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CORE DISCUSSION PAPER
2008/16

Von Neumann-Morgenstern farsightedly stable sets in two-sided matching

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March 2008

Abstract

We adopt the notion of von Neumann-Morgenstern farsightedly stable sets to predict which matchings are possibly stable when agents are farsighted in one-to-one matching problems. We provide the characterization of von Neumann-Morgenstern farsightedly stable sets: a set of matchings is a von Neumann-Morgenstern farsightedly stable set if and only if it is a singleton set and its element is a corewise stable matching. Thus, contrary to the von Neumann-Morgenstern (myopically) stable sets, von Neumann-Morgenstern farsightedly stable sets cannot include matchings that are not corewise stable ones. Moreover, we show that our main result is robust to many-to-one matching problems with responsive preferences.

Keywords: matching problem, von Neumann-Morgenstern stable sets, farsighted stability

JEL Classification: C71, C78

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This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

Gale and Shapley (1962) have proposed the simple two-sided matching model, known as the marriage problem, in which matchings are one-to-one. There are two disjoint sets of agents, men and women, and the problem is to match agents from one side of the market with agents from the other side where each agent has the possibility of remaining single. They have shown that the set of corewise stable matchings is nonempty. A matching is corewise stable if there is no subset of agents who by forming all their partnerships only among themselves (and having the possibility of becoming single), can all obtain a strictly preferred set of partners.[1] Recently, Ehlers (2007) has characterized the von Neumann-Morgenstern stable sets (hereafter, vNM stable sets) in one-to-one matching problems. A set of matchings is a vNM stable set if this set satisfies two robustness conditions: (internal stability) no matching inside the set is dominated by a matching belonging to the set; (external stability) any matching outside the set is dominated by some matching belonging to the set. Ehlers has shown that the set of corewise stable matchings is a subset of any vNM stable sets.

The notions of corewise stability and of vNM stable set are myopic notions since the agents cannot be farsighted in the sense that individual and coalitional deviations cannot be countered by subsequent deviations. A first contribution that reflects the idea that agents are not myopic is the concept of weak stability due to Klijn and Massó (2003). An individually rational matching is weakly stable if for every blocking pair one of the members can find a more attractive partner with whom she/he forms another blocking pair for the original matching. Another interesting contribution is Diamantoudi and Xue (2003) who have investigated farsighted stability in hedonic games (of which one-to-one matching problems are a special case) by introducing the notion of the coalitional largest farsighted conservative stable set which coincides with the largest consistent set due to Chwe (1994).[2] The largest consistent set is based on the indirect dominance relation which captures the fact that farsighted agents consider the end matching that their move(s) may lead to. Diamantoudi and Xue (2003) have shown that in hedonic games with strict preferences core partitions are always contained in the largest consistent set.

In this paper, we adopt the notion of von Neumann-Morgenstern farsightedly stable sets (hereafter, vNM farsightedly stable sets) to predict which matchings are possibly stable when agents are farsighted. This concept is due to Chwe (1994) who has introduced the

[1] We refer to Roth and Sotomayor (1990) for a comprehensive overview on two-sided matching problems (marriage problems and college admissions or hospital-intern problems).

[2] Other approaches to farsightedness in coalition formation are suggested by the work of Xue (1998), or Herings, Mauleon, and Vannetelbosch (2004).

notion of indirect dominance into the standard definition of vNM stable sets. Thus, a set of matchings is a vNM farsightedly stable set if no matching inside the set is *indirectly* dominated by a matching belonging to the set (internal stability), and any matching outside the set is *indirectly* dominated by some matching belonging to the set (external stability).

Our main result is the characterization of vNM farsightedly stable sets in one-to-one matching problems. A set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a corewise stable matching. Thus, contrary to the vNM (myopically) stable sets, vNM farsightedly stable sets cannot include matchings that are not corewise stable ones. In other words, we provide an alternative characterization of the core in one-to-one matching problems. Finally, we show that our main result is robust to many-to-one matching problems with responsive preferences: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a setwise stable matching.

With respect to other farsighted concepts, we show that matchings that do not belong to any vNM farsightedly stable sets (hence, that are not corewise stable matchings) may belong to the largest consistent set. We also provide an example showing that there is no relationship between vNM farsightedly stable sets and the set of weakly stable matchings.

The paper is organized as follows. Section 2 introduces one-to-one matching problems and standard notions of stability. Section 3 defines vNM farsightedly stable sets. Section 4 provides the characterization of vNM farsightedly stable sets in one-to-one matching problems. Section 5 deals with many-to-one matching problems. Section 6 concludes.

2 One-to-one matching problems

A one-to-one matching problem consists of a set of N agents divided into a set of men, $M = \{m_1, ..., m_r\}$, and a set of women, $W = \{w_1, ..., w_s\}$, where possibly $r \neq s$. We sometimes denote a generic agent by i and a generic man and a generic woman by m and w, respectively. Each agent has a complete and transitive preference ordering over the agents on the other side of the market and the prospect of being alone. Preferences are assumed to be strict. Let P be a preference profile specifying for each man $m \in M$ a strict preference ordering over $W \cup \{m\}$ and for each woman $w \in W$ a strict preference ordering over $M \cup \{w\}$: $P = \{P(m_1), ..., P(m_r), P(w_1), ..., P(w_s)\}$, where $P(i)$ is agent i's strict preference ordering over the agents on the other side of the market and himself (or herself). For instance, $P(w) = m_4, m_1, w, m_2, m_3, .., m_r$ indicates that woman w prefers m_4 to m_1 and she prefers to remain single rather than to marry anyone else. We write $m \succ_w m'$ if woman w strictly prefers m to m', $m \sim_w m'$ if w is indifferent between m and

m', and $m \succeq_w m'$ if $m \succ_w m'$ or $m \sim_w m'$. Similarly, we write $w \succ_m w'$, $w \sim_m w'$, and $w \succeq_m w'$. A one-to-one matching market is simply a triple (M, W, P).

A matching is a function $\mu : N \to N$ satisfying the following properties: (i) $\forall m \in M$, $\mu(m) \in W \cup \{m\}$; (ii) $\forall w \in W$, $\mu(w) \in M \cup \{w\}$; and (iii) $\forall i \in N$, $\mu(\mu(i)) = i$. We denote by $\mathcal{M}$ the set of all matchings. Given matching μ, an agent i is said to be unmatched or single if $\mu(i) = i$. A matching μ is *individually rational* if each agent is acceptable to his or her mate, i.e. $\mu(i) \succeq_i i$ for all $i \in N$. For a given matching μ, a pair $\{m, w\}$ is said to form a blocking pair if they are not matched to one another but prefer one another to their mates at μ, i.e. $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. We shall often abuse notation by omitting the brackets to denote a set with a unique element; here we write $\mu(i) = j$ instead of $\mu(i) = \{j\}$. A coalition S is a subset of the set of agents N.[3]

Definition 1 (corewise enforceability) *Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following condition holds*: $\forall i \in S$, *if $\mu'(i) \neq \mu(i)$, then $\mu'(i) \in S$.*

Notice that the concept of enforceability is independent of preferences.

Definition 2 *A matching μ is directly dominated by μ', or $\mu < \mu'$, if there exists a coalition $S \subseteq N$ of agents such that $\mu' \succ \mu$ $\forall i \in S$ and S can enforce μ' over μ.*

Definition 2 gives us the definition of direct dominance. The direct dominance relation is denoted by $<$. A matching μ is corewise stable if there is no subset of agents who by forming all their partnerships only among themselves, possibly dissolving some partnerships of μ, can all obtain a strictly preferred set of partners. Formally, a matching μ is corewise stable if μ is not directly dominated by any other matching $\mu' \in \mathcal{M}$.[4] Let $C(<)$ denote the set of corewise stable matchings. Gale and Shapley (1962) have proved that the set of corewise stable matchings is non-empty. Sotomayor (1996) has provided a non-constructive elementary proof of the existence of stable marriages.[5]

Another concept used to study one-to-one matching problems is the vNM stable set (von Neumann and Morgenstern, 1953), a set-valued concept, that imposes both internal and external stability. A set of matchings is a vNM stable set if (internal stability) no

[3] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion.

[4] Setting $|S| \leq 2$ in the definition of corewise stability, we obtain Gale and Shapley's (1962) concept of pairwise stability that is equivalent to corewise stability in one-to-one matchings with strict preferences.

[5] Roth and Vande Vate (1990) have demonstrated that, starting from an arbitrary matching, the process of allowing randomly chosen blocking pairs to match will converge to a corewise stable matching with probability one in the marriage problem. In relation, Jackson and Watts (2002) have shown that if preferences are strict, then the set of stochastically stable matchings coincides with the set of corewise stable matchings.

matching inside the set is directly dominated by a matching belonging to the set, and (external stability) any matching outside the set is directly dominated by some matching belonging to the set.

Definition 3 *A set of matchings $V \subseteq \mathcal{M}$ is a vNM stable set if*

(i) *For all μ, $\mu' \in V$, $\mu \not> \mu'$;*

(ii) *For all $\mu' \in \mathcal{M} \backslash V$ there exists $\mu \in V$ such that $\mu > \mu'$.*

Definition 3 gives us the definition of a vNM stable set $V(<)$. Let $\mathcal{V}(<)$ be the set of all vNM stable sets. Ehlers (2007) has studied the properties of the vNM stable sets in one-to-one matching problems. He has shown that the set of corewise stable matchings is a subset of vNM stable sets. Example 1 illustrates his main result.

Example 1 (Ehlers, 2005) *Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let P be such that*:

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(w_1)$	$P(w_2)$	$P(w_3)$
w_1	w_2	w_3	m_2	m_3	m_1
w_2	w_3	w_1	m_3	m_1	m_2
m_1	m_2	m_3	w_1	w_2	w_3
w_3	w_1	w_2	m_1	m_2	m_3

Let

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{pmatrix}, \mu' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_3 & w_1 \end{pmatrix}, \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_3 & w_1 & w_2 \end{pmatrix}.$$

It can be shown that the set of corewise stable matchings is $C = \{\mu'\}$ and $V = \{\mu, \mu', \mu''\}$ is the unique vNM stable set.□

In Example 1 the matchings μ and μ'' belong to the unique vNM stable set because μ' does not directly dominate neither μ nor μ'' even though μ and μ'' are not individually rational matchings (either all women or all men prefer to become single). However, farsighted women may decide first to become single in the expectation that further marriages will be formed leading to μ'. The women prefer μ' to μ and once everybody is divorced, men and women prefer μ' to the situation where everybody is single. A similar reasoning can be made for μ'' with the roles of men and women reversed. Then, we may say that (i) μ' farsightedly dominates μ, (ii) μ' farsightedly dominates μ'', and (iii) $V = \{\mu, \mu', \mu''\}$ is not a reasonable candidate for being a vNM farsightedly stable set.

The concept of weak stability due to Klijn and Massó (2003) is a first interesting attempt to reflect the idea that agents are not myopic. A matching μ is weakly stable if for all μ' such that $\mu' > \mu$ with μ' being enforced over μ by S, there exists μ'' such that $\mu'' > \mu$ with μ'' being enforced over μ by coalition T such that:

(i) $T \backslash S \neq \varnothing$, $S \backslash T \neq \varnothing$, and $T \cap S \neq \varnothing$;

(ii) $\mu''(i) \succ_i \mu'(i)$ for some $i \in T \cap S$.

Since any corewise stable matching is a weakly stable matching, the set of weakly stable matchings is non-empty.[6] A matching μ is said to be weakly efficient if there is no other matching μ' such that $\mu'(i) \succ_i \mu(i)$ for all $i \in N$. Klijn and Massó (2003) have shown that any corewise stable matching is weakly efficient. However, there are weakly stable matchings that are not weakly efficient as shown in Example 2.

Following Zhou (1994), Klijn and Massó (2003) have defined the notion of bargaining set for one-to-one matching problems as follows. An objection against a matching μ is a pair (S, μ') where μ' is a matching that can be enforced over μ by S and such that $\mu'(i) \succ_i \mu(i)$ for all $i \in S$. A counterobjection against an objection (S, μ') is a pair (T, μ'') where μ'' is a matching that can be enforced over μ by T such that:

(i) $T \backslash S \neq \varnothing$, $S \backslash T \neq \varnothing$, and $T \cap S \neq \varnothing$;

(ii) $\mu''(i) \succeq_i \mu(i)$ for all $i \in T \backslash S$ and $\mu''(i) \succeq_i \mu'(i)$ for all $i \in T \cap S$.

An objection (S, μ') against a matching is justified if there is no counterobjection against (S, μ'). The bargaining set is the set of matchings that have no justified objections. The set of corewise stable matchings is a subset of the bargaining set. Klijn and Massó (2003) have shown that the bargaining set in one-to-one matching markets coincides with the set of matchings that are both weakly stable and weakly efficient.

Example 2 (Klijn and Massó, 2003) *Let* $M = \{m_1, m_2, m_3\}$ *and* $W = \{w_1, w_2, w_3\}$. *Let* P *be such that*:

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(w_1)$	$P(w_2)$	$P(w_3)$
w_3	w_1	w_2	m_3	m_1	m_2
w_2	w_3	w_1	m_2	m_3	m_1
w_1	w_2	w_3	m_1	m_2	m_3
m_1	m_2	m_3	w_1	w_2	w_3

[6] Setting $|S| \leq 2$ and $|T| \leq 2$ in the above definition of weak stability, we obtain the original concept of *weak stability* introduced by Klijn and Massó (2003). Both definitions are equivalent in one-to-one matchings since essential coalitions are pairs of agents.

Let

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{pmatrix}, \mu' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_3 & w_1 \end{pmatrix}, \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_3 & w_1 & w_2 \end{pmatrix}.$$

It can be shown that μ is a weakly stable matching but μ is not weakly efficient since $\mu'(i) \succ_i \mu(i)$ for all $i \in N$. Thus, μ does not belong to the bargaining set. The bargaining set is equal to $\{\mu', \mu''\}$ which is also the set of corewise stable matchings.$\square$

3 Von Neumann-Morgenstern farsighted stability

The indirect dominance relation was first introduced by Harsanyi (1974) but was later formalized by Chwe (1994). It captures the idea that coalitions of agents can anticipate the actions of other coalitions. In other words, the indirect dominance relation captures the fact that farsighted coalitions consider the end matching that their matching(s) may lead to. A matching μ' indirectly dominates μ if μ' can replace μ in a sequence of matchings, such that at each matching along the sequence all deviators are strictly better off at the end matching μ' compared to the status-quo they face. Formally, indirect dominance is defined as follows.

Definition 4 *A matching μ is indirectly dominated by μ', or $\mu \ll \mu'$, if there exists a sequence of matchings $\mu^0, \mu^1, ..., \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of coalitions $S^0, S^1, ..., S^{K-1}$ such that for any $k \in \{1, ..., K-1\}$,*

(i) $\mu^K \succ \mu^{k-1}$ $\forall i \in S^{k-1}$, *and*

(ii) *coalition S^{k-1} can enforce the matching μ^k over μ^{k-1}.*

Definition 4 gives us the definition of indirect dominance. The indirect dominance relation is denoted by $\ll$. Direct dominance is obtained by setting $K = 1$ in Definition 4. Obviously, if $\mu < \mu'$, then $\mu \ll \mu'$.

Diamantoudi and Xue (2003) have investigated farsighted stability in hedonic games (of which one-to-one matching problems are a special case) introducing the notion of the coalitional largest farsighted conservative stable set which coincides with the largest consistent set due to Chwe (1994).[7]

[7] The largest consistent set exists, is non-empty, and satisfies external stability (i.e. any matching outside the set is indirectly dominated by some matching belonging to the set). But a consistent set does not necessarily satisfy the external stability condition. Only the largest consistent set is guaranteed to satisfy external stability.

Definition 5 *$Z(\ll) \subseteq \mathcal{M}$ is a consistent set if $\mu \in Z(\ll)$ if and only if $\forall \mu'$, S such that S can enforce μ' over μ, $\exists \mu'' \in Z(\ll)$, where $\mu' = \mu''$ or $\mu' \ll \mu''$, such that $\mu(i) \not<_i \mu''(i)$ $\forall i \in S$. The largest consistent set $\Gamma(\ll)$ is the consistent set that contains any consistent set.*

Interestingly, Diamantoudi and Xue (2003) have proved that in hedonic games with strict preferences (i) any partition belonging to the core indirectly dominates any other partition, and (ii) core partitions are always contained in the largest consistent set. Thus, in one-to-one matching markets, for all $\mu' \neq \mu$ with $\mu \in C(<)$ we have that $\mu \gg \mu'$, and $C(<) \subseteq \Gamma(\ll)$. However, the largest consistent set may contain more matchings than those matchings that are corewise stable as shown in Example 3.

Example 3 (Knuth, 1976) *Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let P be such that:*

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(m_4)$	$P(w_1)$	$P(w_2)$	$P(w_3)$	$P(w_4)$
w_1	w_2	w_3	w_4	m_4	m_3	m_2	m_1
w_2	w_1	w_4	w_3	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_2	m_1	m_4	m_3
w_4	w_3	w_2	w_1	m_1	m_2	m_3	m_4
m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4

Let

$$\mu' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_3 & w_2 & w_4 \end{pmatrix}, \ \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_4 & w_2 & w_3 & w_1 \end{pmatrix}.$$

There are ten corewise stable matchings where men (m_1, m_2, m_3, m_4) are matched to women (w_1, w_2, w_3, w_4), (w_2, w_1, w_3, w_4), (w_1, w_2, w_4, w_3), (w_2, w_1, w_4, w_3), (w_2, w_4, w_1, w_3), (w_3, w_1, w_4, w_2), (w_3, w_4, w_1, w_2), (w_3, w_4, w_2, w_1), (w_4, w_3, w_1, w_2), and (w_4, w_3, w_2, w_1), respectively. It can be shown that μ' and μ'' belong to the bargaining set (see Klijn and Massó, 2003) but none of them is a corewise stable matching. We already know that if $\mu \in C(<)$ then $\mu \in \Gamma(\ll)$. We will show that μ' and μ'' belong to the largest consistent set, $\Gamma(\ll)$. We know that if $\mu \in C(<)$ then for all $\widehat{\mu} \neq \mu$ we have that $\mu \gg \widehat{\mu}$. Moreover, we have that (i) for each $i \in N$ there is $\mu \in C(<)$ such that $\mu(i) = \mu'(i)$, and (ii) for each $i \in N$ there is $\mu \in C(<)$ such that $\mu(i) = \mu''(i)$. Hence, for all μ''', S such that S can enforce μ''' over μ', $\exists \mu \in C(<) \subseteq \Gamma(\ll)$, where $\mu''' \ll \mu$, such that $\mu'(i) \not<_i \mu(i)$ $\forall i \in S$. Thus, $\mu' \in \Gamma(\ll)$. Similarly, for all μ''', S such that S can enforce μ''' over μ'', $\exists \mu \in C(<) \subseteq \Gamma(\ll)$, where $\mu''' \ll \mu$, such that $\mu''(i) \not<_i \mu(i)$ $\forall i \in S$; and, $\mu'' \in \Gamma(\ll)$. So, the largest consistent set may contain more matchings than those matchings that are corewise stable.$\square$

Now we give the definition of a vNM farsightedly stable set due to Chwe (1994).

Definition 6 *A set of matchings $V \subseteq \mathcal{M}$ is a vNM farsightedly stable set with respect to P if*

(i) *For all $\mu \in V$, there does not exist $\mu' \in V$ such that $\mu' \gg \mu$;*

(ii) *For all $\mu' \notin V$ there exists $\mu \in V$ such that $\mu \gg \mu'$.*

Definition 6 gives us the definition of a vNM farsightedly stable set $V(\ll)$. Let $\mathcal{V}(\ll)$ be the set of all vNM farsightedly stable sets. Part **(i)** in Definition 6 is the internal stability condition: no matching inside the set is indirectly dominated by a matching belonging to the set. Part **(ii)** is the external stability condition: any matching outside the set is indirectly dominated by some matching belonging to the set. Chwe (1994) has shown that the largest consistent set always contains the vNM farsightedly stable sets. That is, if $V(\ll)$ is a vNM farsightedly stable set, then $V(\ll) \subseteq \Gamma(\ll)$.

We next reconsider the above examples to show that non corewise stable matchings, that belong either to the vNM stable set or to the set of weakly stable matchings or to the largest consistent set, do not survive the stability requirements imposed by introducing farsightedness into the concept of vNM stable sets.

Example 1 (*continue*) Remember that $C(<) = \{\mu'\}$ is the set of corewise stable matchings and $V(<) = \{\mu, \mu', \mu''\}$ is the unique vNM stable set. It is easy to verify that $\mu' \gg \mu$ and $\mu' \gg \mu''$. Let $\mu^0 = \mu$, $\mu^1 = \varnothing$ (all agents are single), $\mu^2 = \mu'$, $S^0 = \{w_1, w_2, w_3\}$, and $S^1 = N$. We have (i) $\mu^2 \succ \mu^0$ $\forall i \in S^0$ and S^0 can enforce μ^1 over μ^0, (ii) $\mu^2 \succ \mu^1$ $\forall i \in S^1$ and S^1 can enforce μ^2 over μ^1. Thus, $\mu^2 \gg \mu^0$ or $\mu' \gg \mu$. Similarly, it is easy to verify that $\mu' \gg \mu''$. Hence, $\{\mu, \mu', \mu''\}$ cannot be a vNM farsightedly stable set, nor can $\{\mu, \mu'\}$ or $\{\mu', \mu''\}$ be a vNM farsightedly stable set since internal stability is violated. Moreover, μ does not indirectly dominates μ' and μ'' does not indirectly dominates μ'. It implies that the sets $\{\mu, \mu''\}$, $\{\mu\}$ or $\{\mu''\}$ cannot be vNM farsightedly stable sets as they violate the external stability condition. In fact, $V(\ll) = \{\mu'\}$ is the unique vNM farsightedly stable set.$\square$

Example 2 (*continue*) Remember that μ is weakly stable. We will show that μ cannot belong to any vNM farsightedly stable sets. We prove it by contradiction. Suppose that μ belong to some $V(\ll)$. Since $\mu \not\gg \mu'$ and $\mu \not\gg \mu''$, μ has to be with μ' and μ'' in $V(\ll)$ in order to satisfy the external stability condition. But, since $\mu' \gg \mu$ and $\mu'' \gg \mu$, $\{\mu, \mu', \mu''\} \subseteq V(\ll)$ would violate internal stability. Hence, $\mu \notin V(\ll)$. In fact, the vNM farsightedly stable sets are $\{\mu'\}$ and $\{\mu''\}$. That is, $\mathcal{V}(\ll) = \{\{\mu'\}, \{\mu''\}\}$.□

Example 3 (*continue*) Remember that μ' and μ'' belong to the largest consistent set but are not corewise stable. First, we show that $\{\mu'\}$ cannot be a vNM farsightedly stable set since the external stability condition would be violated. Indeed, for instance, μ' does not indirectly dominate the matching where men (m_1, m_2, m_3, m_4) are matched to women (w_2, w_1, w_3, w_4). Second, we show that a set composed of μ' and other matching(s) cannot be a vNM farsightedly stable set since the internal stability condition would be violated. Indeed, (i) $\mu \gg \mu'$ if $\mu \in C(<)$; (ii) $\mu' \gg \mu''$ and $\mu' \ll \mu''$; (iii) the matchings where men (m_1, m_2, m_3, m_4) are matched to women (w_1, w_3, w_4, w_2), (w_3, w_1, w_2, w_4), (w_1, w_4, w_2, w_3), (w_1, w_4, w_3, w_2), (w_4, w_1, w_2, w_3), (w_4, w_1, w_3, w_2), (w_2, w_3, w_1, w_4), (w_2, w_3, w_4, w_1), (w_3, w_2, w_1, w_4), (w_3, w_2, w_4, w_1), (w_2, w_4, w_3, w_1), and (w_4, w_2, w_1, w_3), respectively, indirectly dominate μ', but μ' does not indirectly dominate any of these matchings. Thus, the largest consistent set may contain more matchings than those matchings that belong to the vNM farsightedly stable sets.□

4 Main results

From Definition 6, we have that, for $V(\ll)$ to be a singleton vNM farsightedly stable set, only external stability needs to be verified. That is, the set $\{\mu\}$ is a vNM farsightedly stable set if and only if for all $\mu' \neq \mu$ we have that $\mu \gg \mu'$.

In order to show our main results we use Lemma 1 that shows that an individually rational matching μ does not indirectly dominate another matching μ' if and only if there exists a pair $\{i, \mu'(i)\}$ that blocks μ. In other words, μ indirectly dominates μ' if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ.

Lemma 1 *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ is individually rational. Then, $\mu \not\gg \mu'$ if and only if there exists a pair $\{i, \mu'(i)\}$ such that both i and $\mu'(i)$ prefer μ' to μ.*

Proof. Let $B(\mu', \mu)$ be the set of men and women who are strictly better off in μ than in μ'. Accordingly, let $I(\mu', \mu)$ and $W(\mu', \mu)$ be the set of men and women who are indifferent between μ and μ' and worse off in μ than in μ', respectively.

($\Rightarrow$) For μ to indirectly dominate μ' it must be that i or $\mu'(i)$ get divorced along the path from μ' to μ. But both i and $\mu'(i)$ belong to $W(\mu', \mu)$, and then, they will never divorce. Hence $\mu \not\gg \mu'$.

($\Leftarrow$) We will prove it by showing that $\mu \gg \mu'$ if the above condition is not satisfied. Assume that for all pairs $\{i, \mu'(i)\}$ such that $\mu'(i) \neq \mu(i)$, either i or $\mu'(i)$ or both belong to $B(\mu', \mu)$. Notice that every agent i single in μ' that accepts a match with someone else in μ also belongs to $B(\mu', \mu)$ since μ is individually rational. Next construct the following sequence of matchings from μ' to μ: μ^0, μ^1, μ^2 (where $\mu^0 = \mu'$, $\mu^1 = \{\mu^1(i) = i,\ \mu^1(\mu'(i)) = \mu'(i)$ for all $i \in B(\mu', \mu)$, and $\mu^1(j) = \mu'(j)$ otherwise$\}$, and $\mu^2 = \mu$), and the following sequence of coalitions S^0, S^1 with $S^0 = B(\mu', \mu)$ and $S^1 = B(\mu', \mu) \cup \{\mu(i)$ for $i \in B(\mu', \mu)\}$. Then, coalition S^0 can enforce μ^1 over μ^0 and coalition S^1 can enforce μ^2 over μ^1. Moreover, $\mu^2 \succ \mu^0$ for S^0, and $\mu^2 \succ \mu^1$ for S^1 because every mate of $i \in B(\mu', \mu)$ in μ^2 (in μ) also prefers his or her mate in μ^2 to being single in μ^1. Indeed, for every $i \in B(\mu', \mu)$, either $\mu^2(i) \in B(\mu', \mu)$ and hence both prefer μ^2 to μ^1, or $\mu^2(i) \in W(\mu', \mu)$. In this last case, $\mu^2(i)$ must have lost his or her mate in μ^0 and $\mu^0(\mu^2(i))$ must belong to $B(\mu', \mu)$ since otherwise $\mu^0(\mu^2(i))$ and $\mu^2(i)$ would form a blocking pair of μ^2, and this by assumption is not possible. Hence $\mu^2(i)$ must be single in μ^1. Then, since μ^2 is individually rational, $\mu^2(i)$ must prefer accepting his or her mate in μ^2 than remaining single at μ^1. So, we have that $\mu \gg \mu'$. ■

Hence, if μ is individually rational and there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ, then $\mu \gg \mu'$.

Lemma 2 *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ' is individually rational. Then $\mu \gg \mu'$ implies that μ is also individually rational.*

Proof. Suppose not. Then, there exists $i \in N$ that prefers to be single than to be married to $\mu(i)$ in μ. Since $\mu \gg \mu'$ and μ' is individually rational, we have that i was either single at μ' or matched to $\mu'(i) \succ_i i$. But then in the sequence of moves between μ' and μ, the first time i has to move she/he was either matched with $\mu'(i)$ or single and, hence, i cannot belong to a coalition S^{k-1} that can enforce the matching μ^k over μ^{k-1} and such that all members of S^{k-1} prefer μ to μ^{k-1}, contradicting the fact that $\mu \gg \mu'$. ■

The next theorem shows that every corewise stable matching is a vNM farsightedly stable set.

Theorem 1 *If μ is a corewise stable matching, $\mu \in C(<)$, then $\{\mu\}$ is a vNM farsightedly stable set, $\{\mu\} = V(\ll)$.*

Proof. We only need to verify condition (ii) in Definition 6: for all $\mu' \neq \mu$ we have that $\mu \gg \mu'$. Since $\mu \in C(<)$, we know that $\forall \mu' \neq \mu$, $\nexists\, i \in M$ and $j \in W$ such that $\mu'(i) = j$ and $\mu' \succ \mu$ for both i and j. Since μ is individually rational, we have from Lemma 1 that $\mu \gg \mu'$. ■

We have just shown that if $\mu \in C(<)$ then $\{\mu\}$ is a vNM farsightedly stable set.[8] But, a priori there may be other vNM farsightedly stable sets of matchings. We now show that the only possible vNM farsightedly stable sets are singleton sets whose elements are the corewise stable matchings.

Theorem 2 *If $V(\ll) \subseteq \mathcal{M}$ is a vNM farsightedly stable set of matchings then $V(\ll) = \{\mu\}$ with $\mu \in C(<)$.*

Proof. Notice that if $V(\ll) \subseteq C(<)$, then $V(\ll)$ is a vNM farsightedly stable set only if $V(\ll)$ is a singleton set $\{\mu\}$ with $\mu \in C(<)$. From Theorem 1, we know that for all $\mu' \neq \mu$, $\mu \gg \mu'$. Suppose now that $V(\ll) \subsetneq C(<)$. Then, either $V(\ll) \cap C(<) = \varnothing$ or $V(\ll) \cap C(<) \neq \varnothing$.

Suppose first that $V(\ll) \cap C(<) \neq \varnothing$. Then, there exists a matching $\mu \in V(\ll) \cap C(<)$, and we know that for all $\mu' \neq \mu$, $\mu \gg \mu'$ since $\{\mu\}$ is a vNM farsightedly stable set. But, then there exists a matching $\mu' \neq \mu \in V(\ll)$ such that $\mu \gg \mu'$, violating the internal stability condition.

Suppose now that $V(\ll) \cap C(<) = \varnothing$. Then, we will show that $V(\ll)$ is not a vNM farsightedly stable set because either the internal stability condition (condition **(i)** in Definition 6) or the external stability condition (condition **(ii)** in Definition 6) is violated.

Assume $V(\ll) = \{\mu\}$ is a singleton. Since $\mu \notin C(<)$ there exists a deviating coalition S in μ and a matching $\mu' \in \mathcal{M}$ such that $\mu' \succ_i \mu$ for all $i \in S$ and S can enforce μ' over μ. Then, $\mu \not\gg \mu'$ and the external stability condition is violated.

Assume now that $V(\ll)$ contains more than one matching that do not belong to $C(<)$. Take any matching $\mu_1 \in V(\ll)$. Since $\mu_1 \notin C(<)$, there exists at least a pair of agents $\{i, j\}$ such that $\mu_1(j) \neq i$ (or a single agent $\{i\}$) and a matching $\mu_1' \in \mathcal{M}$ such that $\mu_1' \succ \mu_1$ for both i and j (or $\mu_1' \succ \mu_1$ for i), and $\{i, j\}$ (or i) can enforce μ_1' over μ_1, i.e. such that $\mu_1'(j) = i$ (or $\mu_1'(i) = i$). Let $S(\mu_1)$ be the set of blocking pairs of μ_1. Consider the deviation from μ_1 to μ_1' of the subset of blocking pairs $S'(\mu_1) \subseteq S(\mu_1)$, where $S'(\mu_1)$

[8] Diamantoudi and Xue (2003) were first to show that in hedonic games (of which marriage problems are a special case) with strict preferences, any partition belonging to the core indirectly dominates any other partition. Here, we provide an alternative proof of their result for one-to-one matching problems with strict preferences, and in Section 5 we show that this result also holds for many-to-one matching problems with responsive preferences.

contains the maximum number of blocking pairs and is such that the subset $S(\mu_1)\diagdown S'(\mu_1)$ does not contain any blocking pair of μ'_1. In order for $V(\ll)$ being a vNM farsightedly stable set we need that the following conditions are satisfied:

(i) for any other matching $\mu_2 \in V(\ll)$, $\mu_2 \neq \mu_1$, it should be that $\mu_1 \not\gg \mu_2$ and $\mu_2 \not\gg \mu_1$

(ii) for all $\mu' \notin V(\ll)$ there exists $\mu \in V(\ll)$ such that $\mu \gg \mu'$ (in particular, we need that there exists a matching $\mu_2 \in V(\ll)$ such that $\mu_2 \gg \mu'_1$ for each matching, like μ'_1, that can be enforced by any subset of blocking pairs of any matching in $V(\ll)$).

We will show that $V(\ll)$ is not a vNM farsightedly stable set because one of the above conditions is not satisfied. Three different cases should be considered.

1. Assume that $S'(\mu_1)$ contains a blocking pair $\{i\}$ that divorces from $\mu_1(i)$. Consider the deviation of $\{i\}$ from μ_1 to μ''_1 where he or she divorces from $\mu_1(i)$, while the other blocking pairs do not move. Then if $\mu_2 \gg \mu''_1$ (in order for $V(\ll)$ satisfying external stability), we also have that $\mu_2 \gg \mu_1$ since i will never marry someone else and becoming worse off than being single. Hence, the internal stability condition is violated and $V(\ll)$ is not a vNM farsightedly stable set. So $V(\ll)$ cannot contain non-individually rational matchings.

2. Assume that $S'(\mu_1)$ contains at least a blocking pair $\{i, j\}$ that are single at μ_1 but married at μ'_1, i.e., $\mu'_1(i) = j$. Consider the deviation of $\{i, j\}$ from μ_1 to μ''_1 where they get married, while the other blocking pairs do not move. Then if $\mu_2 \gg \mu''_1$ (in order for $V(\ll)$ satisfying external stability), we will show that the internal stability condition is violated. Two sub-cases have to be considered.

2.1. If i and j are still married in μ_2 (or they are married to someone else preferred to j and i respectively), then $\mu_2 \gg \mu_1$ and the internal stability condition is violated.

2.2. On the contrary, assume that one of them, i, has divorced from j (leaving j single in μ_2 like she was in μ_1) to marry to another woman $\mu_2(i)$ preferred to j. But notice that the position (matching) of any other agent different than i and j in μ''_1 is the same than in μ_1, since only i and j married at μ''_1 while they were single at μ_1, and then since $\mu_2 \gg \mu''_1$, we should have that $\{i, \mu_2(i)\} \in S(\mu_1)$. But then, the pair $\{i, j\}$ cannot belong to $S'(\mu_1)$, since j is not the best partner for i. Thus, consider the deviation of $\{i, \mu_2(i)\}$ from μ_1 to μ'''_1 where they get married, while the other blocking pairs do not move. Then, if we have that $\mu_2 \gg \mu'''_1$ (in order for $V(\ll)$ satisfying external stability), we also have $\mu_2 \gg \mu_1$ and the internal stability condition is violated.

3. Assume that all blocking pairs $\{i, j\} \in S'(\mu_1)$ are such that at least one of the blocking partners (or both of them) is married at μ_1 with someone else, $\mu_1(j) \neq i, j$ and now at μ_1' they get married $\mu_1'(j) = i$. Assume that in the deviation from μ_1 to μ_1' all blocking pairs in $S'(\mu_1)$ get married so that at μ_1' no other blocking pair exists ($S(\mu_1) \diagdown S'(\mu_1)$ does not contain any blocking pair of μ_1'). Three sub-cases have to be considered.

3.1. If at μ_2 we have that every blocking pair $\{i, j\} \in S'(\mu_1)$ is such that i and j are still married or they are married to someone else but preferred to j and to i, respectively, then $\mu_2 \gg \mu_1$ and the internal stability condition is violated.

3.2. Assume that at μ_2 no initial blocking pair $\{i, j\} \in S'(\mu_1)$ is still married, and that in each blocking pair $\{i, j\} \in S'(\mu_1)$ we have that i is marrying $\mu_2(i) \neq j$ divorcing from j. Hence, in order to have that $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$ and given Lemma 1 and Lemma 2, we need that there does not exist a pair $\{k, \mu_1'(k)\}$ married at μ_1' (or a single agent k) that blocks μ_2, and that there exists a pair $\{k, \mu_1(k)\}$ married at μ_1 (or a single agent k) that blocks μ_2, with μ_2 individually rational. Notice that the only change between μ_1 and μ_1' is that each blocking pair $\{i, j\} \in S'(\mu_1)$ gets married leaving $\mu_1(j)$ (and possibly $\mu_1(i)$) single. Then we will prove that whenever $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$, we have that $\mu_1 \gg \mu_2$, violating the internal stability condition. By Lemma 1, we only need to show that there does not exist a pair $\{k, \mu_2(k)\}$ married at μ_2 (or a single agent k) that blocks μ_1.

Since $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$, we have that, for each pair $\{i, j\} \in S'(\mu_1)$, whenever i is better off at μ_2 than at μ_1' (and then better than at μ_1), $\mu_2(i)$ is worse off at μ_2 than at μ_1', because no other blocking pair different from the ones contained in $S'(\mu_1)$ exists at μ_1' (and hence $\mu_2(i)$ is worse off at μ_2 than at μ_1), and some pair $\{\mu_1(j), j\}$ married at μ_1 (with j such that $\mu_1'(j) = i$, and $\{i, j\} \in S'(\mu_1)$) prefers μ_1 to μ_2 and is a blocking pair of μ_2 making $\mu_2 \not\gg \mu_1$. Thus, whenever $\mu_1(j)$ and j are either single or married to someone else at μ_2 they both would prefer μ_1 to μ_2. Also all the remaining initial partners $\mu_1(i)$ (and $\mu_1(j)$) that have been left by i (and by j) with $\{i, j\} \in S'(\mu_1)$, when single at μ_2, prefer μ_1 to μ_2 because otherwise μ_1 would not be individually rational. Whenever some initial blocking partner $\mu_1(i)$ (and $\mu_1(j)$) is not single at μ_2 but married to someone else $\mu_2(\mu_1(i))$ (married to $\mu_2(\mu_1(j))$), either $\mu_1(i)$ or $\mu_2(\mu_1(i))$ (either $\mu_1(j)$ or $\mu_2(\mu_1(j))$) or both should prefer μ_1 to μ_2 because otherwise the pair $\{\mu_1(i), \mu_2(\mu_1(i))\}$ (the pair $\{\mu_1(j), \mu_2(\mu_1(j))\}$) would have been also a blocking pair at μ_1'. So, we have shown that every pair of agents matched (every single agent at μ_2) at μ_2 and not matched (not single) at μ_1 that contains one of the initial deviating players in $S'(\mu_1)$ or one of the players initially matched at μ_1 to some $i \in S'(\mu_1)$ is such that one of the mates prefers μ_2 to μ_1 while the other prefers μ_1 to μ_2.

Since $\mu_2 \gg \mu_1'$ we have, by Lemma 1, that there is no blocking pair of μ_2 at μ_1', and then, every pair of agents $\{k, l\}$ such that $\mu_2(l) = k$ and $\mu_1'(l) \neq k$ with $\mu_1'(k) = \mu_1(k)$ and $\mu_1'(l) = \mu_1(l)$, is such that if one of the mates prefers μ_2 to μ_1' (and hence, prefers μ_2 to μ_1) then the other prefers μ_1' to μ_2 (and hence, prefers μ_1 to μ_2), and all single agents k at μ_2 that are married at μ_1', with $\mu_1'(k) = \mu_1(k)$, prefer μ_1' to μ_2 (and hence, prefer μ_1 to μ_2). Moreover, since the pairs $\{i, j\} \in S'(\mu_1)$ are the only blocking pairs at μ_1, every pair of agents $\{k, l\}$ such that $\mu_2(l) = k$ and $\mu_1'(k) \neq l$ with $\{k, l\} \in S(\mu_1) \diagdown S'(\mu_1)$, is such that if one of the mates prefers μ_2 to μ_1 then the other prefers μ_1 to μ_2 because otherwise they would have been a blocking pair at μ_1, and all single agents at μ_2 that belong to $S(\mu_1) \diagdown S'(\mu_1)$, that are married at μ_1' (and hence, are married at μ_1) prefer μ_1' to μ_2 (and hence, prefer μ_1 to μ_2) because otherwise μ_1' (and hence, μ_1) would not be individually rational.

Thus, we have proved that when $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$, we have that there is a pair $\{\mu_1(j), j\}$ (with j such that $\mu_1'(j) = i$, and $\{i, j\} \in S'(\mu_1)$) that prefers μ_1 to μ_2 and that every other pair of agents matched at μ_2 and not matched at μ_1 is such that one of the mates prefers μ_2 to μ_1 while the other prefers μ_1 to μ_2. So, there does not exist a pair $\{i, \mu_2(i)\}$ matched at μ_2 that blocks μ_1, and then by Lemma 1 we have $\mu_1 \gg \mu_2$, violating the internal stability condition.

3.3. Finally, consider the case in which μ_2 contains some but not all initial blocking pairs from μ_1 contained in $S'(\mu_1)$. Then, consider the deviation from μ_1 to μ_1'' such that $\mu_1'' \succ \mu_1$ by all the initial blocking pairs belonging to $S''(\mu_1) \subsetneq S'(\mu_1)$ that are still married at μ_2 and that can be enforced by such blocking pairs from μ_1. Since $\mu_2 \gg \mu_1''$ (in order for $V(\ll)$ satisfying external stability), we will have that $\mu_2 \gg \mu_1$ violating the internal stability condition. ■

5 Many-to-one matching problems

A many-to-one matching problem consists of a set of N agents divided into a set of hospitals, $H = \{h_1, ..., h_r\}$, and a set of medical students, $S = \{s_1, ..., s_s\}$, where possibly $r \neq s$. For each hospital $h \in H$ there is a positive integer q_h called the quota of hospital

h, which indicates the maximum number of positions to be filled. Let $Q = \{q_h\}_{h \in H}$. Each hospital $h \in H$ has a strict, transitive, and complete preference relation over the set of medical students S and the prospect of having its position unfilled, denoted h. Hospital h's preferences can be represented by a strict ordering of the elements in $S \cup \{h\}$; for instance, $P(h) = s_1, s_2, h, s_3, ...$ denotes that hospital h prefers to enroll s_1 rather than s_2, that it prefers to enroll either one of them rather than leave a position unfilled, and that all other medical students are unacceptable. Each medical student $s \in S$ has a strict, transitive, and complete preference relation over the hospitals H and the prospect of being unemployed. Student s's preferences can be represented by a strict ordering of the elements in $H \cup \{s\}$; for instance, $P(s) = h_2, h_1, h_3, s, ...$ denotes that the only positions the student would accept are those offered by h_2, h_1, and h_3, in that order. Let $P = (\{P(h)\}_{h \in H}, \{P(s)\}_{s \in S})$. A many-to-one matching market is simply (H, S, P, Q).

Definition 7 *A matching μ is a mapping from the set $H \cup S$ into the set of all subsets of $H \cup S$ such that for all $s \in S$ and $h \in H$:*

(a) Either $|\mu(s)| = 1$ and $\mu(s) \subseteq H$ or else $\mu(s) = s$.

(b) $\mu(h) \in 2^S$ and $|\mu(h)| \leq q_h$.

(c) $\mu(s) = \{h\}$ if and only if $s \in \mu(h)$.

We denote by $P^*(h)$ the preference relation of hospital h over sets of students. We assume that $P^*(h)$ is responsive to $P(h)$. That is, whenever $\mu'(h) = \mu(h) \cup \{s\} \setminus \{s'\}$ for $s' \in \mu(h)$ and $s \notin \mu(h)$, then h prefers $\mu'(h)$ to $\mu(h)$ (under $P^*(h)$) if and only if h prefers s to s' (under $P(h)$). Under this condition, as in Roth and Sotomayor (1990), we can associate to the many-to-one matching problem a one-to-one matching problem in which we replace hospital h by q_h positions of h denoted by $h^1, h^2, ..., h^{q_h}$. Each of these positions has preferences over individuals that are identical with those of h. Each student's preference list is modified by replacing h, wherever it appears on his or her list, by the string $h^1, h^2, ..., h^{q_h}$ in that order. That is, if s prefers h_1 to h_2, then s prefers all positions of h_1 to all positions of h_2, and s prefers h_1^1, to all the other positions of h_1.

In many-to-one matching problems, it makes sense to distinguish between setwise enforceability and corewise enforceability.[9]

Definition 8 (setwise enforceability) *Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following condition holds*: $\forall i \in S$, *if $\mu'(i) \neq \mu(i)$, then $\mu'(i) \setminus (\mu'(i) \cap \mu(i)) \in S$.*

[9] Corewise enforceability has already been defined in Definition 1. Obviously, setwise enforceability and corewise enforceability are equivalent in one-to-one matching problems.

Depending on the notion of enforceability used (setwise or corewise), we obtain the setwise direct dominance relation ($\overset{s}{<}$) or the corewise direct dominance relation ($\overset{c}{<}$), the set of setwise stable matchings ($C(\overset{s}{<})$) or the set of corewise stable matchings ($C(\overset{c}{<})$), the vNM setwise stable sets ($V(\overset{s}{<})$) or the vNM corewise stable sets ($V(\overset{c}{<})$). A matching μ is setwise stable if there is no subset of agents who by forming new partnerships only among themselves, possibly dissolving some partnerships of μ, can all obtain a strictly preferred set of partners. A matching of the many-to-one matching problem is setwise stable if and only if the corresponding matchings of the associated one-to-one matching problem is (setwise) stable (see Roth and Sotomayor, 1990). However, this result does not hold for corewise stability.

In many-to-one matching problems with responsive preferences, the indirect dominance relation is invariant to the notion of enforceability in use. Indeed, if μ is indirectly dominated by μ' under corewise enforceability, it is obvious that μ is indirectly dominated by μ' under setwise enforceability. In the other direction, if μ is indirectly dominated by μ' under setwise enforceability, then μ is indirectly dominated by μ' under corewise enforceability because if for some $i \in S$ we have that $\mu'(i) \cap \mu(i) \neq \varnothing$, then i could first become "single" and then match with $\mu'(i)$, instead of matching directly with $\mu'(i) \backslash (\mu'(i) \cap \mu(i))$.

Lemma 3 *A matching μ is indirectly dominated by μ' in a many-to-one matching problem if and only if μ is indirectly dominated by μ' in the associated one-to-one matching problem.*

Proof. If μ is indirectly dominated by μ' in the associated one-to-one matching problem, then there exists a sequence of matchings $\mu^0, \mu^1, ..., \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of coalitions $S^0, S^1, ..., S^{K-1}$ consisting only of individual students or hospitals, or of student-hospital pairs and such that for any $k \in \{1, ..., K-1\}$, $\mu^K \succ \mu^{k-1}$ $\forall i \in S^{k-1}$, and coalition S^{k-1} can enforce the matching μ^k over μ^{k-1}. But then, μ is indirectly dominated by μ' in the many-to-one matching problem by the deviations of the sequence of coalitions consisting of the same singletons or pairs.

In the other direction, if μ is indirectly dominated by μ' in the many-to-one matching problem by the deviations of a sequence of coalitions $S^0, S^1, ..., S^{K-1}$, then the fact that $\mu^K(i) \succ \mu^{k-1}(i)$ $\forall i \in S^{k-1}$ implies that, if i is a hospital h, then there exists a student s in $\mu^K(h) \setminus \mu^{k-1}(h)$ and a σ in $\mu^{k-1}(h) \setminus \mu^K(h)$ such that $s \succ_h \sigma$. (Otherwise, $\sigma \succeq_h s$ for all σ in $\mu^{k-1}(h) \setminus \mu^K(h)$ and s in $\mu^K(h) \setminus \mu^{k-1}(h)$, and this would imply $\mu^{k-1}(h) \succeq_h \mu^K(h)$, by repeated application of the fact that preferences are responsive and transitive.) So s is in S^{k-1} and s prefers $h = \mu^K(s)$ to $\mu^{k-1}(s)$. Thus, the coalition $\widehat{S}^{k-1}$ formed by the pair $\{h, s\}$ would deviate from μ^{k-1} in order to end at μ^K. Then, if the coalition $\widehat{S}^{k-1} = \{h, s\}$ does not coincide with S^{k-1} (i.e. with the deviating coalition from μ^{k-1} to μ^k that likes to end at μ^K), once the pair $\{h, s\}$ has moved from μ^{k-1} to $\widehat{\mu}^{k-1}$, then the fact that $\mu^K(i) \succ \widehat{\mu}^{k-1}(i)\ \forall i \in S^{k-1} \setminus \widehat{S}^{k-1}$ implies that, if i is a hospital h', then there exists a student s' in $\mu^K(h') \setminus \widehat{\mu}^{k-1}(h')$ and a σ' in $\widehat{\mu}^{k-1}(h') \setminus \mu^K(h')$ such that $s' \succ_{h'} \sigma'$. So s' is in $S^{k-1} \setminus \widehat{S}^{k-1}$ and s' prefers $h' = \mu^K(s')$ to $\widehat{\mu}^{k-1}(s')$. Thus, the coalition $\widetilde{S}^{k-1}$ formed by the pair $\{h', s'\}$ would deviate from $\widehat{\mu}^{k-1}$ in order to end at μ^K. And so on until every deviating pair contained in S^{k-1} has moved and we are in μ^k. So, every coalitional deviation from μ^{k-1} to μ^k in the sequence of deviations from μ to μ' could be replaced by a sequence of deviations of the deviating pairs contained in S^{k-1}, and μ is indirectly dominated by μ' in the corresponding one-to-one matching problem. ■

From Theorem 1 and Theorem 2, using Lemma 3, we obtain the following corollary.

Corollary 1 *In a many-to-one matching problem with responsive preferences, a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a setwise stable matching.*

Thus, our characterization of the vNM farsightedly stable set for one-to-one matching problems extends to many-to-one matching problems with responsive preferences. This result contrasts with Ehlers (2007) who has shown that there need not be any relationship between the vNM corewise stable sets of a many-to-one matching problem and its associated one-to-one matching problem. Example 4 illustrates our main result for many-to-one matching problems with responsive preferences: vNM farsightedly stable sets only contain setwise stable matchings. Thus, if there is a matching of the many-to-one matching problem that is corewise stable but not setwise stable, then this matching is never a vNM farsightedly stable set.

Example 4 (Ehlers, 2005) *Consider a many-to-one matching problem and its associated one-to-one matching problem with* $H = \{h_1, h_2\}$, $S = \{s_1, s_2, s_3, s_4\}$, $q_{h_1} = 2$, $q_{h_2} = 1$, *and* P *such that*:

$P(h_1^1)$	$P(h_1^2)$	$P(h_2)$	$P(s_1)$	$P(s_2)$	$P(s_3)$	$P(s_4)$
s_1	s_1	s_2	h_2	h_1^1	h_1^1	h_1^1
s_2	s_2	s_1	h_1^1	h_1^2	h_1^2	h_1^2
s_3	s_3	s_3	h_1^2	h_2	h_2	h_2
s_4	s_4	s_4	s_1	s_2	s_3	s_4
h_1^1	h_1^2	h_2				

Let

$$\mu = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & s_4 \\ s_2 & s_3 & s_1 & s_4 \end{pmatrix}, \mu' = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & s_4 \\ s_1 & s_3 & s_2 & s_4 \end{pmatrix}, \widetilde{\mu} = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & s_3 \\ s_2 & s_4 & s_1 & s_3 \end{pmatrix},$$

$$\widetilde{\mu}' = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & s_3 \\ s_1 & s_4 & s_2 & s_3 \end{pmatrix}, \widetilde{\mu}'' = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & s_1 \\ s_3 & s_4 & s_2 & s_1 \end{pmatrix}, \widehat{\mu} = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & s_4 \\ s_1 & s_2 & s_3 & s_4 \end{pmatrix}$$

Ehlers (2005) has shown that $V(\overset{c}{<}) = \{\mu, \mu', \widetilde{\mu}, \widetilde{\mu}', \widetilde{\mu}''\}$ *in this many-to-one matching problem, while* $V(\overset{c}{<}) = \{\mu, \mu'\}$ *in its associated one-to-one matching problem. Notice that* $V(\overset{s}{<}) = \{\mu, \mu'\}$ *in both the many-to-one matching problem and its associated one-to-one matching problem. The set of setwise stable matchings is* $C(\overset{s}{<}) = \{\mu\}$*. However,* $\mu' \in C(\overset{c}{<})$ *in the many-to-one matching problem, since* μ' *is not directly dominated via any coalition (for instance, the coalition* $\{h_1, s_2\}$ *that prefers* $\widehat{\mu}$ *to* μ' *cannot enforce* $\widehat{\mu}$ *over* μ'*, and the members of the coalition* $\{h_1, s_1, s_2\}$ *that can enforce* $\widehat{\mu}$ *over* μ' *do not all prefer* $\widehat{\mu}$ *to* μ'*). Now, applying our results we have that* $V(\ll) = \{\mu\}$ *in both the many-to-one matching problem and its associated one-to-one matching problem. Contrary to the direct dominance relation,* μ *indirectly dominates* μ'*. Indeed, the sequence of deviations is as follows. First,* s_2 *leaves hospital* h_2*; second, hospital* h_2 *hires student* s_1*; third, hospital* $h_1 = \{h_1^1, h_1^2\}$ *hires student* s_2 *(either directly when using setwise enforceability, or first leaving student* s_3 *and then hiring both students* s_2 *and* s_3 *when using corewise enforceability).*□

6 Conclusion

We have characterized the vNM farsightedly stable sets in one-to-one matching problems: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a corewise (hence, setwise) stable matching. Thus, we have provided an alternative characterization of the core in one-to-one matching problems. Finally, we have shown that our main result is robust to many-to-one matching problems with responsive preferences: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a setwise stable matching.

Acknowledgments

Financial support from Spanish Ministerio de Educación y Ciencia under the project SEJ2006-06309/ECON, support from the Belgian French Community's program Action de Recherches Concertée 03/08-302 and 05/10-331 (UCL), and support of a SSTC grant from the Belgian State - Belgian Science Policy under the IAP contract P6/09 are gratefully acknowledged.

References

[1] Chwe, M.S., "Farsighted Coalitional Stability," *Journal of Economic Theory* 63, 299–325 (1994).

[2] Diamantoudi, E. and L. Xue, "Farsighted Stability in Hedonic Games," *Social Choice and Welfare* 21, 39-61 (2003).

[3] Ehlers, L. , "Von Neumann-Morgenstern Stable Sets in Matching Problems," CIREQ Discussion Paper 12-2005 (2005).

[4] Ehlers, L. , "Von Neumann-Morgenstern Stable Sets in Matching Problems," *Journal of Economic Theory* 134, 537-547 (2007).

[5] Gale, D. and L. Shapley, "College Admissions and the Stability of Marriage," *American Mathematics Monthly* 69, 9-15 (1962).

[6] Harsanyi, J.C., "An Equilibrium-Point Interpretation of Stable Sets and a Proposed Alternative Definition," *Management Science* 20, 1472-1495 (1974).

[7] Herings, P.J.J., A. Mauleon and V. Vannetelbosch, "Rationalizability for Social Environments," *Games and Economic Behavior* 49, 135–156 (2004).

[8] Jackson, M.O. and A. Watts, "The Evolution of Social and Economic Networks," *Journal of Economic Theory* 106, 265–295 (2002).

[9] Klijn, F. and J. Massó, "Weak Stability and a Bargaining Set for the Marriage Model," *Games and Economic Behavior*, 42, 91-100 (2003).

[10] Knuth, D., *Marriages Stables*, Les Presses de l'Université de Montréal, Montreal, Canada (1976).

[11] Roth, A.E. and M. Sotomayor, *Two-Sided Matching*, Econometric Society Monographs 18, Cambridge University Press, Cambridge, UK (1990).

[12] Roth, A.E. and J.H. Vande Vate, "Random Paths to Stability in Two-Sided Matching," *Econometrica* 58, 1475-1480 (1990).

[13] Sotomayor, M., "A Non-Constructive Elementary Proof of the Existence of Stable Marriages," *Games and Economic Behavior* 13, 135-137 (1996).

[14] Von Neumann, J. and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press (1944).

[15] Xue, L., "Coalitional Stability under Perfect Foresight," *Economic Theory* 11, 603–627 (1998).

[16] Zhou, L., "A New Bargaining Set of an N-person Game and Endogenous Coalition Formation," *Games and Economic Behavior* 6, 512-526 (1994).

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