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MEASURING THE ERROR OF DYNAMIC HEDGING: A LAPLACE TRANSFORM APPROACH

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Abstract

Using the Laplace transform approach, we compute the expected value and the variance of the error of a hedging strategy for a contingent claim when trading in discrete time. The method applies to a fairly general class of models, including Black-Scholes, Merton's jump-diffusion and Normal Inverse Gaussian, and to several interesting strategies, as the Black-Scholes delta, the Wilmott's improved-delta and the local optimal one. With this approach, also transaction costs may be treated. The results obtained are not asymptotical approximations but exact and efficient formulas, valid for any number of trading dates. They can also be employed under model mispecification, to measure the influence of model risk on a hedging strategy.

1 Introduction

The object of this paper is the measurement of the hedging error due to trading in discrete time, usually referred to as the "discretization error". Most of the financial models for pricing and hedging derivatives assume that trading is possible in continuous time. Of course, such an assumption does not hold for practical applications. For example, the widely used Black-Scholes delta hedging strategy produces a discretization error even if all other assumptions of the model are met. The discretization error depends on the path followed by the underlying asset until maturity and hence, even computing the variance of the error on a claim as simple as a European call, can be a very hard task. The practical importance of such a computation is self-evident, since it provides a way to measure the risk involved with discrete trading and, consequently, to quantify a compensation for it.

Quantifying the discretization error associated to an hedging strategy is a problem that is relevant both from a practical and a theoretical point of view and has been addressed by many papers in the literature. Hayashi and Mykland [8] use a weak convergence argument to derive the asymptotic distribution of the hedging error as the number of trades goes to infinite. Some approximating formulas for the variance have been obtained, under the assumption of small trading intervals and for the log-normal model, by Kamal and Derman [10], by Mello and Neuhaus [15] and, in presence of transaction costs, by Toft [21]. The fact that such approximations hold for vanishing time intervals constitutes an important limitation to their application, since in this case the error would also vanish. Moreover (to the best of our knowledge) the error associated with such approximations has never been measured. Our results may also be useful to assess this point.

The discretization error becomes even more important in presence of transaction costs. The pioneer study in the Black-Scholes setting is the paper of Leland [13], where a heuristic argument lead to find a hedging volatility, transaction costs adjusted, when the level of transaction cost does not depend on the number of trading intervals, which is perhaps the most interesting case from a practical point of view. This result was then used by Toft [21] to find the approximation of the variance cited above. Following a conjecture of Leland, Lott [14] proved that, when the transaction cost level goes to zero as the inverse of the square root of the number of trading intervals, the hedging error converges to zero. More generally, Kabanov and Safarian [9] were able to show that this convergence result holds when the transaction cost level goes to zero as the inverse of any power within the interval $(0, 0.5)$. In the case of constant level transaction costs considered by Leland, Kabanov and Safarian proved that the argument given by Leland was only heuristic: in fact, they showed that the hedging error, in this case, does not converge to zero as the number of trading intervals goes to infinity.

A related, very important, problem is that of determining a strategy that minimizes the variance of the hedging error in an incomplete market. An extremely rich branch of the financial literature flourished after the seminal papers of Föllmer and Sondermann [5]. Schweizer [19] contains a review of the main results and contributions. The general solution in a discrete setting was found by Schweizer [20], who provided a characterization of the optimal strategy and a general formula for the optimal variance. However, an explicit computation for practical application is usually quite burdensome. For this reason, some algorithms useful for actual implementation have been proposed, for example by Bertsimas et al. [1] , with a dynamic programming approach or by Wilmott [22], with an independent approach, specific to the Black-Scholes model and based on second order approximation, who proposed a very easily implementable trading strategy.

A breakthrough in the problem of determining an efficient way to compute optimal strategies and their associated variances was proposed by Hubalek et al. [7] and by Černý [2]. Their idea is to consider contingent claims whose payoff function can be written as an inverse Laplace transform. They showed that, under quite general assumption on the dynamics of the underlying, it is possible to compute the optimal strategy and its variance as an inverse Laplace transform of a function that depends on the claim and on the underlying process. This represents a relevant contribution from a practical point of view, since inverse Laplace transform can be evaluated very efficiently with standard numerical algorithms.

The present paper follows such approach, with the main objective of determining an efficient way to compute the first two moments of the distribution of the hedging error, in presence of transaction costs, for "sub-optimal" strategies, such as the standard Black-Scholes delta or Wilmott hedge ratio. The formulas that we obtain are valid for any fixed number of trading dates, whereas all previous formulas are asymptotic approximations. Equipped with our results, we are able to assess the precision of the approximations, that hold under much more restrictive assumptions on the model and on the claim, like those of Kamal and Derman [10] or by Toft [21]. Moreover, our result can be applied to measure the performance of a hedging strategy under model misspecification. For example, in the case of a trader that detects a market implied volatility higher than what she expects and wishes to exploit it. In this case, we can measure the expected performances of the hedging strategies in terms of Sharpe ratios. From a practical point of view this could help a trader to choose which trading strategy to adopt.

The rest of the paper is composed as follows: Section 2 contains the general setting and defines the class of strategies, that we call "compatible", whose hedging error will be measured. Section 3 contains the main result, while Section 4 shows how to extend the same techniques to transaction costs. We provide some details of the numerical implementation of our results in Section 5, showing some applications in Section 6. Section 7 concludes..

2 Compatible hedging strategies

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in(0,1,\ldots,N)}, P)$ be a filtered probability space. We consider a one-dimensional process

$$S_n = S_0 \exp(X_n),$$

where the process $X = (X_n)$ for $n = 0, 1, \ldots, N$, satisfies

1. X is adapted to the filtration $(\mathcal{F}_n)_{n\in(0,1,\ldots,N)}$,
2. $X_0 = 0$,
3. $\Delta X_n = X_n - X_{n-1}$ has the same distribution for $n = 1, \ldots, N$,
4. ΔX_n is independent from $\mathcal{F}_{n-1}$ for $n = 1, \ldots, N$.

We denote the moment generating function of X_1 by $m(z)$. We assume that $E[S_1^2] < \infty$ so that the moment generating function $m(z)$ is defined at least for complex z with $0 \leq Re(z) \leq 2$. Moreover, we exclude the case when S is a deterministic process. We suppose, without loss of generality, that the risk-free rate is zero or, equivalently, that S represents a discounted price.

Following the approach proposed by Hubalek et al. [7] we consider European contingent claims written on S with maturity T and payoff $H = f(S_N)$, where $f : (0, \infty) \to \mathbb{R}$ is of the form

$$f(s) = \int s^z \Pi(dz), \tag{2.1}$$

for some finite complex measure Π on a strip in the complex plane. Condition (2.1) states that the payoff function can be written as an inverse Laplace tranform. For instance, for a European call option with strike price $K > 0$, the function $(s-K)^+$ may be written as

$$(s-K)^+ = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz,$$

for an arbitrary $R > 1$ and for each $s > 0$. For more details and other examples of integral representation of payoff functions we refer to [7].

Let $\vartheta = (\vartheta_n)$, for $n = 1, \dots, N$, be an admissible trading strategy [1] with cumulative gains $G_n(\vartheta) = \sum_{k=1}^{n} \vartheta_k \Delta S_k$. Note that, because of the assumption of a null interest rate, the money market account does not contribute to the cumulative gain. The hedging error of the strategy is

$$\varepsilon(\vartheta, c) = H - c - G_N(\vartheta).$$

The random variable ϑ_n may be interpreted as the number of shares of the underlying asset held from time $n-1$ up to time n. If there exists a riskless asset, the strategy ϑ determines a unique self-financing portfolio and the hedging error $\varepsilon(\vartheta, c)$ may be viewed as the net loss one can suffer at maturity if one starts with the initial capital c and follows the strategy. The problem is to evaluate its expected value $E[\varepsilon(\vartheta, c)]$ and its variance $\mathrm{var}(\varepsilon(\vartheta, c))$.

It is well known that, for each initial endowment c, there exists a strategy $\xi^{(c)}$ which minimizes the expected square value of the hedging error (see for instance [20]). The optimal strategy and its variance are effectively computed by Hubalek et al. [7] under the same assumptions on the process of the underlying as the present paper. It is also possible (see [20]) to compute the optimal c. However the most widely used trading strategy in practice is still the Black-Scholes delta hedging strategy. In this paper we will compute expected value and variance of the error of delta hedging and we will compare it to other possible strategies, in particular to the optimal one.

First of all we note that the Black-Scholes price at time n for claims satisfying Condition (2.1) can be computed as

$$C_n^{bs} = E_n^{bs}[H] = E_n^{bs}\left[\int S_N^z \Pi(dz)\right],$$

where E_n^{bs} is the Black-Scholes risk neutral expectation conditional to $\mathcal{F}_n$, which assumes i.i.d. and normal log-return, with (annual) volatility σ. By

[1] An admissible strategy is a predictable process such that the cumulative gains are square-integrable, see [7] or [20].

Fubini's theorem, we can exchange the expected value with the integral in the complex variable to get

$$\begin{aligned} C_n^{bs} &= \int E_n^{bs}\left[S_N^z\right]\Pi(dz) = \int S_n^z E_n^{bs}\left[\exp(z(\Delta X_{n+1} + \ldots \Delta X_N))\right]\Pi(dz) \\ &= \int S_n^z m^{bs}(z)^{N-n}\Pi(dz), \end{aligned}$$

where

$$m^{bs}(z) = \exp\left(\left(-\frac{\sigma^2}{2}z + \frac{\sigma^2}{2}z^2\right)\frac{T}{N}\right).$$

Therefore, the units of underlying held from time $n-1$ to time n, when following the Black-Scholes delta hedging strategy, are

$$\Delta_n = \frac{\partial C_{n-1}^{bs}}{\partial S_{n-1}} = \int m^{bs}(z)^{N-n+1}\frac{\partial S_{n-1}^z}{\partial S_{n-1}}\Pi(dz) = \int z m^{bs}(z)^{N-n+1} S_{n-1}^{z-1}\Pi(dz)$$

Motivated by this computation, we give the following

Definition 2.1 *A hedging strategy* ϑ *is* **compatible** *with a contingent claim with a payoff function satisfying Condition (2.1) if it is of the form*

$$\vartheta_n = \int \vartheta(z)_n \Pi(dz), \tag{2.2}$$

with $\vartheta(z)_n = f^\vartheta(z)_n S_{n-1}^{z-1}$*, where* $f^\vartheta(z)_n$ *is a function of the complex variable* z *which does not contain* S_k *for any* k*.*

We remark that our method continues to hold even in the case of strategies that are not compatible with a given contingent claim, as long as Condition (2.2) is satisfied. In fact, as it will be made clear in the following, our approach can be adopted, for instance, to compute the variance of any dynamic strategy satisfying (2.2). However, for the sake of a clearer exposition, since we are focusing on hedging, we preferred to connect each strategy to its associated contingent claim.

The Black-Scholes delta hedging is not a unique case; other interesting strategies which are compatible are:

- the local optimal strategy, that is the strategy which minimizes the variance of the next period costs (for a formal definition, see [20]). This is

  $$\xi_n = \int \xi(z)_n \Pi(dz) = \int f^\xi(z)_n S_{n-1}^{z-1}\Pi(dz),$$

  where $f^\xi(z)_n$ is given in Theorem 2.1 in [7].

- the "improved-delta" strategy proposed by Wilmott ([22]) for the Black-Scholes model, that is an easily implementable approximation to the local optimal strategy

$$\Delta_n^w = \Delta_n + \frac{T}{N}(\mu - \frac{1}{2}\sigma^2)\Gamma_n S_{n-1} =$$
$$= \int S_{n-1}^{z-1}\left(zm^{bs}(z)^{N-n+1} + \frac{T}{N}(\mu - \frac{1}{2}\sigma^2)z(z-1)m^{bs}(z)^{N-n+1}\right)\Pi(dz),$$

  where μ and σ are respectively the drift and the volatility of the process. The expression of Δ_n^w has been obtained in an analogous way as for the delta, since the gamma of the claim Γ_n is the second derivative of C_{n-1}^{bs} with respect to S_{n-1}.

Of course, not all strategies are compatible. The most important example of a non-compatible strategy is the optimal one.

The delta and the improved-delta strategies are conceived for a log-normal process. Nevertheless, they may also be considered, and our results will still apply, when the underlying is not log-normal; in this case, a sensible choice for parameters μ and σ would be to fit mean and variance of the log-returns. We will show one such example in Section 6.

3 Measuring the discretization error

To assess the risk of the discretization error of compatible strategies one can compute its variance and expected value. Černý [2] and Hubalek et al. [7] computed the variances for global and local optimal strategies. We will generalize their results to sub-optimal strategies and in presence of model risk, that is when the adopted strategy is obtained by a model that is not the data generating process. Later on, transaction costs will also be included in the picture.

The hedging error of a strategy which is compatible with a contingent claim satisfying Condition (2.1) has the following integral representation

$$\begin{aligned}\varepsilon(\vartheta, c) &= H - c - \sum_{k=1}^{N} \vartheta_k \Delta S_k \\ &= \int \left(H(z) - \sum_{k=1}^{N} \vartheta(z)_k \Delta S_k\right)\Pi(dz) - c, \qquad (3.3)\end{aligned}$$

where $H(z) = S_N^z$.

The following theorem gives the expected value and the variance of the hedging error of a compatible strategy for a given initial capital c.

Theorem 3.1 *Let ϑ be a strategy which is compatible with a contingent claim H and let c be its initial value, then*

$$E[\varepsilon(\vartheta, c)] = \int S_0^z \left[m(z)^N - (m(1)-1) \sum_{k=1}^{N} f^\vartheta(z)_k m(z)^{k-1} \right] \Pi(dz) - c \quad (3.4)$$

and

$$E[\varepsilon(\vartheta, 0)^2] = \int\int S_0^{y+z}(v_1(y,z) - v_2(y,z) - v_3(y,z) + v_4(y,z))\Pi(dz)\Pi(dy), \quad (3.5)$$

where

$$v_1(y,z) = m(y+z)^N,$$

$$v_2(y,z) = \sum_{k=1}^{N} f^\vartheta(y)_k m(y+z)^{k-1} m(z)^{N-k}(m(z+1) - m(z)),$$

$$v_3(y,z) = \sum_{k=1}^{N} f^\vartheta(z)_k m(y+z)^{k-1} m(y)^{N-k}(m(y+1) - m(y)),$$

$$\begin{aligned} v_4(y,z) &= (m(2) - 2m(1) + 1) \sum_{k=1}^{N} f^\vartheta(y)_k f^\vartheta(z)_k m(z+y)^{k-1} + \\ &+ (m(1)-1) \sum_{k<j} \sum_{j=2}^{N} f^\vartheta(y)_k f^\vartheta(z)_j m(y)^{j-1-k} m(z+y)^{k-1}(m(y+1) - m(y)) + \\ &+ (m(1)-1) \sum_{j<k} \sum_{k=2}^{N} f^\vartheta(y)_k f^\vartheta(z)_j m(z)^{k-1-j} m(z+y)^{j-1}(m(z+1) - m(z)). \end{aligned}$$

Therefore, the variance of the hedging error is

$$\mathrm{var}(\varepsilon(\vartheta, c)) = \mathrm{var}(\varepsilon(\vartheta, 0)) = E[\varepsilon(\vartheta, 0)^2] - E[\varepsilon(\vartheta, 0)]^2.$$

Proof. Given (3.3), we have, by Fubini's Theorem,

$$E[H-\sum_{k=1}^N \vartheta_k \Delta S_k] = \int E[S_N^z - \sum_{k=1}^N f^\vartheta(z)_k S_{k-1}^{z-1}\Delta S_k]\Pi(dz) =$$

$$= \int \left\{ E[S_0^z \exp(z(\Delta X_1 + \ldots \Delta X_N))] - \sum_{k=1}^N f^\vartheta(z)_k E[S_{k-1}^{z-1}\Delta S_k] \right\} \Pi(dz) =$$

$$= \int S_0^z \left\{ m(z)^N - \sum_{k=1}^N f^\vartheta(z)_k E[\exp((z-1)(\Delta X_{k-1}+\ldots+\Delta X_1)) \right.$$
$$\times[\exp(\Delta X_k + \ldots + \Delta X_1) - \exp(\Delta X_{k-1}+\ldots+\Delta X_1)]]\} \,\Pi(dz) =$$

$$= \int S_0^z \left\{ m(z)^N - \sum_{k=1}^N f^\vartheta(z)_k \right.$$
$$\times E[(\exp(z(\Delta X_{k-1}+\ldots+\Delta X_1)+\Delta X_k) - \exp(z(\Delta X_{k-1}+\ldots+\Delta X_1))]\} \,\Pi(dz) =$$

$$= \int S_0^z \left[m(z)^N - \sum_{k=1}^N f^\vartheta(z)_k m(z)^{k-1}(m(1)-1) \right] \Pi(dz).$$

which is (3.4). To prove (3.5) we need to compute

$$E[(H-\sum_{k=1}^N \vartheta_k \Delta S_k)^2] =$$

$$= E[\int (H(z) - \sum_{k=1}^N \vartheta(z)_k \Delta S_k)\Pi(dz) \int (H(y) - \sum_{k=1}^N \vartheta(y)_k \Delta S_k)\Pi(dy)] =$$

$$= E[\int\int (H(z) - \sum_{k=1}^N \vartheta(z)_k \Delta S_k)(H(y) - \sum_{k=1}^N \vartheta(y)_k \Delta S_k)\Pi(dz)\Pi(dy)] =$$

$$= \int\int E[(H(z) - \sum_{k=1}^N \vartheta(z)_k \Delta S_k)(H(y) - \sum_{k=1}^N \vartheta(y)_k \Delta S_k)]\Pi(dz)\Pi(dy).$$

Let us compute all the expectations needed:

$$\begin{aligned} E[H(z)H(y)] &= S_0^{y+z}\exp(z(\Delta X_N+\ldots+\Delta X_1)+y(\Delta X_N+\ldots+\Delta X_1)) = \\ &= S_0^{y+z} m(y+z)^N = S_0^{y+z} v_1(y,z). \end{aligned}$$

$$E[H(z)\sum_{k=1}^{N}\vartheta(y)_k\Delta S_k] =$$

$$= \sum_{k=1}^{N} f^{\vartheta}(y)_k E[S_N^z S_{k-1}^{y-1}\Delta S_k] =$$

$$= \sum_{k=1}^{N} f^{\vartheta}(y)_k S_0^{z+y} E[\exp(z(\Delta X_N + \ldots + \Delta X_1))\exp((y-1)(\Delta X_{k-1} + \ldots + \Delta X_1))$$

$$\times [\exp(\Delta X_k + \ldots + \Delta X_1) - \exp(\Delta X_{k-1} + \ldots + \Delta X_1)]] =$$

$$= S_0^{z+y}\sum_{k=1}^{N} f^{\vartheta}(y)_k$$

$$\times \{E[\exp((y+z)(\Delta X_{k-1} + \ldots + \Delta X_1))\exp(z(\Delta X_N + \ldots + \Delta X_k))\exp(\Delta X_k)] +$$

$$- E[\exp((y+z)(\Delta X_{k-1} + \ldots + \Delta X_1))\exp(z(\Delta X_N + \ldots + \Delta X_k))]\} =$$

$$= S_0^{z+y}\sum_{k=1}^{N} f^{\vartheta}(y)_k \left[m(y+z)^{k-1}m(z)^{N-k}m(z+1) - m(y+z)^{k-1}m(z)^{N-k+1}\right] =$$

$$= S_0^{z+y} v_2(y,z).$$

Analogously, computing the expectation

$$E[H(y)\sum_{k=1}^{N}\vartheta(z)_k\Delta S_k]$$

one gets $S_0^{y+z} v_3(y,z)$.

The last term is

$$E[\sum_{k=1}^{N}\vartheta(z)_k\Delta S_k\sum_{j=1}^{N}\vartheta(y)_j\Delta S_j] =$$
$$= \sum_{k=1}^{N}\sum_{j=1}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j E[S_{k-1}^{z-1}S_{j-1}^{y-1}\Delta S_k\Delta S_j] =$$
$$= S_0^{y+z}\sum_{k=1}^{N}\sum_{j=1}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j$$
$$\times E[\exp((z-1)(\Delta X_{k-1}+\dots\Delta X_1))\exp((y-1)(\Delta X_{j-1}+\dots\Delta X_1))$$
$$\times\exp(\Delta X_{k-1}+\dots\Delta X_1)(\exp(\Delta X_k)-1)$$
$$\times\exp(\Delta X_{j-1}+\dots\Delta X_1)(\exp(\Delta X_j)-1)] =$$
$$= S_0^{y+z}\sum_{k=1}^{N}\sum_{j=1}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j$$
$$\times E[\exp(z(\Delta X_{k-1}+\dots\Delta X_1))\exp(y(\Delta X_{j-1}+\dots\Delta X_1))$$
$$\times(\exp(\Delta X_k)-1)(\exp(\Delta X_j)-1)].$$

The last sum may be computed separating the cases $k=j$, $k<j$ and $k>j$ as

$$\sum_{k=j}\sum_{k=1}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_k m(y+z)^{k-1}(m(2)-2m(1)+1) +$$
$$+\sum_{k<j}\sum_{j=2}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j m(y+z)^{k-1}(m(y+1)-m(y))(m(1)-1) +$$
$$+\sum_{k>j}\sum_{k=2}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j m(y+z)^{j-1}(m(z+1)-m(z))(m(1)-1),$$

which is $v_4(y,z)$. □

Theorem 3.1 states that the expected value and the variance of the hedging error may be represented respectively as one- and two-dimensional inverse Laplace transforms. Although the formulas look a bit involved, they can be easily evaluated numerically. In Section 5 we will give some details on their implementation and discuss the precision of the algorithm used.

A similar argument can be applied to compute higher order moments of the hedging errors, that can be useful to get more information on the probability distribution.

We remark that for Theorem 3.1 to hold it is not necessary that the compatible strategy is consistent with the model. One can, for instance, consider the case where the data generating process is the Black-Scholes process with a certain drift and volatility, while the strategy is based on different estimates. Or it may be the case that the data generating model is the Merton jump-diffusion model ([16]), while the strategy is conceived according to the Black-Scholes world, perhaps by fitting mean and variance of the returns.

A more general form of Theorem 3.1 holds if the increments ΔX_n are not identically distributed. In this case, all of the computations would go through and one would get similar results by substituting all the powers of $m(\cdot)$ in the formulas with suitable products of the moment generating functions of each ΔX_n, for $n = 1, \dots, N$. Such a generalization may have interesting application to interest rate sensitive derivatives, where most of the model adopted for hedging and pricing consider a non-constant volatility for the underlying.

Since a compatible strategy does not depend on the initial value endowment, the expected value of the error produced by such a strategy with initial value c can be obtained by simply subtracting c from the expected value of the same strategy with zero endowment. For the same reason, the variance of the error produced by a compatible strategy does not depend on c. On the other hand, the optimal strategy does indeed depend on the initial capital. We will now prove, in our setting, a result that measures the influence of the initial capital c on the expectation and the variance of the optimal strategy. It gives an immediate way to compute the expected value of the optimal strategy for a given c, provided that one knows the optimal initial endowment V_0, that is the value of c that minimizes the expectation of the square of the discretization error (namely, the solution of Problem (3.1) in [20]). It also shows that the variance of an optimal strategy does not depend on c.

Proposition 3.1 *Let ξ^c be the optimal, N-step, strategy for a contingent claim H with an initial endowment c and let $\varepsilon(\xi^c, c)$ be its hedging error.*

Then

$$E[\varepsilon(\xi^c, c)] = (V_0 - c)\left(\frac{m(2) - m(1)^2}{m(2) - 2m(1) + 1}\right)^N \tag{3.6}$$

where V_0 *is the optimal initial capital. Moreover, the variance of* $\varepsilon(\xi^c, c)$ *does not depend on* c.

Proof. From Corollary 2.5 in [20] it follows that

$$\begin{aligned} E[\varepsilon(\xi^c, c)] &= E[H\tilde{Z}^0] - cE[\tilde{Z}^0] \\ &= (V_0 - c)E[\tilde{Z}^0] \end{aligned}$$

where (using the fact that we are in the case of a deterministic mean-variance payoff),

$$\tilde{Z}_0 = \prod_{k=1}^{N}(1 - \alpha_k \Delta S_k)$$

with

$$\alpha_k = \frac{E[\Delta S_k|\mathcal{F}_{k-1}]}{E[\Delta S_k^2|\mathcal{F}_{k-1}]}.$$

Setting

$$\lambda = \frac{(m(1) - 1)}{m(2) - 2m(1) + 1},$$

we have

$$E[\tilde{Z}^0] = \prod_{k=1}^{N}(1 - \lambda(m(1) - 1)) = \left(\frac{m(2) - m(1)^2}{m(2) - 2m(1) + 1}\right)^N,$$

from which we get (3.6).

To prove the statement on the variance, recall that, from Theorem 4.4 in [20],

$$E[(H - c - G_N(\xi^{(c)})^2] = (V_0 - c)^2 \prod_{k=1}^{N}(1 - \alpha_k E[\Delta S_k|\mathcal{F}_{k-1}]) + \mathrm{var}(H - G_N(\xi^{V_0}))$$
$$= (V_0 - c)^2 \left(\frac{m(2) - m(1)^2}{m(2) - 2m(1) + 1}\right)^N + \mathrm{var}(H - G_N(\xi^{V_0}))$$
$$= E[H - c - G_N(\xi^{(c)}]^2 + \mathrm{var}(H - G_N(\xi^{V_0})),$$

where we used (see (2.8) in [20]),

$$E[\tilde{Z}^0] = E[(\tilde{Z}^0)^2]$$

□

This concludes our exposition for the case of markets without transaction costs, whose influence on the hedging error will be studied in the next section.

4 Transaction Costs

In this section we show how to compute the expectation and the variance of the hedging error of compatible strategies in presence of transaction costs. We consider the case of proportional transaction costs, that is we indicate by S the mid-price of the underlying, so that the bid and ask prices are given, respectively, by $S(1-k/2)$ and $S(1+k/2)$. The transaction cost at time t_n to change the position from ϑ_n to ϑ_{n+1} is

$$\mathrm{TC}_n \;=\; \frac{k}{2} S_n |\vartheta_{n+1} - \vartheta_n|.$$

In line with other authors (e.g. Kabanov and Safarian [9]), we set $TC_N = 0$, because this last transaction cost depends upon contract specification. Also, we do not consider the transaction cost in $n = 0$, as it is known. The hedging error of a strategy ϑ in presence of transaction costs is equal to

$$\varepsilon^{\mathrm{tc}}(\vartheta, c) = \varepsilon(\vartheta, c) + \sum_{n=1}^{N} TC_n.$$

To compute the mean and the variance of $\varepsilon^{\mathrm{tc}}(\vartheta, c)$ we have to compute the mean and the variance of $\sum_{n=1}^{N} \mathrm{TC}_n$ as well as its covariance with the trading gains $\vartheta_n \Delta S_n$ and with the final payoff H. For the sake of a shorter exposition, since the computations follow the approach of the previous section, we briefly sketch the procedure and refer the reader to the Appendix for more details.

Let us first observe that the transaction costs can be written as

$$\mathrm{TC}_n = \frac{k}{2} S_n \left[2\mathbf{1}_{\vartheta_{n+1} > \vartheta_n} (\vartheta_{n+1} - \vartheta_n) + (\vartheta_{n+1} - \vartheta_n) \right], \tag{4.7}$$

where $\mathbf{1}_A$ denotes the indicator function of the set A. The second term in the above expression can be treated exactly as in the previous section, hence we shall concentrate on the first one. Let us make the assumption that ϑ_n is a monotone function of the value of the underlying. Let us suppose that it is an increasing function (the case of a decreasing function being analogous), that is

$$\vartheta_{n+1} > \vartheta_n \Longleftrightarrow S_n > S_{n-1}.$$

This is, for instance, the case of the Delta hedging strategy for a call option. Hence

$$\vartheta_{n+1} > \vartheta_n \Longleftrightarrow \Delta X_n > 0,$$

therefore

$$\mathbf{1}_{\vartheta_{n+1} > \vartheta_n} \left(\vartheta_{n+1} - \vartheta_n\right) = \mathbf{1}_{\Delta X_n > 0} \vartheta_{n+1} - \mathbf{1}_{\Delta X_n > 0} \vartheta_n.$$

We are then concerned with terms of the form

$$S_n \mathbf{1}_{\Delta X_n > 0} \vartheta_{n+j} = S_n \int \mathbf{1}_{\Delta X_n > 0} f^{\vartheta}(z)_{n+j} S_{n+j-1}^{z-1} \Pi(dz), \tag{4.8}$$

with $j = 0$ or 1.

The expected value of a term in (4.8) can be written as inverse Laplace transform of

$$\begin{aligned}
&E\left[\mathbf{1}_{\Delta X_n > 0} f^{\vartheta}(z)_{n+j} S_n S_{n+j-1}^{z-1}\right] = \\
&= E\left[f^{\vartheta}(z)_{n+j} S_{n-1} S_{n-1}^{z-1} \mathbf{1}_{\Delta X_n > 0} \exp\left((j(z-1)+1)\Delta X_n\right)\right] = \\
&= f^{\vartheta}(z)_{n+j} S_0^z E\left[\exp\left(z \sum_{i=1}^{n-1} \Delta X_i\right)\right] \\
&\times E\left[\mathbf{1}_{\Delta X_n > 0} \exp\left((j(z-1)+1)\Delta X_n\right)\right] = \\
&= f^{\vartheta}(z)_{n+j} S_0^z m(z)^{n-1} m^+(j(z-1)+1),
\end{aligned}$$

where

$$m^+(z) = E\left[\mathbf{1}_{\Delta X_1 > 0} \exp\left(z \Delta X_1\right)\right].$$

The expected value of a term of (4.7) non-involving the indicator function gives a similar contribution, namely

$$E\left[f^{\vartheta}(z)_{n+j} S_n S_{n+j-1}^{z-1}\right] = f^{\vartheta}(z)_{n+j} S_0^z m(z)^{n-1} m(j(z-1)+1).$$

Summing up the four terms, one gets that the expected value of all the transaction costs can be written as the sum for n from 1 to $N-1$ of terms

$$\frac{1}{2} k \int S_0^z m(z)^{n-1} \left[f^{\vartheta}(z)_{n+1}(2m^+(z) - m(z)) - f^{\vartheta}(z)_n (2m^+(1) - m(1))\right] \Pi(dz).$$

The function $m^+(z)$ depends on the model considered for the underlying. For example, in the Black-Scholes model one has

$$m^+(z) = \frac{1}{2}\exp\left(\left(\mu z + \frac{\sigma^2}{2}z^2\right)\frac{T}{N}\right)\left[1 - \operatorname{erfz}\left(-\frac{\sigma z\sqrt{\frac{T}{N}}}{2}\right)\right],$$

where $\operatorname{erfz}(z)$ is the error function

$$\operatorname{erfz}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}dy.$$

The computations involved for the variance-covariance terms of $\varepsilon^{tc}(\vartheta, c)$ proceed along the same lines and are relegated to the Appendix for the sake of brevity.

5 Numerical implementation

There are at least two possible approaches to compute Equations (3.4), (3.5) and those involving transaction costs: numerical integration and inversion of Laplace transform. We followed the second approach, implementing the algorithms in MATLAB. The formulas we wish to compute involve one- and two-dimensional Laplace transforms. A list of the available MATLAB codes for the one-dimensional case can be found in [11]. For the one dimensional case (computation of expected values) we used "invlap.m" constructed by [6], based on the method by [4], which is accurate and fast. We wrote a code based on Formula (2.11) in [3] for the bi-dimensional case (to compute second moments).

The parameters of the algorithm in the code invlap.m for the one-dimensional inversion are two: the pole with largest real part of the function to be inverted and a tolerance parameter which essentially gives the distance from the largest pole of the vertical line of integration. The largest pole has to be given correctly: for instance, in the case of a call option, the largest pole is 1. The default value for the tolerance parameter (10^{-9}) gives, in our experience, rather accurate results.

As explained in [3], the two-dimensional algorithm depends on six parameters, A_1, A_2, l_1,l_2, n and m, and has three possible sources of error: the aliasing error, the roundoff error and the truncation error. Each error is controlled by two parameters. We found that $A_1 = A_2 = 30$ and $l_1 = l_2 = 1$ are good choices to get small, respectively, aliasing and roundoff errors. The parameters n and m are the most relevant for our computations; in fact, these are the parameters that control the Euler approximation to the infinite sums.

To give an idea of the results produced by the algorithm, we consider an at-the-money European call option with maturity $T = 0.25$ years, hedged only once, at time 0, using the Black-Scholes delta. We assume that the underlying process is a Geometric Brownian motion with drift $\mu = 0.1$, volatility $\sigma = 0.4$ and that the initial price is $S_0 = 100$. In this case, the expected value and the variance of the hedging error produced by this static strategy (the units of underlying are 0.5398) can be explicitly computed. The expected value, for an initial capital equal to the Black-Scholes price of the option (7.9655), is 0.062723168, the variance is 39.10233. The one-dimensional algorithm produces an expected value equal to 0.062723143 with largest pole set to 1 and default tolerance parameter.

To compute the variance we use our code to invert a double dimensional Laplace transform. In Table 1 we show the results produced by the algorithm as a function of m and n, the most important pair of parameters, keeping $A_1 = A_2 = 30$ and $l_1 = l_2 = 1$. We report the difference (multiplied by 10^4) between the computed variance and the exact one. Note that the number of terms in the approximating sum computed by the algorithm is $n + m$, hence the computational burden increases with n and m. From this example (but it is our general impression), it appears that a higher accuracy is reached by increasing n, while larger m's provide just a marginal improvement.

6 Applications

In this section we employ the methodology developed above to illustrate some applications related to hedging options with and without transaction costs. The purpose is more that of giving a hint on the potential applications of our approach than that of providing a deep exam of the specific problems presented. The applications we present here are: checking the precision of some well known approximating formulas, comparing ex-ante the effectiveness of different strategies for different models, measuring ex-ante the hedging error of an option trader who tries to speculate on the implied volatility.

m	n	err
20	40	494.21
50	50	156.21
50	100	33.91
100	100	20.01
100	200	4.31
200	200	2.51
20	400	1.01
50	400	0.91
100	400	0.71
20	600	0.31

Table 1: Absolute error (multiplied by 10^4) of variance of the hedging error of the delta strategy computed with the two-dimensional inversion algorithm for an at-the money call option with number of trading dates $N = 1$ as a function of m and n. The other parameters used by the algorithm are $A_1 = A_2 = 30$, $l_1 = l_2 = 1$.

6.1 Approximating formulas

Here we shall assess the precision of some approximations for the variance of the hedging error. Toft [21] provided a useful formula for computing an approximate value of the variance of the discretization error produced in the Black-Scholes model when hedging a European call option using the standard delta strategy. The formula, an approximation as the number of trading dates N goes to infinite, reads as follows

$$\begin{aligned} \mathrm{var}(\varepsilon(\Delta, c)) &\approx \frac{1}{2}\sigma^4 \left(\frac{T}{N}\right)^2 S_0^4 \Gamma_0^2 \sum_{i=0}^{N-1} g(t_i), \\ g(t) &= \sqrt{\frac{T^2}{T^2 - t^2}} \exp\left(2\mu t - 2d_1 \frac{(\mu - r)t}{\sigma\sqrt{T}} - \frac{(\mu - r)^2 t^2}{\sigma^2 T}\right) \times \\ &\quad \exp\left(\left[d_2^2 + 2d_2 \frac{(\mu - r)t}{\sigma\sqrt{T}} - \frac{(\mu - r)^2 t^2}{\sigma^2 T}\right] \frac{t}{T + t}\right), \end{aligned} \tag{6.9}$$

where Γ_0 is the option's gamma computed at time 0, and d_1 and d_2, the usual quantities in the Black-Scholes formula, are also computed at time 0.

A very popular approximation, proposed by Kamal an Derman [10], involving the option's vega κ_0 at time 0, is

$$\mathrm{var}(\varepsilon(\Delta, c)) \approx \frac{\pi}{4N}\sigma^2\kappa_0^2. \tag{6.10}$$

The top panel of Figure 1 represents the relative error of the two approximating formulas of standard deviation for hedging dates $N = [1, 3, 5, 7, 10, 13, 26, 39, 52, 65]$. The parameters used are $S_0 = 100$, $r = 0$, $\mu = 0.05$, $\sigma = 0.5$, $T = 1$, $K = 100$. We see that, in this case, the approximation (6.9) underestimate the standard deviation while (6.10) overestimate it. The error of the first formula is above 4% when the trading intervals are fewer than 10 but goes under 2% as N gets greater than 26. Formula (6.10) gives similar, if not better results, especially for small values of N.

In the Black-Scholes setting, Toft [21] finds an approximate value for the variance of the hedging error in presence of transactions costs, which is a generalization of (6.9). In particular, he shows an approximating formula for the variance of transactions costs valid if the hedging volatility is Leland's transactions costs adjusted volatility, namely

$$\bar{\sigma} = \sigma\sqrt{1 + \frac{\sqrt{2/\pi}k}{\sigma\sqrt{\frac{T}{N}}}}.$$

Denoting with $\bar{\Gamma}_n$ the adjusted Gamma of the option at time n, introduced by Leland [13], the variance of each transaction cost is approximated by

$$\mathrm{var}(TC_n) = \frac{1}{4}\Psi_{0,n}k^2\left(1 - \frac{2}{\pi}\right)\left(\frac{N}{T}\right)^2, \tag{6.11}$$

where the term $\Psi_{0,n} = E\left[\bar{\Gamma}_n S_n^2\right]$ is explicitly computed in [21]. This formula is based on the approximation, proposed by Leland [13], of the transaction cost at time n

$$TC_n \approx \frac{1}{2}k\bar{\Gamma}_n S_n^2 |\frac{S_{n+1} - S_n}{S_n}|. \tag{6.12}$$

However, Kabanov and Safarian [9] proved that, when k does not depend on N, as in Leland's and Toft's setting, approximating the total transaction costs by simply summing up the approximating terms (6.12) produces an error which remains bounded even when N goes to infinity. In the bottom panel of Figure 1, we represent the relative error of approximation (6.11) of standard deviation of the transaction cost at time $N-1$. We considered up to a high number of trading dates, $N = [6, 12, 24, 36, 48, 60, 90, 120, 240, 360, 720]$, to show that, even for a single term, as far as variance is concerned, the error is indeed non-vanishing. To compute the total variance of transaction costs, one should sum up all the variances and the covariances between transaction costs at different times. The general result of [9] is that, when the level k of transaction cost is constant, the hedging error does not converge to zero as N goes to infinity. However, the hedging error does converge to zero when k is of the form $k_0 N^{-\alpha}$, with $\alpha \in (0, 1/2)$, and $k_0 > 0$. This convergence result holds also for $\alpha = 1/2$, as it was proven by Lott [14].

6.2 Comparing the strategies

Now we want to compute the mean and the variance of the errors produced by different strategies to hedge a European call option without transaction costs. We suppose that the initial value of the strategy c is equal to the Black-Scholes value. To provide an ex-ante measurement of the performances of the strategies we compute the expected values and the standard deviations of their final shortfalls. As it was shown above, the initial capital c only influences the expected values, leaving the standard deviations unaffected, also for the optimal strategy.

The natural goal for the hedger is to get a negative expected loss (i.e. a gain), hopefully with a small variance. A possible way to take both objectives into account is to compute the Sharpe index of the strategy $s(\vartheta, c) = \frac{-E[\varepsilon(\vartheta,c)]}{\sqrt{\mathrm{var}(\varepsilon(\vartheta,c))}}$.

We compare the following strategies:

1. The Black-Scholes delta hedging strategy;
2. The "improved delta" strategy (Wilmott [22]);
3. The local optimal strategy ;
4. The optimal strategy .

For all of the instances we analysed, the improved-delta strategy looked almost indistinguishable from the local optimal one, and therefore we decided not to include it in the figures.

We consider an at-the-money European call option with maturity $T = 0.25$ years where the initial price of the underlying asset is $S_0 = 100$. We assume that a trader believes that the underlying follows a geometric Brownian motion with mean $\mu \approx 0.1451$ and volatility $\sigma \approx 0.4379$. The trader has the choice between different trading strategies with different numbers of trading dates, namely $N = (1, 3, 5, 7, 10, 13, 26, 39, 52, 65)$. We suppose that the price of the option is $c = 8.7176$, corresponding to the volatility σ. The trader sells the option and invest all the money in the hedging strategy.

We consider two cases for the data generating process of the underlying:

1. Geometric Brownian motion with parameters μ and σ (i.e. the trader is adopting the correct model)
2. Merton jump-diffusion process with normally distributed jumps, with parameters of the Geometric Brownian motion $\mu' = 0.05$ and $\sigma' = 0.3$, intensity of the jumps $\lambda = 10$, and mean and standard deviation of the jumps respectively $\nu = 0$ and $\tau = 0.1$. Note that, with such choice of parameters, the trader, although using an incorrect model, is estimating the correct values for the mean and the standard deviations of the returns.

The results of the first case are reported in Figure 2. In the top panel we represent expected values of the total loss for the strategies considered as functions of the number of trading dates N. In the middle panel we plot the ratio between the standard deviation of each strategy and the minimal variance (achieved by the optimal strategy). We note that, as the number of trading dates increases, the means and the variances of all the strategies go to zero, as expected since the model becomes complete in the limit. The expected value for the standard Black-Scholes delta is positive and quite different from other strategies. The reason for that is that the positive μ is ignored by the delta hedging strategy but taken into account by all others. This is also reflected by the Sharpe indexes in the bottom panel of Figure 2 showing the worse ratio attained by the Delta hedging strategy.

In the second case we assume that the trader follows a strategy based on the Black-Scholes model, while the data generating process is the jump-diffusion Merton model. This is an application of Theorem 3.1 to the case of a strategy based on incorrect modeling assumptions. Analysis like this one may offer an insight on the influence of model risk on the performances of different hedging strategies. The trader may adopt the delta strategy or the local optimal strategy, both based on the Black-Scholes model with the observed parameters. Notice that we cannot analyze the performance of the optimal strategy based on a model other than the data generating one, because that would not be a compatible strategy. However, the local optimal

Black-Scholes strategy is a good proxy for the optimal one. Alternatively, if the trader had a perfect knowledge of the data generating process she/he could use either the local or the global optimal strategy. As usual, we use the optimal strategy to serve as a benchmark.

The results of the second case are represented in Figure 3. In this case, the model is not complete in the limit and therefore, neither the expected values (top panel) nor the standard deviations of the strategies go to zero. The smallest value (2.87) of the standard deviation is achieved for $N = 65$ by the optimal strategy. Interestingly enough, the local optimal Black-Scholes strategy, performs better, with respect to the Sharpe index (bottom panel), than the standard delta strategy also in the case of model mispecification. In particular, the standard deviation of the delta strategy is consistently 2% higher than that of the optimal one, while that of the log-normal local optimal stays under 2%, as it is shown in Figure 3, middle panel.

6.3 Exploiting the personal views

Now we assume that a trader has a view on the future values of the volatility of the underlying and wants to implement a profitable strategy. In particular, we suppose that the market price of the option considered in the previous section is $c = 5.9785$, corresponding to an implied volatility $\sigma_0 = 0.3$. The trader believes that the underlying asset follows a Geometric Brownian motion with a lower volatility and therefore she/he sells the option and hedges it using the market implied volatility σ_0.

The results of this experiment are in Figure 4. The top panel reports the Sharpe indexes of the delta hedging and of the local optimal strategy, assuming a drift rate $\mu_0 = 0.1$ and a number of trading dates $N = 10$, as the actual volatility σ ranges from 0.1 to 0.5. As expected, the Sharpe index is positive when σ is lower than σ_0, that is when the views of the trader are confirmed. We note that the local optimal strategy gives a consistently slightly better performance. When $\sigma = \sigma_0$ the Sharpe index of the delta is negative (-0.0052), while that of the local optimal strategy is positive (0.0099). In this case we can compute the Sharpe index of the optimal strategy, which turns out to be 0.01. It is evident that the greater influence on the performance of the strategy is due to the difference between the hedging and the realized volatility, rather than to the strategy followed.

The lower panel of Figure 4 represents the influence of the drift on the Sharpe ratios of the two strategies. It appears that the Sharpe ratio of the local optimal strategy is consistently better than that of the delta hedging when μ is not zero. In the martingale case the performances of the two strategies are very similar.

The analysis is concluded by Figure 5 that shows the influence of both the actual volatility σ and the actual drift μ on the Sharpe index of the local optimal strategy, assuming a hedging volatility $\sigma_0 = 0.3$ and a hedging drift $\mu_0 = 0$. It is evident that the influence of the volatility is much stronger than that of the drift.

7 Conclusions

Using the inverse Laplace transform, we are able to measure the error of a hedging strategy for a contingent claim as measured in terms of expected value and variance. The contingent claim must be of European type with a payoff representable as an inverse Laplace transform. The strategies for which the method can be adopted must be compatible. i.e. they must have an integral representation too.

Our analysis applies to several interesting strategies, as the Black-Scholes delta, the improved-delta and the local optimal one, and to a fairly general class of models, including Black-Scholes, Merton's jump-diffusion and Normal Inverse Gaussian. The method may also be applied in presence of transaction costs. A relevant contribution of our results is that they are not asymptotical approximations but exact and efficient formulas, valid for any number of trading dates. Through our approach we were able to asses the precision of existing approximating formulas, as those proposed by [21] and [10] for the variance of the hedging error. We found that, for the cases examined, as the number of trading intervals increases, such formulas provide good approximations when transaction costs are not taken into consideration; however, in presence of transaction costs, the errors remain bounded away from zero, as theoretically shown by Kabanov and Safarian [9].

We showed some possible applications of our findings. In particular, we compared the performance of different strategies under various model settings, taking as a benchmark the optimal-variance strategy and as a main performance measure the Sharpe index. The computations may be done also under model mispecification, hence we measure the influence of model risk on hedging strategies. From the cases examined resulted that the delta hedging strategy, the simplest and most used of the dynamic strategies is always over-performed by the other strategies.

We also quantified, always in terms of the expected gain (or loss) over units of risk, within the Black-Scholes model, the effect of a wrong forecast of the drift and the volatility of the underlying on the performance of the hedging. Our partial analysis showed that a wrong choice of model parameters has a stronger impact on the hedging performances than the choice of the particular strategy adopted.

The Laplace transform approach is a very promising and effective tool for the quantitative measure of the risk involved in dynamic strategies. We believe that possible fields of extensions of the present approach include multi-variate analysis, moved-based hedging strategies and processes with autocorrelated increments.

8 Appendix

Here we give the basic expressions for the second moment of $\varepsilon^{\mathrm{tc}}(\vartheta, c)$, the hedging error inclusive of transaction costs. Starting from Theorem (3.1), it remains to compute the variance of $\sum_{n=1}^{N} TC_n$ and its covariance with $\varepsilon(\vartheta, 0)$. We base ourselves on (4.7) and (4.8). For the covariance of transaction costs we have to compute terms like:

$$\phi_c(n, m, h, k, i, j) = E\left[\mathbf{1}^i_{\Delta X_n > 0} \vartheta_{n+h} S_n \mathbf{1}^j_{\Delta X_m > 0} \vartheta_{m+k} S_m\right], \tag{8.13}$$

for h, k, i and j equals 0 or 1, $n, m = 1, \dots N-1$, with say $n > m$. For the variances of single transaction costs, we just need

$$\phi_v(n, h, k) = E\left[\vartheta_{n+h} S_n \vartheta_{n+k} S_n\right], \tag{8.14}$$

for h and k equals 0 or 1 and $n = 1, \dots, N-1$. As for the covariance with $\varepsilon(\vartheta, 0)$, we need terms like

$$\psi(n, m, k, j) = E\left[S_n \Delta S_n \mathbf{1}^j_{\Delta X_m > 0} \vartheta_{m+k} S_m\right], \tag{8.15}$$

for k and j equals 0 or 1 and $n = 1, \dots, N$, $m = 1, \dots, N-1$, and

$$\psi_h(m, k, j) = E\left[H \mathbf{1}^j_{\Delta X_m > 0} \vartheta_{m+k} S_m\right], \tag{8.16}$$

for k and j equals 0 or 1 and $m = 1, \dots, N-1$. Suitably summing up all the terms, we can easily compute that second moment. First, let us define

$$m_a(z) = \begin{cases} m^+(z) & a = 1; \\ m(z) & a = 0. \end{cases}$$

Since ϑ is compatible, we have

$$\begin{aligned}\phi_c(n,m,h,k,i,j) &= \int S_0^{y+z} f^{\vartheta}(z)_{n+h} f^{\vartheta}(y)_{m+k} \times \\ & m(y+z)^{m-1} m(z)^{n-m-1} \times \\ & m_i(h(z-1)+1) m_j(k(y-1)+z+1)\Pi(dz)\Pi(dy),\end{aligned}$$

while

$$\begin{aligned}\phi_v(n,h,k) &= \int S_0^{y+z} f^{\vartheta}(z)_{n+h} f^{\vartheta}(y)_{n+k} \times \\ & m(y+z)^{n-1} m(h(z-1)+k(y-1)+2)\Pi(dz)\Pi(dy).\end{aligned}$$

Then we have

$$\psi(n,m,k,j) = \int S_0^{y+z} f^{\vartheta}(z)_n f^{\vartheta}(y)_{m+k} A(n,m,k,j)\Pi(dz)\Pi(dy),$$

where

$$A(n,m,k,j) = \begin{cases} m(y+z)^{m-1} m(z)^{n-m-1} \times & \\ (m(1)-1) m_j(z+k(y-1)+1) & n > m; \\ m(y+z)^{n-1} m(y)^{m-n-1} \times & \\ (m(y+1)-m(y)) m_j(k(y-1)+1) & n < m; \\ m(z+y)^{n-1} \times & \\ (m_j(k(y-1)+2) - m_j(k(y-1)+1)) n = m. & \end{cases}$$

Finally,

$$\begin{aligned}\psi_h(m,k,j) &= \int S_0^{y+z} f^{\vartheta}(y)_{m+k} \times \\ & m(y+z)^{m-1} m(z)^{N-m} \times \\ & m_j(z+k(y-1)+1)\Pi(dz)\Pi(dy).\end{aligned}$$

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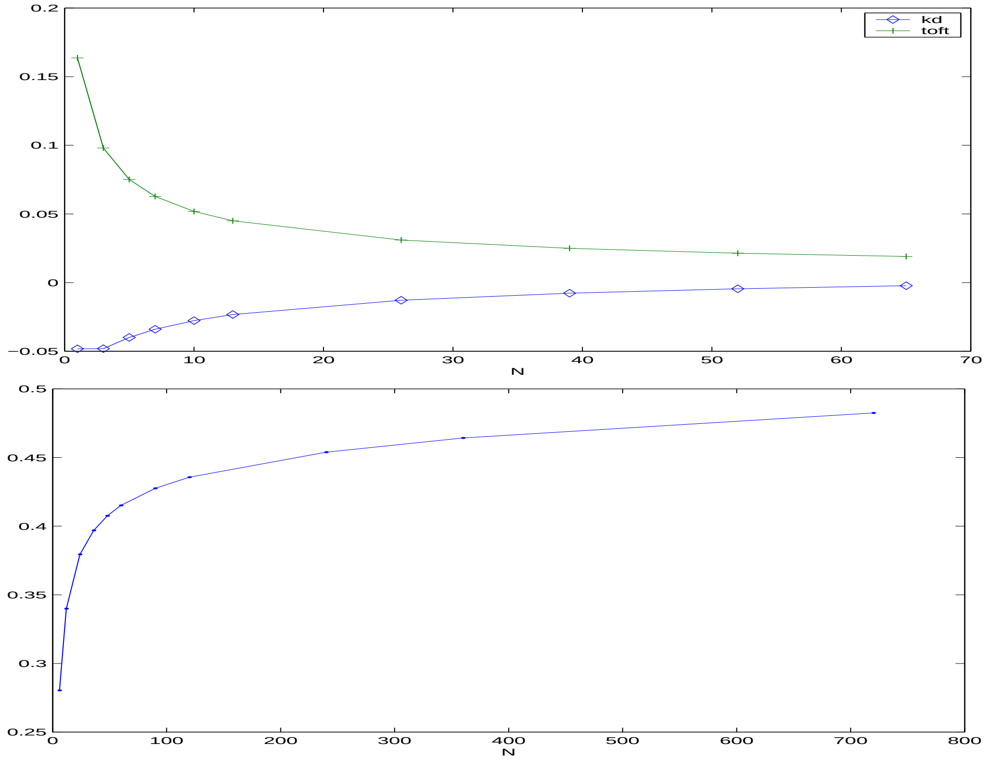

Figure 1 *Approximation errors of asymptotic formulas. Black-Scholes model with* $S_0 = 100$*,* $r = 0$*,* $\mu = 0.05$*,* $\sigma = 0.5$*, European call option with* $K = 100$*,* $T = 1$*. Top: relative error (1-approx/exact) of Toft's and Kamal-Derman's vega approximations of the standard deviation of the hedging error as a function of the number of trading intervals (*$N = [1, 3, 5, 7, 10, 13, 26, 39, 52, 65]$*). Bottom: relative error (1-approx/exact) of Toft's approximation of the single standard deviation of last transaction cost as a function of the number of trading intervals (*$N = [6, 12, 24, 36, 48, 60, 90, 120, 240, 360, 720]$*).*

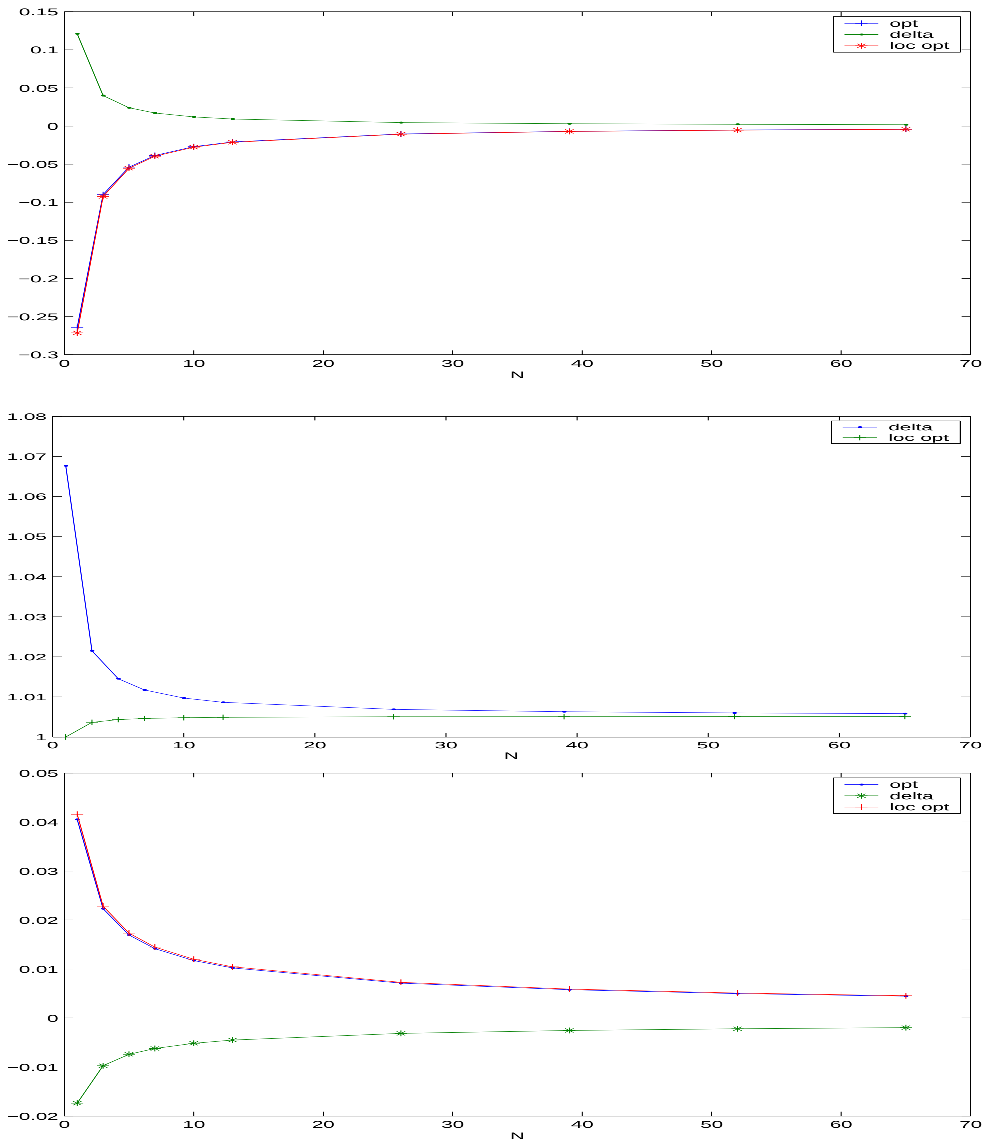

Figure 2 *Three measures of the hedging error of different strategies as the number of trading dates increases when the model (Black-Scholes) is correct. Expected value (top), ratio of standard deviation with respect to the optimal one (middle) and Sharpe index (bottom) of hedging error.*

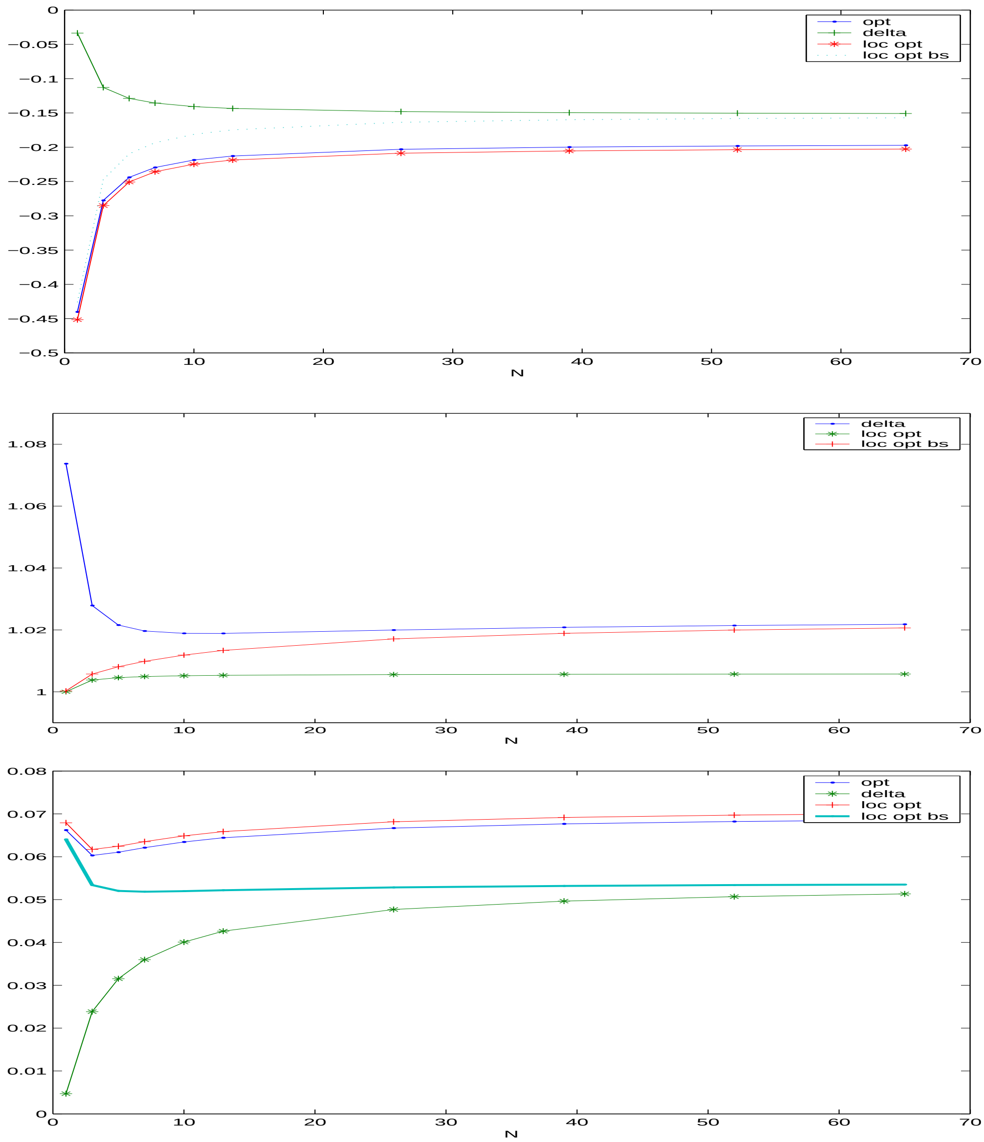

Figure 3 *Three measures of the hedging error of different strategies when the hedging model is incorrect. The delta and the local optimal bs strategies are based on the Black-Scholes model, the local optimal and the optimal ones, are based on the data generating process that is Merton's jump-diffusion. The panels represent: expected value (top), ratio of standard deviation with respect to the optimal one (middle) and Sharpe index (bottom) of hedging error of different strategies as the number of trading dates increases.*

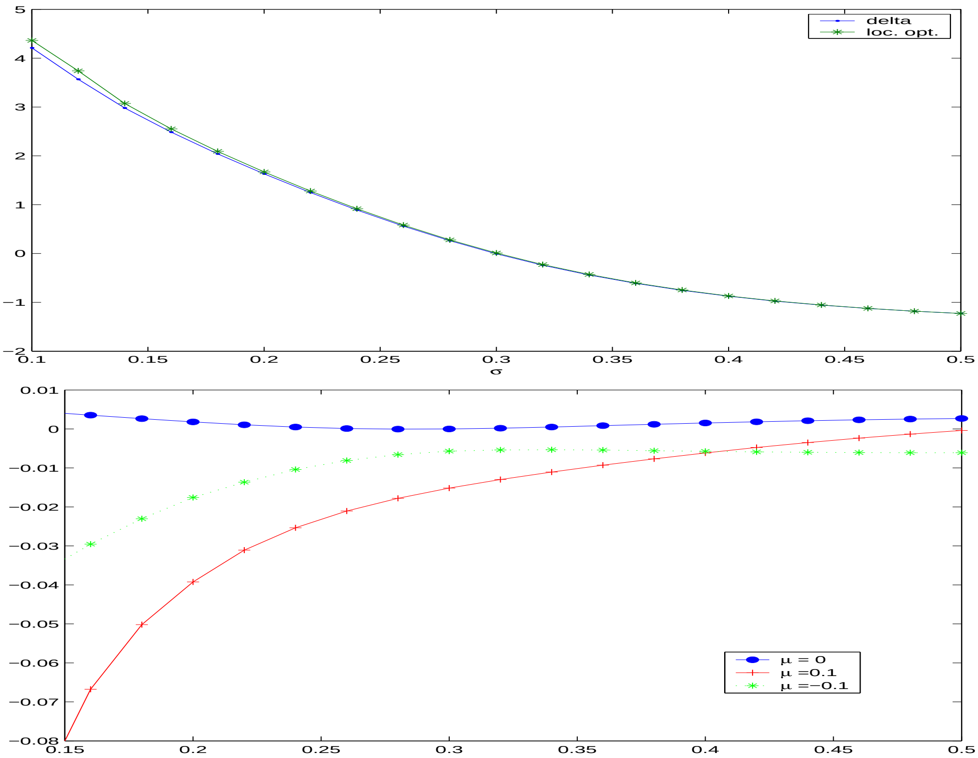

Figure 4 *Influence of the actual parameters on the performances of delta hedging and of the local optimal strategies based on personal views on the volatility parameter σ. Black-Scholes model, with $S_0 = K = 100$ and number of trading dates $N = 10$. Top: Sharpe index of hedging error of delta and local optimal strategies as a function of σ, the actual volatility of the Black-Scholes process. The strategies are constructed with $\sigma_0 = 0.3$ and $\mu_0 = 0.1$ (assuming that the value of μ_0 has been correctly estimated) . Bottom: differences between Sharpe indexes of hedging error of delta and of local optimal strategies as a function of σ for different values of μ. The strategies are constructed with $\sigma_0 = 0.3$ and the correct value of μ. The three curves correspond to $\mu = 0, 0.1, -0.1$*

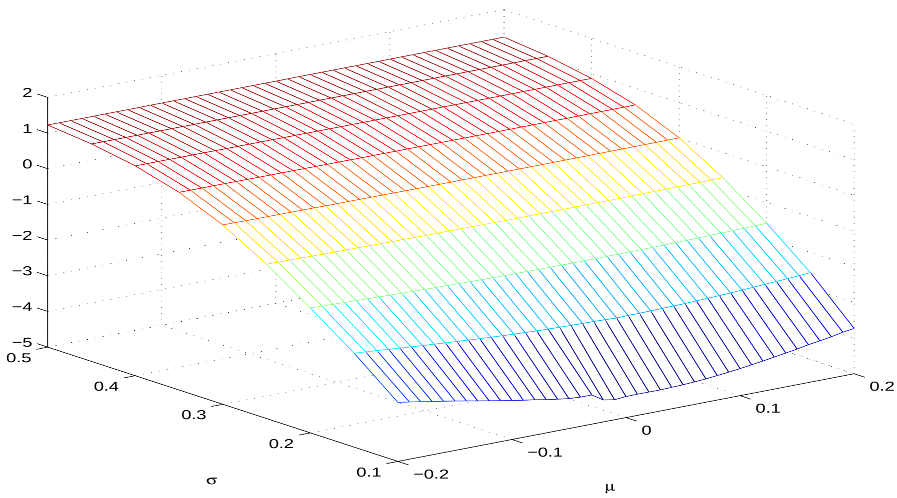

Figure 5 *Dependence of the Sharpe index of a local optimal strategy on the actual parameters* μ *and* σ*. Black-Scholes model, with* $S_0 = K = 100$*, number of trading dates* $N = 10$*. The strategy is constructed assuming* $\sigma_0 = 0.3$ *and* $\mu_0 = 0$*.*

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