# with Two-Sided Uncertainty

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The role of strategic delay is analyzed in an infinite-horizon alternating-offer model of bargaining. A buyer and seller are engaged in the trade of a single object. Both bargainers have private information about their own preferences and are impatient in that delaying agreement is costly. An equilibrium is constructed in which the bargainers signal the strength of their bargaining positions by delaying prior to making an offer. A bargainer expecting large gains from trade is more impatient than one expecting small gains, and hence makes concessions earlier on. Trade occurs whenever gains from trade exist, but due to the private information, only after costly delay.

Panmunjom, Korea—(UPI)—The American general and the North Korean general glared at each other across the table and the only sound was the wind howling across the barren hills outside their hut.

Maj. Gen. James B. Knapp, negotiator for the United Nations Command (UNC), was waiting for Maj. Gen. Ri Choonsun of the Democratic People's Republic of North Korea to propose a recess.

They sat there, arms folded, for 4 1/2 hours. Not a word. Finally, Gen. Ri got up, walked out and drove away.

—*Evening Bulletin,* Philadelphia (11 April 1969)

## 1. INTRODUCTION

Two crucial variables in any bargaining setting are information and time. Consider, for example, the negotiations of a seller and a buyer over the price of a house in a market with few alternative traders of similar houses. Each trader knows his own valuation of the house, but can only make a rough assessment of the other's valuation. Due to the traders' private information, the bargainers are uncertain as to whether gains from trade exist. Thus, some learning must occur before the traders can reach an agreement. The learning, however, is complicated by the fact that each trader has an incentive to convince the other that the gains from trade are smaller than they truly are. Both bargainers are aware of the other's incentive to deceive and thus are not so easily misled. The seller cannot simply say, "This house is worth a great deal to me, so you had better be willing to pay a high price if you want to have it". Words must be backed up by actions. If the seller wants to convince the buyer that her valuation is high, she must take actions that a low-valuation seller would find unattractive.

Time provides a means for this strategic communication to occur. A seller with a low valuation expects large gains from trade and so is less willing to delay agreement than a high-valuation seller who stands to lose little from delay. Hence, the traders are able to communicate their private information by revealing their willingness to delay agreement.

This paper analyzes an infinite-horizon, alternating-offer model of bargaining related to the one analyzed in Rubinstein (1982).[1] There are two main differences in the extensive form. First, unlike in Rubinstein's model in which the time between offers is fixed, here I follow Admati and Perry (1987) and allow each trader to delay making offers. This ability to delay offers enables each bargainer to commit to not revising or rescinding an offer until a counter-offer is made. Second, who makes the first offer is endogenous: both traders have the option of making the initial offer.

This paper extends Admati and Perry's (1987) analysis of bargaining with one-sided uncertainty and two possible types to a setting of two-sided uncertainty and a continuum of types. The importance of analyzing this case rests on the premise that bargainers typically are uncertainty about each others' reservation values. A second premise is that bargaining is more realistically modelled by alternating offers; whereas, most studies of bargaining with two-sided uncertainty allow only one party to make offers (Ausubel and Deneckere (1988, 1991), Cho (1990b), and Cramton (1984a)). This paper differs from two other papers that allow alternating offers in a setting of two-sided uncertainty. Chatterjee and Samuelson (1987) restrict the players' types and offers to come

[1] More recently, several authors have studied related models involving incomplete information. For models with one-sided uncertainty and one-sided offers, see Ausubel and Deneckere (1989a, b), Fudenberg, Levine, and Tirole (1985), Gul, Sonnenschein, and Wilson (1986), Hart (1988), Sobel and Takahashi (1983) and Vincent (1989), for models with one-sided uncertainty and alternating offers, see Admati and Perry (1987), Bikhchandani (1988), Cramton (1991), Cramton and Tracy (1990), Grossman and Perry (1986), Gul and Sonnenschein (1988), and Rubinstein (1985); for models with two-sided uncertainty and one-sided offers, see Ausubel and Deneckere (1988, 1991), Cho (1990b), Cramton (1984a), and Fudenberg and Tirole (1983), for models with two-sided uncertainty and alternating offers, see Chatterjee and Samuelson (1987) Cho (1990a), and Perry (1986). Reviews of the literature are found in Binmore, Osborne, and Rubinstein (1990), Kennan and Wilson (1989, 1991), and Osborne and Rubinstein (1990).

from two-point sets. Cho (1990a) examines a class of equilibria that only exists when the supports of the seller's valuation and the buyer's valuation are sufficiently disjoint. Here I consider a continuum of types with overlapping supports; the parties are uncertain whether gains exist.

The contribution of this paper is the construction of a sequential equilibrium for bargaining with two-sided uncertainty. The equilibrium I construct, while perhaps appealing, is only one of a continuum of possible equilibria. The multiplicity stems from the vast possibilities for communication in this rich information setting. No attempt is made at characterizing the set of all sequential equilibria. Thus, rather than determining how rational traders must negotiate, the results here simply specify one form rational behaviour may take when bargainers are faced with two-sided uncertainty.

A virtue of the equilibrium is its simplicity. Regardless of the distribution of uncertainty, the timing and magnitude of the initial offer are determined from a numerical integration subject to a single constraint. The timing and magnitude of subsequent offers are given by two simple formulae, which are independent of the distribution of uncertainty. This simplicity comes from two features of the construction: (1) information is revealed from the timing of the offers, which is determined from simple incentive constraints, and (2) all the offers are Rubinstein (1982) full-information offers for a specified pair of trader types. A second virtue of the equilibrium, stemming from feature (2), is that the parties settle at terms that are fair given the information revealed by the bargaining. Neither party has an incentive to renegotiate the terms once trade has occurred.

The equilibrium can be described intuitively as follows. Initially, both bargainers delay negotiations by refusing to make an offer. As time passes, each becomes more pessimistic about the magnitude of the gains from trade, since if the gains were larger the other trader would already have made an offer. Eventually either the traders realize that gains from trade do not exist, or the less patient bargainer (say the seller) makes an offer, which reveals her valuation. The buyer then either accepts the revealing offer immediately or delays making a counter-offer so as to convince the seller to accept a lower price. If the buyer does made a counter-offer, it is accepted by the seller immediately. Thus, if gains from trade exist, an agreement is reached after at most two offers.

The equilibrium has the property that a trader's delay in making an offer fully reveals the trader's private information. Hence, *delay is* used as the sole signalling device: offers are simply chosen to be optimal given the beliefs induced by the strategic delay. The reason the traders choose to use delay as the signalling device, rather than price, is that it is a more efficient signal of strength. Price is a less effective signal, since it is easier for low-valuation sellers to imitate the price behaviour of high-valuation sellers (Admati and Perry (1987)).

Both on and off the equilibrium path, the information structure evolves as follows: (i) prior to either trader making an offer there is two-sided uncertainty, (ii) after the first offer, there is one-sided uncertainty, and (iii) after a counter-offer is made, there is full information. Given this form of communication, the equilibrium can be constructed by analyzing the three phases of the game, beginning with the full-information subgame, then the subgame with one-sided uncertainty, and finally the subgame with two-sided uncertainty.[2] The construction relies on Rubinstein (1982) for the analysis of the complete-information subgame, and Admati and Perry (1987) for the analysis of the subgame with one-sided uncertainty. The initial phase with two-sided uncertainty is a game of timing similar to the war of attrition.[3]

The existence of an equilibrium of this form in stationary strategies depends on the assumption that after a trader has made an offer she must wait until the other trader makes a counter-offer before she can revise her previous offer. That is, a bargainer is unable to bargain against herself. One justification for this assumption is that a trader would find it unprofitable to revise her offer, if doing so adversely affects her reputation. If the buyer interprets a drop in the seller's offer as a sign of weakness, and so he expects further concessions and becomes more demanding, then the seller may be better off not revising her offer (Cramton (1984b)). The difficulty with allowing this reputational story is that it can be used to support a large set of equilibria (Ausubel and Deneckere (1988)).

The problem that appears if the traders are unable to delay offers and the time between offers is arbitrarily small is demonstrated in Gul, Sonnenschein, and Wilson (1986) for the case of one-sided offers, and Gul and Sonnenschein (1988) for the case of alternating offers. They show that with one-sided uncertainty every sequential equilibrium, in which the informed player's strategy is stationary, has the property that as the time between offers shrinks to zero, the delay to agreement vanishes. Moreover, when the time between offers is arbitrarily small the uninformed trader (say the seller) loses all of her bargaining power and her offers quickly collapse to her reservation value.[4]

[2] The term *subgame is* used throughout this paper in a non-standard way to refer to what would be a proper subgame if it were not for the private information about valuations. The term *subform* has been used by others.

[3] See, for example, Hendricks, Weiss and Wilson (1988) and Hendricks and Wilson (1988) for an analysis of games of timing with complete information, Milgrom and Weber (1985) for a model with incomplete information, and Osborne (1985) for an application of games of timing to a bargaining problem.

[4] Unlike in Gul and Sonnenschein (1988), 1 am assuming that the lowest possible buyer valuation is less than or equal to the seller's valuation. In general, the seller's offer falls to the price she would offer if it were common knowledge that the buyer had the lowest possible

Faced with a payoff of zero if one reveals one's valuation, a trader clearly has no incentive to make a revealing offer. Thus, for the model with a fixed and arbitrarily small time between offers, a stationary sequential equilibrium cannot involve completely revealing offers, unless the supports of the traders' valuations do not overlap.[5] If a stationary equilibrium does exist, it must entail a more complicated form of revelation, such as a partition equilibrium where each offer partitions the set of remaining trader types. Unfortunately, construction of such an equilibrium has proved elusive.

Admati and Perry (1987) show that allowing the traders to delay making offers averts this "no delay" result.[6] They prove that, when the seller's valuation is known and there are two possible types of buyers whose valuations are sufficiently different, there is a sequential equilibrium that involves delay with positive probability, even as the minimum time between offers shrinks to zero. Moreover, if we restrict attention to sequential equilibria satisfying two weak assumptions, then the equilibrium path is unique and fully revealing. As we shall see here, when this model is extended to the case of a continuum of traders, this separating equilibrium continues to exist and results in a positive payoff to the uninformed party.

Wilson (1986), in a different game, extends the equilibrium constructed here to a setting of many buyers and sellers in an oral bid-ask market. A key insight of his analysis is that discounting is no longer needed to motivate trade, since competition among the traders is a sufficient source of impatience.

The paper is organized as follows. I begin by describing the model (Section 2) and the equilibrium beliefs and strategies (Section 3). Next, the equilibrium is derived, beginning with the analysis after two or more offers have been made (Section 4), then the analysis after only one offer has been made (Section 5), and finally the analysis prior to any offers (Section 6). In Section 7, I turn to an example in which the traders' beliefs are uniformly distributed, so that the equilibrium strategies can be derived explicitly. Proofs are relegated to the appendix.

## 2. FORMULATION

A buyer and seller are bargaining over the price of an object that is worth S to the seller and $B$ to the buyer. Let $S$ refer to a seller with valuation $S$ and $B$ refer to a buyer with valuation $B$. Each trader knows his own valuation, but only knows the distribution from which the other trader's valuation is drawn. I analyse a symmetric, private-valuation model. The buyer's valuation $B \in [0, 1]$ is given by the distribution $F(a)=\Pr\{B \le a\}$. The seller's valuation $S \in [0,1]$ is drawn independently (and symmetrically) from $F(1 - a) = \Pr\{S \ge a\}$. It is assumed that $F$ has a positive density $f$ on [0, 1].

Both bargainers prefer agreement today to the same agreement tomorrow, and the form of this impatience is given by a positive discount rate $r$. An outcome of the game, denoted by $\langle t, p\rangle$, specifies the time and price of trade. If the players trade at time $t$ for the price $p$, then the payoff to $S$ is $e^{-rt}(p - S)$ and $B$'s payoff is $e^{-rt}(B - p)$. The traders seek to maximize their expected payoffs: neither risk aversion nor wealth effects are present. The discount rate r and the probability distribution $F$ are common knowledge.

As in Admati and Perry (1987), the players alternate making offers with a minimum time of $t^0 = -(1/r)\log(d)$ between offers. Hence, $d$ is the discount factor from one period of delay. Initially, both traders have the option of making the first offer or terminating negotiations (at time $t \ge -t^0$) If the traders happen to make initial offers at the same time, then a fair coin is flipped to determine which offer stands as the initial offer. After an offer is made, the other trader has three possible responses: (1) a counter-offer, in which case the game continues, (2) acceptance, in which case the game ends with trade occurring at the offered price, or (3) termination, in which case the game ends with both traders receiving a payoff of 0. The response can occur at any time after the minimum time between offers has passed. Suppose that trader $T \in \{S, B\}$ makes the first offer $p_1$ after a delay of $\Delta_1$, and that in round $i$ the offer $p_i$ is made after a delay $\Delta_i$ *beyond* the minimum time $t^0$ between offers. Then after $n$ rounds of play ($n \ge 1$), the history $h^n$ is given by $\{T, (\Delta_i, p_i)_{i=1,\dots,n}\}$, and prior to either player making an offer the history is $h^0 = \{\Delta\}$, where $\Delta$ is the time past since the game began at $t = -t^0$. Throughout the paper, when I refer to an offer being accepted or a counter-offer being made "immediately" or "without delay," I mean that the action is taken with no *additional* delay beyond the minimum time between offers (i.e. $\Delta_i = 0$)

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valuation greater than or equal to the seller's valuation. Although the results in Gul and Sonnenschein (1988) and Gul, Sonnenschein, and Wilson (1986) cannot be applied immediately here, since this paper has a different information structure, a result similar to their "no delay" result does apply in the subgame with one-sided uncertainty with stationary strategies in the Rubinstein model.

[5] If the valuation supports are disjoint, then a stationary separating equilibrium can exist. This is because the trader who makes the first revealing offer can profitably make an offer that is acceptable to all possible trader types. Hence, trade occurs with probability one at the moment the first equilibrium offer is made, provided the time between offers is sufficiently small.

[6] An alternative approach to avoid the "no delay" result is to drop the requirement that the informed trader employs a stationary strategy, as is done in Cramton (1984*b*). Ausubel and Deneckere (1989*a*, *b*) show that this leads to a Folk Theorem in bargaining with one-sided uncertainty when there is no gap between the uninformed trader's valuation and the support of the informed trader's valuation.

A pure strategy $p_S$ for $S$ specifies, after each history $h^n$ at which it is $S$'s turn to move, a time delay $\Delta_{n+1}$ and whether to terminate negotiations, accept $p_n$, or make a counter-offer $p_{n+1}$. Similarly, define $p_B$ to be a pure strategy for $B$, and let $p = \{p_S, p_B \; \forall \; (S, B) \in [0, 1]^2\}$ be a profile of strategies for the traders. (Only pure strategies are considered.) The strategies $p$ result in an outcome $\{t(S, B), p(S, B)\}$, which depends on the traders' valuations $(S, B)$. "No trade" is represented by $t = \infty$. Since all actions are publicly observed, $S$'s belief about $B$'s valuation, which is initially independent of $S$, remains independent of $S$ after any history $h^n$. We can therefore denote the system of beliefs after any history $h^n$ by $m = \{F_B(\cdot \mid \cdot), F_S(\cdot \mid \cdot)\}$, where $F_B(\cdot \mid h^n)$ *is* the belief that $S$ has about $B$'s valuation conditional on the history $h^n$ and $F_S(\cdot \mid h^n)$ is $B$'s history contingent belief about $S$'s valuation.

A *sequential equilibrium* (Kreps and Wilson (1982)) for this game is a pair $(p, m)$ of strategies and beliefs, such that after every history $h^n$ each player's strategy is optimal given the other's strategy and his current beliefs about the other's valuation, and the beliefs are consistent with Bayes' rule.

The discounting form of time preferences plays an important role in the model. First, it implies that the time preferences are stationary, so that it is possible to determine a stationary equilibrium, in which the traders' strategies do not depend on time directly. Second, the discounting form means that delay is a credible signal of the strength of one's bargaining position: a buyer with a high valuation is more impatient than a buyer with a low valuation. This is because the traders' preferences satisfy the following single-crossing property.

The buyer's utility function $u(B; t, p) = e^{-rt}(B - p)$ *is* said to satisfy the *single-crossing property* if utility is strictly monotone in p and the slope of the indifference curve in $\langle t, p \rangle$ outcome space is strictly monotone in $B$. This means that the indifference curve of buyer $B'$ is everywhere steeper than the indifference curve of buyer $B<B'$, *so* that the indifference curves cross only once.

**Lemma 1.** *The traders' preferences satisfy the single-crossing property.*

There are three useful implications of the single-crossing property. First, if $B'$ ³ $B$, then $B'$ expects in equilibrium to settle no later than $B$. This follows from the revelation principle and incentive compatibility. Second, for the maximization problem in which a trader selects the optimal delay (time of trade) as a function of his private information, the first-order condition together with monotonicity of the choice function imply the second-order condition. Third, satisfaction of the first- and second-order conditions is sufficient for global optimality.

## 3. EQUILIBRIUM BELIEFS AND STRATEGIES

Before stating the equilibrium, it is helpful to define three functions that determine the equilibrium offer, the acceptance decision, and the delay decision as a function of current beliefs.

Equilibrium offers after any history are the Rubinstein (1982) full-information offers between the type revealed by the offer and the most patient type expected to accept the offer. Rubinstein (1982) shows that if it is common knowledge that $S$'s valuation is $S$ and $B$'s valuation is $B$, the alternating-offer game with a fixed time between offers has a unique subgame-perfect equilibrium outcome in which the bargainers trade immediately at the price $p(S, B)$ if $S$ makes the first offer and $p(B, S)$ if $B$ makes the first offer, where

$$p(S,B) = \frac{dS+B}{1+d} \qquad \text{and} \qquad p(B,S) = \frac{dB+S}{1+d}. \qquad (1$$

Hence, the offeror receives a payoff of $(B - S)/(1 + d)$ and the other receives $d(B - S)/(1 + d)$. The Rubinstein prices are such that each trader is indifferent between trading at the other's offer immediately or trading at his own offer after a one-period delay:

$$B - p(S, B) = d(B - p(B, S)) \qquad \text{and} \qquad p(B, S) - S = d(\, p(S, B) - S).$$

Thus, in our setting if the seller thinks the buyer's value is $b$ and the buyer thinks the seller's value is $s$, then along the equilibrium path the seller makes the offer $p(s, b)$. The buyer $b$ accepts this offer, since his best alternative is to counter immediately with $p(b, s)$ (the seller refuses all offers less than $p(b, s)$), which yields the same payoff as accepting $p(s, b)$ today.

Now suppose that $S$ has revealed $s$, but $B$'s value is still uncertain. Then, $B$'s acceptance decision reveals information to $S$: a less patient buyer accepts the offer, whereas a more patient buyer rejects the offer. Let $\tilde{b}(s,p)$ be the buyer type that is indifferent between accepting or rejecting the off p when $S$'s valuation has been revealed to be $s$. We will show that $\tilde{b}$ 's best alternative is to counter immediately with the offer $p(\tilde{b},s)$, which the seller accepts. Therefore, $\tilde{b}$ must be indifferent between $p$ today and $p(\tilde{b},s)$ tomorrow:

$$\tilde{b} - p = d(\tilde{b} - p(\tilde{b}, s)) = \frac{d}{1+d}(\tilde{b} - s).$$

Solving for $\tilde{b}$ yields $\tilde{b}(s, p)$, which is shown below together with the analogous function for the seller:

$$\tilde{b}(s, p) = (1+d)p - ds \qquad \text{and} \qquad \tilde{s}(b, p) = (1+d)p - db. \tag{2}$$

A buyer $B < \tilde{b}(s, p)$ prefers to delay before making the revealing off $p(B, s)$. Suppose that $S$ infers $B$'s value is $b(\Delta|s, \tilde{b})$ if $B$ delays $\Delta$ before making the offer $p(b, s)$ and $S$ thinks that $B \geq \tilde{b}$ would have accepted $S$'s offer. The length of delay $b = b^{-1}(B|s, \tilde{b})$ required to signal $B$ credibly is given by the incentive constraint

$$e^{-r}[B - p(b(b), s)] = \max_{\Delta} e^{-r\Delta}[B - p(b(\Delta), s)],$$

which says that $B$ does best by delaying for $b$ before making the offer $p(B, s)$.

Taking the derivative of $B$'s payoff with respect to $\Delta$ yields the separable first-order differential equation

$$\frac{db}{d\Delta} = -\frac{r}{d}(b - s),$$

with initial condition $b(0|s, \tilde{b}) = \tilde{b}$. The optimal delay is then found by integration:

$$b(B|s, \tilde{b}) = \int_{\tilde{b}}^{B} -\frac{d}{r}\frac{db}{b - s} = -\frac{d}{r}\log\frac{B - s}{\tilde{b} - s}. \tag{3}$$

By the single-crossing property of Lemma 1, since $b(\cdot|s, \tilde{b})$, and hence $b(\cdot|s, \tilde{b})$, are strictly decreasing, (3) is necessary and sufficient for the buyer's optimization problem. The function $b(B|s, \tilde{b})$ and its inverse $b(\Delta|s, \tilde{b})$ together with the analogous functions $s(S|b, \tilde{s})$ and $s(\Delta|b, \tilde{s})$ for the seller are given below:

$$\begin{aligned} b(B|s, \tilde{b}) &= -\frac{d}{r}\log\frac{B - s}{\tilde{b} - s} \qquad & s(S|b, \tilde{s}) &= -\frac{d}{r}\log\frac{b - S}{b - \tilde{s}} \\ b(\Delta|s, \tilde{b}) &= (\tilde{b} - s)e^{-r\Delta/d} + s \qquad & s(\Delta|b, \tilde{s}) &= b - (b - \tilde{s})e^{-r\Delta/d}. \end{aligned} \tag{4}$$

The offer functions in (1), the acceptance functions in (2), and the delay functions in (4) are used throughout the paper.

The equilibrium path can now be stated. Initially, both bargainers delay negotiations by refusing to make an offer. As time passes, each becomes more pessimistic about the magnitude of the gains from trade, since if the gains were larger the other trader would already have made an offer. Let $s(\Delta)$ denote the valuation of the seller that makes an initial offer after a delay of $\Delta$ and let $s(S) = s^{-1}(S)$ be the delay until $S$ makes an initial offer if $B$ does not make one first. Similarly, define $b(\Delta)$ and $b(B) = b^{-1}(B)$. Less patient trader types trade earlier, so $s(\cdot)$ and $s(\cdot)$ are increasing functions and $b(\cdot)$ and $b(\cdot)$ are decreasing functions. After a delay of $\Delta$, $B$ thinks that $S$'s valuation is no less than $s(\Delta)$, and so $B$ terminates negotiations if $B < s(\Delta)$, since gains from trade do not exist. Hence, after a delay of $\Delta$ without an offer or termination, $S$ thinks $B$'s valuation is in $[s(\Delta), b(\Delta)]$: a more patient buyer ($B < s(\Delta)$) would have terminated negotiations before $\Delta$ and a less patient buyer ($B > b(\Delta)$) would have made an offer before $\Delta$. $S$'s belief, then, is the truncated prior:

$$F(B|\Delta) = \frac{F(B) - F(s(\Delta))}{F(b(\Delta)) - F(s(\Delta))} \qquad \text{for} \qquad s(\Delta) \leq B \leq b(\Delta).$$

Similarly, $B$ believes $S$'s valuation is in $[s(\Delta), b(\Delta)]$ after a delay of $\Delta$.

Eventually either the more patient trader realizes that gains do not exist or the less patient trader makes an offer. Thus, there are three possibilities in the equilibrium path, illustrated in Figure 1, depending on Nature's realization of the valuations $S$ and $B$: (1) If $S \geq B$, then no offers are ever made—the more patient trader learns through the other's delay that gains do not exist, and so terminates negotiations. (2) If S<1/2 and $S < 1 - B$, then $S$ is less patient than $B$ and makes the initial offer $p(S,\tilde{b})$ after a delay of $s(S)$, where $\tilde{b}$ *is* the most patient buyer type to accept the offer. (3) If $B > 1/2$ and $B > 1- S$, then $B$ *is* less patient than $S$ and makes the initial offer $p(B,\tilde{s})$ after a delay of $b(B)$, where $s$ is the most patient seller type to accept the offer. In case (2), $B$ accepts the offer without delay if $B \geq \tilde{b}$. Otherwise, $B$ rejects the offer and reveals his valuation to be $B$ by delaying $b(B|S,\tilde{b})$ beyond the minimum delay $t^0$ before making the offer $p(B, S)$. In case (3), $S$ accepts the initial offer immediately provided $S \leq \tilde{s}$. Otherwise, $S$ waits for $s(S|B,\tilde{s})$ beyond the minimum delay and then makes the offer $p(S, B)$. In either case, the counter-offer is accepted without delay. Although negotiations end in finite time (almost surely) after at most two offers, the equilibrium outcome depends critically on the traders' strategic *option* to make alternating offers forever.

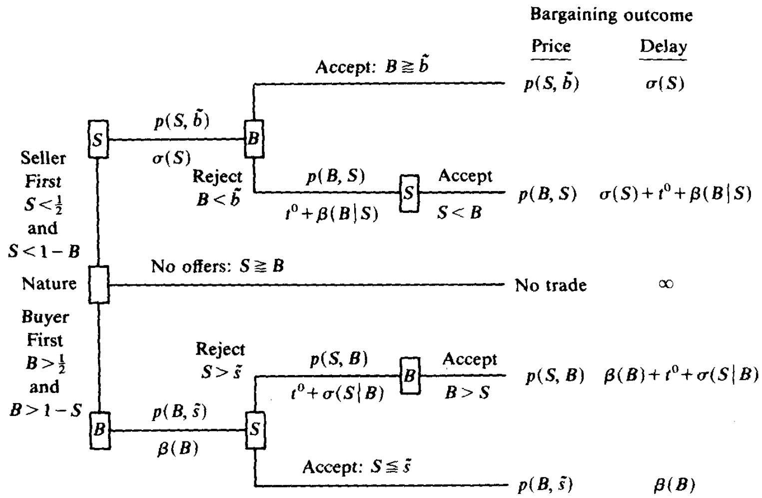


FIGURE 1
Description of the equilibrium path

Figure 2 displays the equilibrium outcome as a function of the traders' valuations when $F$ is the uniform distribution and the minimum time between offers is arbitrarily small. The outcome takes on five different forms, depending on whether trade occurs ($B > S$), whether the buyer makes the first offer ($B > 1 - S$), and whether the initial offer is accepted without delay (the gains from trade are sufficiently large). The equilibrium has a number of intuitive properties: trade is delayed with probability one, but eventually occurs whenever gains from trade are present;[7] trade occurs sooner, the larger are the gains from trade; the trader with the larger expectation of the gains from trade makes the initial offer; and the initial offer is only accepted if the gains are sufficiently large.

Figure 3 shows the equilibrium path for an example with $S = 0{\bullet}1$ and $B = 0{\bullet}4$. Here $S$ makes the first offer $p(S,\tilde{b})$ after a delay of $s(S)$. $B$ rejects the offer and delays $b(B|S,\tilde{b})$ before making the offer $p(B, S)$, which $S$ accepts without delay.

The strategies and beliefs off the equilibrium path have the same stationary structure as on the equilibrium path. The strategies only depend on the current beliefs and the most recent offer. And the posterior beliefs following an offer depend only on the prior belief and the amount of delay before the offer was made. This stationarity greatly simplifies the verification of the equilibrium.

[7] The result that all the gains are realized ex post requires that all bargaining costs are due to discounting. If there are direct costs to bargaining, such as an hourly opportunity cost, then a bargainer would terminate negotiations if the expected gains are too small (Cramton (1991)). In the extreme case, with direct costs but no discounting, then delay cannot be used to signal information, so the bargaining ends after a single take-it-or-leave-it offer (Perry (1986)).

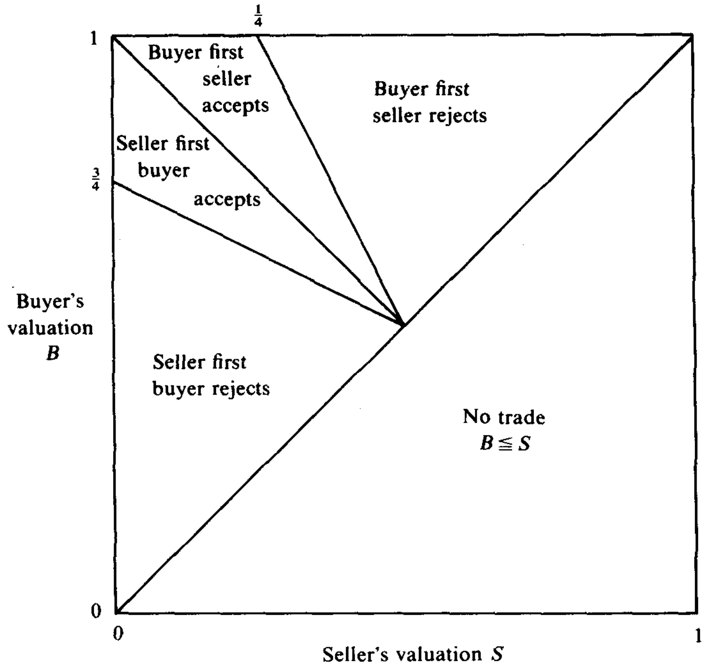


FIGURE 2
Equilibrium outcome as a function of the traders' valuations

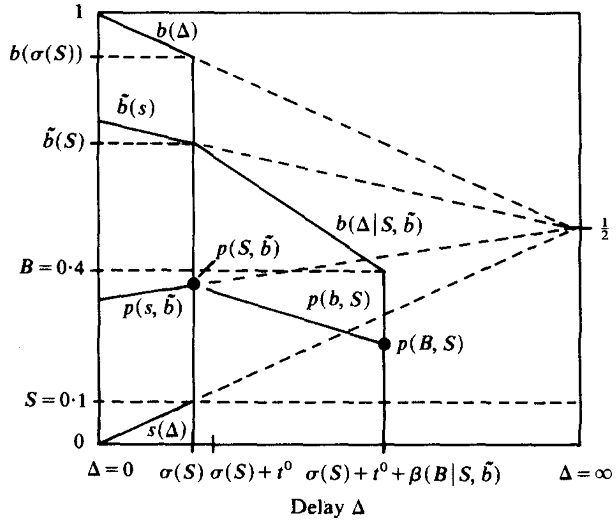


FIGURE 3
An equilibrium path with $S = 0{\cdot}1$ and $B = 0{\cdot}4$

We now state the strategies and beliefs in each of the three phases of the game. For simplicity, it is assumed that the traders never make offers that are more attractive than their revealed valuations. Such offers are dominated by making the appropriate Rubinstein offer. The beliefs and strategies given below are for the equilibrium path in which $S$ makes the first offer (if any) and then $B$ responds with acceptance or a counter offer. The beliefs and strategies for the other possible path, in which $B$ makes the first offer, are symmetric.

*Phase* 0: *No offers have been made*

($m0$) *Belief of S.* If $B$ has yet to make an offer after a delay $\Delta$, $S$ believes that $B$'s valuation is in $[s(\Delta), b(\Delta)]$. If $B$ makes the initial offer $p$ after a delay $\Delta$, $S$ believes that $B$'s valuation is $b(\Delta)$ with probability one. Then use ($m1$).

($p0$) *Strategy of S.* If $S \geq 1/2$, wait for $B$ to make the first offer, but terminate negotiations if the delay $\Delta > b(S)$. If $S < 1/2$, make the offer $p(S,\tilde{b}(S))$ after a delay of $s(S)$, if $B$ does not make an offer first. Then follow ($p1$).

*Phase* 1: *One offer has been made*

Assume that $S$ revealed $s$ with an offer $p > s$ made when $S$ believed $B$'s valuation to be in $[s', b]$. Let $b^0 = \min\{b, \tilde{b}(s,p)\}$. It is now $B$'s turn to move.

($m$1) *Belief of S.* In response to a counter-offer by $B$ after a delay of $\Delta$, $S$ believes that $B$'s value is $b(\Delta \mid s, b^0)$ with probability one. Then use ($m$2).

($p$1) *Strategy of B. B*'s response to $S$'s offer $p$ that reveals $s$ is:

(i) If $B \le s$, terminate negotiations.

(ii) If $B \ge b^0$, accept $p$ without delay provided $B - p \ge d[B - p(b^0, s)]$, and otherwise counter-offer with $p(b^0, s)$ without delay.

(iii) If $s < B < b^0$, then counteroffer $p(B, s)$ after delaying $b(B|s, b^0)$.

Then follow ($p$2).

*Phase* 2: *Two or more offers have been made*

Suppose previous offers have revealed valuations to be s and $b$. Further suppose that $B$ has just made $S$ an offer of $p$. Let $s^0 = \max\{s, \tilde{s}(b,p)\}$.

($m$2) *Belief of B.* If $S$ counter-offers after a delay of $\Delta$ in response to $B$'s offer $p$, then $B$ infers that $S$'s valuation is $s(\Delta|b, s^0)$.

($p$2) *Strategy of S. S*'s response to $B$'s offer of $p < b$ *is:*

(i) If $S \ge b$, then terminate negotiations.

(ii) If $S \le s^0$, then accept $p$ immediately provided $p - S \ge d[p(s^0, b) - S]$, and otherwise counter with $p(s^0, b)$ without delay.

(iii) If $s^0 < S < b$, then delay $s(S|b, s^0)$ before making the offer $p(S, b)$.

There are several things to note about the beliefs ($m$2) and strategies ($p$2). First, the fact that trade occurs at the Rubinstein price once valuations have been revealed is required by the beliefs ($m$2), since immediate counter-offers in response to Rubinstein offers do not alter beliefs (see Admati and Perry (1987) for details). Nonetheless, ($m$2) and ($p$2) do satisfy the spirit of the Cho-Kreps intuitive criterion, since beliefs are revised (as they must be) following a credible signal of strength. (The qualifier "in the spirit of " is used here, since the refinements by Cho and Kreps (1987) and Cho (1987) are not formally defined for models with two-sided signalling and a continuum of actions.) In addition, the beliefs ($m$2) have a desirable property: if a trader mistakenly reveals that she is weaker than she truly is by making an offer too early, it is the trader's best response to *correct* the mistake by delaying her next offer so as to reveal the truth. This property simplifies the derivation of the optimality of the strategies off the equilibrium path.

It remains to determine the functions $s(S)$, $b(S)$, $\tilde{b}(S)$, and $\tilde{s}(B)$, and verify that the above strategies and beliefs form an equilibrium.

## 4. SUBGAME AFTER BOTH PARTIES HAVE MADE OFFERS

We begin by analyzing the subgame after both traders have made revealing offers. This subgame is related to Rubinstein's (1982) complete information analysis. There is, however, a substantial difference between the information structure in Rubinstein (1982), where valuations are common knowledge, and in the subgame after both parties have made revealing offers. The difference is that in the common knowledge case, beliefs are fixed, whereas here beliefs may change as a result of future actions the traders take. Consequently, it is possible to sustain an equilibrium other than the Rubinstein solution by allowing the traders to threaten with beliefs.

One way to guarantee the Rubinstein outcome in this subgame is to assume that beliefs stay fixed at the revealed values (see for example Bikhchandani (1992), Grossman and Perry (1986), and Rubinstein (1985)). The problem with this restriction is that by fixing beliefs one does not allow a trader to correct a past mistake. (Of course in equilibrium, "mistakes" are never made. Nonetheless, it is important to argue that strategies and beliefs off the equilibrium path are reasonable.) If the seller mistakenly reveals that her valuation is much lower than it truly is, then the buyer may demand a price that is below the seller's true valuation. If beliefs are fixed, there is nothing that the seller can do to convince the buyer of the truth. Indeed fixing beliefs violates the spirit of the

Cho-Kreps (1987) intuitive criterion, since delay provides the seller with a credible means of convincing the buyer of the truth.

To avoid this problem but still guarantee the Rubinstein outcome in this subgame, we assume that beliefs stay fixed at the revealed values, *unless* an offer is delayed or an offer that should have been acceptable is rejected.

**Proposition 1.** In the subgame after the seller has revealed her valuation to be $s$ and the buyer has revealed his valuation to be b, the beliefs ($m2$) and strategies ($p2$) form an equilibrium. Along the equilibrium path trade occurs without delay at a price $p(s, b)$ if it is the seller's turn to make an offer or $p(b, s)$ if it is the buyer's turn.

## 5. SUBGAME AFTER ONE PARTY HAS MADE AN OFFER

Now suppose that only one of the traders' valuations has been revealed. In particular, suppose that the seller has revealed her valuation to be $s$ with an offer $p$ and the buyer's valuation is on $[s', b]$. We wish to derive the separating equilibrium in this subgame. Let $b^0 = \min\{b, \tilde{b}(s,p)\}$. If $B$ does not immediately accept the offer $p$, then $S$ infers that $B < b^0$, so $b(\Delta|s, b^0)$ is the valuation of the buyer who delays $\Delta$ before making an offer in response to the offer $p$ by a seller who has revealed the valuation $s$. Figure 4 displays the equilibrium path for this subgame: the buyer accepts $p$ immediately if $B \geq \tilde{b}$, and otherwise delays making an offer until he has convinced the seller of the strength of his position.

**Proposition 2.** *In the subgame after the seller has revealed s with an offer p and the buyer's valuation is in* $[s', b]$, *the beliefs* ($m1$) *and* ($m2$) *and strategies* ($p1$) *and* ($p2$) *form an equilibrium. Along the equilibrium path, B accepts p without delay if* $B \geq \tilde{b}(s,p)$, *and otherwise counter-offers with* $p(B, s)$ *after a delay of* $b(B|s,\tilde{b})$. *This offer is accepted without delay by the seller.*

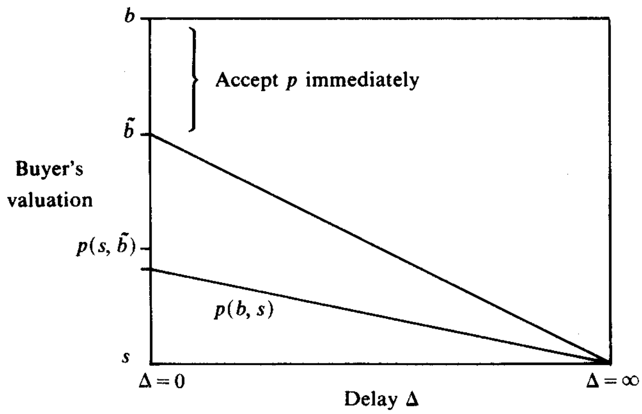


FIGURE 4
Equilibrium path with one-sided uncertainty

## 6. SUBGAME BEFORE ANY OFFERS

The initial subgame is a game of timing in which neither trader has made a revealing offer. The traders' strategies, $s(S)$ and $b(B)$, determine when they are to make an offer as a function of their valuations. We wish to construct a separating equilibrium, so that the time of a trader's offer is a monotone function of his valuation. Only the seller's optimization problem is analyzed below; the analysis for the buyer is analogous. Here, we focus on the limiting case as the minimum time between offers goes to zero ($d \to 1$). The case with $d < 1$ is treated in the appendix.

We begin by determining $S$'s optimal offer at time $\Delta$, given that the buyer infers that her valuation is $s$ with probability one. To guarantee a unique solution to $S$'s optimization problem, we make the following assumption about the distribution $F$.

(F) The distribution $F$ has a positive density $f$ on $[0, 1]$, such that for all $s \in [0, 1/2)$ and $b \in (s, 1 - s)$,

$$F(b) - F(s) \le f(b)(b - s).$$

For example, (F) is satisfied by the distribution $F(u) = u^a$ for $a \ge 1$.

**Proposition 3.** *Suppose* (*F*) *holds and* $d \to 1$. *If S makes an initial offer after a delay of* Δ *and S believes that* $B \in [0, b]$, *then along the equilibrium path S makes an initial offer of* $p(s,\tilde{b})$ *where* $\tilde{\ }$ , ) *uniquely satisfies*

$$F(b) - F(\tilde{b}) = \int_s^{\tilde{b}} \left(\frac{B-s}{\tilde{b}-s}\right)^2 dF(B).$$

Finally, we determine the function $s(\cdot)$, which follows from the seller's incentive constraint. The binding incentive constraint is that a less patient trader will imitate a more patient trader by delaying too long (this follows from the single-crossing property). Rather than follow the proposed equilibrium strategy by making a revealing offer after a delay of $s(S)$, $S$ has the option of delaying $\Delta > s(S)$, *so* as to convince the buyer that her valuation is $s(\Delta) > S$. A necessary condition for $s(\cdot)$ to be part of an equilibrium is that such a deviation by $S$ is unprofitable. The equilibrium strategies are found by determining the function $s(\Delta)$ such that the best response of $S$ to $b(\cdot) = 1 - s(\cdot)$ is to make a revealing offer at time $s(S)$.

**Theorem.** Suppose (*F*) holds. If the time between offers is sufficiently small, then the strategies (*pi*) and beliefs (*mi*), $i = 0, 1, 2$, form an equilibrium. In the limit as $d \to 1$, the initial delay for $S$ is

$$s(S) = \int_0^S \frac{F(1-s) - F(\tilde{b}(s)) + f(1-s)[1-s-\tilde{b}(s)]}{r[F(1-s) - F(\tilde{b}(s))][\tilde{b}(s) - s]} ds,$$

*where* $\tilde{b}(s) \in (s, 1-s)$ *uniquely solves*

$$F(1-s) - F(\tilde{b}) = \int_s^{\tilde{b}} \left(\frac{B-s}{\tilde{b}-s}\right)^2 dF(B)$$

## 7. EXAMPLE AND CONCLUSION

When both traders' valuations are uniformly distributed on [0,1], it is possible to derive explicit solutions to the equations that define an equilibrium.

**Proposition 4.** *Suppose the traders' valuations are uniformly distributed on* [0, 1]. *Then S delays* $s(s)$ *before making an initial offer*; *seller* $s(\Delta)$ *makes the initial offer* $p(\Delta)$ *after a delay of* Δ, *which is immediately accepted by the buyer* $B \ge \tilde{b}(\Delta)$, *where*

$$s(S) = -\frac{g}{r}\log(1-2S) \quad p(\Delta) = \frac{a}{1+d} + \left(1 - \frac{2a}{1+d}\right)s(\Delta)$$
$$s(\Delta) = \tfrac{1}{2}\left(1 - e^{-r\Delta/g}\right) \qquad \tilde{b}(\Delta) = a - (2a-1)s(\Delta)$$
$$\frac{1}{2a} = 1 - \frac{d^2}{2+d} \qquad g = \frac{4d(2-d^2)}{2+d}.$$

Figure 5 depicts the traders' equilibrium strategies in the initial subgame. If the seller has not yet made a revealing offer, the buyer $b(\Delta) = 1 - s(\Delta)$ makes the offer $1 - p(\Delta)$, which is accepted without delay by the seller if $S \le \tilde{s}(\Delta) = 1 - \tilde{b}(\Delta)$.

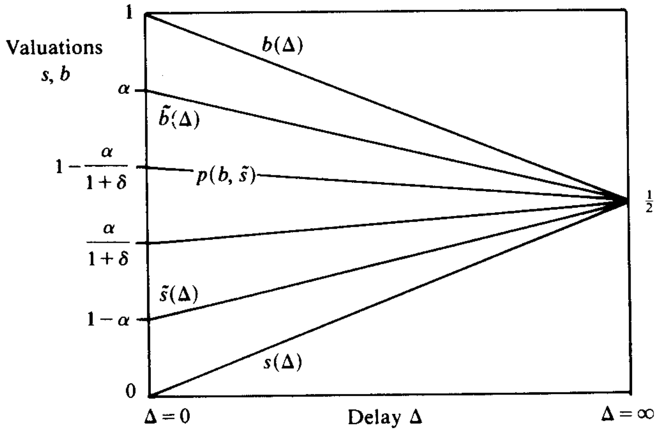


FIGURE 5

Time and magnitude of first revealing offer

It is interesting to compare the expected payoffs obtained in this dynamic game with what the players would get if they were able to commit to playing the ex ante efficient trading mechanism. Myerson and Satterthwaite (1983) prove that the ex ante efficient mechanism is a static mechanism that results in either immediate trade or no trade at all. Moreover, for the uniform example, they show that the simultaneous-offer game studied by Chatterjee and Samuelson (1983) is ex ante efficient. What sort of ex ante loss in payoffs do the traders suffer from being unable to commit to the static trading rule? A little algebra (see the appendix) shows that the ex ante gains from trade are reduced by a factor of $\frac{2}{3}$, due to the traders' inability to commit to the static mechanism, when the time between offers is made arbitrarily small. It, however, is not the case that all trader types prefer the static game over the dynamic game. Indeed, 43% of the trader types ($S > 0{\cdot}57$) are better off with the outcome from the dynamic game.

A second comparison is between this equilibrium and the one-sided-offer models. Ausubel and Deneckere (1988) show that the additional efficiency loss of the equilibrum here relative to the ex ante efficient mechanism is not a necessary implication of a sequential trading procedure. In particular, when stationarity is relaxed, then the game with one-sided offers has an equilibrium that yields the ex ante efficient payoffs in the limit as the time between offers goes to zero. In stark contrast, the equilibria identified in the one-sided-offer models of Cramton (1984a) and Cho (1990b) have the property that if the valuation distributions for the seller and the buyer completely overlap, then in the limit as the time between offers goes to zero, the expected gains realized in equilibrium go to zero. All the gains from trade are consumed by delay of infinite duration. This negative result follows because, due to a strong stationarity assumption, the equilibria satisfy the Coase conjecture in the subgame with one-sided uncertainty. (Ausubel and Deneckere (1991) prove this "no trade" result more generally.) An implication of the "no trade," result is that the equilibrium payoffs in Cramton (1984a) and Cho (1990b) are quite sensitive to the time between offers. Whether the bargainers can make offers once a day, once an hour, or once a minute has a first-order affect on their payoffs. In contrast, the equilibrium payoffs here are relatively insensitive to the minimum time between offers. For example, with uniform uncertainty and a discount rate of 10% per year, whether offers can be made every day, hour, or minute alters the equilibrium payoffs by less than 0 1%.

Comparisons with the models with alternating offers (Chatterjee and Samuelson (1987) and Cho (1990a)) are not possible. First, it is not clear how Chatterjee and Samuelson's analysis extends from two types to a continuum of types. Second, the class of equilibria studied by Cho does not exist when the parties are uncertain about whether gains from trade are present.

The virtue of this equilibrium is its absence of *ex post regret.* Here all the gains from trade are realized ex post; whereas, in the static game, the negotiation ends with positive probability in a state in which both bargainers know that gains are possible, and yet they are forced to walk away from the bargaining table. A second virtue is that the outcome is *ex post fair*. If the parties' private information is revealed during the bargaining, they settle at the terms of the full information game. Hence, neither trader has an incentive to renegotiate the deal based on information available after settlement.

# APPENDIX

***Proof of Lemma* 1.** For $B$. we have $u(B;\, t,\, p)=e^{-rt}(B - p)$. Hence, $B$'s payoff is strictly decreasing in $p$, since $\P u/\P p = -\, e^{-rt} < 0$. The slope of the indifference curve is $-(\P u/\P t)/(\P u/\P p) = -r\,(B-p)$, which is strictly decreasing in $B$. The proof for the seller is identical.||

***Proof of Proposition* 1.** Suppose, without loss of generality, previous offers have revealed valuations to be $s$ and $b$. Further suppose that the buyer has just made the seller an offer of $p$. Let $s^0 = \max\{s,\ \tilde{s}(b, p)\ \}$.

We need to show that the beliefs ($m2$) and strategies ($p2$) constitute an equilibrium. That is, $S$'s strategy is optimal given the belief and strategy of $B$ when $S$ believes that $B = b$. Only single-period deviations need to be checked in order to determine if the seller's strategy is optimal, since the seller's optimization problem is a dynamic programming problem with discounting and bounded reward (Blackwell (1965)).

*Case (i).* If $S \geq b$, then gains from trade do not exist (almost surely), so that $S$ cannot do better than 0.

*Case (ii).* Suppose $S \leq s^0$. An immediate counteroffer by $S$ signals that $S = s^0$. Hence, the counter-offer $p(s^0, b)$ *is* accepted by $B$ with probability one, so $S$ will accept $p$ only if $p - S' \geq d[p(s^0, b) - S]$. Moreover, $S$ will accept $p$ if $p - S' \geq d[\ p(s^0, b) - S]$ provided offering $p(s^0, b)$ without delay is the best counter-offer $S$ can make. Suppose $S$ offers $p'$ after a delay of $\Delta$, so that $B$ believes $S$'s valuation is $s = s(\Delta|b, s^0)$. Then $S$'s optimal offer is $p(s, b)$*:* offering $p' < p(s, b)$ yields less, since both $p'$ and $p(s, b)$ are accepted with probability one; similarly, offering $p' > p(s,b)$ yields less, since then $B$ counters immediately with $p(b,s) < p(s,b)$, which $S$ accepts, since under ($m2$) beliefs are not altered by the counter-offer and only single-period deviations need be considered. It remains to determine the optimal delay, given that $p(s, b)$ is the optimal offer after a delay of $\Delta$. $S$'s payoff from offering $p(s, b)$ after a delay of $\Delta$ is $u(\Delta) = e^{-r\Delta}\,[p(s, b) - S]$ where

$$p(s,b) = \frac{ds+b}{1+d} \quad \text{and} \quad s = b-(b-s^0)e^{-r\Delta/d}.$$

The marginal utility of waiting is then

$$\frac{du}{d\Delta} = -re^{-r\Delta}[b-S-(b-s^0)e^{-r\Delta/d}].$$

By inspection, $du/d\Delta < 0$ for all $\Delta > 0$ when $S \leq s^0$, so that the optimal delay is $\Delta = 0$. Hence, (ii) is optimal.

*Case* (iii). Finally, suppose $s^0 < S < b$. We need to show that, by rejecting $p$, $S$ can do no better than by offering $p(S, b)$ after a delay of $s(S|b, s^0)$. As in (ii) the optimal offer is $p(s, b)$ if $S \leq s$ and $B$ believes $S$'s valuation is $s$; hence $p(S, b)$ *is* the optimal offer after a delay of $s(S|b, s^0)$. The delay $\Delta = s(S|b, s^0)$ is optimal when $S > s$, since

$$\frac{du}{d\Delta} \gtreqless 0 \quad \text{if} \quad \Delta \lesseqgtr s\,(S|b,s^0).$$

Thus, (iii) is optimal.

Along the equilibrium path $S = s$ and $B = b$, so trade occurs with probability one at the price $p(s, b)$ if it is $S$'s turn to make an offer or $p(b, s)$ if it is $B$'s turn to make an offer.||

*Proof of Proposition* 2. We must show that the strategies ($p1$) and ($p2$) and beliefs ($m1$) and ($m2$) form an equilibrium in the subgame after $S$ has revealed $s$ with an offer $p$ and $B$'s valuation is in $[s', b]$.

*Case* ($i$). $B \leq s$ can do no better than 0, so termination is optimal.

*Case* (ii). Suppose $B \geq b^0$. From ($m1$), if $B$ makes a counter-offer after a delay of $\Delta$, $S$ believes $B$'s valuation is $b(\Delta|s, b^0)$. But then from Proposition 1, $B$'s best counter-offer is $p(b^0, s)$ without delay. Hence, $B$ is best off accepting $p$ without delay if $B - p \geq d[B - p(b^0, s)]$, and (ii) is optimal.

*Case* (iii). Suppose $s < B < b^0$. From ($m1$), delaying $\Delta$ before responding to the seller's offer reveals the buyer valuation to be $b(\Delta|s, b^0)$. But then we enter the subgame with complete information analyzed in Proposition 1. In this subgame, $B$'s best response is to offer $p(B, s)$ after a delay of $b(B|s, b^0)$, which is accepted without delay by the seller. Hence, (iii) is optimal.

Along the equilibrium path, $s \leq B \leq b$ and $b^0 = \tilde{b}$. Thus, (ii) becomes accept $p$ if $B \geq \tilde{b}(s, p)$, since $B \geq \tilde{b}(s, p)$ if and only if $B \geq p \geq d[B - p(\tilde{b}\ , s)]$.

The propositions of Section 6 consider the limiting case as the minimum time between offers goes to zero ($d \to 1$). Here we analyse the case for $d < 1$. In order to guarantee a unique solution to the first-order condition that determines the initial offer, we make the following assumption about the distribution of uncertainty.

(F$d$) The distribution $F$ has a positive, differentiable density $f$ on [0,1], such that for all $s \in [0, \tfrac{1}{2})$ and $b \in (s, 1 - s)$

$$d^3\frac{F(b)-F(s)}{f(b)(b-s)} \leq \frac{1+d^3}{1+d} + (1-d)\left[1+(b-s)\frac{f'(b)}{f(b)}\right].$$

Notice that the second term on the right-hand-side converges to 0 as the minimum time between offers goes to 0. Hence, in the limit as $d \to 1$, (F$d$) simplifies to (F).

**Lemma 2.** *Suppose if the seller makes an initial offer after a delay of* $\Delta$ *the buyer infers that the seller's valuation is* $s$*, where* $s \geq S$. *If* $S$ *believes that* $B \in [0, b]$*, then* $S$ *makes an initial offer of* $p(s,\ \tilde{b}\ )$ *where*

$$\tilde{b}(S|s,b) \in \arg\max_{u\in[s,b]} \tilde{u}(S,u|s,b)$$

*and*

$$\tilde{u}(S,\tilde{b}|s,b) = [F(b) - F(\tilde{b})][p(s,\tilde{b}) - S + \int_s^{\tilde{b}} d[p(B,s) - S\left(\frac{B-s}{\tilde{b}-s}\right)^d dF(B).$$

*Moreover, along the equilibrium path* $\tilde{b} \in (s, b)$ *satisfies*

$$F(b) - F(\tilde{b}) - (1-d^2)(\tilde{b}-s)f(\tilde{b}) = \int_s^{\tilde{b}} d^3\left(\frac{B-s}{\tilde{b}-s}\right)^{1+d} dF(B), \tag{5}$$

*and if* (F*d*) *holds the solution to* (5) *is unique.*

*Proof of Lemma 2.* Faced with the beliefs and strategies from Propositions 1 and 2, the seller chooses the revealing offer *p* to maximize her utility. Equivalently, *S* can choose $\tilde{b}$ to maximize her utility, since Proposition 2 defines a one-to-one relationship between *p* and $\tilde{b}$; namely, $p = p(s, \tilde{b})$. From Proposition 2, *p* is accepted immediately if $B \in [\tilde{b}, b]$, and otherwise, since $S \leq s$, *S* accepts the offer $p(B, s)$ after her payoff has been discounted by the factor

$$de^{-rb(B|s,\tilde{b})} = d\left(\frac{B-s}{\tilde{b}-s}\right)^d.$$

Thus, *S*'s expected utility from offering $p(s, \tilde{b})$ *is* given by $\tilde{u}(S, b|s, b)$. Since $\tilde{u}$ is continuous on [s, *b*], a maximum exists.

Along the equilibrium path $S = s$, so the seller's marginal utility (multiplied by 1 + *d*) from offering $\tilde{b}$ *is* given by

$$(1+d)^2\tilde{u}'(s,\tilde{b}|s,b) = F(b) - F(\tilde{b}) - (1-d^2)(\tilde{b}-s)f(\tilde{b}) - \int_s^{\tilde{b}} d^3\left(\frac{B-s}{\tilde{b}-s}\right)^{1+d} dF(B).$$

Note that $\tilde{u}'$ is positive at $\tilde{b} = s$ and negative at $\tilde{b} = b$; hence, the maximum occurs at an interior point and must satisfy the first-order necessary condition (5). Now $\tilde{u}''$ is given by

$$(1+d)^2\tilde{u}''(s,\tilde{b}|s,b) = (d^2 - d^3 - 2)f(\tilde{b}) - (1-d^2)(\tilde{b}-s)f(\tilde{b}) + d^3(1+d)\int_s^{\tilde{b}} \frac{(B-s)^{1+d}}{(\tilde{b}-s)^{2+d}} dF(B).$$

Since

$$\int_s^{\tilde{b}} \frac{(B-s)^{1+d}}{(\tilde{b}-s)^{2+d}} dF(B) < \frac{F(\tilde{b}) - F(s)}{\tilde{b}-s},$$

$\tilde{u}'' < 0$ if (F*d*) is satisfied. Thus, $\tilde{u}''$ is strictly decreasing over $\tilde{b} \in [s, b]$ and (5) has a unique solution, which is the global maximizer of *S*'s payoff. ||

*Proof of Proposition 3.* This follows immediately from Lemma 2 by letting *d* go to 1. ||

**Lemma 3.** *If* (F*d* ) *holds and* $dc(s) + g(s) + h(s) > 0$ *for all* $s \in [0, \tfrac{1}{2})$, *then the strategies* (*pi*) *and beliefs* (*mi*), *i* = 0, 1, 2, *constitute a sequential equilibrium, where*

$$\begin{aligned} s(S) &= \int_0^S \frac{dc(s) + g(s) + h(s)}{rc(s)[\tilde{b}(s) - s]} ds \\ c(s) &= F(1-s) - F(\tilde{b}(s)) - (1-d)[\tilde{b}(s) - s]f(\tilde{b}(s)) \\ g(s) &= \frac{d}{1+d} f(1-s)[d(1-2s) - (2\tilde{b}(s) - 1)] \\ h(s) &= d^2(1-d)\int_s^{\tilde{b}(s)} \left(\frac{b-s}{\tilde{b}(s)-s}\right)^d dF(b), \end{aligned} \tag{6}$$

*and* $\tilde{b}(s) \in (s, 1-s)$ *satisfies* (5).

*Proof of Lemma* 3. Assume that (F*d*) holds so that $\tilde{b}(s)$ *is* well-defined. We begin by computing the utility $U(S, \Delta)$ to *S* from making a revealing offer after a delay of $\Delta \geq s(S)$, given that the buyer is following the strategy $b(\cdot)$ and he expects the seller to be playing $s(\cdot)$. Three events involving trade are possible depending on the buyer's valuation, as shown in Figure 6:

(1) For $B \in [s(\Delta), \tilde{b}(s(\Delta))]$, *S* reveals first and B makes a counter offer after some delay.
(2) For $B \in [\tilde{b}(s(\Delta)), b(\Delta)]$, *S* reveals first and B immediately accepts *S*'s offer.
(3) For $B \in [b(\Delta), b(s(S))]$, *B* reveals first.

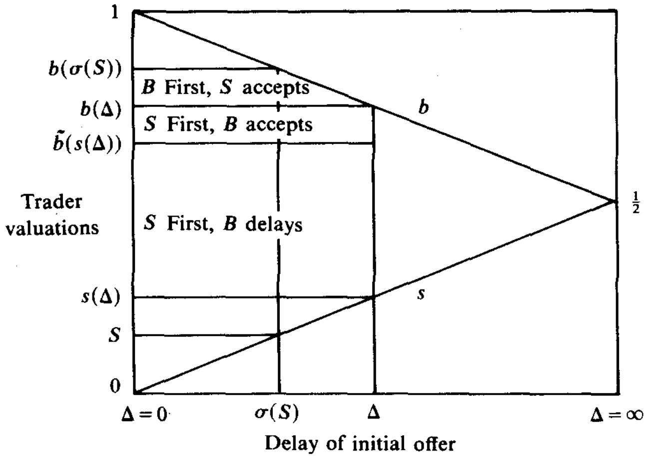


FIGURE 6

***S* deviation from waiting too long**

In what follows, first derivatives with respect to Δ are denoted by ′, as in $s' \equiv ds/d\Delta$.

*Case* 1. If $B \in [s, \tilde{b}(s)]$, then *S*'s offer is rejected by *B* and the counter-offer $p(B, s)$ is accepted after payoffs have been discounted by an additional factor of

$$d\left(\frac{B-s}{\tilde{b}-s}\right)^{d}$$

resulting in the ex post payoff to *S* of

$$de^{-r\Delta}[p(B,s)-s]\left(\frac{B-s}{\tilde{b}-s}\right)^{d}.$$

*Case* 2. If $B \in [\tilde{b}(s), b]$, then *S*'s offer of $p(s, \tilde{b})$ after a delay of Δ is accepted by *B* for an ex post payoff of $e^{-r\Delta}[p(s, \tilde{b}) - S]$.

*Case* 3. If $B \in [b, b(s(s))]$, then *B* reveals first with the offer $p(B, \tilde{s})$ after a delay of $b(B)$. This offer is accepted without delay by *S*, since $S \leq \tilde{s}(s(b(B)))$ for all $B \in [b, b(s(S))]$. Hence, the ex post payoff to *S* is $e^{-rb(B)}[p(B, \tilde{s}) - S]$.

The seller's expected utility is then found by integrating over the buyer's potential valuations for which trade takes place:

$$(1+d)U(S,\Delta) = \int_{s}^{\tilde{b}(s)} x(B,\Delta)dB + y\Delta + \int_{b}^{b(s(s))} z(B)dB \tag{7}$$

where

$$x(B,\Delta) = de^{-r\Delta}\left(\frac{B-s}{\tilde{b}-s)}\right)^{d}[dB+s-(1+d)S]f(B)$$

$$y(\Delta) = e^{-r\Delta}[F(b)-F(\tilde{b}(s))][\tilde{b}(s)+ds-(1+d)S]$$

$$z(B) = e^{-rb(B)}[dB+\tilde{s}(s(b(B))-(1+d)S]f(B).$$

A necessary condition for the seller's strategy to be a best response is that the marginal utility of waiting is zero at $S = s(\Delta)$:

$$\left.\frac{\partial U}{\partial \Delta}(S,\Delta)\right|_{S=S(\Delta)} = 0.$$

Taking the derivative of (7) with respect to Δ and substituting $S = s$ yields

$$(1+d)U'(s,\Delta) = \int_{s}^{\tilde{b}} x'(B,\Delta)dB + \tilde{b}'x(\tilde{b},\Delta) - b'z(b) + y'(\Delta)$$

where

$$x'(B,\Delta) = e^{-r\Delta} d\left(\frac{B-s}{\tilde{b}-s}\right) f(B)\left[(1-d^2)s' - d(B-s)\left(r - d\,\frac{s'-\tilde{b}'}{\tilde{b}-s}\right)\right]$$

$$y'(\Delta) = e^{-r\Delta}[(F(b)-F(\tilde{b}))(\tilde{b}'+ds'-r(\tilde{b}-s)) + (\tilde{b}-s)(b'f(b)-\tilde{b}'f(\tilde{b}))]$$

$$x'(\tilde{b},\Delta) = e^{-r\Delta} d^2(\tilde{b}-s)f(\tilde{b})$$

$$z(b) = e^{-r\Delta}(db+\tilde{s}-(1+d)s)f(b).$$

Collecting terms and multiplying by $e^{-r\Delta}$ results in the first-order condition

$$\begin{aligned}0 &= [F(b)-F(\tilde{b})][\tilde{b}'+ds'-r(\tilde{b}-s)]\\ &+b'f(b)[\tilde{b}-\tilde{s}-d(b-s)] - \tilde{b}'f(\tilde{b})(1-d^2)(\tilde{b}-s)\\ &-\left(\frac{r}{d}(\tilde{b}-s)-s'+\tilde{b}'\right)\int_s^{\tilde{b}} d^3\left(\frac{B-s}{\tilde{b}-s}\right)^{1+d} dF(B)\\ &+d(1-d^2)s'\int_s^{\tilde{b}}\left(\frac{B-s}{\tilde{b}-s}\right)dF(B).\end{aligned}$$

Substituting the first-order condition (5) and multiplying by *d* then yields

$$\begin{aligned}0 &= [F(b)-F(\tilde{b})][d(1+d)s'-(1+d)r(\tilde{b}-s)] - ds'(1-d^2)(\tilde{b}-s)f(\tilde{b})\\ &+db'f(b)[\tilde{b}-\tilde{s}-d(b-s)] + r(1-d^2)(\tilde{b}-s)^2 f(\tilde{b})\\ &+d^2(1-d^2)s'\int_s^{\tilde{b}}\left(\frac{B-s}{\tilde{b}-s}\right)^d dF(B).\end{aligned}$$

Finally, by dividing by $1 + d$ and substituting $b = 1 - s$ and $\tilde{s} = 1 - \tilde{b}$ (required by symmetry), we get the differential equation

$$\begin{aligned}0 &= [ds'-r(\tilde{b}-s)][F(1-s)-F(\tilde{b}) - ds'(1-d^2)(\tilde{b}-s)f(\tilde{b})\\ &+\frac{d}{1+d}s'f(1-s)[d(1-2s)-(2\tilde{b}-1)] + d^2(1-d)s'\int_s^{\tilde{b}}\left(\frac{B-s}{\tilde{b}-s}\right)^d dF(B).\end{aligned}$$

Solving for s’ yields the separable first-order differential equation

$$s' = \frac{rc(s)[\tilde{b}(s)-s]}{dc(s)+g(s)+h(s)}. \tag{8}$$

The initial condition is $s(0) = 0$, which says that the most impatient seller makes an initial offer without delay. It is not possible for the seller with valuation 0 to delay before making an initial offer in a symmetric equilibrium, since this would imply that that the buyer with valuation I must delay as well, in which case seller 0 is better off making an initial offer without delay.

The optimal delay is then found by integrating (8), which yields (6). By the single-crossing property, this solution is necessary and sufficient for the seller’s optimization problem if $s' > 0$ for all $s \in [0, \tfrac{1}{2})$. First, note that $\tilde{b} \in (s, 1 - s)$, so $\tilde{b} - s > 0$. To see that $c(s) > 0$, substitute (5) into $c(s)$ to yield

$$c(s) = d(1-d)(\tilde{b}-s)f(\tilde{b}) + \int_s^{\tilde{b}}\left(\frac{b-s}{\tilde{b}-s}\right)^{1+d} dF(b) > 0.$$

Hence, the numerator in (8) is positive and $s' > 0$ provided the denominator $dc(s) + g(s) + h(s)$ is positive.

The strategies ($p0$) and beliefs ($m0$) defined by (6), therefore, form a Nash equilibrium in the inital subgame. Since every history $h^0 = \{\Delta\}$ in this subgame is reached with positive probability and subsequent behaviour is optimal as determined by Propositions 1 and 2, the strategies and beliefs form a sequential equilibrium as well. ||

*Proof of Theorem.* First, note that (F) implies that (F*d*) is satisfied for *d* sufficiently close to 1. It remains to show that $dc(s) + g(s) + h(s) > 0$ when *d* is sufficiently close to 1. This follows since $c(s) > 0$, $h(s) > 0$, and $g(s) > 0$ because $1 - 2s > 2\tilde{b}(s) - 1$. Thus, we can apply Lemma 3. ||

*Proof of Proposition* 4. We begin by finding the value of $\tilde{b}(s)$ that maximizes $\tilde{u}(s, \tilde{b} | s, b)$. Since (F*d*) is satisfied for the uniform distribution for all $d > 0$, the global maximizer is the unique solution to (5). Substituting $F(B) = B$ and $S = s$ into (5) yields

$$b - 2\tilde{b} + s + 2d^2 \frac{\tilde{b} - s}{2+d} = 0,$$

which after solving for $\tilde{b}$ results in

$$\tilde{b}(s) = ab + (1-a)s = a - (2a-1)s \quad \text{where} \quad \frac{1}{2a} = 1 - \frac{d^2}{2+d}.$$

Thus, $\tilde{b}$ is a convex combination of $s$ and $b = 1 - s$, since $a \in [\frac{1}{2}, \frac{1}{4}]$ for all $d \in [0,1]$.

Substituting $\tilde{b}$ into (8) yields (after much simplification) the differential equation

$$s' = \frac{r}{2g}[1 - 2s(\Delta)], \tag{9}$$

where

$$g = \frac{4d(2-d^2)}{2+d}.$$

Given the initial condition $s(0) = 0$, the unique solution to (9) is found by integration:

$$s(S) = \int_0^S \frac{2}{r(1-2s)} ds = -\frac{g}{r}\log(1-2S).$$

Hence,

$$s(\Delta) = \tfrac{1}{2}(1 - e^{-r\Delta/g}). \tag{10}$$

Since $s(\Delta)$ *is* increasing for all $d \in (0,1]$, (10) is necessary and sufficient for $S$'s optimization problem by the single-crossing property. ||

*Calculation of the traders' ex ante and interim utility*

From Figure 2, four cases must be considered, depending on who makes the first offer and whether this offer is accepted immediately or a counter-offer is made. The seller's ex post utility $u_i(S, b)$ in each of the four regions involving trade ( $i = 1, \ldots, 4$) is given below.

1. Seller First, Buyer Delays: $S \le B \le a - (2a-1)S$.

$$u_1(S,B) = \frac{d^2 a^{-d}}{1+d}(B-S)^{1+d}(1-2S)^{g-d}$$

2. Seller First, Buyer Accepts: $a - (2a-1)S < B < 1 - S$.

$$u_2(S,B) = \frac{a}{1+d}(1-2S)^{1+g}$$

3. Buyer First, Seller Accepts: $1 - S \le B \le \min\{1, (a-S)/(2a-1)\}$.

$$u_3(S,B) = (B-S)(2B-1)^g - \frac{a}{1+d}(2B-1)^{1+g}$$

4. Buyer First, Seller Delays: $\min\{S, (a-S)/(2a-1)\} \le B \le 1$.

$$u_4(S,B) = \frac{da^{-d}}{1+d}(B-S)^{1+d}(2b-1)^{g-d}$$

The interim utility $U(S)$ of $S$ is determined by integrating over all buyer types. Thus, if we define

$$U_i(S,u,w) = \int_u^w u_i(S,B)dB,$$

and let $x = a - (2a-1)S$ and $y = (a-S)/(2a-1)$, the interim utility of $S$ can be calculated as follows:

$$U(S)=\begin{cases} U_1(S,S,x)+U_2(S,x,1-S)+U_3(S,1-S,1) & \text{if } S\le 1-a \\ U_1(S,S,x)+U_2(S,x,1-S)+U_3(S,1-S,y)+U_4(S,y,1) & \text{if } 1-a<S<a \\ U_4(S,S,1) & \text{if } S\ge a. \end{cases}$$

By symmetry, $B$'s interim utility is given by $U(1 - B)$.

In the limiting case as $d \to 1$, then $a = 1/g = \frac{3}{4}$, and $U(S)$ can be determined analytically to be

$$U(S)=\begin{cases} \frac{1}{1120}[141-240S+69(1-2S)^{10/3}] & \text{if } S\le \frac{1}{4} \\ \frac{1}{1120}[2(140S^2-220S+89)-3(23-12(2)^{1/3})(1-2S)^{10/3}] & \text{if } \frac{1}{4}<S<\frac{1}{2} \\ \frac{1}{560}[140S^2-220S+89-9(2S-1)^{10/3}] & \text{if } S\ge \frac{1}{2}. \end{cases}$$

(Nose that each of the functions $u_i(s, b| d)$ converge uniformly to $u_i(s, b|d = 1)$ as $d \to 1$, so $U(S|d)$ converges to $U(S|d = 1)$ as $d \to 1$.) Integrating $U(S)$ over $S \in [0,1]$ yields an ex ante utility of $\frac{6}{64}$ in the dynamic game. The interim utility $U_S(S)$ of the ex ante efficient *static* mechanism is given by

$$U_S(S)=\begin{cases} \frac{1}{2}\left(\frac{3}{4}-S\right)^2 & \text{if } S\le \frac{3}{4} \\ 0 & \text{if } S\ge \frac{3}{4}. \end{cases}$$

Integrating $U_S(S)$ over $S \in [0, 1]$, yields an ex ante utility of $\frac{9}{64}$ in the static game.

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