Then by Corollary 3 h is differentiable at p, $\partial h(p) \in U$ and $\partial h(p) = D(p, \partial h(p))$.

We now claim that the derivative function $p \mapsto \partial h(p)$ is constant. Indeed, let $p \in \mathbb{R}^n_{++}$, and let $w^* = \partial h(p)$. Then $w^* = D(p, w^*)$. By weak complementarity (via Lemma (4.4)) there exists an open set of p' such that $w^* = D(p', w^*)$. By the second item of Corollary 3, $\partial h(p') = w^*$ for every such p'. In particular, every value in the image of $p \mapsto \partial h(p)$ is achieved on a set with non-empty interior. From the separability of $\mathbb{R}^n_{++}$ it follows that the image of the function $p \mapsto \partial h(p)$ is countable. Since the function $p \mapsto \partial h(p)$ is continuous (as a derivative of a smooth concave function) it must be constant. Assume that $\partial h(p) = w_0$ for every $p \in \mathbb{R}^n_{++}$ for some $w_0 \in U$. By definition of the subgradient, it follows that $p \cdot w_0 \leq p \cdot w$ for every $w \in U$ and every $p \in \mathbb{R}^n_{++}$. By continuity, the later inequality follows for every $\mathbb{R}^n_+$. Therefore $w_0 \leq w$ for every $w \in U$. Since U is upper comprehensive and $w_0 \in U$ we know that if $w_0 \leq w$ then $w \in U$. Therefore $U = w_0 + \mathbb{R}^n_{++}$ is a translated orthant. The rest of the argument is the same as in the previous theorem.

5. Proof of Theorem 3

Fix $\hat{x}$ in the interior of consumption space. Denote by $\nabla u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}\right)$. Note that

$$\hat{x} = D\left(\nabla u(\hat{x}), \nabla u(\hat{x}) \cdot \hat{x}\right).$$

Let $p = \nabla u(\hat{x})$. We calculate p_1' such that $(\hat{x}_1 + \epsilon, \hat{x}_2)$ lies on the budget line for (p_1', p_2) with income $p \cdot \hat{x}$. So $p_1'(\hat{x}_1 + \epsilon) + p_2\hat{x}_2 = p_1\hat{x}_1 + p_2\hat{x}_2$. Conclude

$$\frac{p_1'}{p_2} = \frac{\hat{x}_1}{\hat{x}_1 + \epsilon} m(\hat{x}).$$

The rest of the argument is illustrated in Figure 6. Since $p_1' < p_1$, complementarity implies that $D(p_1', p_2, I)$ lies weakly to the northwest of $(\hat{x}_1 + \epsilon, \hat{x}_2)$ on the budget line. By the strict convexity of u, $u(y) > u(\hat{x}_1 + \epsilon, \hat{x}_2)$ for any y that lies between $D(p_1', p_2, I)$ and $(\hat{x}_1 + \epsilon, \hat{x}_2)$ on the budget line. Hence, if u does not achieve its maximum on the budget line at $(\hat{x}_1 + \epsilon, \hat{x}_2)$, it is increasing as we move northwest on the budget line. So the product $\nabla u \cdot v$, of the gradient of u with any vector pointing northwest, is nonnegative. This gives $m(\hat{x}_1 + \epsilon, \hat{x}_2) \leq \frac{p_1'}{p_2}$, so

$$m(\hat{x}_1 + \epsilon, \hat{x}_2) \leq \frac{\hat{x}_1}{\hat{x}_1 + \epsilon} m(\hat{x}).$$

Since $\epsilon > 0$ was arbitrary, and since the two sides of the inequality are equal at $\epsilon = 0$, we can differentiate with respect to ϵ and evaluate at $\epsilon = 0$ to obtain

$$\frac{\partial m(\hat{x})}{\partial \hat{x}_1} \frac{1}{m(\hat{x})} \leq \frac{-1}{\hat{x}_1}$$

The proof of the second inequality is analogous.

6. Proof of Theorem 1

6.1. **Preliminaries.** For $p \in \mathbb{R}^2_{++}$ let $L(p) = \{x \in \mathbb{R}^2_+ | p \cdot x = 1\}$.

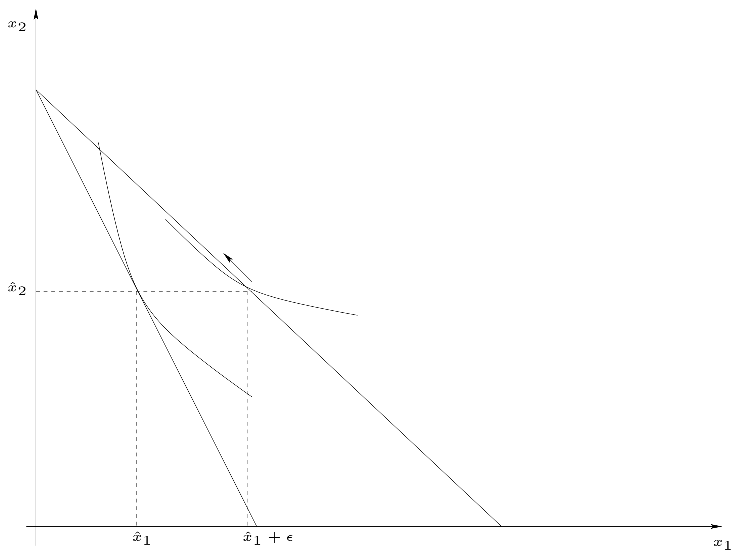

FIGURE 6. Illustration for the proof of Theorem 3.

For $x \in \mathbb{R}$ let $\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$.

The following lemmas are obvious.

Lemma 6.1. *Let $a, b, b' \in \mathbb{R}^2_+$ such that $a \cdot b = a \cdot b' = 1$. Then*

(1) *$sgn(b_1 - b'_1) \cdot sgn(b_2 - b'_2) \leq 0$.*
(2) *If $a \gg 0$ and $b \neq b'$ then $sgn(b_1 - b'_1) \cdot sgn(b_2 - b'_2) = -1$.*

Lemma 6.2. *Let $a, b \in \mathbb{R}^2$ such that $a \gg 0$ and $b > 0$. Then $a \cdot b > 0$.*

Lemma 6.3. *Let $a, b, c \in \mathbb{R}^2$ such that $a \gg 0$. If $a \cdot b \leq a \cdot c$ and $b_i \geq c_i$ for $i \in \{1, 2\}$ then $b_j \leq c_j$ for $j = 3 - i$.*

For $p \in \mathbb{R}^2_{++}$ and $x \in \mathbb{R}_+$ such that $p_j x \leq 1$ let $X_i(p, x) = (1 - p_j x)/p_i$ where $j = 3 - i$. Then $X_i(p, x)$ is the i-th coordinate of the element of $L(p)$ whose j-th coordinate is x. Note that, when $p, p' \in \mathbb{R}^2_{++}$ and $p \cdot (x_i, x_j) = 1$, $X_i(p \wedge p', x_j)$ is well defined; this will be a recurrent use of X_i in the sequel.

Lemma 6.4. *Let $p, p' \in \mathbb{R}^2_{++}$ and $x, x' \in \mathbb{R}_+$ and $i \in \{1, 2\}$ such that $p_j x, p'_j x' \leq 1$, and let $j = 3 - i$. Then*

(1) *$p_i X_i(p, x) \leq 1$ and $x = X_j(p, X_i(p, x))$*
(2) *If $p \leq p'$ then $X_i(p, x') \geq X_i(p', x')$.*
(3) *If $x' < x$ then $X_i(p, x') > X_i(p, x)$*

Lemma 6.5. *If $p \in \mathbb{R}^2_{++}$ and $x \in \mathbb{R}^2_+$, $i \in \{1, 2\}$ and $j = 3 - i$. Assume that $p_j x_j \leq 1$. Then*

(1) *$p \cdot x \leq 1$ iff $x_i \leq X_i(p, x_j)$.*
(2) *$p \cdot x \geq 1$ iff $x_i \geq X_i(p, x_j)$*

Note that Statements 1 and 2 in Lemma 6.5 are not equivalent.

Lemma 6.6. *Let $p, q \in \mathbb{R}^2_{++}$ such that $q_i \geq p_i$ for some product $i \in \{1, 2\}$, and let $x, y \in L(p)$. If $q \cdot y \geq 1$ and $x_i \geq y_i$ then $q \cdot x \geq 1$.*

Proof. Since $x_i \geq y_i$ and $x_i \leq 1/p_i$ (as $x_i p_i \leq x \cdot p = 1$) it follows that $x_i = \lambda y_i + (1 - \lambda) 1/p_i$ for some $0 \leq \lambda \leq 1$. Since $y \in L(p)$, $\lambda y + (1 - \lambda) e^i \in L(p)$, where $e^i \in \mathbb{R}^2_{++}$ is given by $e^i_i = 1/p_i$ and $e^i_j = 0$ for $j = 3 - i$. Then, $x = \lambda y + (1 - \lambda) e^i$, as there is only one element of $L(p)$ with i-th component x_i. Therefore

$$q \cdot x \geq \min(q \cdot y, q \cdot e^i) = \min(q \cdot y, q_i/p_i) \geq 1,$$

as desired. ∎

Lemma 6.7. *Let $p, q \in \mathbb{R}^2_{++}$ such that $q_i > p_i$ for some product $i \in \{1, 2\}$ and assume that $q \cdot x \leq 1$ for some $x \in L(p)$. Then $p_j \geq q_j$ for $j = 3 - i$.*

Proof. If $p_j < q_j$ then $q - p \gg 0$ and therefore

$$q \cdot x - p \cdot x = (q - p) \cdot x > 0$$

By Lemma 6.2, but this is a contradiction since $q \cdot x \leq p \cdot x = 1$. ∎

6.2. **The conditions are necessary.** We now prove that the conditions in Theorem 1 are necessary. Let $D : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be a decreasing demand function that satisfies the weak axiom of revealed preference. Let $p, p' \in \mathbb{R}^2_{++}$.

To prove that D satisfies Condition 1 note first that from the monotonicity of D it follows that

$$D(p) \vee D(p') \leq D(p \wedge p'). \tag{9}$$

Therefore

$$(p \wedge p') \cdot (D(p) \vee D(p')) \leq (p \wedge p') \cdot D(p \wedge p') = 1,$$

where the inequality follows from (9) and monotonicity of the scalar product in the second argument.

To prove that D satisfies Condition 2 assume that $p' \cdot D(p) \leq 1$ and, say, that $p'_1 > p_1$. We want to show that $D(p')_2 \geq D(p)_2$. Let $p'' = \frac{1}{p' \cdot D(p)} p'$. Then $p' \leq p''$ and $p'' \cdot D(p) = 1$. In particular, it follows from the last equality and the weak axiom of revealed preference that $p \cdot D(p'') \geq 1$. Let $x = D(p)$ and $x'' = D(p'')$. Then $p \cdot x = p'' \cdot x'' = p'' \cdot x = 1$ and $p \cdot x'' \geq 1$. Therefore

$$\begin{aligned} 0 \geq p \cdot x + p'' \cdot x'' - p \cdot x'' - p'' \cdot x = (p - p'') \cdot (x - x'') = \\ (p_1 - p''_1) \cdot (x_1 - x''_1) + (p_2 - p''_2) \cdot (x_2 - x''_2). \end{aligned} \tag{10}$$

Since $p''_1 \geq p'_1 > p_1$ and $p'' \cdot x = p \cdot x$ we get from Lemma 6.1 that $p''_2 \leq p_2$. Assume, by way of contradiction, that $x''_2 < x_2$. Then, since $p'' \cdot x'' = p'' \cdot x$ and $p'' \gg 0$ it follows from Lemma 6.1 that $x_1 < x''_1$, in which case the sum in the right hand side of (10) is strictly positive (since $p''_2 \leq p_2$, $p''_1 > p_1$,$x''_2 < x_2$ and $x_1 < x''_1$), which leads to a contradiction. It follows that $x''_2 \geq x_2$, i.e. $D(p'')_2 \geq D(p)_2$. By monotonicity of D it follows that $D(p') \geq D(p'')$. Hence

$$D(p')_2 \geq D(p'')_2 > D(p)_2,$$

as desired.

6.3. **The conditions are sufficient.** A *data point* is given by a pair $(p, x) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{+}$ such that $p \cdot x = 1$.

Definition 1. *A pair $(p, x), (p', x') \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{+}$ of data points is* permissible *if the following conditions are satisfied:*

(1) $(p \wedge p') \cdot (x \vee x') \leq 1$.
(2) *If $p' \cdot x \leq 1$ and $p'_i > p_i$ for some product $i \in \{1, 2\}$ then $x'_j \geq x_j$ for $j = 3-i$.*
(3) *If $p \cdot x' \leq 1$ and $p_i > p'_i$ for some product $i \in \{1, 2\}$ then $x_j \geq x'_j$ for $j = 3-i$.*

Let us say that a partial demand function $P : D \to \mathbb{R}^2_{++}$ is *permissible* if $(p, D(p)), (p', D(p'))$ is a permissible pair for every $p, p' \in P$ Using this terminology, a partial demand function $D : P \to \mathbb{R}^2_{++}$ satisfies the conditions of Theorem 1 iff it is permissible

Monotonicity is a consequence of permissibility:

Lemma 6.8. *If $(p, x), (p', x') \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{+}$ is a permissible pair of data points and $p \leq p'$ then $x' \leq x$.*

Proof. If $p \leq p'$ then $p \wedge p' = p$ and therefore it follows from Condition 1 of Definition 1 that $p \cdot (x \vee x') \leq 1$. But $p \cdot x = 1$ and therefore

$$p \cdot (x \vee x' - x) = p \cdot (x \vee x') - p \cdot x \leq 0.$$

Since $x \vee x' - x \geq 0$ it follows from the last inequality and Lemma 6.2 that $x \vee x' - x = 0$, i.e. $x' \leq x$, as desired. ∎

The weak axiom of revealed preference is a consequence of permissibility:

Lemma 6.9. *If $(p, x), (p', x') \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{+}$ is a permissible pair of data points and $p' \cdot x < 1$ then $p \cdot x' > 1$.*

Proof. We show that $p \cdot x' \leq 1$ implies $p' \cdot x \geq 1$. Assume that $p \cdot x' \leq 1$. If $p' \geq p$ then $p' \cdot x \geq p \cdot x = 1$ and we are done. Let $p' \not\geq p$. Assume w.l.o.g. that $p_1 > p'_1$. By Condition 3 of Definition 1 it follows that $x_2 \geq x'_2$. Also, since

$$(p - p') \cdot x' = p \cdot x' - p' \cdot x' \leq 0$$

and since $x' > 0$ it follows from Lemma 6.2 that it cannot be the case that $p-p' \gg 0$. Therefore $p_2 \leq p'_2$. Let $x'' \in \mathbb{R}^2_{++}$ be such that $x''_2 = x'_2$ and $p \cdot x''_2 = 1$; that is $x'' = (X_1(p, x'_2), x'_2)$; note that $X_1(p, x'_2)$ is well defined because $p_2 x'_2 \leq p'_2 x'_2 \leq 1$. Since $p \cdot x' \leq 1 = p \cdot x''$ and $x''_2 = x'_2$ it follows from Lemma 6.3 that $x''_1 \geq x'_1$. Therefore $x'' \geq x'$, and, in particular, $p' \cdot x'' \geq p' \cdot x' \geq 1$. Since $x_2 \geq x'_2 = x''_2$ and $p_2 \leq p'_2$ it follows from Lemma 6.6 that $p' \cdot x \geq 1$ as desired. ∎

The following lemma provides an equivalent characterization of permissible pairs. Unlike the previous characterization, here the roles of p and p' are not symmetric. For fixed p and p', the lemma states the restrictions on x' (the demand at p') such that the pair $(p, x), (p', x')$ is permissible assuming that x is already given. Recall Figure 4(a). From the lemma we see that every good induces one restriction on x'. If the good is cheaper for p' (as is the good that corresponds to the vertical axis in Figure 4(a)) then it induces an inequality of type 1 – an upper bound on the demand for that good. This is the line A–A in the figure. If the good is more expensive for p' (as is the good that corresponds to the horizontal axis in Figure 4(a)) then it induces an inequality of type 2 or 3, depending on whether x is a possible consumption at the new price p'. In the figure, since x is not possible in the new price, we get an inequality of type 3 – an upper bound on the demand for that product. This is the line B–B in the figure.

Lemma 6.10. *A pair $(p, x), (p', x')$ is permissible iff the following conditions are satisfied for every product $i \in \{1, 2\}$ and $j = 3 - i$.*

(1) *If $p'_i \leq p_i$ then $x'_i \leq X_i(p \wedge p', x_j)$.*
(2) *If $p'_i > p_i$ and $p' \cdot x \leq 1$ then $x'_j \geq x_j$.*
(3) *If $p'_i > p_i$ and $p' \cdot x > 1$ then $x'_i \leq x_i$.*

The proof of Lemma 6.10 requires some auxiliary results, presented here as Claims 6.12, 6.11, and 6.13.

Claim 6.11. *If $(p, x), (p', x')$ is a pair of data points and $(p \wedge p') \cdot (x \vee x') \leq 1$ then $x'_i \leq X_i(p \wedge p', x_j)$*

Proof. Let $i \in \{1,2\}$ and $j = 3 - i$. Let $y \in \mathbb{R}^2_{++}$ be such that $y_j = x_j$ and $y_i = x'_i$. Then

$$(p \wedge p') \cdot y \leq (p \wedge p') \cdot (x' \vee x) \leq 1,$$

where the first inequality follows from the fact that $y \leq x' \vee x$. In particular, it follows from the last inequality and Lemma 6.5 that

$$x'_i = y_i \leq X_i(p \wedge p', y_j) = X_i(p \wedge p', x_j),$$

as desired. ∎

Claim 6.12. *For every $p, p' \in \mathbb{R}^2_{++}$ and $x \in L(p)$ the set of all $x' \in L(p')$ such that $(p \wedge p') \cdot (x \vee x') \leq 1$ is a subinterval of $L(p')$*

Proof. The function $x' \mapsto (p \wedge p') \cdot (x \vee x')$ is concave since the inner product is monotone and linear and since

$$x \vee \big(\lambda\alpha + (1-\lambda)\beta\big) \leq \lambda(x \vee \alpha) + (1-\lambda)(x \vee \beta)$$

for every $\alpha, \beta \in \mathbb{R}^2_{++}$ and every $0 \leq \lambda \leq 1$. ∎

Claim 6.13. *If $(p, x), (p', x')$ is a permissible pair such that $x_1 < x'_1$ and $x_2 > x'_2$ then $p_1 > p'_1$ and $p_2 < p'_2$.*

Proof. We show that any other possibility leads to a contradiction. Note first that Lemma 6.8 implies $x \geq x'$ if $p \leq p'$, and $x \leq x'$ if $p \geq p'$. Both cases contradict the hypotheses on x and x'.

Second, suppose that $p_1 < p'_1$ and $p_2 > p'_2$. Consider the following three cases.

- If $p' \cdot x \leq 1$, then $x'_2 \geq x_2$ by Condition 2 of Definition 1.
- If $p \cdot x' \leq 1$, then $x_1 \geq x'_1$ by Condition 3 of Definition 1.
- If $p' \cdot x > 1$ and $p \cdot x' > 1$ then

$$0 < p \cdot x' + p' \cdot x - p \cdot x - p' \cdot x' = (p - p') \cdot (x' - x) =$$
$$(p_1 - p'_1) \cdot (x'_1 - x_1) + (p_2 - p'_2) \cdot (x'_2 - x_2) < 0$$

The first inequality follows from the fact that $p \cdot x = p' \cdot x' = 1$ and $p \cdot x', p' \cdot x > 1$. The last inequality follows because, in each product, one multiplier is negative and one is positive.

All three cases contradict the hypotheses on x and x'. The only possibility left is $p_1 > p'_1$ and $p_2 < p'_2$, as desired. ∎

We now prove Lemma 6.10

Proof. We consider separately the possible positions of p, p', x, up to symmetry between the products.

Case 1: $p \ll p'$. We show first that the conditions in the lemma imply permissibility. Since $p \ll p'$ then $p' \cdot x = p \cdot x + (p' - p) \cdot x > 1$ (the inequality follows from Lemma 6.2) and, by Condition 3 in the lemma $x' \leq x$.

Since $p \leq p'$, $x' \leq x$ implies that $(p \wedge p') \cdot (x \vee x') = p \cdot x = 1$. So Condition 1 in the definition of permissibility is satisfied. In addition, $x' \leq x$ implies that Condition 3 is satisfied. We show Condition 2: If $p' \cdot x \leq 1$ and $p'_i > p_i$, then $p \cdot x = 1$ implies that $x'_i = x_i = 0$ and that $p'_j = p_j$ for $j = 3 - i$. Then $x'_2 = 1/p'_2 = 1/p_2 = x_2$. So Condition 2 is satisfied.

Now we show that permissibility implies the conditions in the lemma. Condition 1 in the lemma follows from Claim 6.11. Condition 3 holds because Lemma 6.8 implies $x' \leq x$. Finally, Condition 2 follows from Condition 2 in the definition of permissibility.

Case 2: $p' \leq p$. For each i, $p'_i \leq p_i$. So $x'_i \leq X_i(p', x_j)$ by Condition 1 of the lemma, as $p' = p' \wedge p'$. But $x'_i = X_i(p', x'_j)$, so $X_i(p', x'_j) \leq X_i(p', x_j)$. Since X_i is monotone decreasing in x_j (item 3 of Lemma 6.4), $x_j \leq x'_j$. This shows that $x \leq x'$. The rest of the argument is analogous to the previous case.

Case 3: $p_1 < p'_1, p_2 > p'_2$ and $p' \cdot x \leq 1$. Let

$$A = \{x' \in L(p') | x'_2 \geq x_2, (p \wedge p') \cdot (x \vee x') \leq 1\}.$$

Note that A is the set of all x' such that the pair $(p, x), (p', x')$ is permissible. Let

$$B = \{x' \in L(p') | x'_2 \geq x_2, x'_2 \leq X_2(p \wedge p', x_1)\}$$

be the set of all x' such that the pair $(p, x), (p', x')$ satisfies the conditions of Lemma 6.10. We have to prove that $A = B$. From Claim 6.11 we get that $A \subseteq B$. For the other direction, note that the set B is the closed interval whose endpoints are the unique points y, z in $L(p')$ such that $y_2 = x_2$ and $z_2 = X_2(p \wedge p', x_1)$. Since, by Claim 6.12, A is an interval, it is sufficient to prove that $y, z \in A$.

Since $p' \cdot x \leq 1$ it follows that $x_1 \leq y_1$ and therefore $x \leq y$ and $x \vee y = y$ and therefore

$$(p \wedge p') \cdot (x \vee y) = (p \wedge p') \cdot y \leq p' \cdot y = 1,$$

and thus $y \in A$.

Now,

$$z_2 = X_2(p \wedge p', x_1) \geq X_2(p, x_1) = x_2 \text{ and}$$
$$z_1 = X_1(p', z_2) \leq X_1(p' \wedge p, z_2) = X_1(p' \wedge p, X_2(p' \wedge p, x_1)) = x_1,$$

where the inequalities follow from item 2 of Lemma 6.4. It follows that $x \vee z = (x_1, z_2)$. Since $z_2 = X_2(p \wedge p', x_1)$ it follows that $(p \wedge p') \cdot (x \vee z) = 1$ and therefore $z \in A$.

Case 4: $p_1 < p'_1, p_2 > p'_2$ and $p' \cdot x > 1$. Note that, in this case, the conditions in the lemma are equivalent to $x'_1 \leq x_1$ and $x'_2 \leq X_2(p \wedge p', x_1)$.

We show first that permissibility implies the latter conditions. We need to show that $x'_1 \leq x_1$, as Claim 6.11 gives $x'_2 \leq X_2(p \wedge p', x_1)$. First, if $p \cdot x' \leq 1$ then by Condition 3 of the definition of permissibility $x'_1 \leq x_1$. Second, let $p \cdot x' \nleq 1$. Then $p' \cdot x > 1$ and $p \cdot x' > 1$ imply $x' \ngeq x$ and $x \ngeq x'$. Now, $x'_1 > x_1$ and $x'_2 < x_2$ imply, by Claim 6.13 that $p'_1 < p_1$ and $p'_2 > p_2$. So it must be that $x'_1 < x_1$ and $x'_2 > x_2$. Thus, either way we get that $x'_1 \leq x_1$.

We now show that the conditions imply permissibility. Let $y = (x_1, X_2(p \wedge p', x_1))$; so $(p \wedge p') \cdot y = 1$. Note that $x_2 = X_2(p, x_1) \leq X_2(p \wedge p', x_1)$, so $x \leq y$. The conditions are equivalent to $x' \le y$. So we obtain

$$(p \wedge p') \cdot (x \vee x') \le (p \wedge p') \cdot (x \vee y) \le (p \wedge p') \cdot y = 1.$$

Thus Condition 1 of the definition of permissibility is satisfied. Condition 2 in the definition follows from Condition 2 in the lemma. Finally, Condition 3 in the definition is satisfied since $x'_1 \le x_1$. ∎

The proof of Theorem 1 is based on the following lemma:

Lemma 6.14. *Let P be a finite subset of $\mathbb{R}^2_{++}$ and let $D : P \to \mathbb{R}^2_+$ be a permissible partial demand function. Let $p' \in \mathbb{R}^2_{++}$. Then D can be extended to a permissible partial demand function over $P \cup \{p'\}$.*

Proof. For $p \in P$ and $x = D(p)$ let $\mathcal{A}(p)$ be the set of all $x' \in L(p')$ such that the pair $(p, x), (p', x')$ is permissible. We have to prove that $\bigcap_{p \in P} \mathcal{A}(p)$ is nonempty. From Lemma 6.10, $\mathcal{A}(p)$ is a sub-interval of $L(p')$. It is then sufficient to show that for any p^a and p^b in P, $\mathcal{A}(p^a) \cap \mathcal{A}(p^b) \neq \emptyset$, as any collection of pairwise-intersecting intervals has nonempty intersection (an easy consequence of Helly's Theorem, for example see Rockafellar (2006), Corollary 21.3.2).

Thus we fix p^a and p^b in P. From Lemma 6.10, $\mathcal{A}(p^a)$ and $\mathcal{A}(p^b)$ are defined by a set of inequalities, one inequality for each product. We have to show that the intersection of the solution sets for these inequalities is nonempty. Note that two inequalities that correspond to the same products are always simultaneously satisfiable.

Case 1: $p'_1 \le p^a_1$ and $p'_2 \le p^b_2$. Let $y \in \mathbb{R}^2_{++}$ be given by $y_1 = X_1(p^a \wedge p', x^a_2)$ and $y_2 = X_2(p^b \wedge p', x^b_1)$. We have to prove that $L(p') \cap \{x' | x' \le y\}$ is nonempty, or equivalently that $p' \cdot y \geq 1$. Indeed,

$$\begin{aligned} p' \cdot y &= p'_1 \cdot y_1 + p'_2 \cdot y_2 \\ &= (p'_1 \wedge p^a_1) \cdot y_1 + (p'_2 \wedge p^b_2) \cdot y_2 \\ &= 2 - \sum_{(i,j)\in\{(a,2),(b,1)\}} (p'_j \wedge p^i_j) \cdot x^i_j \\ &\geq 2 - \sum_{(i,j)\in\{(a,2),(b,1)\}} (p^j_j \wedge p^i_j) \cdot (x^i_j \vee x^j_j) \\ &= 2 - (p^a \wedge p^b) \cdot (x^a \vee x^b) \\ &\geq 1. \end{aligned}$$

The second equality above follows from the fact that $p'_1 \leq p^a_1$ and $p'_2 \leq p^b_2$. The third equality follows from the fact that $(y_1, x^a_2) \in L(p' \wedge p^a)$, so $(p'_1 \wedge p^a_1) \cdot y_1 = 1 - (p'_2 \wedge p^a_2) \cdot x^a_2$, and similarly for (x^b_1, y_2). The first inequality is because $p'_1 \leq p^a_1$ and $p'_2 \leq p^b_2$. The last inequality is because $(p^a, x^a), (p^b, x^b)$ is permissible.

Case 2: $p'_1 > p^a_1$ and $p' \cdot x^a \leq 1$, while $p'_2 > p^b_2$ and $p' \cdot x^b \leq 1$. Let $y = (x^b_1, x^a_2)$. We have to prove that $L(p') \cap \{x' | x' \geq y\}$ is nonempty. Or, equivalently, that $p' \cdot y \leq 1$. If $y \leq x^a$ or $y \leq x^b$ then we are done. Suppose then that $y \nleq x^a$ and $y \nleq x^b$; hence that $x^a_2 > x^b_2$ and $x^b_1 > x^a_1$. In this case it follows from Claim 6.13 that $p^a_1 > p^b_1$ and $p^a_2 < p^b_2$. Since we assumed that $p'_2 > p^b_2$ it follows that $p'_2 > p^a_2$. Since we assumed that $p'_1 > p^a_1$ it follows that $p' \gg p^a$, which contradicts $p' \cdot x^a \leq 1$ (since $p^a \cdot x^a = 1$).

Case 3: $p'_1 > p^a_1$ and $p' \cdot x^a > 1$, while $p'_2 > p^b_2$ and $p' \cdot x^b > 1$. Let $y = (x^a_1, x^b_2)$. We prove that $L(p') \cap \{x' | x' \leq y\}$ is nonempty. Or, equivalently, that $p' \cdot y \geq 1$. If $y \geq x^a$ or $y \geq x^b$ then we are done. Suppose then that $y \ngeq x^a$ and $y \ngeq x^b$, so that $x^a_2 > x^b_2$ and $x^b_1 > x^a_1$. In this case it follows from Claim 6.13 that $p^a_1 > p^b_1$ and $p^a_2 < p^b_2$. Therefore $p^a \wedge p^b = (p^b_1, p^a_2)$ and $x^a \vee x^b = (x^b_1, x^a_2)$. It follows that

$$p' \cdot y = p'_1 \cdot y_1 + p'_2 \cdot y_2 \geq p^a_1 \cdot x^a_1 + p^b_2 \cdot x^b_2 =$$
$$2 - p^b_1 \cdot x^b_1 - p^a_2 \cdot x^a_2 = 2 - (p^a \wedge p^b) \cdot (x^a \vee x^b) \geq 1,$$

The first inequality follows from the assumption that $p'_1 > p^a_1$ and $p'_2 > p^b_2$. The second equality follows from $p^i \cdot x^i = 1$, $i = a, b$. The last inequality follows from permissibility (Condition 1 in Definition 1).

Case 4: $p'_1 \leq p^a_1$ and $p'_2 > p^b_2$ and $p' \cdot x^b \leq 1$. Note first that Lemma 6.7 implies $p'_1 \leq p^b_1$. We need there to exist $x' \in L(p')$ with $x'_1 \leq X_1(p^a \wedge p', x^a_2)$ and $x'_1 \geq x^b_1$. That is, we need $x^b_1 \leq X_1(p^a \wedge p', x^a_2)$. Or, equivalently, that $(p^a \wedge p') \cdot y \leq 1$ where $y = (x^b_1, x^a_2)$. If $y \leq x^a$ then $(p^a \wedge p') \cdot y \leq p^a \cdot x^a = 1$. If $y \leq x^b$ then $(p^a \wedge p') \cdot y \leq p' \cdot x^b \leq 1$. The only other possibility is that $y > x^a$ and $y > x^b$, so that $x^a_2 > x^b_2$ and $x^b_1 > x^a_1$. In this case it follows in particular from Claim 6.13 that $p^a_2 < p^b_2$. Now, $p'_1 \leq p^b_1$ implies that $p^a \wedge p' \leq p^b$ and therefore $p^a \wedge p' \leq p^a \wedge p^b$. In addition, in this case, $y = x^a \vee x^b$. Therefore

$$(p^a \wedge p') \cdot y \leq (p^a \wedge p^b) \cdot (x^a \vee x^b) \leq 1$$

the last inequality follows from Condition 1 in Definition 1

Case 5: $p'_1 \leq p^a_1$ and $p'_2 > p^b_2$ and $p' \cdot x^b > 1$. Let $y_1 = X_1(p^a \wedge p', x^a_2)$ and $y_2 = x^b_2$. We have to prove that the set $L(p') \cap \{x' | x' \leq y\}$ is nonempty, or equivalently that $p' \cdot y \geq 1$. If $x^a_1 \geq x^b_1$ then $y \geq x^b$ (since $y_1 = X_1(p^a \wedge p', x^a_2) \geq X_1(p^a, x^a_2) = x^a_1$) and, in particular, $p' \cdot y \geq p' \cdot x^b \geq 1$. If $x^b_2 \geq x^a_2$ then $y_2 \geq X_2(p^a \wedge p', y_1)$ (Since, by Lemma 6.4, $X_2(p^a \wedge p', y_1) = x^a_2$) and therefore $p' \cdot y \geq (p^a \wedge p') \cdot y \geq 1$. The only other possibility is that $x^a_2 > x^b_2$ and $x^b_1 > x^a_1$. In this case it follows from Claim 6.13 that $p^a_2 < p^b_2$ and $p^a_1 > p^b_1$. So $p^a \wedge p^b = (p^b_1, p^a_2)$, and, since $p'_2 > p^b_2$, $p'_2 > p^a_2$. Now,

$$p' \cdot y \geq (p'_1, p^b_2) \cdot (y_1, y_2) =$$
$$(p^b_1, p^b_2) \cdot (x^b_1, y_2) + (p'_1, p^a_2) \cdot (y_1, x^a_2) - (p^b_1, p^a_2) \cdot (x^b_1, x^a_2) \geq 1.$$

Where the last inequality follows from the following observations:

$$(p^b_1, p^b_2) \cdot (x^b_1, y_2) = p^b \cdot x^b = 1.$$

$$(p'_1, p^a_2) \cdot (y_1, x^a_2) = (p^a \wedge p') \cdot (y_1, x^a_2) = 1 \text{ since } y_1 = X_1(p^a \wedge p', x^a_2).$$

$$(p^b_1, p^a_2) \cdot (x^b_1, x^a_2) = (p^a \wedge p^b) \cdot (x^a \vee x^b) \leq 1$$

The last equality follows from $(p_1^b, p_2^a) = (p^a \wedge p^b)$, as we established above. The inequality follows from permissibility.

Case 6: $p_1' > p_1^a$ and $p' \cdot x^a \le 1$ and $p_2' > p_2^b$ and $p' \cdot x^b > 1$. We have to prove that $x_2^a \le x_2^b$. Indeed, from Lemma 6.7 it follows that $p_2' \le p_2^a$. Thus $p_2^a > p_2^b$. If $p^a \cdot x^b > 1$ then by Condition 3 of Lemma 6.10 $x_2^a \le x_2^b$, as desired. If $p^a \cdot x^b < 1$ then, since $p_2^a > p_2^b$, it follows from Condition 2 of Definition 1 that $x_1^a \ge x_1^b$. Since $p' \cdot x^a \le 1 < p' \cdot x^b$ it follows from Lemma 6.3 that $x_2^a \le x_2^b$, as desired. ∎

Finally, we complete the proof of Theorem 1. Let P be a finite subset of $\mathbb{R}^2_{++}$ and let $D : P \to \mathbb{R}^2_+$ be a partial demand function that satisfies the conditions of the theorem, i.e. such that the pair $(p, D(p)), (p', D(p'))$ is permissible for every $p, p' \in P$. Let Q be a countable dense subset of $\mathbb{R}^2_{++}$ that contains P. By Lemma 6.14, D can be extended to a function $D : Q \to \mathbb{R}^2_+$ such that for every $p, p' \in Q$ the pair $(p, D(p)), (p', D(p'))$ is permissible.

In particular, by Lemma 6.8, D is monotone on Q. Extend D to $\mathbb{R}^2_{++}$ by defining $\tilde{D}(p) = \bigwedge_{q \in Q, q \le p} D(q)$ for every $p \in \mathbb{R}^2_{++}$. Since D is monotone, it follows that $\tilde{D}(p) = D(p)$ for $p \in Q$ and that $\tilde{D}$ is monotone. Since $p \cdot D(p) = 1$ for $p \in Q$ it follows that $p \cdot \tilde{D}(p) = 1$ for $p \in \mathbb{R}^2_{++}$. That is, for all $q \in Q$, $q \le p$, $q \cdot \tilde{D}(p) \le q \cdot D(q) = 1$, so that in the limit, $p \cdot \tilde{D}(p) \le 1$. If, in fact, $p \cdot \tilde{D}(p) < 1$, then there exists $q \in Q$, $q \le p$ such that $p \cdot D(q) < 1$; from which we conclude that $q \cdot D(q) \le q \cdot D(p) < 1$, a contradiction. .Therefore, $p \cdot \tilde{D}(p) = 1$.

Now, by Lemma 6.15, $\tilde{D}$ is continuous.

Lemma 6.15. *If a demand function satisfies complementarity, then it is continuous.*

Proof. Let $p^* \in \mathbb{R}^2_{++}$ and $\{p^n\}_{n=1}^{\infty} \subseteq \mathbb{R}^2_{++}$ such that $p^n \to p^*$. First consider the case in which for all n, $p^n \le p^*$. In particular, for all n, $D(p^n) \ge D(p^*)$. Let $\varepsilon > 0$; we wish to show that there exists some N such that for all $i = 1, 2$, $n \ge N$ implies $D_i(p^n) < D_i(p^*) + \varepsilon$. Suppose that there exists no such N and without loss of generality suppose that $D_1(p^{n_k}) > D_1(p^*) + \varepsilon$ for some subsequence. The equality

$p_1^{n_k} D_1(p^{n_k}) + p_2^{n_k} D_2(p^{n_k}) = 1$ implies that

$$\begin{aligned} D_2(p^*) \leq D_2(p^{n_k}) &= \frac{1 - p_1^{n_k} D_1(p^{n_k})}{p_2^{n_k}} \\ &< \frac{1 - p_1^{n_k}(D_1(p^*) + \varepsilon)}{p_2^{n_k}}. \end{aligned}$$

Hence, in the limit we have

$$D_2(p^*) \leq \frac{1 - p_1^*(D_1(p^*) + \varepsilon)}{p_2^*}.$$

But then

$$p_1^* D_1(p^*) + p_2^* D_2(p^*) \leq 1 - p_1^* \varepsilon < 1,$$

contradicting that D is a demand function.

A similar argument holds for $p^n \geq p^*$.

Now suppose that p^n is arbitrary. By monotonicity, we have

$$D(p^* \vee p^n) \leq D(p^n) \leq D(p^* \wedge p^n),$$

and as $p^* \vee p^n \to p^*$ and $p^* \wedge p^n \to p^*$, we conclude that $D(p^n) \to D(p^*)$. ∎

It remains to show that $\tilde{D}$ is rationalizable by a monotone increasing utility. We first establish that $\tilde{D}$ satisfies the weak axiom. So, suppose by means of contradiction that there exists p, p' such that $p \cdot \tilde{D}(p') < 1$ and $p' \cdot \tilde{D}(p) \leq 1$. By monotonicity and continuity of $\tilde{D}$, we may therefore find $q \in Q$, $q \ll p'$ such that $p \cdot \tilde{D}(q) < 1$ and $q \cdot \tilde{D}(p) < 1$. By continuity, there exists $q' \in Q$ such that $q' \cdot \tilde{D}(q) < 1$ and $q \cdot \tilde{D}(q') < 1$. However, Lemma 6.9 implies that $\tilde{D}$ satisfies the axiom on Q, a contradiction. The result then follows from Lemma 6.16

Lemma 6.16. *A continuous demand function satisfying the weak axiom of revealed preference is rationalizable by an upper semicontinuous, monotone increasing, utility.*

Remark. Lemma 6.16 is related to Theorem 12 in Richter (1971) and Theorem 1 in Hurwicz and Richter (1971). Both results require a convex-range assumption. In addition, the monotonicity and upper-semicontinuity hold on the range of demand, not necessarily on consumption space. Richter (1971) obtains a strictly increasing utility on the range of demand. We obtain a utility such that $x \gg y$ implies $u(x) > u(y)$ and $x \geq y$ implies $u(x) \geq u(y)$. Note that the demand function $D(p_1, p_2) = \big(1/(p_1 + p_2), 1/(p_1 + p_2)\big)$, which is rationalized by $u(x_1, x_2) = \min(x_1, x_2)$, admits no utility function that is strictly monotone over $\mathbb{R}^2$. On the role of convex range, see our Example 1.

Proof. Let R be the revealed preference binary relation on $\mathbb{R}^2_+$, so $x\, R\, y$ if there is a p with $x = D(p)$ and $p \cdot y \leq 1$. The weak axiom with two commodities implies that R^τ is acyclic (the strong axiom). Let R^τ be the transitive closure of R; . Then R^τ is a strict partial order.

Let $x, x' \in Y$ with $x\, T\, x'$. We claim that there exists $z \in D(\mathbb{Q}^2_{++})$ and a neighborhood U of x' such that $x\, R\, z$ and $z\, R\, y$ for every $y \in U$. In particular, this will demonstrate two facts. Firstly, there exists a countable R^τ-dense set, and secondly, that R^τ is an upper semicontinuous binary relation.

Suppose first that $p \cdot x' < 1$. Then there exists $q \in \mathbb{Q}^2_{++}$ with $p \leq q$ and $q \cdot x' < 1$. Let $z = D(q)$ and U be a neighborhood of x' such that $q \cdot y < 1$ for every $y \in U$. It follows that $z\, R\, y$ for every $y \in U$. Moreover, as $p \leq q$ and $q \cdot z = 1$, $x\, R\, z$.

Secondly, suppose that $p \cdot x' = 1$. Without loss of generality, suppose that $x_1 < x'_1$ and $x'_2 < x_2$. Choose $w = (1/2)x + (1/2)x'$, so $x, w, x' \in L(p)$. Let $\delta > 0$ be such that $B_\delta(x) \cap B_\delta(w) = \emptyset$, where $B_\delta(x)$ denotes the open ball of radius δ and center x. For all $q_1 < p_1$, let $\hat{p}_2(q_1) = (1/w_2)(1 - q_1 w_1)$ (Note that $w_2 > 0$ since $w_2 > x'_2 \geq 0$). So $\hat{p}_2(q_1) > p_2$ and w is the intersection of $L(p)$ and $L(q_1, \hat{p}_2(q_1))$. Note that if $z \in L(q_1, \hat{p}_2(q_1))$ and $z_1 < w_1$ then $p \cdot z < 1$ and if $z \in L(p)$ with $w_1 < z_1$ then $(q_1, \hat{p}_2(q_1)) \cdot z < 1$. Since demand is continuous, and $x = D(p)$, there exists $\epsilon > 0$ such that $\tilde{p} \in B_\epsilon(p)$ implies $D(\tilde{p}) \in B_\delta(x)$. Fix $q_1 \in \mathbb{Q}$, $q_1 > p_1$, such that $(q_1, \hat{p}_2(q_1)) \in B_\epsilon(p)$. Note that by the choice of δ, and since $D(q_1, \hat{p}_2(q_1)) \in B_\delta(x)$, we have that $D(q_1, \hat{p}_2(q_1)) < w_1$. So $p \cdot D(q_1, \hat{p}_2(q_1)) < 1$. Similarly, $(q_1, \hat{p}_2(q_1)) \cdot x' < 1$. Using continuity of demand again, there is a $q_2 \in \mathbb{Q}_{++}$ close enough to $\hat{p}_2(q_1)$ such that $p \cdot D(q_1, q_2) < 1$ and $(q_1, q_2) \cdot x' < 1$. Let U be a neighborhood of x' such that

$(q_1, q_2) \cdot y < 1$ for every $y \in U$. Set $z = D(q_1, q_2)$. Then $x \, R \, z \, R \, y$ for every $y \in U$, as desired.

Therefore R^τ is a partial strict order over $\mathbb{R}^2_{++}$ admitting a countably dense set (namely $D(\mathbb{Q}^2_{++})$) and such that the lower contour sets are open. By Peleg (1970) [5] there exists an upper semi continuous function $u : \mathbb{R}^2_{++} \to \mathbb{R}$ that rationalizes R^τ. Let $v(x) = \max\{u(y) | y \in \mathbb{R}^2_{++} \text{ and } y \leq x\}$ (note that the maximum is achieved since u is upper semi continuous). Then v is monotone and also rationalizes R^τ.

It remains to show that v is upper semi continuous. Let $\alpha \in \mathbb{R}$ and consider the lower contour set $V = \{x \in \mathbb{R}^2_{++} | v(x) < \alpha\}$. We have to show that V is open. Indeed, let $x \in V$, and let $B = \{y \in \mathbb{R}^2_{++} | y \leq x\}$ and $U = \{x \in \mathbb{R}^2_{++} | u(x) < \alpha\}$. Then $B \subseteq U$ by definition of v, and, moreover, B is compact and U is open (since u is upper semi-continuous). Therefore there exists some $\epsilon > 0$ such that $B_{+\epsilon} \subseteq U$ where $B_{+\epsilon} = \{z \in \mathbb{R}^2_{++} | \exists x \in B \ \|x - z\|_\infty < \epsilon\}$ (here ϵ is the $\|\|_\infty$-distance between the compact set B and the closed set U^c). Let $z \in \mathbb{R}^2_{++}$ be such that $\|z - x\|_\infty < \epsilon$. Then $y \in B_{+\epsilon}$ for every $y \leq z$ and therefore $u(y) < \alpha$ for every such y. It follows that $v(z) < \alpha$. This proves that V is open. ∎

Remark. Theorem 2 follows from lemmas 6.15 and 6.16.

[5] Peleg also assumes a condition on the order that he terms spaciousness. Spaciousness is used to guarantee that the utility representation is continuous; but the rest of his theorem guarantees the existence of an upper semicontinuous binary relation as we have here. One can alternatively use the results of Rader (1963) and Jaffray (1975).

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