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A Characterization of Walrasian Economies of Infinity Dimension

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Abstract

We consider a pure exchange economy, where agent's consumption spaces are Banach spaces, goods are contingent in time of states of the world, the utility function of each agent is not necessarily a separable function, but increasing, quasiconcave, and twice Fréchet differentiable over the consumption space. We characterize the set of walrasian equilibria, by the social weight that support each walrasian equilibria. Using technical of the functional analysis, we characterize this set as a Banach manifold and in the next sections we focuses on singularities.

1 Introduction

We consider an economy where agent's consumption space are Banach spaces, agents will be indexed by $i \in I = \{1, 2, ...n\}$; and X_+ will denote the positive cone of the Banach space X. We do not assume separability in the utility functions $u_i : X_+ \rightarrow R$. Utility functions are in the $C^2(X, R)$ space, i.e. in the set of the functions with continuous second F-derivatives, and increasing. We suppose that for all $x \in X$ the inverse operator $(u_i^{''})^{-1}$ of u_i at x, exists. In this work $C^k(X, Y)$ denote the space of $k-times$ continuously F-differentiable operators from X into Y, and $L(X, Y)$ denote the space of linear and continuous operators from X into Y. By $C^\infty(X, Y)$ we denote the set of functions belonging to $C^k(X, Y)$ for all integer k.

Each consumer has the same consumption space and it will be symbolized by X, the cartesian product of the n consumption spaces is represented by: Ω. So, a bundle set for the i-agent will be symbolized by $x_i \in X$ and an allocation will be denoted by $x = (x_1, x_2, ..., x_n) \in \Omega$. The i-agent endowments will be symbolized by w_i, and $w = (w_1, w_2, ...w_n)$. The total mounts of available goods will be denoted by $W = \sum_{i=1}^n w_i$. All of them contingent goods in time or state of the world.

With the purpose to obtain interior equilibria, we will assume that utilities satisfy at least one of the following two, widely used assumptions in economics, conditions:

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(i) (Inada condition) $\lim u_j'(x) = \infty \; if \; x \to \partial(X_+)$, for each $j = 1, 2, ..., l$ and for each utility function, by $\partial(X_+)$, we denote the frontier of the positive cone. It assumes that marginal utility is infinite for consumption at zero.

(ii) All point in the interior of the positive cone X_+, is preferable to all point in the frontier of this cone.

An economy will be represented by

$$\mathcal{E} = \{u_i, w_i, I\}.$$

As examples of economies with the properties above mentioned, consider those where the consumption set is $X_+ = C_{++}(MR^n)$ and utility functions are $u_i(x) = \int_M U_i(x(t), t)dt$, see [Chichilnisky, G. and Zhou, Y.] and [Aliprantis, C.D; Brown, D.J.; Burkinshaw, O.].

It is well known that the demand function is a good tool to deal with the equilibrium manifold in economies in which consumption spaces are subset of Hilbert spaces, in particular R^l [Mas-Colell, A. (1985)].But unfortunately if the consumption spaces are subsets of infinite dimensional spaces (not a Hilbert space), the demand function may not be a differentiable function [Araujo, A. (1987)]. However it is possible to characterize the equilibrium set from the excess utility function, see for instance [Accinelli, E. (1996)]. This is the Negishi approach. Using this approach it is possible to work in infinite dimensional economies with similar techniques than in the finite dimensional case, and to generalize the result obtained by [Chichilnisky, G. and Zhou, Y.] for smooth infinite dimensional economies with no separable utilities, see [Accinelli, E. (1996)].

In this work, following the Negishi approach, we will characterize the equilibrium set of the economy, as the set of zeroes of the excess utility function $e : \Delta \times \Omega \to R^{n-1}$. So, the equilibrium set will be denoted by

$$\mathcal{E}_q = \{(\lambda, w) \in \Delta \times \Omega : e((\lambda, w) = 0\}$$

Where Δ symbolize the social weight set,

$$\Delta = \left\{ \lambda \in R^n : \sum_{i=1}^{n} \lambda_i = 1, \;\; 0 \leq \lambda_i \leq 1; \forall i \right\},$$

and $w = (w_1, w_2, ...w_n)$ are the initial endowments. Our assumptions on the utilities imply that if the agent has no null endowment, the null bundle set will be not a result of his maximization process, then his relative weight can not be zero. Then without loss of generality, we can consider only cases where $\lambda \in \Delta_+ = int[\Delta]$.

In section (3) we prove that $\mathcal{E}_q$ is a Banach manifold.

Next we will focuses on singularities. In this section we will consider economies which utility functions are in $C^{\infty}(XR)$, certainly this is a strong restriction but it is necessary to analyze singularities from the point of view of the smooth analysis.

Singular economies, in contrast with regular economies, characterize the sudden qualitative and unforeseen changes in the economy. More explicitly, regular economies have locally, the same behavior, this means that in a neighborhood of a regular economy there is not big changes, and all economy in this neighborhood is a regular economy too. If the economy is regular, small changes in the distributions of the endowments do not imply big changes in the behavior of the economy as a system, and the new economy will be a regular economy too but, in a neighborhood of a singular economy small changes in the distribution of the endowments usually, imply big changes in the main characteristics of the economy, for instance its number of equilibria. Our object in this section will be to analyze this kind of economies.

An economy will be singular if the zero is a singular value of the excess utility function of this economy, and as the utilities appear explicitly in the excess utility function, the strong relation between the characteristics of the agent preferences, and the behavior of the economy appear clearly reflected in this function. In spite of to be singular economies from a topological or measure theory point of view a very small set, but it play central role in economics. For instance, the existence of multiplicity of equilibria in an economy is a straightforward result of the existence of singularities in the excess utility function, then its existence depend on characteristics of the utility functions.

There are not many works about singular economies in General Equilibrium Theory, Y. Balasko has several works on singularities, in [Balasko, Y. (1988)], [Balasko, Y. 1997a] and also in [Mas-Colell, A. (1985)] there are characterizations of the singular economies, however the General Equilibrium Theory is indebted with singularities. We hope to make a little collaboration in the long way to pay this debt with this paper.

2 Some of notation and mathematical facts

In this section we recalling some basic mathematical definitions that will be used later. Our main reference for considerations on Functional Analysis is [Zeidler, E. (1993)].

Definition 1 *Let $f : Dom(f) \subseteq X \to Y$ be a mapping between two Banach spaces, (B-spaces) X and Y over K, here $Dom(f)$ is the domain of f, and let $f'(x)$ be the Fréchet derivative (F-derivative) at the point x for the map f*

1. $f' : D(f') \subseteq X \to L(X, Y)$ i.e, $f'(x)$ is a continuous linear map from X to Y.

2. f is called a **submersion** at the point x iff f is a C^1-mapping on a neighborhood of x, if $f'(x) : X \to Y$ is surjective and if the null space

$$Ker(f'(x)) = \{x \in X : f'(x) = 0\},$$

splits X. The null space $Y_1 = Ker(f'(x))$ splits X means that $X = Y_1 \oplus Y_2$ (topological direct sum). f is called submersion on the subset $M \subseteq X$ iff f is a submersion at each $x \in M$.

We will denote the image set of a linear operator $T : X \to Y$ by

$$R(T) = \{y \in Y : \textit{ there exists } x \in X : y = T(x)\},$$

the dimension of $R(T)$ will be denoted by $rankT$, and the codimension of $(R(f))$ will be symbolized as $corankT = dim\,[X/ker(T)]$, where $X/ker(T)$ is the factor space.

3. The point $x \in X$ is called a **regular point** of f iff f is a submersion at x. Otherwise x is called **singular point**.

4. The point $y \in Y$ is called a **regular value** of f iff f^{-1} is empty or consists solely of regular points. Otherwise y is called **singular value**.

5. Let X be a Banach space, it follows that $f : U(x_0) \subset X \to R$ has a singular point at x_0 if an only if $f'(x_0) = 0$. Such point will be nondegenerate if and only if the bilinear form $(h, k) \to f''(x_0)hk$ is nondegenerate.

Definition 2 *A function f is called a* **Morse function** *if every critical point is a no degenerate critical point.*

Theorem 1 (Generalized Morse Lemma) *Let X be a Banach space, and let $f : U(x_0) \subset X \to R$ be a smooth function, $x_0 \in X$ is a no degenerate critical point of f Then there exists a local diffeomorphism ψ (in a neighborhood U_{x_0} of x_0) such that:*

$$f(\psi(y)) = f(x_0) + f''(x_0)y^2/2 \tag{1}$$

is satisfy for all $y \in U_p$.

The following global result is shown in [Zeidler, E. (1993)]:

Theorem 2 Preimage Theorem. *Let X and Y be B-spaces over K (real or complex numbers), if y is a regular valued of the C^k- mapping $f : X \to Y$, with $1 \leq k \leq \infty$, the the set M of all solutions of $f(x) = y$ is a C^k- Banach manifold.*

Recall that a linear map $T : X \to Y$ is called a **Fredholm operator** if and only if is continuous and both numbers the dimension of the $ker(T), dim(Ker(T))$ and the codimension of the rank of $f, codim(R(T))$ are finite. The index of f is defined by: $ind(T) = dim(Ker(T)) - codim(R(T))$.

3 The Negishi approach

The Negishi approach start considering a social welfare function given by: $W_\lambda : \Omega_+ \to R$ defined as:

$$W_\lambda(x) = \sum_{i=1}^{n} \lambda_i u_i(x_i). \tag{2}$$

where u_i is the utility function of the agent indexed by i, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in int[\Delta]$ (each λ_i represents the social weight of the agent in the market), and Ω_+ is the positive cone in the consumption space Ω.

As it is well know if $x^* \in \Omega$ solves the maximization problem of $W_{\lambda^*}(x)$ for a given λ^*, subject to be a factible allocation i.e.,

$$x^* \in \mathcal{F} = \left\{ x \in \Omega_+ : \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} w_i \right\}$$

then x^* is a Pareto optimal allocation . Reciprocally it can be proved that if a factible allocation x^*, is Pareto optimal, then there exists any $\lambda^* \in \Delta$ such that x^*, maximize W_{λ^*}, see [Accinelli, E. (1996)] There exists some Pareto optimal allocation where $x_i^* = 0$ for some $i \in \{1, 2, ..., n, \}$ if each agent has positive no null endowments, these cases are possible if and only if the agents indexed in this subset be out of the market, i.e., if and only if $\lambda_i = 0$. Then we can restrict ourselves, without loss of generality, to consider only cases where $\lambda \in \Delta_+$.

In this way characterized the set of Pareto optimal allocations, our next step is to choose the elements x^* in the Pareto optimal set such that can be supported by a price p and satisfying $px^* = pw_i$ for all $i = 1, 2, ..., n$ i.e., an equilibrium allocation.

Suppose that the aggregate endowment of the economy is fixed, call it W, so $\sum_{i=1}^{n} = w$. We will use the following notation

For any $\lambda \in int\,[\Delta] = \{\lambda \in \Delta : \lambda_i > 0 \ \ \forall i \in I\}$,

$$x(\lambda, W) = argmax \left\{ \sum_{i=1}^{n} \lambda_i u_i(x), \ s.t \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i \right\}. \tag{3}$$

and let $e : \Delta \times \Omega \to R^n$ be the excess utility function, which coordinates are given by:

$$e_i(\lambda, W) = u_i'(x_i(\lambda, W))(x_i(\lambda, W)) - w_i).$$

Here $u_i'(x_i(\lambda, W)) : X \to R$ is the F-differential of the utility $u_i(x_i(\lambda, W))$.

Definition 3 *For fixed utility functions, for each $w \in \Omega$ we define the set*

$$\mathcal{E}q(w) = \{\lambda \ \in \ \Delta_+ : e_w(\lambda) = 0\},$$

it will be called the set of the **Equilibrium Social Weights***.*

In [Accinelli, E. (1996)] it is proved that it is a non-empty set.

Theorem 3 *Let $\lambda \in \mathcal{E}q(w)$, and let $x^*(\lambda)$ be a factible allocation, solution of the maximization problem of W_λ and let $\gamma(\lambda)$ be the corresponding vector of Lagrange multipliers. Then, the pair $(x^*(\lambda), \gamma(\lambda))$ is a walrasian equilibrium and reciprocally, if (p, x) is a walrasian equilibrium then, there exists $\bar{\lambda} \in \mathcal{E}q$ such that x maximize $W_{\bar{\lambda}}$ restricted to the factible allocations set, and p will be the corresponding vector of Lagrange multipliers i.e., $p = \gamma(\bar{\lambda})$.*

The proof can be see in [Accinelli, E. (1996)].

4 The equilibrium set as a Banach manifold

The first order conditions for (3) are ;

$$\begin{array}{c} \lambda_i u_i'(x_i(\lambda, W))) = \lambda_h u_h'(x_h(\lambda, W)), \ \forall h \neq i \\ \sum_{i=1}^n x_i(\lambda, w) = W, \end{array} \tag{4}$$

where $W = \sum_{i=1}^n w_i$. It follows that for each i, the consumption of the i-agent, given by the function $x_i : \Delta \times \Omega \to X$ is, for all $\lambda \in int[\Delta]$ and $w \in \Omega$, a F-differentiable function. We denote by $x_{i,\lambda_j}(\lambda, W)$ and $x_{i,w_j}(\lambda, W)$ de partial F-derivatives with respect to the variable λ_j and w_j respectively, $j \in \{1, 2, ..., n\}$.

The following are well know properties of the excess utility function:

(1) $\lambda e(\lambda, w) = 0$.

2) $e(\alpha\lambda, w) = e(\lambda, w), \forall \alpha > 0$.

See for instance [Accinelli, E. (1996)].

From item (1) it follows that the rank of the jacobian matrix $J_\lambda e(\cdot, w)$ of the excess utility function $e(\cdot, w) : \Delta \to R^n$ is at most equal to $n-1$. And as from item (2) we know that if $e_i(\lambda, w) = 0 \ \forall i = 1, 2, ..., n-1$, then $e_n(\lambda, w) = 0$, we will consider the restricted function $\bar{e} : \Delta \times \Omega \to R^{n-1}$. This is the restricted function obtained from the excess utility function removing one of its coordinates.

The following theorem holds:

Theorem 4 *The equilibrium set $\mathcal{E}q$ is a Banach manifold.*

Proof: To prove this theorem, we will prove the following assertions:

(i) There exist a residual set $\Omega_0 \subseteq \Omega$ such that, the mapping $\bar{e} : int[\Delta] \times \Omega_0 \to R^{n-1}$ is C^1, and zero is a regular value of e i.e. for all $(\lambda, w) \in int[\Delta] \times \Omega_0$, such that $e(\lambda, w) = 0$ the mapping $\bar{e}$ is a submersion.

(ii) For each parameter $w \in \Omega_0$, the mapping $\bar{e}(\cdot, w) : int[\Delta] \to R^{n-1}$ is Fredholm of index zero.

(iii) Convergence of $\bar{e}(\lambda_n, w_n) \to 0$ as $n \to \infty$ and convergence of $\{w_n\}$ implies the existence of a convergent subsequence of $\{\lambda_n\}$ in $int[\Delta]$.

- Then the solution set of $e(\lambda, w) = 0, \quad \lambda \in \Delta, w \in \Omega_0$ is a Banach Manifold, and for each $w \in \Omega_0$, the solution of $e(\lambda, w) = 0$ has at most finitely many solutions λ and these are all regular.

Proof of the step (i): Consider the mapping from $int[\Delta] \times \Omega_0 \to R^{n-1}$ defined by the formula:

$$\lambda, w \to \bar{e}(\lambda, w),$$

where $\bar{e}(\lambda, w)$ is the vector (in R^{n-1}) defined by $n-1$ coordinates of the vector $e(\lambda, w)$.

We need to prove that 0 is a regular value of the restricted excess utility function $\bar{e}$. So the restricted excess utility function $\bar{e}$ is a submersion at each point $(\lambda, w) \in \lambda \times \Omega$, i.e, $\bar{e}'(\lambda, w) : int[\Delta] \times \Omega_0 \to R^{n-1}$ is surjective and the null space $Ker(e'(\lambda, w))$ splits X.

To be true that $\bar{e}$ is a submersion all that is required is that the linear tangent mapping is always onto, or equivalently that the rank of the linear map $\bar{e}'$ will be always equal to $n-1$.

The existence of $\frac{\partial x_i}{\partial w_j}$ *and* $\frac{\partial x_i}{\partial \lambda_h}$ is a consequence of the first order conditions (4) and the hypothesis on the utility functions.

The F-derivative of $\bar{e}(\lambda, \cdot)$ can be write as:

$$\frac{\partial\left(e_1, e_2, \ldots, e_{n-1}\right)}{\partial\left(w_1, w_2, \ldots w_n\right)} = \tag{5}$$

$$\left[\begin{array}{cccc} \bigtriangledown e_1 \frac{\partial x_1}{\partial w_1} & \bigtriangledown e_1 \frac{\partial x_1}{\partial w_2} & \ldots & \bigtriangledown e_1 \frac{\partial x_1}{\partial w_n} \\ \bigtriangledown e_2 \frac{\partial x_2}{\partial w_1} & \bigtriangledown e_2 \frac{\partial x_2}{\partial w_2} & \ldots & \bigtriangledown e_2 \frac{\partial x_2}{\partial w_n} \\ \vdots & \vdots & \vdots & \vdots \\ \bigtriangledown e_{n-1} \frac{\partial x_{n-1}}{\partial w_1} & \bigtriangledown e_{n-1} \frac{\partial x_{n-1}}{\partial w_2} & \ldots & \bigtriangledown e_{n-1} \frac{\partial x_{n-1}}{\partial w_n} \end{array}\right] - \left[\begin{array}{cccc} u_1' & 0 & \ldots & 0 \\ 0 & u_2' & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & u_{n-1}' \end{array}\right]$$

Where $\bigtriangledown e_k(\lambda, w) = u_k''(x_k(\lambda, W))\left[x_k(\lambda, W) - w_k\right] - u_k'(x_k(\lambda, W)$.

We will prove that this matrix has rank $n-1$. To see this suppose that we consider a little change in endowment given by $w(v) = w + v$, where $v = (v_1, v_2, ..., v_n) \in R^n$ is a vector in a small open neighborhood U of zero, such that $v_n = \sum_{i=1}^{n-1} v_i$. The vector v will be thought as a state-independent parameter for redistributions of initial endowments. Observe that $\sum_{i=1}^{n} w_i(v) = \sum_{i=1}^{n} w_i = W$.

The excess utility function for the economy $\mathcal{E}(v) = \{u_i, w(v)_i\}$ will be:

$$e(\lambda, v) = \left(e_1(\lambda, v_1), ..., e_n(\lambda, v_n)\right), \tag{6}$$

where

$$e_i(\lambda, v) = u_i(x_i(\lambda, W))[x_i(\lambda, W) - w_i - v_i].$$

Observe that the allocations that solve (3) for the economies $\mathcal{E}(v)$ and $\mathcal{E}$ are the same.

It is easy to see that:

$$\frac{\partial\left(e_1, e_2, \ldots, e_{n-1}\right)}{\partial\left(v_1, v_2, \ldots, v_n\right)} = -\left[\begin{array}{cccc} u_1 & 0 & \ldots & 0 \\ 0 & u_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & u_{n-1} \end{array}\right] \tag{7}$$

The rank of this matrix is equal to $n-1$, the rank of a matrix is locally invariant, this means that there exists a neighborhood U_v such that the rank of $e'(\lambda, w)$ is equal to $n-1$, for all $w \in U_v$. Moreover, for all $w \in \Omega$ there exists a neighborhood o zero, U_0 such that for all $v \in U_0$, the rank of $\bar{e}(\lambda, w(v))$ is equal to $n-1$. This means that generically in w, $\bar{e}(\lambda, w)$ is surjective, this means that the property is true for each $w \in \Omega_0$ a residual set.

To prove that zero is a regular value for e we need to prove that $Ker(e')$ splits. In our case, as $R(e) = R^{n-1}$, the quotient space $(\Delta \times \Omega_0)/Ker(e')$ has finite dimension, then $codim[Ker(e')] < \infty$ and the splitting property is automatically satisfied, see [Zeidler, E. (1993)].

Proof of the step (ii) We will prove that, $\bar{e}(\cdot, w) : \Delta \to R^{n-1}$ is a Fredholm operator of index zero. This map will be a Fredholm operator if is a C^1-map and if $J_\lambda \bar{e}(\cdot, w) : \Delta \to L(\Delta, R^{n-1})$ is a linear Fredholm operator for each $\lambda \in \Delta$. Where $J_\lambda \bar{e}(\cdot, w) : \Delta \to R^{n-1}$ is the jacobian matrix of $e(\cdot, w)$. The index of $J_\lambda \bar{e}(\cdot, w)$ at λ is

$$ind(J_\lambda \bar{e}(\lambda, w)) = dim(Ker(J_\lambda \bar{e}(\lambda, w))) + codim(R(J_\lambda \bar{e}(\lambda, w))).$$

The operator, $(J_\lambda e(\lambda, w))$ is, for each $w \in \Omega_0$ a finite linear operator from $R^{n-1} \to R^{n-1}$ and then, for each $\lambda \in \Delta$ is a Fredholm map of index zero.

The economies $\mathcal{E} = \{w_i, u_i, I\}$ where $w \in \Omega_0$ will be called **Regular Economies**.

Proof of the step (iii) Note that under the assumptions of our model and as $w_i > 0 \ \forall i$, it is enough to consider the social weights in a compact set $\bar{\Delta} \subset int[\Delta]$, so if $e(\lambda_n, w_n) \to 0$ and $\{w_n\}$ is a convergente sequence, then from the continuity of e and compactness of $\bar{\Delta}$ there exists a convergent subsequence of $\{\lambda_n\}$ in $\bar{\Delta}$. Then the set $\mathcal{E}_q$ of solution of $e(\lambda, w) = 0$ is a C^1 Banach manifold [.]

As a corollary of this theorem, it follows that: There is an open dense subset Ω_0 of Ω such that, for each $w \in \Omega_0$ the equation $e(\lambda, w) = 0, \ \lambda \in int[\Delta]$ has at most finitely many solutions λ of $e_w(\lambda) = 0$. Oddiness of this solutions using differential techniques, is proved in [Accinelli, E. (1996)].

5 Singular economies and its properties

In this section utility functions are fixed and we describe each economy by it excess utility function $e : int[\Delta] \times \Omega \to R^{n-1}$. The equilibria of an economy are described by the state variables $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n), \in \mathcal{E}q(w)$ these equilibrium states change when the parameters $w \in \Omega$ change, these parameters are called external or control parameters. Given w the set of λ such that $e(\lambda, w) = e_w(\lambda) = 0$ determine the state of the system, i.e. the equilibrium in which the system rest. The parameters w describe the dependence of the system on external forces, the action of these forces cause changes in the states of the economy. Generically these changes are no so big, and the new state is similar to the previous one, this is because generically economies are regular. Nevertheless in some cases, a sudden transition resulting from a continuous parameter change, can be shown. This kind of changes is referred to as a **catastrophe**. A catastrophe can take place only in a neighborhood of a singular economy.

A state, or equilibrium $\lambda \in \mathcal{E}q(w)$ such that the corank of the jacobian matrix $J_\lambda e_w$ is positive, will be called **singular or critical equilibrium.** Singular economies will be classified in two big classes:

Definition 4 *The set of singular economies such that:*

1. *for all* $\bar{\lambda} \in \mathcal{E}q(w)$ *the* $corank J_\lambda e_w(\bar{\lambda}) \leq 1$ *and with strict inequality for at least one* $\lambda \in \mathcal{E}q(w)$ *This is the set of* **no degenerate singular economies.** *And the states of equilibria corresponding will be called* **critical no degenerate equilibria.**

2. *And the set of all remain singular economies, it will will be called the set of* **degenerate singular economies.** *An equilibrium* $\bar{\lambda} \in \mathcal{E}q'(w)$ *where* $corank J_\lambda e_w(\bar{\lambda}) > 1$, *will be called a* **degenerate critical equilibrium.**

   The corank of $J_\lambda e_w(\bar{\lambda})$ *is given by:*

$$corank\left[J_\lambda e_w(\bar{\lambda})\right] = (n-1) - dim\left[J_\lambda e_w(\bar{\lambda})\right].$$

In this way we can say that the corank is a measure for the degree of the degeneration of the equilibria.

To clarify these considerations and to justify the introduction to the Catastrophe Theory in economics, let us now consider the following two examples:

Example 1 *Let* $E(W) = \{u_i, w_i; i = 1, 2\}$ *be the set of interchange economies which total endowment* $W = (W_1, W_2)$ *are fixed. This means that:*

$$W_j = w_{1j} + w_{2j},\ j = 1, 2;\quad (*)$$

where w_{ij} *is the initial endowment of agent* i *in the commodity* j*. Initial endowment may be redistributed but the total endowment can not be modified, so the components of* W *are constants.*

The equilibrium set will be symbolized by:

$$\mathcal{V}_W = \{(\lambda, w) \in int[\Delta] \times \Omega, : e(\lambda, w) = 0,\ w_{1j} + w_{2j} = W_j; j = 1, 2\}$$

An equilibrium is a pair (λ, w) such that $e_1(\lambda, w) = 0,\ e_2(\lambda, w) = 0$. As in this example the total supply is fixed, to characterize the equilibrium, we can consider, without loss of generality the initial endowments of the only one agent, for instance the agent indexed by 1. And from the fact that social weight are in the sphere of radius 1, it is enough to consider only one component of λ. So, a pair (λ, w) will be an equilibrium if and only if, $e_1(\lambda_1; w_{11}, w_{12}) = 0$.

Suppose that the excess utility function of the agent 1 is given by:

$$e_1(\lambda_1, w_{11}, w_{12}) = 3W_1\lambda_1 - 3w_{11}(\lambda_1)^{\frac{1}{3}} + w_{12}. \tag{8}$$

In terms of catastrophe theory λ_1 is the state variable and w_1 are the control parameters.

The social equilibria of this economy will be given by the set of pairs (λ, w) such that its components $(\lambda_1, w_{11}, w_{12})$ solve the equation $e_1(\lambda_1, w_{11}, w_{12}) = 0$ and by the corresponding $(\lambda_2, w_{21}, w_{22})$ obtained from the former. The set

$$C_F = \{(\lambda_1, w_{11}, w_{12}) \in \mathcal{V}_W : det J_{\lambda_1} e_1(\lambda_1, w_{11}, w_{12}) = 0\},$$

is the **Catastrophe surface.**

The economies whose endowments are in this surface are the singular economies. In our case this surface is defined by:

$$C_F = \left\{(\lambda_1, w_{11}, w_{12}) \in \mathcal{V}_W : \frac{\partial e}{\partial \lambda_1} = 3W_1 - w_{11}\lambda^{-\frac{2}{3}} = 0\right\}.$$

Explicitly:

$$C_F = \left\{\left(\frac{w_{11}}{3W_1}\right)^{\frac{3}{2}}, \ w_{11}, \ \frac{2w_{11}^{\frac{3}{2}}}{(3W_1)^{\frac{1}{2}}}\right\}.$$

The projection of this set in the space of parameters will be called the **Bifurcation set.** In our case:

$$B_F = \left\{w_{11}, \ \frac{2w_{11}^{\frac{3}{2}}}{(3W_1)^{\frac{1}{2}}}\right\}.$$

This set is represented in the space of parameters, w_{11}, w_{12} by a parabola. By varying the parameters continuously, and crossing this parabola, something unusual happens: *the number of possible states of equilibria associated with the initial endowments w change:* increases or decreases by two.

The number of equilibria is given by the sign of δ where:

$$\delta = 27\left(\frac{w_{11}}{W_1}\right)^2 - 4\left(\frac{w_{12}}{W_1}\right)^3$$

so if:

- $\delta < 0$ associate with w, there exist three regular equilibria.
- $\delta > 0$ there is one regular equilibrium associate with w.
- $\delta = 0$, $w_{11}w_{22} \neq 0$ there exists one critical (or singular) equilibrium and one regular equilibrium.

The additional consideration taken from [Balasko, Y. 1997b]: *the set of regular economies with a unique equilibrium is arc connected in the two agents case,* help us to obtain a good geometric representation of economies. Therefore, the set of economies where $\delta > 0$ is an arc-connected set.

The hessian matrix of the considerate excess utility function (the matrix defined by the second order derivatives of e_w at λ) is singular, this means, as we will see later, that the critical equilibrium is degenerate. So economies with endowments which satisfy $\delta = 0$ are degenerate singular economies.

Example 2 *Consider the economy $E = \{u_{\alpha,i}, w_i, R_+l, i = 1, 2\}$ which utility functions are:*

$$\begin{array}{rl} u_{\alpha,1} = & x_{11} - \frac{1}{\alpha} x_{12}^{-\alpha} \\ u_{\alpha,2} = & x_{21} - \frac{1}{\alpha} x_{21}^{-\alpha} x_{22} \end{array},$$

and endowments $W = w_1 + w_2$.

Following the Negishi approach we begin solving the optimization problem:

$$max W_\lambda(x) = \lambda_1 u_1(x_1) + \lambda_2 u_2(x_2),$$

restricted to the factible set:. $\mathcal{F} = \left\{ x \in R^4_+ : \sum_{i=1}^2 x_i \leq \sum_{i=1}^2 w_i \right\}$

Denoting $\lambda_1 = \lambda$ it follows that $\lambda_2 = 1 - \lambda$. Then we write the excess utility function:

$$e_{uw} = \begin{cases} \left(\frac{1-\lambda}{\lambda}\right)^{\frac{\alpha}{1+\alpha}} - \left(\frac{1-\lambda}{\lambda}\right)^{\frac{1}{1+\alpha}} - w_{12}\left(\frac{1-\lambda}{\lambda}\right) + w_{21} \\ \left(\frac{1-\lambda}{\lambda}\right)^{\frac{-\alpha}{1+\alpha}} - \left(\frac{1-\lambda}{\lambda}\right)^{\frac{-1}{1+\alpha}} - w_{21}\left(\frac{1-\lambda}{\lambda}\right)^{-1} + w_{12} \end{cases}$$

The catastrophe surface is given by:

$$C_F = \left\{ (\lambda, w_{11}, w_{12}) \in \mathcal{V}_W : w_{12} = \frac{\alpha}{1+\alpha} h^{\frac{1}{1+\alpha}} - \frac{1}{1+\alpha} h^{\frac{\alpha}{1+\alpha}} \right\}$$

where $h = \frac{\lambda}{1-\lambda}$.

Then economies E, which endowments are given by $(w_{11}, w_{12}, w_{21}, w_{22})$ verifying

$$W = w_1 + w_2$$

and

$$w_{12} = \frac{\alpha}{1+\alpha} h^{\frac{1}{1+\alpha}} - \frac{1}{1+\alpha} h^{\frac{\alpha}{1+\alpha}}$$

are singular. Solving $e_u(\lambda, w) = 0$ it is easy to see that in all neighborhood of this economies there exist economies with one equilibrium and economies with three equilibria.

6 Catastrophe theory and economic theory

The catastrophe theory can be applied with wide generality in quasiestatical models, (models which equilibria states are modified only by cause of external forces) in which little changes in its parameters cause sudden changes. When the system is a rest in a position of equilibrium the state variables, (λ in our case,) determine the state of the system. The parameters, (initial endowments of the economy) describe the dependence of the system on external forces. The action of these forces usually give raise to sudden jump from an equilibrium position to another, these sudden transitions, when originate from continuous modifications in parameters are referred as catastrophes. In General Equilibrium models this kind of transition only can be obeserved in a neigborhood of a singular economy.

Catastrophe theory shows that it is possible to analyze this kind of transition by means of few canonical forms. The behavior of economies which utility functions give place to the same kind of singularities is locally similar, then it is possible to classify the economies according to the the stereotype in correspondence with its singularities.

We start this section considering the most elemental case of economies with two agents. In this case the equilibrium states can be characterized by only one of the components of the excess utility function, for instance $e_i : int[\Delta] \times \Omega \to R$ where i may be equal to 1 or equal to 2, that is a real function. In this case the main theorem to study singularities is the Generalized Morse theorem [Zeidler, E. (1993)]. This theorem states that locally around a no degenerate singular economy all excess utility function can be transformed to a simple standard form by changing coordinates. There are exactly 3 such forms and these are quadratic forms. To each function corresponds exactly one of these canonical forms.

Later more general cases will be considered.

6.1 Two agents economies

Let $\mathcal{E} = \{u_i, w_i; i = 1, 2\}$ be an interchange economy with two agents and l commodities. The property 2 of definition 4, allow us to characterize the economy by one component of it excess utility function as a function of the initial endowments, and property 1 of the same definition, allow us consider only one of the two social weight. Let $e_i : (0, 1) \times \Omega \to R$ be the excess utility function of the agent indexed by i. The function is defined by $(\lambda_i, w) \to e_i(\lambda_i, w)$.

The characterization of no degenerates critical points in terms of the *hessian matrix* (see section (2))is a confortable condition to characterize singular economies:

Remark 1 *A two agents interchange economy w, is a degenerate singular economy if and only if* *the hessian matrix of* e_{wi} :

$$\partial e_{wi} = \left\{ \frac{\partial e_{wi}(\lambda)}{\partial \lambda_h \lambda_k} \right\},\ h, k = 1, 2..., n.$$

is singular for at least one $\lambda \in \mathcal{E}q(w)$.

The significance of Morse's Lemma is in reducing the family of all smooth functions vanishing at the origin ($f(p) = 0$) in R^n with the origin as a no degenerate critical point, to just $n+1$ simple stereotypes.

Applying this theorem in economic setting it follows that, in a neighborhood $U_{\bar{\lambda}}$ of a social equilibrium $\bar{\lambda}$ of a no degenerate singular economy $\bar{w}$, the excess utilities functions $e_{\bar{w}}$ will behave in similar way for every non degenerate $\bar{w}$ with independence of utilities. Moreover, if given the utility function, there are only no degenerates singular economies, then by smooth coordinate transformation it is possible to reduce the family of all excess utility function to just 3 simple stereotypes, namely:

$$e_{\bar{w}i}(\psi(\lambda)) = \pm\lambda_1^2 \pm \lambda_2^2.$$

The following two theorems, help us to know some characteristics of the nondegenerates singular economies

Theorem 5 *Let* $f : X \rightarrow R$ *be a smooth function with a no degenerate critical point* p. *Then there exists a neighborhood* V *of* p *in* X *such that no other critical point of* f *are in* V, *i. e., no degenerate critical points are isolates.*

So, no degenerate critical points are isolates, and if we considere endowments in a finite subset of Ω there are finite number of they. Moreover, generically in Ω, there exists only one λ such that $e_w(\lambda) = 0$ is a critical no degenerate social equilibrium. This follows as a conclusion of the next theorem:

Theorem 6 *Let* X *be a smooth manifold. The set of Morse functions all of whose critical values are distinct (i.e., if* p *and* q *are distinct critical points of* f *in* X, *then* $f(p) \neq f(q)$*) form a residual set in* $C^\infty(X, R)$.

This means that generically, if the economy $\mathcal{E} = \{u_i, w_i, I\}$ is singular nondegenerate, then there exists only one critical equilibrium $\lambda \in \mathcal{E}q(w)$.

Remark 2 (About singularities and oddness in the number of equilibria) *In terms of the economic theory this means that, generically a singular no degenerate economy* w, *with 2*

Figure 1: Two goods two agents economies

agents has only one critical equilibrium. The oddness of the number of equilibria force that in a neighborhood of the singular economy there are economies $\bar{w}$ with only one $\lambda \in \mathcal{E}q(barw)$ and economies $\bar{\bar{w}}$ with three distinct $\lambda \in \mathcal{E}q(w)$

If we add the hypothesis of 2 commodities, the oddness and the arc-connectedness properties of the regular economies with one equilibrium before mentioned, allows us to show the picture as generically representative of the behavior of this kind of economies.

Finally, the economic interpretation of the above considerations is that:

1) Regular economies have a similar behavior around an equilibrium.

2) The excess utility function of all no degenerate singular economy with two agents, have a similar behavior in a neighborhood of a no degenerate critical equilibrium. And this behavior is characterized by a second order polinomial.

The following question is of major importance for a qualitative understanding of many economical (in general scientific) phenomena: *When does the Taylor expansion up to some order k*

$$j_x^k f(u) = f(x) + f(x)u + ... + f^k u^k/k!$$

provide enough information to understand the local behavior of a function f at x? This means: it would be possible to characterize the behavior of an economy for the Taylor expansion of the excess utility function up to some order k? As we shown above, using the Morse lemma it is possible for a no degenerate two-agent economies.

7 Starting a classification: The S_r classification

We begin this section with an important question of the cathastrophe teory, the k-determination of $C^k(X,Y)$functions and then we will related this topic with the the qualitative behavior of the economies in a neighborhood of an equilibrium. We consider economies with an arbitrary but finite number of consumers and then we focus our attention on two kind of singularities: the folds and the cusps. Finally we will connect the kind of the singularities that it can appear in a particular economy with the number of agent and goods that this economy has.

The first question: The question of the $k-$determination of a function is fundamental to catastrophe theory: *when a function f is determined in a neighborhood of a point x by one of its*

Taylor polynomials at x in the sense that every other function having the same Taylor polynomial coincide with f in a neighborhood of x up to a diffeomorphism? Recall that a map $f \in C^k(X, Y)$, is k-equivalent at a point $x_0 \in X$ to a map $g \in C^k(U, V)$ at a point u_0 if and only if there exist local C^k diffeomorphisms at u_0 and $f(x_0)$ respectively with $\phi(u_0) = x_0$ and $\psi(f(x_0) = g(u_0)$. In this case f and g need only be defined in a neighborhood of x_0 and u_0. Where X, Y, U, and V are Banach-manifolds. There is no obvious relationship between this two kind of equivalence.

Recall that a function cannot be determined by its Taylor polynomial in an arbitrary point x. For instance the functions $f : R^2 \to R,\ f(x, y) = x^2$, and $g : R^2 \to R,\ g(x, y) = x^2 - y^{2l}$, have the same $k-$ th polynomial at $0 \in R^2$ when $l > k/2$ holds, but if if $\phi = (\phi_1, \phi_2)$ is any local diffeomorphism at $0 \in R^2$ then:

$$f(\phi(0, y)) = (\phi_1(0, y))^2 \neq -y^{2l} = g(0, y)$$

is true for nonzero $y \in R$ Thus f is not determined by any of its Taylor polynomials.

Definition 5 *We will say that the economy $\mathcal{E} = \{u_i, w_i, i \in I\}$ is k-equivalent at a $\lambda^0 \in \mathcal{E}q(w)$ to the economy $\mathcal{E}' = \{u_i, w_i', i \in I\}$ at $\lambda^1 \in \mathcal{E}q(w)$ if and only if its respective excess utility functions e_w and e_w' are $k-$ equivalent functions at λ^0 and λ^1 .*

1. Let $f : U(p) \subset X \to Y$ be $C^k(X, Y), k \geq 1$ and X and Y Banach manifolds and let $g = j_k^1(f)$ i.e.,

$$g(u) = f(x) + f'(x)u$$

   If f is submersion or inmersion at x then f is $k-$equivalent to g at 0.

2. If $X = R^n$ and $Y = R^m$ and f is a submersion at x then f is k-equivalent at x to g at 0. Moreover, if $rank f'(x) = r$, then f is k-equivalent at x to $h : X \to Y$ with

$$h(x_1, ..., x_n) = (x_1, ..., x_r, 0...0)$$

This means that the excess utility function of all regular economy is $k-determined$, i.e, locally, in a neighborhood of an equilibrium, they have the same qualitative behavior.

A function $f : U(x) \subset X \to R^m$ is called $k-$ determined if and only if for each function $g : U(x) \subset X \to R^m$ with the same Taylor polinomial of degree $k j_p^k f$, there exists a local C^∞ diffeomorphism $\phi \in R^n$ such that $g(\phi(u)) = j_x^k f(u)$ in a neighborhood of x.

Roughly speaking, a function f will be k-determined if all function which differ from f only in terms of order higher than k behave qualitatively like the $k-$th Taylor polinomial of f. This means that the Taylor expansion up to order k completely determines f and its perturbations with terms of order higher than k. So, if the excess utility function of a given economy, e_w is k-determined, then all economy wich excess utility function have the same Taylor polinomial up to order k, show the same qualitative behavior than the former.

We will look at what is called the $k-jet$ of that function at $p \in Dom(f)$, and then we will show some characteristic of the set of singularities of each clase of functions identified in this way.

Definition 6 Jet Bundles: *Let X and Y be n and m dimensional, smooth manifolds and $f,\ g : X \to Y,\ f(x) = g(x) = y$ be smooth functions. Consider the following equivalence relation: $f \sim_k g$ will mean that the $k-th$ Taylor expansion of f coincides with the $k-th$ expansion of g at x. The equivalence class of f at x under this relation is called the* **k-jet of f at** x, *and will be denoted by $J^k(f)_x$.*

- By the symbol $D^h f$ we represent the set of partial derivatives such that: $\frac{\partial^{|h|} f}{\partial x_1^{h_1} ... \partial x_n^{h_n}}$ where $|h| = \sum_{j=1}^{n} h_j,\ h_i \geq 0\ \ i = 1, 2, ..., n$.
- Let $J^k(X, Y)_{x,y}$ denote the set of equivalence classes under $\sim_k$ at p of mapping $f : X \to Y$ where $f(x) = y$.
- An element $\sigma \in\ J^k(X, Y) = \cup_{(x,y) \in X \times Y} J^k(X, Y)_{x,y}$, is called **k-jet** where $f(x) = y$.
- let $J^k(X, Y) = \cup_{(x,y) \in X \times Y} J^k(X, Y)_{x,y}$ (disjoint union). Then $J^k(X, Y)$ is the set of all k-jet with source X and target Y.

Theorem 7 *Let X and Y be smooth manifolds with $n = dimX$ and $m = dimY$. Then, $J^k(X, Y)$ is a smooth manifold with:*

$$dimJ^k(X, Y) = m + n + dim(B^k_{n,m}),$$

where $B^k_{n,m}$ is the space of formed by the direct sum of polynomial in $n-$variables with degree $\leq k$.

The object of our analysis is the excess utility function, and the social equilibria. Obviously its critical values can be other than zero, but our interest is focused at the origin, because only this value have an economical means: The preimagen of zero by e is the set of the social equilibria. Then we are interested in consider the class $J^k(X, Y)_{(\lambda,w),0}$ that is, the $k-$jet σ with source $(\lambda, w) \in X = int[\Delta] \times \Omega$ and target $0 \in Y = R^{n-1}$.

Remark 3 (Notation) *To avoid future possible mistakes arose from the notation, from now on we will represent the jacobian matrix of a mapping f at p by the symbol: $(\partial f)_x$.*

Let $\sigma \in J^1(X,Y)$; then σ defines a unique linear mapping of $T_xX \to T_yY$, where x is the source of σ and y is the target of σ. Let f be a representative of σ in $C^\infty(X,Y)$. Then $(\partial f)_x$ is that linear mapping. Define $rank(\sigma) = rank(\partial f)_p$ and $corank(\sigma) = \mu - rank\sigma$, where $\mu = min(dimX, dimY)$. Let

$$S_r = \left\{\sigma \in J^1(X,Y) : corank(\sigma) = r\right\}.$$

This is the subset of the equivalence classes under $\sim_1$ in $C^\infty(X,Y)$ such that the $corank(\partial f)_p = r$ where p is the source of σ. The subset S_r is a submanifold of $J^1(X,Y)$ with

$$codim\ S_r = (n-\mu+r)(m-\mu+r),$$

see [Golubistki, M. and Guillemin,V.(1973)].

As we said above our interest is the class $\sigma \in J^1(int[\Delta] \times \Omega, R^{n-1})$ with source (λ, w) and target $0 \in Y = R^{n-1}$. It follows that: $dimX = (n-1) + nl$ and $dimY = n-1$, then $\mu = n-1$. So, $codimS_r = (nl + r)r$.

The set of singularities of $f : X \to Y$ where the rank of it jacobian matrix drops by r i.e., the set $x \in X$ where $rank(\partial f)_x = min(dimX, dimY) - r$ is represented by the symbol: $S_r(f) = (j^1 f)^{-1}(S_r)$. Then $S_r(f)$ will be, generically, a manifold of the same codimension that (S_r), [Golubistki, M. and Guillemin,V.(1973)].

As $codimS_r(f) = dimX - dimS_r(f) \geq 0$ there is a relation between the kind of singularities possible for each $f \in C^\infty(X,Y)$ and the dimension of the manifold.

Applying this concepts to economics, $S_r(e)$ is the set of critical points of e where the jacobian matrix of e drops rank by r. This set is a manifold and the set of critical social equilibria is the subset of $(\lambda, w) \in S_r(e) : e(\lambda, w) = 0$. For instance, $S_1(e)$ is the set of no degenerate critical social equilibria that is, the set of pairs $(\lambda, w) \in int[\Delta] \times \Omega$ such that $e(\lambda, w) = 0$.

It follows that, there exists a relation between the number of agents and commodities and the form of possible singularities. In others words, the excess utility function could have only some types of singularities, and these will be determined by the number of commodities and consumer in the economy, to be more precise, the codimension of S_r depend on the number of goods and agents, and its dimension depend only on the agent number but do no depend on the number of goods.

Then, $codimS_r(f) > |dimX - dimY|$, then $dimS_r(f) < dimY$. Applying this observation to economics, where: $X = int[\Delta] \times \Omega,\ \ Y = R^{n-1}$ and f is the excess utility function e, it follows that: if n is the number of consumers of the economy then, $dimS_r(e) < n-1$. In cases where $n = 2$ we obtain that singular economies are generically isolates points in Ω.

It is important to remind that the topology used in theorems about transversality of maps in $C^{\infty}(X,Y)$ is the Withney topology, this is a very strong topology, therefore if a proposition is satisfy generically in a topological space with the Whitney topology, is indeed satisfy in quite large sense and is a strong result.

The next example clarify these considerations:

Example 3 *If the economy have two goods and two consumers we have that* $dimX = 5$, $dimY = 1$ *and* $codimS_r = (4+r)r$ *so, e could have only singularities of kind* S_1 *and* S_0*. Note that if* $r = 1$ *we obtain that critical social equilibria are isolate points.*

Now we will show **some characteristic of S_1 singularities:**

7.1 The Fold and the Cusp in economics

Definition 7 (Submersions with Folds) *Let X and Y be a smooth manifolds with* $dimX \geq dimY$ *Let* $f : X \to Y$ *be a smooth mapping, such that* $J^1 f$ *is transversal to* S_1*. Then a point* $p \in S_1(f)$ *is called fold point if:*

$$T_x S_1(f) + Ker(\partial f)_x = T_x X.$$

Definition 8 *We say that a map is one generic if* $J^1 f$ *is transversal to* S_1*. This is a generic situation. [Golubistki, M. and Guillemin, V.(1973)].*

Where S_1 is the submanifold of $J^1(X,Y)$ of jets of corank 1, then $S_1(f) = (j^1 f)^{-1}(S_1)$ is a submanifold of X with $codimS_1(f) = codim(S_1) = k+1$ where $k = dimX - dimY$. Note that is $x \in S_1(f)$ then $dimKer(\partial f)_x = k+1$. That is, the tangent space to $S_1(f)$ and the kernel of $(\partial f)_x$ have complementary dimensions.

Therefore, $codimS_1(e) = nl+1$ it follows that if $(\lambda, w) \in S_1(e)$ then, $dimKer(\partial e)_{(\lambda,w)} = nl+1$ where n is the number of agents and l the number of commodities.

The next theorem characterize the local behavior of a submersion with folds near a fold, similar to the Morse theorem for real function. (Recall that if X and Y are manifolds, and $f : X \to Y$ is differentiable mapping, with rank$(\partial f)_p$ the maximum possible, is a submersion if $dimX \geq dimY$.)

Theorem 8 *Let* $f : X \to Y$ *be a submersion with folds and let p be in* $S_1(f)$*. Then there exist coordinates* $x_1, x_2, ..., x_n$ *centered at* x_0 *and* $y_1, y_2, ...y_n$ *centered at* $f(p)$ *so that in these coordinates f is given by:*

$$(x_1, x_2, ..., x_n) \to (x_1, x_2, ..., x_{m-1}, x_m^2 \pm ... \pm x_n^2)$$

This theorem is proved in [Golubistki, M. and Guillemin,V.(1973)].

Taking a particularly simple example of 2-manifolds (manifolds with dimension equal 2), we see the reason for the nomenclature fold point. In this case the normal form is given by: $(x_1, x_2) \rightarrow (x_1, x_2^2)$. This transformation is obtained by means of the following steps:

1) Map the (x_1, x_2) map onto the parabolic cylinder, (x_1, x_2, x_2^2),

2) then, project onto the (x_1, x_3) plane.

An example: 3-agent economies:

Let X and Y be 2-manifolds and let $f : X \rightarrow Y$ be a one generic mapping. By our computation $codim S_1(f) = 1$ in X, and S_2 does not occur, since it would to have codimension 4. Let p be a point in $S_1(f)$ and $q = f(p)$. One of the following two situations can occur:

(a) $T_p S_1(f) \oplus Ker(\partial f)_p = T_p X$.

(b) $T_p S_1(f) = Ker(\partial f)_p$

Remark 4 *Whitney proved that if f belongs to $C^\infty(X, Y)$ generically the only singularities are folds and simple cusp.*

Note that only if the interchange economy has 3 agents and fixed initial endowment the excess utility function is a mapping between 2-manifolds, $e_w : int[\Delta] \rightarrow R^2$.

Let $\bar{\lambda} = (\bar{\lambda_1}, \bar{\lambda_2}, \bar{\lambda_3}) \in \Delta$ be a singular social equilibrium for the economy w.

i) In the first case (item (a)) applying 8 one can choose a system of coordinates (λ_1, λ_2) centered at $(\bar{\lambda 1}, \bar{\lambda 2}) \in S_1(e_w)$ and (e_1, e_2) centered at $e_w(\bar{\lambda}) = 0$ such that e_w is a fold: $(\lambda_1, \lambda_2) \rightarrow (\lambda_1, \lambda_2^2)$.

ii) If (b) holds the situation is considerable more complicated. Generically singularities, in this case, are simple cusps. In this case one can find coordinates $(\bar{\lambda_1}, \bar{\lambda_2})$ centered at $e(\bar{\lambda})$ such that:

$$(\bar{\lambda}_1, \bar{\lambda}_2) \rightarrow (\bar{\lambda}_1, \bar{\lambda}_1 \bar{\lambda}_2 + \bar{\lambda}_2^3).$$

In a neighborhood of a cusp or a fold there exist regular economies with different number of equilibrium. Recall that, at the moment of through a singularity the changes in the number of equilibria appear.

7.2 The $S_{r,s}$ singularities

Let $f : X \to Y$ be one generic. We will denote by $S_{r,s}(f)$ the set of points where the map $f : S_r(f) \to Y$ drops by rank s. Analogous to the S_r it si possible to build:

$$S_{r,s} \subset \{\sigma \in J^2(X,Y) : corank(\sigma) = r\}.$$

Note that $x \in S_{r,s}(f)$ if and only if $x \in S_r(f)$ and the kernel of $(\partial f)_x$ intersects the tangent space to $S_r(f)$ in a subspace a s dimensional subspace. From $dimS_r(e) < n-1$ it follows that in cases of economies where $n = 2$ the singularities are $S_{1,0}(e)$ folds, or $S_{1,1}(e)$ cusps.

Using the Transversality Theorem in [Golubistki, M. and Guillemin,V.(1973)] is proved that j^2f is generically transversal to $S_{r,s}$ and then the sets $S_{r,s}$ are submanifolds in $J^2(X,Y)$ and like in the case of $S_r(f)$,

$$S_{r,s}(f) = (j^2(f))^{-1}(S_{r,s}).$$

In the cited work the dimension of $S_{r,s}(f)$ is computed.

Generically $S_{r,s}(f)$ are submanifolds in X whose dimensions are given by:

$$dimS_{r,s}(f) = dimX - r^2 - \mu r - \left(codimS_{r,s}(f)\ in\ S_r(f)\right). \tag{9}$$

where

$$codimS_{r,s} = \frac{m}{2}x(k+1) - \frac{m}{2}(k-s)(k-s+1) - s(k-s), \tag{10}$$

where

- $m = dimY - dimX + k$
- $k = r + max(dimX - dimY, 0)$.

[Golubistki, M. and Guillemin,V.(1973)]

In this way we see that the set of possible singularities in economics are strongly relate with the number of agents and commodities, then it follows that some kind of singularities appear only if the number of agents are big enough.

7.3 Singularities and its relations with the dimension of the economy

Applying to economies with a finite number of agents and commodities, we obtain that:

- $k = r + nl$ and $m = r$

- where n is the number of agents, l is the number of commodities an r is the codimension of (∂e) in the singularity.

So, generically, we obtain substituting in (10) that:

$$codimS_{r,s}(e) = (r + nl)(rs - s) - \frac{rs}{2} + \frac{rs}{2} - \frac{rs^2}{2} + s^2.$$

Substituting $k = nl + r$ in (10) it follows:

$$codimS_{r,s}(e) = nl(rs - s) + r^2 s - \frac{rs^2}{2} + \frac{rs}{2} + s^2.$$

From (9) we obtain that

$$dimS_{r,s}(e) = nl(1 - r - rs + s) + (n - 1) + rs\left(-r + \frac{s}{2} - \frac{1}{2}\right) - s^2 - r^2.$$

In particular for $S_{1,1}$ it follows that $codimS_{11} = 2$ and $dimS_{11} = n - 2$ holds.

From (9) and (10) it follows that generically, singularities like$S_{1,2}(e)$ only could appear if the number of consumer is $n > 4$, because $n > 4$ is a necessary condition to be $dimS_{1,2}(e) \geq 0$

8 Conclusions

The introduction of the excess utility function in equilibrium analysis allow us to work in infinite dimensional economies in a similar way than in finite dimensional models and, on the other hand, to improve our understanding of the way in which equilibria depend upon economic parameters (initial endowments) and shows the strong relation existing between utilities and the behavior of the economic system.

This function reflects the weight of consumers in the markets, and show the changes in their relative weights when the initial endowments change. Near a regular economy these changes are smooth and there is not qualitative changes, but around a singularity sudden and big changes occur. The economic weight of the agents change drastically, overthrowing the existent order. The uncertainty in the behavior of the economy is a direct result of the existence of singular economies. If there would not be singularities, economics would be a science with perfect prediction, without sense and perfectly bored

Nevertheless, most part of the literature in economics have focused on regular economies whose equilibria change smoothly according to the changes in the endowments. The study of the discontinuous behavior requires to consider singularities, this led us to the catastrophe theory. This theory refers to drastic changes, however to be sudden, abrupt and unexpected the catastrophe theory show that these changes have a similar substratum that allows us to do a classification according its geometric representation. So, the study of singularities require catastrophe theory and the theory of mapping and their singularities, in this way one might have an approximation to understanding the forms of the unexpected changes in economics.

A final consideration: The excess utility function allows us to extend the analysis of singularities for economies with finite dimensional consumption spaces, to infinite dimensional economies. Then also in these cases, the catastrophe theory may be a gate to begin to understand the behavior of an economical system with infinitely many goods in a neighborhood of a singularity.

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