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EXPLICIT FORMULAS FOR THE MINIMAL VARIANCE HEDGING STRATEGY IN A MARTINGALE CASE

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Explicit formulas for the minimal variance hedging strategy in a martingale case

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Abstract

We explicitly compute the optimal strategy in discrete time for a European option and the variance of the corresponding hedging error under the hypothesis that the underlying is a martingale following a Geometric Brownian motion.

1 Introduction

This paper is devoted to the computation of an explicit formula for the optimal hedging strategy, and its associated variance, for a given contingent claim when the underlying asset follows a geometric Brownian motion with time-varying volatility, the trading is restricted to a given set of dates and the objective is to minimize the variance of the total hedging error. Figlewski [4], in an accurate study on the practical consequences of the most important assumptions underlying the Black-Scholes model, computed by simulation the sample variances on different cases of delta hedging strategies, concluding that: "It is apparent that, simply by rebalancing discretely instead of continuously, we have departed markedly from the theoretical world of Black-Scholes".

We are concerned with the classical problem of minimizing the hedging risk in an incomplete market, proposed in a seminal paper by Föllmer and Sondermann [5]. Schweizer [12] contains a review of the main results and contributions. The main reference for the problem in discrete time is Schweizer [11]. That paper shows that, under a *non-degeneracy* condition for the underlying process, there exists a unique solution and proposes a characterization of the optimal strategy and its variance. Although the problem has been theoretically solved, the effective computation of the optimal strategy and of the minimal variance is usually quite burdensome.

Because of the practical importance of the problem, approximating formulas to compute the variance of a delta-hedging strategy have been proposed, for example, by Kamal and Derman [8], by Toft [13] and, more recently, by Hayashi and Mykland [7]. Such formulas measure the discretization risk of a hedging strategy and can be used by a trader to correct the price (bid or ask) of a derivative. Their major drawback is that they are all asymptotical, i.e. they work better as the number of trading dates increases, that is exactly when the discretization risk vanishes.

Another stream of studies mostly devoted to practical applications is concerned with the actual computation of optimal trading strategies under specific modeling assumptions. Some, like Bertsimas et al. [2] or Primbs and Yamada [9], propose algorithms based on backward induction. Others, like Hubalek et al. [6] and Černý [3], determine the Laplace transforms of the optimal strategy and of its associated variance for a rather general class of models and claims. Laplace transforms must be numerically inverted to recover the required quantities. With a similar methodology and in the same setting, Angelini and Herzel [1] determine the Laplace transform of the variance of the error produced by a standard delta hedging strategy.

In this paper we derive exact formulas for the optimal hedge ratio and its variance for a specific model. To the best of our knowledge, such quantities are usually computed through some kind of numerical algorithms, and this is the first study where closed form formulas are presented. We believe that our results can be of general interest for several reasons: the case considered (an extension of the Black-Scholes model with time-varying volatility) is widely used in many applications; the formulas are easy to implement, hence the optimal hedge ratio can be employed as a valid substitute to the standard Black-Scholes delta; the knowledge of the variance of the total error can be useful for measuring and managing the hedging risk. Moreover, our simple results can serve as a benchmark to check the accuracy, in a particular case, of the approximating formulas and the numerical algorithms valid for more general settings. More details and examples of applications are provided in Section 4.

Our results hold under the strong assumption that the underlying process is a martingale, namely that the drift of the process is zero. As a first consequence, the computed strategy, which, in general, is only locally optimal, is, in this case, also globally optimal. More importantly, the value process of the optimal portfolio is also a martingale and, since the underlying follows a geometric Brownian motion, it may be readily computed. When the drift is not zero the value of the optimal portfolio is a martingale only with respect to the so called *minimal martingale measure*, and it cannot be determined in closed form. In this case one has to turn to numerical algorithms, like the one proposed by Hubalek et al. [6], via the inversion of a Laplace transform, or by Bertsimas et al. [2], via a recursive procedure involving numerical approximation of expectations. While the assumption of a zero drift could be considered as unrealistic, it is well known that statistical estimates of the drift are less reliable than those of the volatility. Hence it is a common practice, at least for short time horizons, to set the drift equal to zero, falling in the case here considered.

The rest of the paper is organized as follows: Section 2 sets the problem and gives a brief overview of the main theoretical results. In Section 3 we derive the closed formulas for the hedging strategy and its variance for our specific setting. Section 4 analyzes the results and shows some of the possible applications.

2 The optimal hedging strategy

We consider the problem of hedging a European option with maturity T and strike price K written on a underlying asset or index S. The payoff of the option is indicated by H. We assume that trading takes place only on a finite set of times $\{t_1, \dots, t_{N-1}\}$ between time $t_0 = 0$ and time $t_N = T$ and that the price process S is a martingale.

We suppose that there exists a riskless asset and, without loss of generality, that the risk-free rate r is zero. In fact, the results obtained will still hold after a change of numeraire, by substituting all the prices entering in the formulas (including the strike price) by their discounted values.

Let V_k be the value at time t_k of a portfolio composed by the underlying and the riskless asset and let ξ_k, $k = 1, \dots, N$, be the units of asset S held from time t_{k-1} up to time t_k. A trading strategy is defined by the two dimensional process $(V_k, \xi_{k+1}), k = 0, \dots, N-1$ and by its terminal value V_N. The cumulative cost C_k necessary to follow the strategy up to time t_k is given by the difference between the value V_k and the cumulative trading gain

$$C_k = V_k - \sum_{i=1}^{k} \xi_k \Delta S_k,$$

for $k = 1, \dots, N$, where $\Delta S_k = S_k - S_{k-1}$. We denote by C_0 the initial cost of the strategy so that $V_0 = C_0$. A strategy is *self-financing* if the cumulative cost process C is constant. It is *mean self-financing* if the process C is a martingale.

There are several alternatives available to a trader who wants to hedge the risk of the contingent claim when perfect replication with a self-financing strategy is not possible. One possibility is to determine a strategy with final value $V_N = H$ and such that the local costs $C_{k+1} - C_k$ are minimized in the mean square sense, that is

$$\min_{\xi_k, V_{k-1}} E_{k-1}[(C_k - C_{k-1})^2] \tag{2.1}$$

where E_{k-1}, $k = 1, \dots, N$, denotes expectation conditional to the information available at time t_{k-1}. This is the approach proposed in the seminal paper of Föllmer and Sondermann [5].

For each time t_k, the objective function is equal to

$$E_{k-1}\left[(V_k - V_{k-1} - \xi_k \Delta S_k)^2\right] = \mathrm{var}_{k-1}(V_k - \xi_k \Delta S_k) + E_{k-1}\left[V_k - \xi_k \Delta S_k - V_{k-1}\right]^2$$

Note that the first term does not depend on V_{k-1}. Therefore it is optimal to choose V_{k-1} so that

$$V_{k-1} = E_{k-1}\left[V_k - \xi_k \Delta S_k\right]. \tag{2.2}$$

This implies that the cumulative cost process of an optimal strategy for (2.1) is *mean self-financing*. Moreover, since the price process S is a martingale, the optimal value process V is also a martingale; in particular $V_{k-1} = E_{k-1}[H]$.

Therefore the optimal solution to (2.1) is given by (2.2) and by

$$\xi_k = \frac{\mathrm{cov}_{k-1}(V_k, \Delta S_k)}{\mathrm{var}_{k-1}(\Delta S_k)} = \frac{E_{k-1}[V_k \Delta S_k]}{E_{k-1}[\Delta S_k^2]}. \tag{2.3}$$

Hence the optimal value is

$$\min_{\xi_k} \mathrm{var}_{k-1}(V_k - \xi_k \Delta S_k) = \mathrm{var}_{k-1}(V_k) - \xi_k^2 E_{k-1}\left[\Delta S_k^2\right], \tag{2.4}$$

with ξ_k defined in (2.3). Expression (2.4) follows from

$$E_{k-1}\left[\xi_k V_k \Delta S_k\right] = \xi_k^2 E_{k-1}[\Delta S_k^2]$$

and the fact that V is a martingale. It is remarkable that the optimal strategy does not depend on the initial capital C_0 invested. Such a strategy may be determined by backward recursion, starting from the terminal value $V_N = H$.

A different approach to the problem tries to determine the self-financing strategy that minimizes the second moment of the final shortfall $E_0[(H - V_N)^2]$. Schweizer [11] shows that such a problem admits a unique solution when the process S satisfies a rather general "non-degeneracy" condition. In the martingale case the two approaches are equivalent, that is the optimal strategy, self-financing strategy is still given by (2.2) and by (2.3) [1]. The computation of the optimal strategy and of the minimal variance is usually a non trivial task. Hubalek et al. [6] show how to compute them by using a methodology based on the Laplace transform. Their approach, although not widely general, covers a number of important cases. In the next section we will study in detail a particular case where explicit computations are indeed possible.

[1] For a self-financing strategy it is necessary to invest also in the riskless asset. The units of the riskless asset are determined from relations (2.2) and (2.3).

3 Computing the strategy and its variance

In this section we explicitly compute the optimal hedging strategy in the case of European call option and the variance of the corresponding hedging error. Analogous computations can be made in the case of a put option.

We suppose that the price S_k of the underlying at t_k is the martingale process

$$S_k = S_{k-1} \exp\left(-\frac{1}{2}\sigma_{k-1,k}^2(t_k - t_{k-1}) + \sigma_{k-1,k}\sqrt{t_k - t_{k-1}} Z_k\right), \tag{3.5}$$

where $\sigma_{0,1}, \ldots, \sigma_{N-1,N}$ are deterministic parameters and Z_k are i.i.d standard gaussian variables. Prices S_k can be interpreted as discrete observations of the continuous process S

$$dS_t = \sigma(t) S_t dW_t,$$

with W standard Brownian motion and $\sigma(t)$ a deterministic function. In this case the volatility parameters of the discrete process would be given by

$$\sigma_{k-1,k}^2 = \frac{1}{(t_k - t_{k-1})} \int_{t_{k-1}}^{t_k} \sigma(u)^2 du.$$

The first crucial point of the computation is that, since S is a martingale, the value of the optimal portfolio V_k is a martingale too. In particular, $V_k = E_k[H]$. The second relevant ingredient is that, since the underlying is log-normal, V_k, for $k = 1, \ldots, N-1$, is given by the Black-Scholes formula.

Let us denote the Black-Scholes formula at time t

$$V(t, s, \sigma) = sN\left(d_1(t, s, \sigma)\right) - KN\left(d_2(t, s, \sigma)\right),$$

where $N(\cdot)$ is the standard normal distribution function,

$$d_1(t, s, \sigma) = \frac{\log(s/K) + \frac{1}{2}\sigma^2(T - t)}{\sigma\sqrt{T - t}}$$

and

$$d_2(t, s, \sigma) = d_1(t, s, \sigma) - \sigma\sqrt{T - t}.$$

Hence,

$$V_k = V(t_k, S_k, \sigma_{k,N}) = E_k[\max\{S_N - K, 0\}], \tag{3.6}$$

where

$$\sigma_{l,m}^2 = \frac{1}{(t_m - t_l)} \sum_{j=l+1}^{m} \sigma_{j-1,j}^2 (t_j - t_{j-1}),$$

for $l = 0, \ldots, N-1$ and $m = l+1, \ldots, N$. We set $\sigma_{l,l} = 0$, for $l = 0, \ldots, N$. If the volatility of the underlying process is constant, $\sigma_{k-1,k} = \sigma$ for all $k = 1, \ldots, N$, then $\sigma_{l,m} = \sigma$ for all l and all $m > l$.

Before stating the main results, we define, for $l = 0, \ldots, N-1$ and $m \geq l$, the quantity

$$\begin{aligned} & A_{l,m}(s_1, s_2) \\ &= s_1 s_2 e^{\sigma_{l,m}^2 (t_m - t_l)} N \Big(d_1(t_l, s_1, \sigma_{l,N}) + \rho_{l,m} \sigma_{l,N} \sqrt{T - t_l}, \\ & d_1(t_l, s_2, \sigma_{l,N}) + \rho_{l,m} \sigma_{l,N} \sqrt{T - t_l}, \rho_{l,m} \Big) \\ & -K s_1 N \Big(d_1(t_l, s_1, \sigma_{l,N}), d_2(t_l, s_2, \sigma_{l,N}) + \rho_{l,m} \sigma_{l,N} \sqrt{T - t_l}, \rho_{l,m} \Big) \\ & -K s_2 N \Big(d_1(t_l, s_2, \sigma_{l,N}), d_2(t_l, s_1, \sigma_{l,N}) + \rho_{l,m} \sigma_{l,N} \sqrt{T - t_l}, \rho_{l,m} \Big) \\ & +K^2 N \left(d_2(t_l, s_1, \sigma_{l,N}), d_2(t_l, s_2, \sigma_{l,N}), \rho_{l,m} \right), \end{aligned} \tag{3.7}$$

where $N(x_1, x_2, \rho)$ is the cumulative distribution function of the bivariate normal variable with correlation ρ and $\rho_{l,m} = \frac{\sigma_{l,m}^2 (t_m - t_l)}{\sigma_{l,N}^2 (T - t_l)}$. Note that when $l = m$ we have $\rho_{l,l} = 0$ because we defined $\sigma_{l,l} = 0$, and the cumulative distribution function of the bivariate normal is, in this case, the product of the cumulative distribution functions of two univariate standard normal. Hence

$$A_{l,l}(s_1, s_2) = V(t_l, s_1, \sigma_{l,N}) V(t_l, s_2, \sigma_{l,N}).$$

Now we can state the main results of the paper. The first one provides an expression for the optimal hedge and the minimal local variance.

Proposition 3.1 *Consider a European call option with maturity T and strike K. Then the optimal hedge at time t_{k-1}, for any $k = 1, \ldots, N$, is*

$$\xi_k = \frac{V\left(t_{k-1}, S_{k-1} e^{\sigma_{k-1,k}^2 (t_k - t_{k-1})}, \sigma_{k-1,N}\right) - V(t_{k-1}, S_{k-1}, \sigma_{k-1,N})}{S_{k-1} (e^{\sigma_{k-1,k}^2 (t_k - t_{k-1})} - 1)} \tag{3.8}$$

and the optimal local variance is

$$\min_{\xi_k} \operatorname{var}_{k-1}(V_k - \xi_k \Delta S_k) = A_{k-1,k}(S_{k-1}, S_{k-1}) - V_{k-1}^2 - \xi_k^2 S_{k-1}^2 \left(e^{\sigma_{k-1,k}^2 (t_k - t_{k-1})} - 1 \right). \tag{3.9}$$

The second result provides an expression for the variance of the shortfall up to time t_n of the optimal self-financing strategy.

Proposition 3.2 *Consider a European call option with maturity T and strike K. Then the expected value and the variance of the hedging error up to time t_n, for any $n = 1, \ldots, N$, corresponding to the optimal strategy are given by*

$$E_0\left[V_n - C_0 - \sum_{k=1}^{n} \xi_k \Delta S_k\right] = V_0 - C_0 = 0; \tag{3.10}$$

$$\begin{aligned}
&\mathrm{var}_0\left(V_n - \sum_{k=1}^{n} \xi_k \Delta S_k\right) = A_{0,n}(S_0, S_0) - V_0^2 - \sum_{k=1}^{n} \frac{1}{e^{\sigma^2_{k-1,k}(t_k - t_{k-1})} - 1} \\
&\times \left\{A_{0,k-1}(S_0, S_0) + A_{0,k-1}\left(S_0 e^{\sigma^2_{k-1,k}(t_k - t_{k-1})}, S_0 e^{\sigma^2_{k-1,k}(t_k - t_{k-1})}\right)\right. \\
&\left. -2A_{0,k-1}\left(S_0, S_0 e^{\sigma^2_{k-1,k}(t_k - t_{k-1})}\right)\right\}.
\end{aligned} \tag{3.11}$$

The case of major interest is when $n = N$ so that $V_n = H$. Nevertheless, the variance of the strategy up to a certain time may also turn out to be useful, for instance, to monitor the running costs and check the ongoing performances of a strategy. For instance, one could derive a confidence interval for the hedged position at time t_1, next re-hedging date. If at that time the realized shortfall $V_1 - C_0 - \xi_1 \Delta S_1$ falls within the confidence interval, then the strategy is behaving as predicted by the model, otherwise one should probably start questioning some or all of the modeling assumptions.

The proof of both propositions are by direct computation. The most tedious computations are relegated in the following

Lemma 3.1 *Under the hypotheses of Proposition 3.1 and 3.2, for any $l = 0, \ldots, N-1$ and $m \geq l$, we have*

$$E_l[S_m V(t_m, S_m, \sigma_{m,N})] = S_l V\left(t_l, S_l e^{\sigma^2_{l,m}(t_m - t_l)}, \sigma_{l,N}\right); \tag{3.12}$$

$$E_l\left[V\left(t_m, S_m e^x, \sigma_{m,N}\right) V\left(t_m, S_m e^y, \sigma_{m,N}\right)\right] = A_{l,m}(S_l e^x, S_l e^y). \tag{3.13}$$

For convenience of the reader, we will indicate all the steps involved in the computations. The major ingredients for proving Lemma 3.1 are the following integrals (see Toft [13]):

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2+Bz)} N\left(Dz+E\right) dz \\
&= e^{\frac{1}{8}B^2} N\left(\sqrt{\frac{1}{1+D^2}}\left(E-\frac{BD}{2}\right)\right); \qquad (3.14) \\
&\int_{-\bar{z}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2+Bz)} dz \\
&= e^{\frac{1}{8}B^2} N\left(-\bar{z}-\frac{B}{2}\right); \qquad (3.15) \\
&\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2+Bz)} N\left(Dz+E\right) N\left(Dz+G\right) dz \\
&= e^{\frac{1}{8}B^2} N\left(\sqrt{\frac{1}{1+D^2}}\left(E-\frac{BD}{2}\right),\right. \\
&\left.\sqrt{\frac{1}{1+D^2}}\left(G-\frac{BD}{2}\right), \frac{D^2}{1+D^2}\right). \qquad (3.16)
\end{aligned}$$

Equation 3.15 is just a re-arrangement of Formula (48) in [13] obtained by completion of the square.

Proof of Lemma 3.1. To prove (3.12) for $m < N$, because of (3.6), we have to show that:

$$E_l[S_m^2 N\left(d_1(t_m, S_m, \sigma_{m,N})\right)] = S_l^2 e^{\sigma_{l,m}^2(t_m-t_l)} N\left(d_1(t_l, S_l e^{\sigma_{l,m}^2(t_m-t_l)}, \sigma_{l,N})\right) \qquad (3.17)$$

and

$$E_l[S_m N\left(d_2(t_m, S_m, \sigma_{m,N})\right)] = S_l N\left(d_2(t_l, S_l e^{\sigma_{l,m}^2(t_m-t_l)}, \sigma_{l,N})\right). \qquad (3.18)$$

To prove (3.17) we write,

$$
\begin{aligned}
& d_1(t_m, S_m, \sigma_{m,N}) \\
&= \frac{\log(\frac{S_l}{K}) - \frac{1}{2}\sigma_{l,m}^2(t_m - t_l) + \sigma_{l,m}\sqrt{t_m - t_l}Z + \frac{1}{2}\sigma_{m,N}^2(T - t_m)}{\sigma_{m,N}\sqrt{T - t_m}} \\
&= \frac{\sigma_{l,m}\sqrt{t_m - t_l}}{\sigma_{m,N}\sqrt{T - t_m}}Z + \frac{\log(\frac{S_l}{K}) - \frac{1}{2}\sigma_{l,m}^2(t_m - t_l) + \frac{1}{2}\sigma_{m,N}^2(T - t_m)}{\sigma_{m,N}\sqrt{T - t_m}} \\
&= \frac{\sigma_{l,m}\sqrt{t_m - t_l}}{\sigma_{m,N}\sqrt{T - t_m}}Z + \frac{\log(\frac{S_l}{K}) - \sigma_{l,m}^2(t_m - t_l) + \frac{1}{2}\sigma_{l,N}^2(T - t_l)}{\sigma_{m,N}\sqrt{T - t_m}},
\end{aligned}
$$

where Z is a standard normal variable and the equalities are intended in distribution. So that, setting

$$D = \frac{\sigma_{l,m}\sqrt{t_m - t_l}}{\sigma_{m,N}\sqrt{T - t_m}} \tag{3.19}$$

and

$$E = \frac{\log(\frac{S_l}{K}) - \sigma_{l,m}^2(t_m - t_l) + \frac{1}{2}\sigma_{l,N}^2(T - t_l)}{\sigma_{m,N}\sqrt{T - t_m}}, \tag{3.20}$$

we get

$$
\begin{aligned}
& E_l[S_m^2 N\left(d_1(t_m, S_m, \sigma_{m,N})\right)] \\
&= S_l^2 e^{-\sigma_{l,m}^2(t_m - t_l)} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 4\sigma_{l,m}\sqrt{t_m - t_l}z)} N\left(Dz + E\right) dz.
\end{aligned}
$$

Hence we can apply Equation (3.14) with $B = -4\sigma_{l,m}\sqrt{t_m - t_l}$ to get

$$E_l[S_m^2 N\left(d_1(t_m, S_m, \sigma_{m,N})\right)] = S_l^2 e^{\sigma_{l,m}^2(t_m - t_l)} N\left(\sqrt{\frac{1}{1 + D^2}}\left(E - \frac{BD}{2}\right)\right).$$

To conclude we have

$$
\begin{aligned}
\sqrt{\frac{1}{1 + D^2}}\left(E - \frac{BD}{2}\right) &= \frac{\log(\frac{S_l}{K}) + \frac{1}{2}\sigma_{l,N}(T - t_l) + \sigma_{l,m}^2(t_m - t_l)}{\sigma_{l,N}\sqrt{T - t_l}} \\
&= d_1(t_l, S_l e^{\sigma_{l,m}^2(t_m - t_l)}, \sigma_{l,N}).
\end{aligned}
$$

As for (3.18), we have that, in distribution,

$$
\begin{aligned}
&d_2(t_m, S_m, \sigma_{m,N}) \\
&= \frac{\log(\frac{S_l}{K}) - \frac{1}{2}\sigma_{l,m}^2(t_m - t_l) + \sigma_{l,m}\sqrt{t_m - t_l}Z - \frac{1}{2}\sigma_{m,N}^2(T - t_m)}{\sigma_{m,N}\sqrt{T - t_m}} \\
&= \frac{\sigma_{l,m}\sqrt{t_m - t_l}}{\sigma_{m,N}\sqrt{T - t_m}}Z + \frac{\log(\frac{S_l}{K}) - \frac{1}{2}\sigma_{l,N}^2(T - t_l)}{\sigma_{m,N}\sqrt{T - t_m}}.
\end{aligned}
$$

Setting D as in (3.19),

$$
E' = \frac{\log(\frac{S_l}{K}) - \frac{1}{2}\sigma_{l,N}^2(T - t_l)}{\sigma_{m,N}\sqrt{T - t_m}}
$$

and

$$
B' = -2\sigma_{l,m}\sqrt{t_m - t_l},
$$

we find

$$
\begin{aligned}
&E_l[S_m N\left(d_2(t_m, S_m, \sigma_{m,N})\right)] \\
&= S_l e^{-\frac{1}{2}\sigma_{l,m}^2(t_m - t_l)} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 + B'z)} N\left(Dz + E'\right) dz \\
&= S_l N\left(\sqrt{\frac{1}{1 + D^2}}\left(E' - \frac{B'D}{2}\right)\right) \\
&= S_l N\left(\frac{\log(\frac{S_l}{K}) - \frac{1}{2}\sigma_{l,N}(T - t_l) + \sigma_{l,m}^2(t_m - t_l)}{\sigma_{l,N}\sqrt{T - t_l}}\right) \\
&= S_l N\left(d_2(t_l, S_l e^{\sigma_{l,m}^2(t_m - t_l)}, \sigma_{l,N})\right).
\end{aligned}
$$

When $m = N$ the computation is slightly different and uses integral (3.15).

$$
\begin{aligned}
&E_l[S_N \max\{S_N - K, 0\}] \\
&= S_l^2 e^{-\sigma_{l,N}^2(T - t_l)} \int_{-d_2(t_l, S_l, \sigma_{l,N})}^{+\infty} \frac{e^{\frac{1}{2}(z^2 - 4\sigma_{l,N}\sqrt{T - t_l}z)}}{\sqrt{2\pi}} dz - K S_l N\left(d_1(t_l, S_l, \sigma_{l,N})\right) \\
&= S_l^2 e^{\sigma_{l,N}^2(T - t_l)} N\left(d_2(t_l, S_l, \sigma_{l,N}) + 2\sigma_{l,N}\sqrt{T - t_l}\right) - K S_l N\left(d_1(t_l, S_l, \sigma_{l,N})\right) \\
&= S_l^2 e^{\sigma_{l,N}^2(T - t_l)} N\left(d_1(t_l, S_l, \sigma_{l,N}) + \sigma_{l,N}\sqrt{T - t_l}\right) \\
&-K S_l N\left(d_2(t_l, S_l, \sigma_{l,N}) + \sigma_{l,N}\sqrt{T - t_l}\right) \\
&= S_l V\left(t_l, S_l e^{\sigma_{l,N}(T - t_l)}, \sigma_{l,N}\right).
\end{aligned}
\tag{3.21}
$$

Now we will prove Equation (3.13). Consider first the case $m = N$. Note that, since $\rho_{l,N} = 1$,

$$A_{l,N}(S_l e^x, S_l e^y) = S_l e^x V(t_l, S_l e^y e^{\sigma_{l,N}^2 (T-t_l)}, \sigma_{l,N}) - KV(t_l, S_l e^y, \sigma_{l,N}),$$

when $x \leq y$ and analogously for $x \geq y$. Let us do for instance the case $x \leq y$. We have

$$\begin{aligned}
&E_l \left[\max\{S_N e^x - K, 0\} \max\{S_N e^y - K, 0\}\right] \\
&= E_l \left[(S_N e^x - K) \max\{S_N e^y - K, 0\}\right] \\
&= E_l \left[S_N e^x \max\{S_N e^y - K, 0\}\right] - KE_l \left[\max\{S_N e^y - K, 0\}\right].
\end{aligned}$$

Similarly to the computation that lead to (3.21) we get

$$E_l \left[S_N e^x \max\{S_N e^y - K, 0\}\right] = S_l e^x V\left(t_l, S_l e^y e^{\sigma_{l,N}(T-t_l)}, \sigma_{l,N}\right).$$

This proves (3.13) for $m = N$ and $x \leq y$.

We will end by briefly showing how to get (3.13) for $m < N$. We have to calculate four pieces:

$$\begin{aligned}
&E_l \left[V\left(t_m, S_m e^x, \sigma_{m,N}\right) V\left(t_m, S_m e^y, \sigma_{m,N}\right)\right] \\
&= E_l \left[S_m^2 e^x e^y N\left(d_1(t_m, S_m e^x, \sigma_{m,N})\right) N\left(d_1(t_m, S_m e^y, \sigma_{m,N})\right)\right] \\
&-KE_l \left[S_m e^x N\left(d_1(t_m, S_m e^x, \sigma_{m,N})\right) N\left(d_2(t_m, S_m e^y, \sigma_{m,N})\right)\right] \\
&-KE_l \left[S_m e^y N\left(d_1(t_m, S_m e^y, \sigma_{m,N})\right) N\left(d_2(t_m, S_m e^x, \sigma_{m,N})\right)\right] \\
&+K^2 E_l \left[N\left(d_2(t_m, S_m e^x, \sigma_{m,N})\right) N\left(d_2(t_m, S_m e^y, \sigma_{m,N})\right)\right].
\end{aligned}$$

Arguing in a similar manner as in the proof of (3.12), using this time Equation (3.16), one gets exactly $A_{l,m}(S_l e^x, S_l e^y)$. For instance, let us compute the first term:

$$\begin{aligned}
&E_l \left[S_m^2 e^x e^y N\left(d_1(t_m, S_m e^x, \sigma_{m,N})\right) N\left(d_1(t_m, S_m e^y, \sigma_{m,N})\right)\right] \\
&= S_l^2 e^{-\sigma_{l,m}^2 (t_m - t_l)} e^x e^y \\
&\times \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(z^2 - 4\sigma_{l,m}\sqrt{t_m - t_l} z)}}{\sqrt{2\pi}} N\left(Dz + E + \frac{x}{\sigma_{m,N}\sqrt{T - t_m}}\right) \\
&\times N\left(Dz + E + \frac{y}{\sigma_{m,N}\sqrt{T - t_m}}\right) dz,
\end{aligned}$$

where D and E are defined respectively in (3.19) and (3.20). Hence, by Equation (3.16), we get

$$E_l\left[S_m^2 e^x e^y N\left(d_1(t_m, S_m e^x, \sigma_{m,N})\right) N\left(d_1(t_m, S_m e^y, \sigma_{m,N})\right)\right]$$
$$= S_l^2 e^{\sigma_{l,m}^2(t_m-t_l)} e^x e^y$$
$$\times N\left(d_1\left(t_l, S_l e^x, \sigma_{l,N}\right) + \frac{\sigma_{l,m}^2(t_m-t_l)}{\sigma_{l,N}\sqrt{T-t_l}},\right.$$
$$\left. d_1\left(t_l, S_l e^y, \sigma\right) + \frac{\sigma_{l,m}^2(t_m-t_l)}{\sigma_{l,N}\sqrt{T-t_l}}, \frac{\sigma_{l,m}^2(t_m-t_l)}{\sigma_{l,N}^2(T-t_l)}\right).$$

which is the first term of $A_{l,m}(S_l e^x, S_l e^y)$. □

Proof of Proposition 3.1. Here we will use Lemma 3.1 for $l = k-1$ and $m = k$, for $k = 1, \ldots, N$. At each time t_{k-1}, the optimal hedge ratio is given by (2.3). The denominator is

$$E_{k-1}[\Delta S_k^2] = S_{k-1,k}^2(e^{\sigma_{k-1}^2(t_k-t_{k-1})} - 1).$$

The numerator is

$$E_{k-1}[V_k \Delta S_k] = E_{k-1}[S_k V_k] - S_{k-1} E_{k-1}[V_k].$$

By (3.6) we have $E_{k-1}[V_k] = V_{k-1} = V(t_{k-1}, S_{k-1}, \sigma_{k-1,N})$ and, from Equation (3.12) of Lemma 3.1,

$$E_{k-1}[S_k V_k] = S_{k-1} V\left(t_{k-1}, S_{k-1} e^{\sigma_{k-1,k}^2(t_k-t_{k-1})}, \sigma_{k-1,N}\right).$$

Expression (3.9) for the optimal local variance is obtained from the general expression (2.4) using (3.13) in Lemma 3.1 to compute $E_{k-1}[V_k^2]$. □

Note that the optimal hedge for a European put option is simply given by $\xi^p = \xi^c - 1$, where ξ^c is that of a call with same strike and same maturity. This is easily seen by using put-call parity in (2.3).

Proof of Proposition 3.2. Here we will use Lemma 3.1 with $l = 0$ and $m = k-1$ for $k = 1, \ldots, n$. Equation (3.10) is obvious because the process S is a martingale so that $E_0[\xi_k \Delta S_k] = E_0[\xi_k E_{k-1}[\Delta S_k]] = 0$. As for (3.11) we have

$$\text{var}_0\left(V_n - \sum_{k=1}^n \xi_k \Delta S_k\right) = E_0[V_n^2] - E_0[V_n]^2$$
$$-2\sum_{k=1}^n E_0[\xi_k V_n \Delta S_k] + \sum_{k=1}^n E_0[\xi_k^2 \Delta S_k^2],$$

since, for $j < k$, $E_0[\xi_j \xi_k \Delta S_j \Delta S_k] = E_0[\xi_j \Delta S_j \xi_k E_{k-1}[\Delta S_k]] = 0$. Because of (2.3) we have that

$$E_0[\xi_k V_n \Delta S_k] = E_0[\xi_k E_{k-1}[V_k \Delta S_k]] = E_0[\xi_k^2 E_{k-1}[\Delta S_k^2]],$$

so that

$$E_0\left[(V_n - \sum_{k=1}^{n} \xi_k \Delta S_k)^2\right] = E_0[V_n^2] - \sum_{k=1}^{n} E_0\left[\xi_k^2 E_{k-1}[\Delta S_k^2]\right].$$

Because of Proposition 3.1, the sum on the right hand side is equal to

$$\sum_{k=1}^{n} E_0\left[\frac{\left(V\left(t_{k-1}, S_{k-1}e^{\sigma^2_{k-1,k}(t_k-t_{k-1})}, \sigma_{k-1,N}\right) - V(t_{k-1}, S_{k-1}, \sigma_{k-1,N})\right)^2}{(e^{\sigma^2_{k-1,k}(t_k-t_{k-1})} - 1)}\right]. \tag{3.22}$$

Equation (3.11) follows now by suitably applying Equation (3.13) of Lemma 3.1. □

For the computation of the variance of the hedging error in the case of a put option one should go trough all the computations in an analogous way.

The case when S is not a martingale, say with drift μ, is much more complicated, because the value of the optimal portfolio is not given anymore by the Black-Scholes formula. It is given indeed by $V_k = \hat{E}_k[H]$ where the expectation is taken with respect to the minimal martingale measure (see for instance [11]). In this case it is only possible to explicitly compute the last locally optimal hedge as

$$\xi_N = \frac{V\left(t_{N-1}, S_{N-1}e^{(\mu+\sigma^2_{N-1,N})(t_N-t_{N-1})}, \sigma_{N-1,N}\right) - V(t_{N-1}, S_{N-1}, \sigma_{N-1,N})}{S_{N-1}e^{\mu(t_N-t_{N-1})}(e^{\sigma^2_{N-1,N}(t_N-t_{N-1})} - 1)}.$$

This may be done using again integral (3.15) to compute the general solution in (2.3). This also holds when one hedges only at initial time and then keeps the position up to maturity, namely for $N = 1$.

Notice that, by expanding Formula (3.8) in the Taylor polynomial of second order in the variable s with initial point S_{k-1} we get

$$\xi_k \approx N\left(d_1(t_{k-1}, S_{k-1}, \sigma_{k-1,N})\right) + \frac{1}{2}\Gamma(t_{k-1}, S_{k-1}, \sigma_{k-1,N}) S_{k-1}(e^{\sigma^2_{k-1,k}(t_k-t_{k-1})} - 1), \tag{3.23}$$

where $\Gamma(t_{k-1}, S_{k-1}, \sigma_{k-1,N})$ obviously stands for the gamma of the option at time t_{k-1}. This, approximating $e^{\sigma^2_{k-1,k}(t_k - t_{k-1})} - 1$ with $\sigma^2_{k-1,k}(t_k - t_{k-1})$ if sufficiently "small", is exactly the formula, in the martingale case, given by Wilmott in [14].

4 Applications

The first question one may ask is how big a difference there is between the optimal hedge ratio and the usual Black Scholes delta. Figure 1 represents the relative difference (in percent) between the optimal ratio and the Black-Scholes delta as a function of volatility and moneyness. The claim to be hedged is a call with maturity one year, with ten hedging times. We note that the optimal ratio is always greater than the delta. This is the case because delta hedging, designed as it is for continuous rebalancing, underestimates the variance of the position. Note also that the difference increases with volatility and decreases with moneyness.

To study the influence of the number of trading dates, we represented in Figure 2 the same percentage difference between Black-Scholes delta and the optimal strategy as a function of the number of rebalancing N for three different moneyness. We considered a one year call option with volatility $\sigma = 0.5$. As it should be expected the difference is decreasing with N and, as also seen in the previous Figure, with moneyness.

A practical application of the formula that computes the variance of the hedging error is as follows: suppose that a trader, who can hedge only at a finite number of times, wants to price an option in a Black-Scholes setting. The trader can compute the variance of the total error of the optimal hedging strategy and then compute an option price such that the final payoff of the hedged portfolio is positive within a given confidence interval (assuming a normal distribution for the error). Figure 3 shows the increments in price necessary to get a 95% confidence interval. Note that they are higher for out of the money options and that they are not negligible at all, even as the number of trading dates is sufficiently great. The discretization error is well known to most of the traders, who often use a convenient approximating formula due to Kamal and Derman [8], involving the option's vega κ_0 at time 0, namely $\frac{\pi}{4N}\sigma^2\kappa_0^2$. Such formula is an heuristic approximation, as the number of trading dates goes to infinite, of the variance of the Delta hedging strategy. It works better for at-the-money options. Although Formula (3.11) refers to the optimal strategy, it is nevertheless interesting to compare the two. Figure 4 represents the relative differences (in percent) in the estimated standard deviation. Note that for out of the money options the difference is negative, that is the Kamal-Derman approximation computes a variance that is smaller than the optimal one, which is of course impossible. This would imply, for instance, that using the Kamal-Derman formula to estimate the variance of the error and consequently adjust the price would lead to an underpricing of the option.

As a last application of the optimal hedging formulas we consider the case of a market with a humped volatility term structure, represented in the top panel of Figure 5. Such structures are quite common in the world of interest rate derivatives like caps or swaptions. We considered an at-the-money option with maturity $T = 3$ years and $N = 3, 10, 50$ trading dates. The corresponding initial optimal hedges are $\xi_1^{(3)} = 0.6032; \xi_1^{(10)} = 0.5832; \xi_1^{(50)} = 0.5776$, that should be compared to a Black-Scholes delta $\Delta = 0.5763$. Figure 5, bottom panel, shows the running variance of the hedging error as computed by Proposition 3.2. To compare the minimal variance to that produced by the standard Black-Scholes delta one possibility is to simulate a number of trajectories of the underlying and to compute the sample variance of the error (another possibility would be to compute the Laplace anti-transform of a function of Delta, see [1]). By simulating 10000 paths of the underlying we obtained, in the case of $N = 3$ trading dates, a sample variance for the total error at maturity of the Delta hedging strategy equal to 50.3786. The variance of the optimal strategy as computed by Proposition 3.2 is 49.1579, while the corresponding sample variance is 49.1765, with 95% confidence interval $(48.7483, 49.6104)$. Therefore in this case the reduction in variance obtained by following the optimal strategy instead of the Black-Scholes hedging is between two and three percentage points. For this example, the difference is rather small and therefore a traders will most likely make the conservative choice of using the standard delta hedging strategy. However, also in this case the possibility of computing exactly the minimal variance will help the trader to assess a price and the risk manager to set adequate capital requirements.

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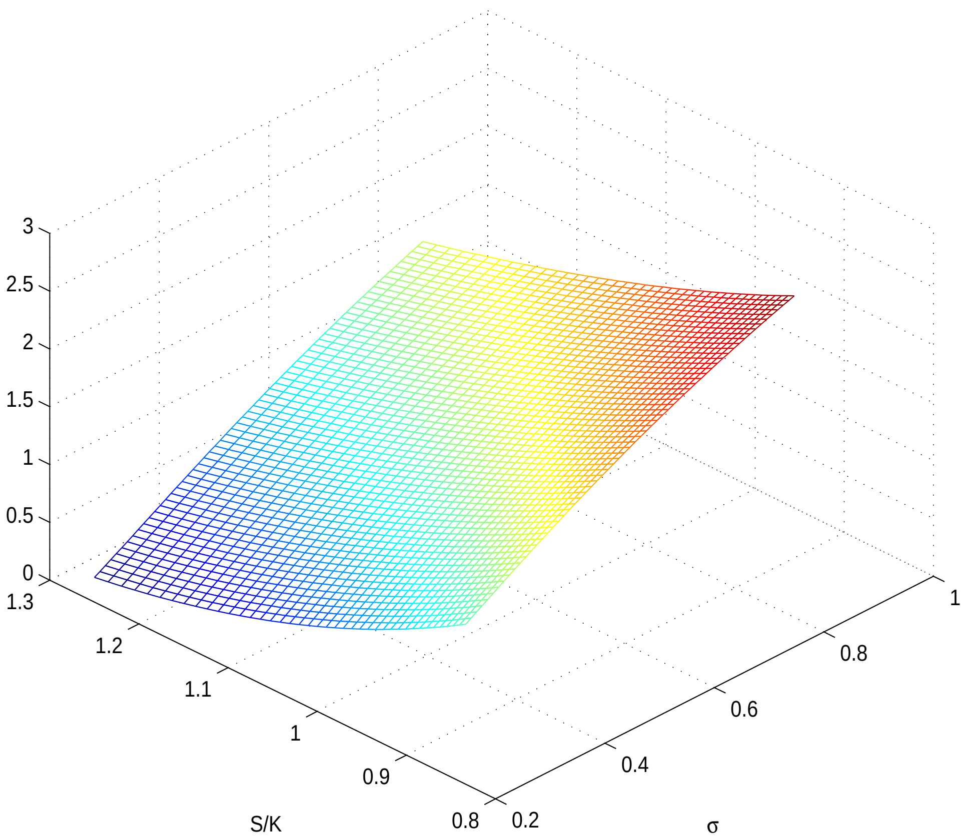

Figure 1 *Relative difference (in percent) between the optimal ratio and the Black-Scholes delta as a function of volatility* σ *and moneyness* S/K*. The claim to be hedged is a call with maturity one year, with ten hedging times*

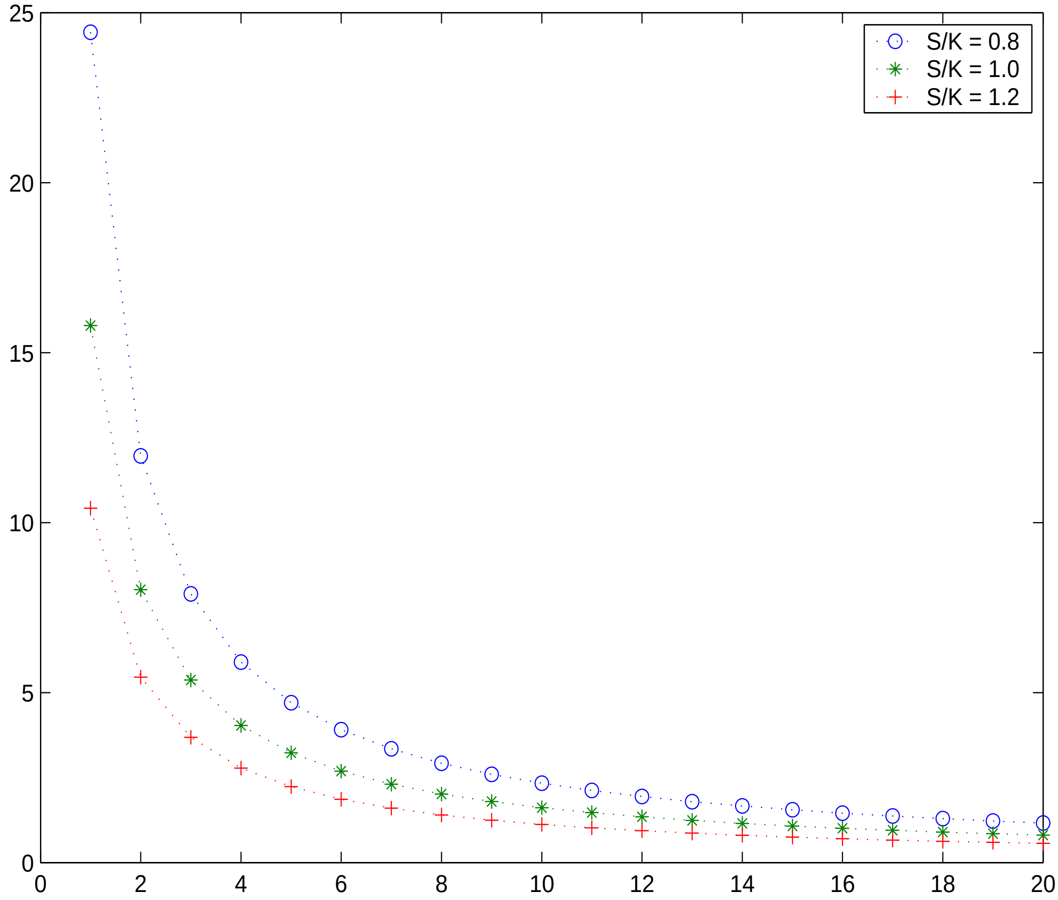

Figure 2 *Relative difference (in percent) between the optimal ratio and the Black-Scholes delta as a function of the number of trading dates for different moneyness. The claim to be hedged is a call with maturity one year and a constant volatility* $\sigma = 0.5$

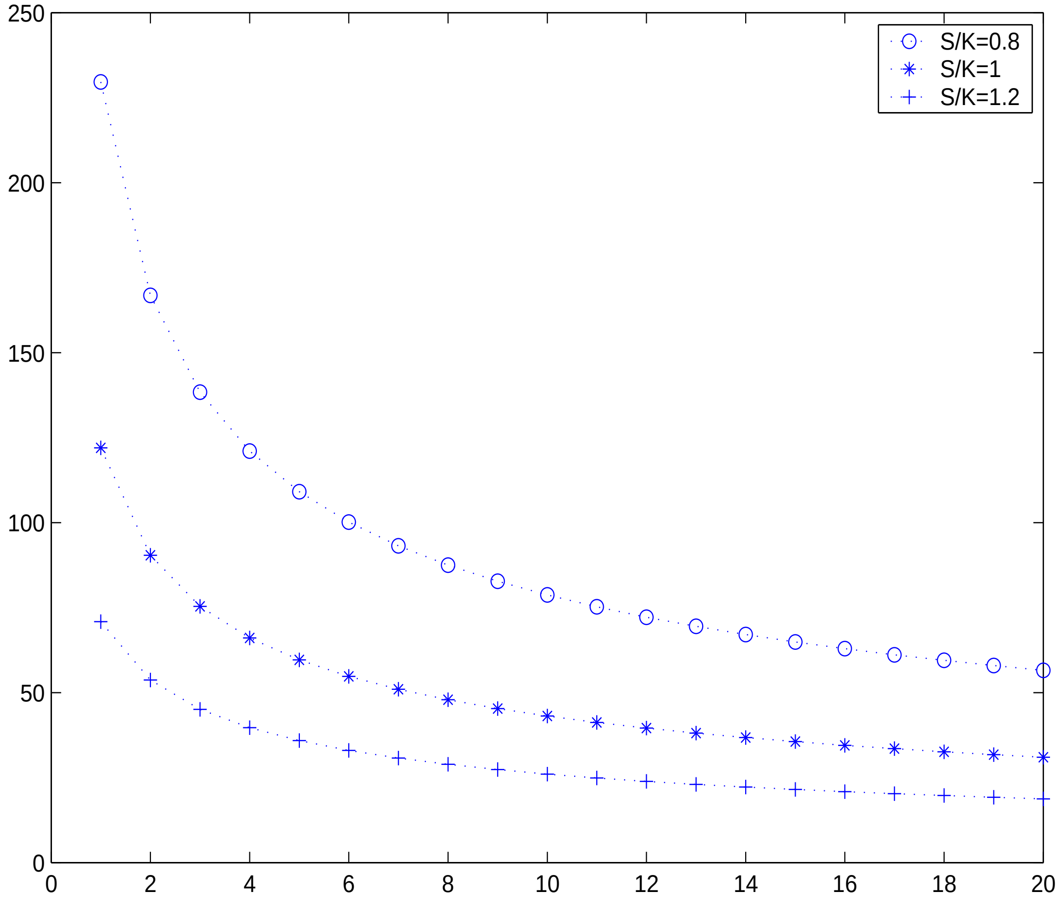

Figure 3 *Increment in price (in percent) of a call option as a function of the number of trading dates if the option writer (who follows an optimal hedging strategy) wants to get a positive payoff with a 95% confidence interval.* $T = 1, \sigma = 0.5$

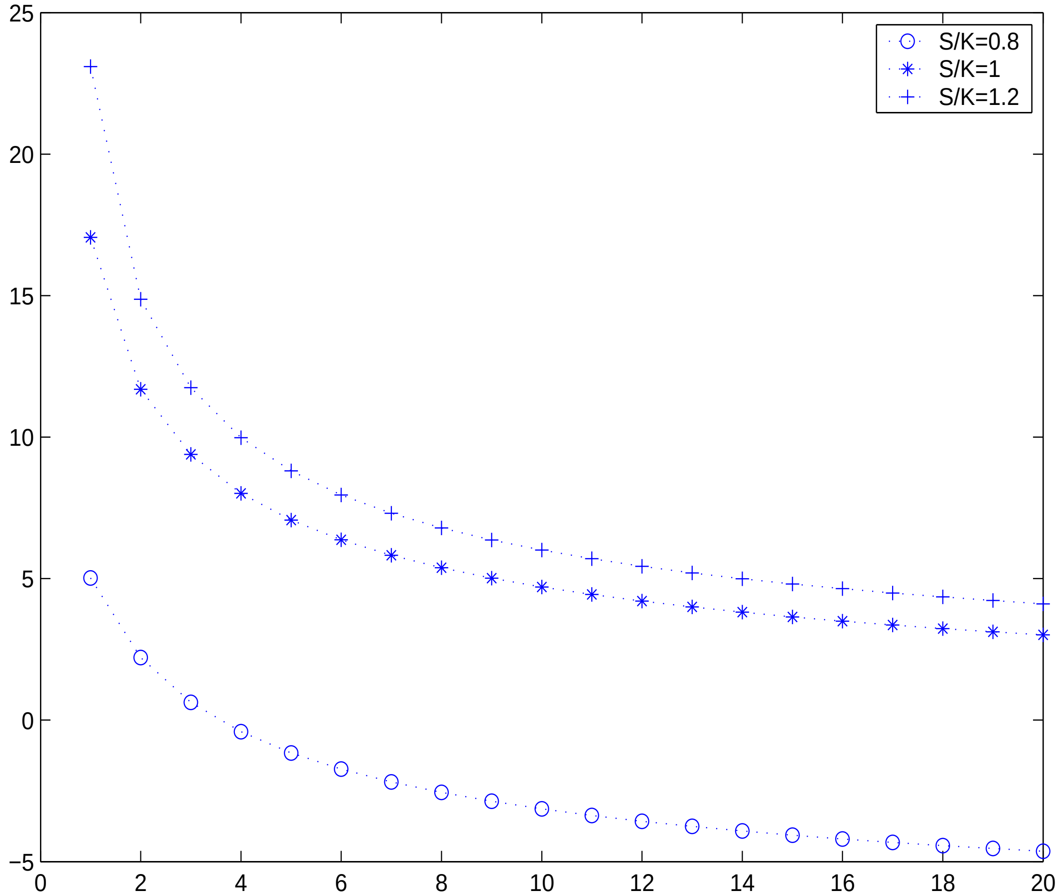

Figure 4 *Relative difference (in percent) of the Derman-Kamal approximation of standard deviation of the hedging error of the Black-Scholes delta strategy and the optimal standard deviation (dk/opt-1) as a function of the number of trading dates.* $T = 1, \sigma = 0.5$

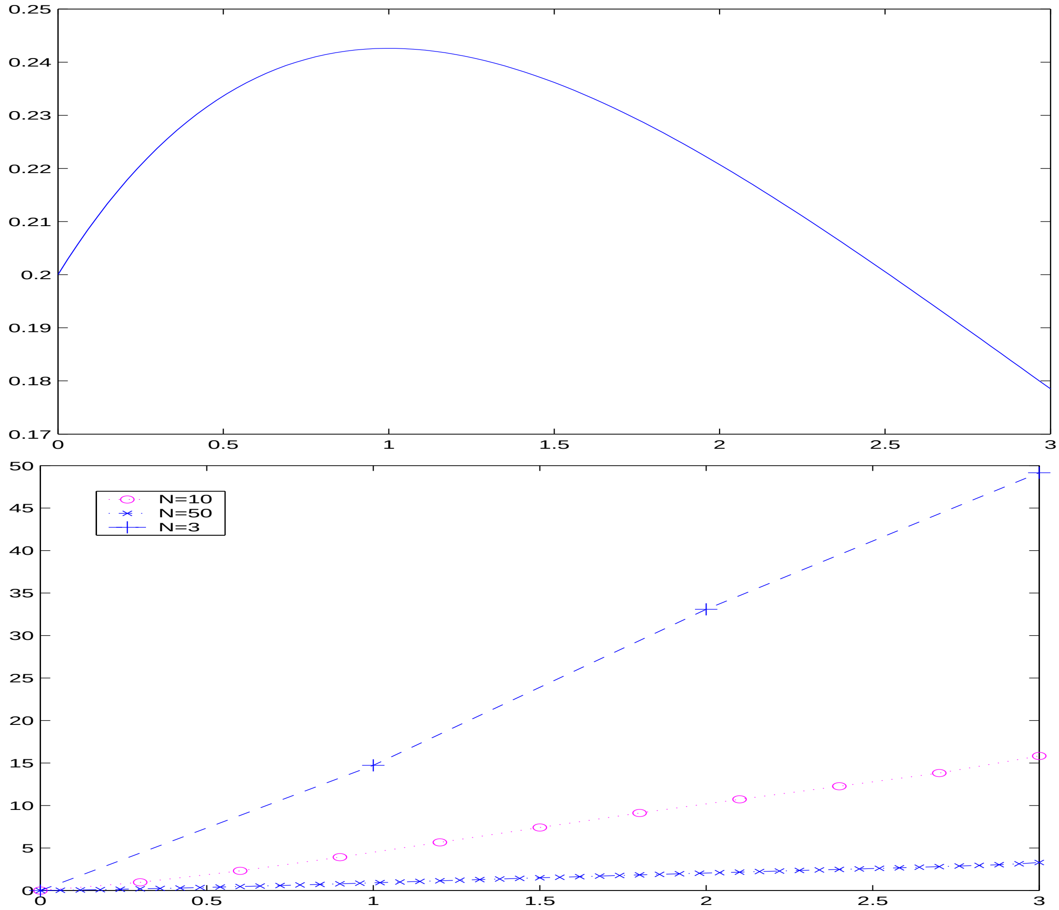

Figure 5 *Hedging with a humped volatility term structure. The volatility structure $\sigma(t)$ is represented on the left. The running variance of the minimal hedging error for different numbers of trading dates is on the right.*

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