AN ESSAY ON DECISION THEORY WITH IMPERFECT RECALL

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Budapest
September 1999

KTK/IE Discussion Papers 1999/5.
Institute of Economics Hungarian Academy of Sciences

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Keywords: decision theory, bounded rationality, imperfect recall, strategy

Published by the Institute of Economics Hungarian Academy of Sciences. Budapest, 1999.
With financial support the Hungarian Economic Foundation

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Abstract

In this paper, I seek to establish a framework in which solutions to imperfect recall decision problems can be suitably examined. I introduce a strategy concept which is an extension of the standard concept employed since von Neumann and Morgenstern, and show how it may provide optimal solutions to problems which feature forgetting. For a technical analysis, I provide a characterization of imperfect recall extensive forms, a crucial input into future studies on the properties of optimal extended strategies. Also, I discuss further issues in decision theory with imperfect recall, including the prospects of induced forgetting when preferences change during the problem.

Összefoglaló

A tanulmányban a döntéselmélet olyan átfogó vizsgálatára teszek kísérletet, amely figyelembe veszi a döntéshozó esetleges felejtéséből adódó problémákat is. Áttekintem, milyen új elméleti keretben vizsgálható a döntéselmélet ezen ága és javaslatot teszek egy kibővített stratégia fogalom bevezetésére.

I.

1. Economic Situations with Imperfect Recall

1. It may be not difficult to argue that there are situations, purely economic or merely having economic relevance, where the forgetting previously held crucial information plays a salient role. One could refer to the following story. It is natural that governments facing elections are interested in convincing voters that things went generally well during their tenure. But since most of the time there are some things which did not go very well, a government seeking reelection may adopt the strategy of blaming other agents for the failures. These other agents could well be agencies or institutions over which the government has some control, but only a limited control. So the government could undertake to insinuate that it was not able to improve on a certain policy outcome just because of the independence of those institutions. Now, since governments are after all responsible for the overall good management of the life of the political community, the emphasis on independence is then concomitant to a silence about at least some aspects of their true relationship to these agencies. It even could have been the case that at the beginning of the electoral term the government had enthusiastically supported the independence of the agency in question for some other reason. We should also acknowledge that it is notoriously difficult to offer a clear-cut and acute description of what the independence of a given government agency or public authority really amounts to. One can even go as far as asserting that such relationships are fairly elusive, even inherently ambiguous. Thus we can also say: quite often governments engage in deliberate switching between radically different interpretations of what their relationship to certain agencies is or was, and offer before the elections the interpretation which is the most expedient for them.

For a more concrete example, consider the scenario when the government, in concordance with its overall efforts to manage the economy, makes the central bank of the country independent in some legislative sense, or adopts and advertises policies which facilitate the independent policy-making of the central bank. This could be induced by an intention to lower the inflationary expectations of the public; reasons for adopting such measures are well-known. So if the government later, during the election campaign, undertakes to blame the stubborn self-interest of the central bank for a bad outcome, it must count on the likely forgetting (or lack of thorough understanding) on the part of the voters of what the original reasons for having the given relationship with the central bank were.

Much ingenuity and care have been devoted recently to the formulation of this and kindred situations in terms of a strategic game between the government and the electorate, the latter pictured as a judiciously composed aggregate of opinions and interests. Now, if one accepts that the electorate, while making a choice at the election, exhibits not only imperfect knowledge but also imperfect recall about the political events in the preceding term, and that governments under some circumstances are interested in taking advantage of the fact that voters have imperfect recall, then there arises the need to model properly this situation as a game which contains forgetful players in its specification.[1]

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[1] More generally, one can claim that the formal modelling of any situation featuring blaming or scapegoat creation (which can be seen as specific instances of the problem

2. The situation presented above involves undoubtedly one of the largest game conceivable. It suffices to mention that in such a game-theoretical model the whole electorate of a country should be treated as a player in a strategic setting, that the duration of the game is very long, and that the web of actions is fantastically intricate. I would like then next to point to a situation which could be termed as very small and which involves bargaining between two agents. Here the rules are precise, and the boundaries of the situation are crisp. Suppose[2] that two agents, an employer and an employee, find themselves in a dispute over wages to be paid to the employee. One way to settle such a dispute is to submit their claims to a court of arbitration. Suppose also that the prevalent rules for arbitering over wage disputes prescribe that there are two subsequent stages available for the parties for reaching an agreement. A first arbitrator studies the case and offers terms for a settlement. Next, the parties make a decision about whether to accept the settlement or not. If they do not, they turn to a second arbitrator whose decision they have to accept as binding. Now if we take the view, or rather assume, that arbitrators have an interest in making an impartial decision (possibly because of their strive to maintain a reputation of always making neutral, unbiased, and wise

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of responsibility allocation), ought to involve the stipulation of forgetful players. Focal scenarios include those in which an agent seeks to manipulate the forgetfulness of others and those in which a group of agents tries to overcome forgetting by establishing rules for the allocation of responsibility; and there are many more. I discuss problems in modelling responsibility allocation at somewhat more length in Part III of my (1996): Institutions for Monetary Management, Delegation and Accountability. Mimeo, Princeton University. There I do not discuss forgetting in any depth, but suggest that any adequate model of responsibility allocation has to transcend the so-called Harsányi doctrine. See *John Harsányi* (1967–68): Games with Incomplete Information Played by `Bayesian' Players. *Management Science* 14; 159–182, 320–334, 486–502.

[2] See *Orley Ashenfelter*, *James Dow* and *Daniel Gallagher* (1986): "Arbitration and Negotiation Behavior under an Appellate System", mimeo., Princeton University

judgements), and that the second arbitrator has an access only to the offer the first one made but not the information on which the reasons for the offer were grounded – this situation of dispute settlement can be seen as one featuring imperfect recall. Both of the arbitrators want to make the same right decision, but the second one does not know what was known by the first one, who acted upon information strictly relevant to the case and also relied on the knowledge of the rules for the whole arbitration procedure. So one can suitably represent the team of the two arbitrators as one agent in a strategic game that loses information in the course of that game.

3. For those who cannot be content with drawing up just some model of the economic situations outlined above, but also feel that the challenge of incorporating forgetting players in those models should be met because of the crucial role forgetting plays in the scenarios, there seems to be no readily available paradigm to turn to. Indeed, the literature on decision theory with imperfect recall is very small, and the literature on game theory with imperfect recall is even smaller.[3] And I think it is fair to add that few

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[3] Recent papers include Steve Alpern (1988): Games with Repeated Decisions *SIAM Journal of Control and Optimization* 26, 2: 468–477; and his (1991): Cycles in Extensive Form Perfect Information Games *Journal of Mathematical Analysis and Applications* 159, 1: 1-17; J. L. Ferreira, Itzhak Gilboa, and Michael Maschler (1992): Credible Equilibria in Games with Utilities Changing during the Play, mimeo., Northwestern University (later sections); Kenneth Binmore (1992): *Fun and Games*. Heath: Lexington; pp. 456–458. It has been recognized that the literature on repeated games played by automata is also relevant here, see Ariel Rubinstein (1986): Finite Automata Play the Repeated Prisoner's Dilemma. *Journal of Economic Theory* 39, 1: 83–96. Ariel Rubinstein and Dilip Abreu (1988): The Structure of Nash Equilibrium in Repeated Games with Finite Automata. *Econometrica* 56, 6: 1259–1281; Ehud Lehrer (1988): Repeated Games with Stationary Bounded Recall Strategies. *Journal of Economic Theory* 46, 1: 130–144. On automata, see the remarks in §28 below. Consider also Robert Aumann and Sylvain Sorin (1989): Cooperation and Bounded Recall. *Games and Economic Behavior* 1, 1: 5–39; and James Dow (1991): Search Decisions with Limited Memory. *Review of Economic Studies* 58, 1: 1–14.

of the contributions to this small literature could be straightforwardly put into work in an economic context.

I will not embark on the task of identifying and modelling economic situations in which forgetting plays a pivotal part in this essay, which is on decision theory with imperfect recall in general. The reason for giving a draft of some situations – viewed as relevant, real, and robust – which ought to admit forgetting agents was to emphasize that efforts spent on decision or game theory with imperfect recall do not seek their sole ultimate rewards in checking yet an other perturbation of the core framework of formal decision theory, but in the prospect of providing tools for a satisfactory treatment of some important economic phenomena, including the two mentioned in the previous point – among numerous others.

That is, just like while motivating a preoccupation with models of bounded rationality, one has to stress that the ultimate rationale for developing and hopefully applying models which go beyond the core model lies not predominantly in the demand of presenting a total picture of human decision making. This is not necessary for the analysis of economic situations. What we need is a satisfactory model of human decision making, just good enough to capture aspects which ought to enter into the examination of a particular scenario, if that examination wishes to meet reasonable standards of adequacy. We are interested in modelling forgetting not because we are in the predicament of providing a perfect model of human decision making with imperfect recall, but because we cannot miss the modelling of situations where forgetting is central. In the examples of the two previous points, one cannot satisfice oneself with an attempt to formulate the most parsimonious model which gives some explanation of what is going on. Forgetting is in the essence of these examples. Is it not the same aim of saving the phenomena which is expressed below by the founders of game theory: "(Economic models) must be similar to reality in those respects which are essential in the investigation at hand... Similarity is needed to make the operation significant"?[4]

2. Decision Theory with Imperfect Recall

4. A significant number of works on decision theory which address the phenomenon of forgetting were written in the early fifties, the era of the first wave of the systematization and clarification of the ideas in the book of von *Neumann* and *Morgenstern*. Indeed, it was a landmark of this period, the famous article of *Kuhn*[5] which both settled the definition of games with perfect recall and at the same time, unintentionally perhaps, endowed games featuring forgetting with the status of awkward exceptions. At the same time, his paper offered a new set of mathematical objects to serve as the canonical model of games and therewith achieved a certain regimentation of the thought of von *Neumann* and *Morgenstern*. Other worked which analyzed imperfect recall, like those of *Thompson, Dalkey, Isbell*, and somewhat later *Aumann*[6] remained in relative obscurity despite their worthy contents.

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[4] John von Neumann and Oskar Morgenstern (1944), (1947): *Theory of Games and Economic Behavior* Princeton: Princeton University Press, p. 32.

[5] Harold W. Kuhn (1953): Extensive Games and the Problem of Information. *Contributions to the Theory of Games* Vol. II, edited by H. Kuhn and W. Tucker. Princeton: Princeton University Press, pp. 193–218.

[6] For papers related to the issue of imperfect recall from this period see G. L. Thompson (1953): Signalling Strategies in n-Person Games. *Contributions to the Theory of Games* Vol. II, edited by H. Kuhn and W. Tucker, Princeton: Princeton University Press, pp. 267–277; Norman Dalkey (1953): Equivalence of Information Patterns and

One problem with modelling forgetting is that it begs the question about the identity of players in a game. Indeed, von *Neumann* and *Morgenstern* mention imperfect recall not in the context of the issue of what one individual player can know during a game,[7] but in connection with the challenge of modelling the card game *Bridge*. Teammates in *Bridge* have identical interests, but they are compelled to make choices alternatingly, not seeing each others' deals. An individual player, when it is his turn to move, is imperfectly informed about some of the past events which were observed by his partner. Now, von *Neumann* and *Morgenstern* insist that *Bridge* is a two-player game. *Kuhn* also raises the issue of the identity of players in imperfect recall situations. He proposes to decompose a player into a collection of "agents" identified by occasions to make a choice. This is, in fact, motivated by making sense of *Bridge*[8] and by the need for clarifying his conception of information sets. He sees this decomposition as natural for perfect recall games, and adds that it is

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Essentially Determinate Games. *Contributions to the Theory of Games* Vol. II, edited by H. Kuhn and W. Tucker, Princeton: Princeton University Press, pp. 217–243; J. R. Isbell (1957): Finitary Games. *Contributions to the Theory of Games* Vol. III, edited by M. Drescher, W. Tucker, and P. Wolfe, Princeton: Princeton University Press, pp. 79–96; and Robert Aumann (1964): Mixed and Behavior Strategies in Infinite Extensive Games. *Advances in Game Theory*, edited by M. Drescher, L. Shapley, and W. Tucker, Princeton: Princeton University Press, pp. 62–650. See also R. Duncan Luce and Howard Raiffa (1957): *Games and Decisions* 2nd ed., Dover; pp. 159–163.

[7] See Von Neumann–Morgenstern: *Theory of Games and Economic Behavior*, p. 53, 79. *Cf.* Luce-Raiffa *Games and Decisions*, pp. 160–161.

[8] Cf. *Kuhn* "Extensive Games and the Problem of Information": "(The) seeming plethora of agents is occasioned by the possibly complicated state of information of our players who may be forced by the rules to forget facts which they knew earlier in a play. (It has been asserted by von *Neumann* that *Bridge* is a two-person game in exactly this manner)", p. 195.

exactly in imperfect recall situations when it is somewhat not clear how these agents make up a player. In these cases they make up a team.[9]

One wonders whether such a hesitation even in the identification of the concept of players in an imperfect recall context could not only have aggravated the difficulties and could not have discouraged prematurely the engagement with the issue head on.

5. The subsequent development of game theory saw virtually all papers and textbooks routinely sidestepping, or if not swiftly abandoning, the case of imperfect recall. Thus when *Piccione* and *Rubinstein*[10] took up the issue again, they almost had to start the discourse from the state it was left in the fifties. They set out to catalog the difficulties which may have prevented others to write on this topic. Their paper is conceptual, the emphasis is more on the explication of these difficulties than on a comprehensive formal analysis of imperfect recall problems, based on some stance on what the right treatment of then would be.

But as already a first reading of their work reveals, there was more behind the intermittent silence than neglect and preoccupation with the

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[9] See *Kuhn*, "Extensive Games and the Problem of Information", pp. 199–200, 211–215. And also "...each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves. This obviates the use of agents; indeed, the only games that do not have perfect recall are those, such as *Bridge*, which include the description of the agents in their verbal rules.", p. 213.

[10] *Michele Piccione* and *Ariel Rubinstein* (1994): On the Interpretation of Decision Problems with Imperfect Recall, mimeo, University of British Columbia and Tel-Aviv University. There is by now a series of papers for which the work of *Piccione* and *Rubinstein* serves as a starting point: these include *Pierpaolo Battigalli* (1995): Time Consistency, Sequential Rationality, and Rational Inferences in Decision Problems with Imperfect Recall, his (1996): Dynamic Consistency and Imperfect Recall. Both mimeo, Princeton University; Joseph Y. Halpern (1995), (1996): On Ambiguities in the Interpretation of Game Trees. Both versions mimeo., IBM Research Division; and Robert Aumann, Sergiu Hart, and Motty Perry (1995): The Absent-Minded Driver. In: *Theoretical Aspects of Rationality and Knowledge* Vol. VI, edited by Y. Shoham, San

fashionable ideas of the day. They make it evident that the difficulties in the analysis of imperfect recall are not simply due to technical complexities or the vagueness surrounding the concept of players. Many concepts, techniques and approaches which serve as cornerstones for decision and game theory as they stand do not work very well in the presence of imperfect recall.

As a natural first step, their attention was limited to decision theory. It could be asserted that they made five main observations about the interpretation of decision theory with imperfect recall. The first registers the need of employing behavioral strategies to solve some imperfect recall problems. This result has been already pointed out by *Isbell*[11] but *Piccione* and *Rubinstein* identify additional ambiguities in interpreting behavioral strategies in imperfect recall contexts. Second, they point out that imperfect recall could generate instances of time inconsistency, the nature of which is totally different from instances when time inconsistency is due to preference changes.[12] Third, urged by the previous observation, they examine the possibility of interpreting imperfect recall problems as the interaction of several temporal selves. This, too, leaves substantial ambiguities in the analysis. Fourth, they discuss how to model the beliefs of the decision maker while he is in the middle of the problem. Finally, they consider the case when the decision maker may even forget his own

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Francisco: Morgan Kaufmann, pp. 97–116. I unfortunately have not had the chance to consult the last of these papers.

[11] See *Isbell:* Finitary Games

[12] The first appearance of the concept of time inconsistency in formal decision theory could very well be in Robert Strotz (1956–57): Myopia and Inconsistency in Dynamic Utility Maximization *Review of Economic Studies* 23, 2: 165–180. There time consistency is due to changing preferences. See also Thomas Schelling (1985): Enforcing Rules on Oneself *Journal of Law, Economics, and Organization* 1, 2: 357–374.

strategy, and therewith yet an other set of interpretational dilemmas appears.

6. I submit that these five ambiguities are all tied to a further one, the ambiguity in the interpretation of the strategy concept in situations riddled with imperfect recall. While at this point this claim cannot be substantiated, the following simple example, itself drawn from the work of *Piccione* and *Rubinstein*[13] can give a suitable illustration.

Figure 1 exhibits a decision problem with imperfect recall, where at the information set I_3 the decision maker forgot what the previous chance move was, something he could have known at either I_1 or I_2. Now suppose that the strategy he formed at the beginning prescribes to do L at I_3. Then if he would end up being at I_2, he should opt for O there. However, if at d_4 in I_3 he could indeed do R, then he should not take O at I_2. But as in the standard treatment of a strategy the same action has to be prescribed for each of the histories in an information set, he at I_2 cannot hope that later at d_4 in I_3 the right decision will be made. Therefore at I_2 there is a reason to change the strategy which had been formulated at the beginning. Suppose that this was indeed possible. Then is it not the case that at I_3 he can deduce from the fact that the strategy has changed where he is exactly, at d_3 or at d_4? So can we allow for changing strategy in the middle of the problem? What can the decision maker know about his later ability to comply with such a change? No matter what the answer to these and various other questions concerning strategies in imperfect recall problems are, we can be sure that they do not even arise in a perfect recall context.

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[13] *Piccione–Rubinstein:* On the Interpretation of Decision Problems with Imperfect Recall. *Example* 2.

Thus the specification of the identity of the players is far from being the only difficulty in analyzing decisions and games with forgetting. The standard concept of strategy is intimately connected to a certain view on rationality and to the case of perfect recall. They stand and fall together. The current essay, indeed, grounds its approach in the analysis of the concept of the strategy, it presents results which are sensitive to the exact formulation of what a strategy is. It proposes an extension to the strategy concept introduced by von *Neumann* and *Morgenstern* and used ever since then, in order to examine what the solution to decision problems featuring forgetting is.

7. *Piccione* and *Rubinstein* at one point wonder whether there would not be a need for a new analytical framework in which imperfect recall problems could be analyzed. The current essay does not claim to provide such a new framework.

However, it does start the discussion with a view on decision theory in extensive form which is to secure the frame for the present analysis, a frame which would not allow ambiguities in interpretation during the subsequent examinations. This loads the presentation with an account of what formal decision theory, and more specifically decision theory in extensive form, is. This burden is taken up because it seems to be clear that short of constructing a new analytical framework, if one wishes to engage with the issues raised in *Piccione* and *Rubinstein* one has to make an attempt at alleviating the ambiguities identified by them.

So this essay provides one perspective on imperfect recall decision problems, and this perspective will underwrite one sort of analysis. Of course, other approaches are also possible and promising. I will try to

anticipate some of these, but this will only take the form of a polemics against potential criticism to the views advocated here (§ 31–34).[14]
The perspective in this essay is comprehensive enough to address all the five ambiguities pointed out by *Piccione* and *Rubinstein*. It is not true, however, that it will manage to extinguish all ambiguities, it will, in fact, create new ones.

Beyond the analysis of the concept of strategy and the introduction of extended strategies, I will also try to demarcate the boundary between imperfect recall problems as they relate to individuals as opposed to teams (to save more of the phenomena). I also discuss the importance of deliberation in decision theory. While doing all this I will attempt to reduce speculations about how to model epistemic or doxastic states of decision makers to a minimum, in this respect this work is fairly old-fashioned. In addition, I report a class of imperfect recall problems which has not been so far identified in the literature (see for this §21).

I present the framework for the analysis of decision problems with imperfect recall in *Part II*. *Part III* contains a classification of decision problems with imperfect recall and a characterization of how these decision problems relate to each other. *Part IV* discusses the concept of strategy, presents the notion of extended strategies, and illustrates how these can provide solutions to many important problems in which imperfect recall appears. Finally, *Part V* comments on decision problems with changing preferences and offers concluding remarks.

[14] From among other approaches, I would like to call attention to the employment of the concept of 'signalling information set'. See Thompson: Signalling Strategies in n-Person Games. (von Neumann–Morgenstern, *Theory of Games and Economic*

II.

3. Decision Problems in Extensive Form

8. The discussion of decision problems with imperfect recall demands an intertemporal setting. This will, as a matter of fact, lead to considerations which do not appear in an analysis of one-shot decision problems, including the treatment of the temporal order of decisions, the persistence of certain objects in the problem, and contingent possibilities for making certain choices. Also, it is quite convenient to view a decision problem as a game featuring only one agent, since then one is enabled to refer to an extensive form decision problem.

Let us say that any presentation of what a decision problem in extensive form is has to be preceded by a First Story, which can properly anchor the analysis. The First Story proposed in this essay has the following three stipulations.

First, there is an Ex Ante state in which the decision maker is confronted with the problem and learns everything which can be known about it, by anyone conceivable. This stipulation responds to calls for basing the discussion of decisions and games in how participants view the problem. The Ex Ante state here accounts for all what we know about the agent's perception of the whole situation.[15]

Behavior pp. 51–54), and to the comprehensive treatment in Halpern: On Ambiguities in the Interpretation of Game Trees.

[15] See *Ariel Rubinstein* (1991): Comments on the Interpretation of Game Theory. *Econometrica* 59, 4: 909–924; especially section 5. Consider also the other stipulations in this respect.

Second, an inactive agent called the Analyst is instituted. The Analyst knows everything about the decision problem that can be known about it, and this extends to the time after the Ex Ante state elapses. This implies that she knows in the Ex Ante state everything the decision maker knows and may know more later on. Also, she plays the role of the umpire as well and ensures that the rules for the decision problem as presented to the decision maker in the Ex Ante state are kept. In general, the figure of the Analyst represents the limits of the analysis.[16]

Third, the problem is an isolated one, it is not one instance of a recurrent set of identical problems.[17]

It is important to see then what Second Stories could at all amount to. One may construct Second Stories with one or more of the following features. First, the decision maker could find himself in the middle of a problem the boundaries of which are not firmly set. Second, the knowledge of the Analyst about the problem could be itself incomplete, specified appropriately. Third, the problem could be part of a repeated series of the same problem where the relationship between the problems could enter as relevant considerations for the decision maker while deciding upon a course of actions. Of course, this last case is in itself fairly familiar unless we insist that the other stipulations are also in place.

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[16] Cf. von Neumann and Morgenstern: *Theory of Games and Economic Behavior*. pp. 8 and 49; Kenneth Arrow (1951): *Social Choices and Individual Values*, New York: John Wiley, p. 2.

[17] The last stipulation is the only one which can be seen as an obvious simplification. It also forestalls the dichotomy between the "eductive" and "evolutionary" points of view introduced by *Kenneth Binmore* (1987–88): Modelling Rational Players I-II. *Economics and Philosophy* 3: 179–214, 4: 9–56. Having made the first stipulation, we are compelled to concentrate on the performance of a decision maker in one given situation.

This essay does not even attempt to model Second Stories, and certainly not because of limitations of space. Even the construction of such Second Stories appears to be a formidable challenge, I suggest that any effort in this direction has to tackle first what is known as the "Harsányi doctrine"[18].

9. Next, within the framework of this First Story I propose a tripartite decomposition of an extensive form decision problem. A formal presentation of this decomposition will be given shortly, here I give a summary only.

This decomposition acknowledges three parts. The first is the physical problem or rules. So, the physical problem is the description of all the feasible sequences of events during the problem, including both the actions of the decision maker and the moves of chance. It also contains the specification of the outcomes, given any such feasible sequence of events. It can thus be naturally regarded as rules, a complete description of what can be done and what any series of events leads to.

The second is the description of the desires of the decision maker. These take a very simple form in the present account, given that it is posited that the decision maker has a subjective preference ordering over any feasible sequences of events.

Finally, one has to treat the beliefs of the decision maker before and during the problem. This part is riddled with special difficulties. It is only one of the hard issues that each and every entity appearing in the decision problem can be a possible object of beliefs for the decision maker. An other source of problems is that as time unfolds his beliefs change and also relate to the beliefs of himself at other points in time, beliefs about the

[18] *Harsányi*, "Games with Incomplete Information Played by 'Bayesian' Players"

beliefs about the beliefs, and so on in the usual way. Given this vast array of possible objects of belief, and their significance for problems involving forgetting, I will proceed very cautiously. I will make an attempt at keeping track of all the possibilities.

Before giving more details on the contents of these three parts, I would like to clarify that I do not regard this decomposition as significant in itself, other discussions could well proceed differently.[19] So there are two reasons why I came to present it. On the one hand, it helps to distinguish later the conative and the epistemic sources of time inconsistency (§23). Also, it allows the analytical move of deflating the epistemic capabilities of the decision maker (§27).

10. The description of the physical problem starts by positing the set of possible histories H in a decision problem (which has generic elements $h \in H$).[20] These histories are constructed as sequences of individual basic actions, themselves elements of the set A. So an individual history h is a sequence $(a^k)_{k=1}^K$ where the superscript k locates an individual action in the sequence. Thus, for example, a^k marks out the basic action a_i, where $a_i \in A$. Then we say that a_i is part of the history h. So we can regard the set A as a set of types of actions, and their occurrence in a sequence individuates them as an action token. The set of action tokens is denoted by $\overline{A}$, and thus we can also say that an action token a^k is part of a certain history h. The reason for making this distinction will be made apparent in §14 below.

[19] In the account of *Richard Jeffrey*, the discussion is based on preferences about truth of propositions concerning probability and desirability. Thus from that point of view, there is not much to gain from the discernment of the lines which divide the three components of the current decomposition. See his (1983): *The Logic of Decision* 2nd ed., Chicago: University of Chicago Press

The set H is assumed to be finite here[21]. It further has to meet the following two requirements. First, $0 \in H$, that is the empty sequence called the initial history is an element of H. Second, if $(a^k)_{k=1}^{K} \in H$ and $(a^k)_{k=1}^{K} \neq 0$, then $(a^k)_{k=1}^{K-1} \in H$.

Finally, if for a $h = (a^k)_{k=1}^{K} \in H$ there is no a_i such that $(h, a_i) \in H$, then that history is called a terminal history. The set of terminal histories is denoted by Z, this set then represents all the courses of action available to the decision maker. (Then note that this approach implicitly makes simultaneity of moves, which may arise even in a decision problem, a non-issue.)

11. It seems to be useful to embed the formulation above into an other one which admits the mathematical object of a graph, more specifically a tree (for the current purposes a connected graph without cycles).[22] In this second formulation, the basic primitive object is a finite tree $\Gamma = \langle H, \overline{A} \rangle$. The vertices of this tree correspond to the elements of H, the edges correspond to the set of action tokens $\overline{A}$. The initial history $0 \in H$ will be represented by the root of the tree.

From this it follows that edges represent individuated actions, and two distinct edges may stand for the same action from the set A. We can naturally write $h' = (h, a_i)$, where a_i is the name of the action attached to

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[20] This part of the presentation of the physical problem corresponds to the approach recommended by Martin Osborne and Ariel Rubinstein (1994): *A Course in Game Theory* Cambridge: MIT Press; pp. 89–90, 200–202.

[21] As far as I can see, the admission of an infinite set of histories would not lead to any conceptual difficulties. However, many technical problems would be introduced by such an admission. These would force, for example, a rethinking of the characterization of perfect recall problems, and therewith affect arguments about optimal solutions to decision problems with forgetting.

[22] *Cf.* Von Neumann–Morgenstern *Theory of Games and Economic Behavior* pp. 65–66, 77–79; and Kuhn, *Extensive Games and the Problem of Information*

the edge (a mathematical object) adjacent to both h and h'. Notice that histories became separate entities here, by being vertices, but the elements in the set H can be identified as sequences of actions as well. In this geometrical picture we can see a sequence of actions construing a history as the sequence of edges from the root of the tree to the history in question as well.

The reason for availing ourselves to the tree formulation is convenience. For the purposes of the technical analysis reported in *Part III* of this essay, the simple graph theoretical notions forthcoming with the concept of the tree seem to be quite expedient in negotiating the difficulties of presenting arguments where the complex object H is involved.

12. Next, let the $A(h) = (a_i \mid (h, a_i) \in H)$ denote the set of feasible actions after history h. We can redefine terminal histories as histories for which $A(h)$ is empty. It is further required that $\forall h \in H \setminus Z, A(h)$ is non-singleton.

A player assignment function $R : H \setminus Z \rightarrow \{chance, DM\}$, where DM denotes the decision maker, divides further the histories in $H \setminus Z$. The interpretation of this function $R(\cdot)$ is immediate, it prescribes the action of either chance (Nature) or the decision maker after each non-terminal histories. $R(\cdot)$ essentially partitions the non-terminal histories, histories when chance is on the move are elements of the set C, histories when the decision maker is on the move are elements of the set D. The set D could be called the set of decision histories (vertices).

For each history in C there is an assignment of a (strictly positive) probability with which the feasible actions after that history could occur, and these probabilities are known to the decision maker in the Ex Ante state and will be never forgotten. We do not need to formalize, or even discuss this further, and since no substantial role will be played by this probability assignment here we can denote these probabilities by f_c and just leave them like that. Sometimes I will distinguish chance moves by the symbol α.

So the physical problem can be summarized now as a tuple $\langle H, R, f_c \rangle$. Note that this is only a shorthand for the full characterization by the tuple $\langle \Gamma, R, f_c \rangle$ or $\langle H, \overline{A}, R, f_c \rangle$. Below, I will always use $\langle H, R, f_c \rangle$, for convenience.

13. The description of the desires upon which the decision maker acts takes a very simple form. As it has been already mentioned, I assume throughout that desires are comprised of preferences over terminal histories. Next, a function, $u : Z \rightarrow \Re$, will be posited which attaches to each terminal history a utility index. Recalling that H is finite reveals that the sidestepping of a more primitive construction of preferences by the direct positing of utility indices is very natural.

It will be further assumed that preferences do not change during the course of the problem. Hence the decision maker is moved by a unabating drive to get to the terminal history the reaching of which is judged by him in the Ex Ante state as the most capable of satisfying his desires. At the end of this essay, §§35–37 contain a short discussion of imperfect recall problems with changing preferences.

4. Beliefs in an Extensive Form Decision Problem

14. The last part of the current decomposition of an extensive form decision problem should specify the beliefs of the decision maker. As it has been already mentioned, this specification is necessarily very involved: in principle any entity appearing in the whole decision problem could be an object of belief. To start with the Ex Ante state, the objects of the beliefs there include the description of preferences. Concerning these, it is insisted that the preferences described in §13 are the true preferences of the decision maker, the possibility of self-deception in this respect is excluded. And since it is assumed that these preferences do not change during the problem, beliefs about preferences will be not subject of the current discussion. The *ab initio* separation of desires and beliefs and the extremely reduced representation of the desires assure the autonomy of the conative impulses. Similarly, the objective probabilities of chance moves are always known by the decision maker, so beliefs concerning these will be not treated. In this way, we can concentrate on the remaining objects of belief.

Still in the Ex Ante state, the decision maker has to be endowed with beliefs concerning the physical problem. Note, however, that beliefs at the Ex Ante state present themselves in a straightforward manner: it coincides with the full description of the problem as far as we, or the Analyst, can know this. But one has to recognize beliefs after the Ex Ante state expired as well, while the decision maker is in the middle of the problem. Since previous assumptions made beliefs about preferences and probabilities made them unproblematic, we can confine our attention to beliefs of the decision maker concerning his position in the physical problem.

The standard formulation of these beliefs in formal decision theory is in terms of information sets. The concept of the information set has two parts. First, it has a formal specification, defining information sets as members of a partition **I** (with generic element *I*) on the set of decision vertices D. (Denote by $|I|$ the number of histories in a given information set I.) Second, it has a conceptual part. This stipulates that if the decision maker is at a history h, he will not be able to distinguish among the histories which are contained in that element of **I** of which h is a member. Further, for the same reasons, the decision maker cannot be able to distinguish individual actions as identified by the history at which they have to be committed. If this was not so, histories could be identified by the actions available. Therefore we have to concede that the decision maker chooses from among action types at a given non-singleton information set. This requires that for all h and h' in an information set I, $A(h) = A(h')$. For the sake of consistency, it is also useful to stipulate that a given type of action a_i cannot occur at more than one information set; that is there is no $h \in I$ and $h' \in I'$, $I \neq I'$, such that $a_i \in A(h)$ and $a_i \in A(h')$. But the standard conceptual interpretation of information sets transcends the above formal restriction, stipulating that the decision maker is capable of seeing through the whole problem after any history with the possible exception of discerning the exact history he is at. This surplus meaning of the conceptual part can be brought out by the fact that the decision maker may forget the physical rules themselves. After some histories, he could be confused about what lies ahead in the problem. This points to the possibility of some "wild" decision problems with imperfect recall.

15. It is worthwhile to clarify what was meant by "wild" problems in the previous point. These problems are wild in the sense that they refer to situations in which the decision maker, while in the middle of the problem, forgets not only which location he is at, but also misrepresents the remainder of the problem as it appears at certain locations.

Consider first the following example, which is constructed on the basis of an example by *Geanakoplos*[23]. A decision maker has a choice of making a bet now or later. The desirability of the bet depends on the realization of one of three possible chance moves, labelled as α, β and γ. The *a priori* probability of each of them is $\pi(\alpha) = \pi(\gamma) = \frac{2}{7}$, and $\pi(\beta) = \frac{3}{7}$. If he would not bet at any time, he gets payoff 0. If he bets, the payoffs are -1 in case of α and γ, and 1 in case of β. Now he also knows in the Ex Ante state that if he postpones betting, he will regard α and β as possible after α occurred, will know that β occurred if it has, and will regard β and γ as possible after γ has occurred. If he bets now, his rewards are determined by the identity of the realized chance move. An attempt to represent this decision problem is shown on *Figure 2*. Note that the usual symbolism for indicating information sets is amended here. A quick glance at this problem shows that if the decision maker postpones betting, he will wish to bet under all circumstances. However, in the Ex Ante state the expected payoffs from betting now are higher than from betting later. Clearly, at history *h*, for example, the decision maker is unable to *h*. Thus, at that point, not only can he not identify his location in the problem, but he has conflicting views about what the problem is.

An other wild problem was identified by *Ariel Rubinstein*[24]. He describes an agent who has to drive home at night on a highway with which he is unfamiliar. The situation is shown on *Figure 3-a*. If he wants to get to *C* for sure, he may take the route without intersections, but that is

[23] *John Geanakoplos* (1989): "Game Theory without Partitions, and Applications to Speculation and Consensus", mimeo., Yale University, p. 9.

[24] See *Rubinstein*: "Comments on the Interpretation of Game Theory", pp. 915-917.

assumed to be very long. If he takes the shorter highway, he may get confused at point *h* whether that is the first or the second exit. *Rubinstein* proposes one representation of this decision problem, reproduced on *Figure 3-b*. Here with a certain *exogenous* probability the driver knows where he is, otherwise he thinks that he is still at intersection *h'*. Here again, the agent is not only uncertain about his exact location in the problem, but also misrepresents the underlying problem: by introducing the exogenous chance move and therewith misconceiving the true situation.

Clearly, there is no limit to the confusion the Decision Maker may endure during the problem, if there is no boundary to what the DM may believe after certain histories.

16. So it will be assumed below that these sorts of wild problems cannot occur in a decision problem. Further, the distinction between the formal definition of an information set and its conceptual interpretation will be exploited. The formal part will be always retained, but the validity of the conceptual part will be suspended. Some reasons for this separation will be provided in §27.

There are still other objects of belief which ought to be correctly identified and then specified. Let me mention here some of them, others I will simply ignore. In the Ex Ante state and later after each history reached, there could be beliefs about beliefs at other histories or in the Ex Ante state. This induces further beliefs about beliefs about beliefs, and so on[25]. It is tempting to conclude that the proper formal treatment of these intrapersonal beliefs should enlist the resources of temporal and epistemic logic, working towards a satisfactory theory of intertemporal common knowledge. But here, instead, I will make efforts to make this issue irrelevant by deflating the epistemic prowess of the decision maker (again in § 27).

Finally, we should not forget about interim beliefs concerning the strategy the deci-sion maker decided to employ in the Ex Ante state, this issue is postponed until §§23-28.

Thus, with the exception of beliefs about some remaining entities in the problem, the description of the whole extensive form decision problem is now complete. This can be summarized by the tuple $\langle H, R, f_c, \mathbf{I}, u\rangle$. Let us say that the tuple $\Delta = \langle H, R, \mathbf{I}\rangle$.stands for the extensive form. (Note that this definition is different from the standard one in that it omits f_c. This omission is justified by the fact that no substantial role is played by these probabilities in the current discussion.)

III.

5. A Classification of Imperfect Recall Extensive Forms

17. In a study of decision problems with imperfect recall, there should be an interest in giving an exact identification of them. Note that decision problems can be classified in terms of the properties of the extensive form. More pertinently, we can define a decision problem with imperfect recall in terms of these properties.

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[25] This is made an explicit theme in *Michael Bacharach* (1991): "Backward Induction and Beliefs about Oneself", mimeo., Oxford University; but his perspective on this

In order to give proper definitions, we have to first introduce some auxiliary notions. Let us identify a set of relations on the object $\langle H, R, \mathbf{I}\rangle$. The first of these is the *initial subhistory* relation, denoted by P. It is defined on the set H as: $h'Ph$ if and only if when $h = (a^k)_{k=1}^{K}$, $h' = (a^k)_{k=1}^{L}$ for some

$L < K$. We also write $h' \in P(h)$. The inverse of this relation is denoted by S, and hSh' if and only if $h'Ph$. We write $h \in S(h')$ accordingly. In graph-theoretical terms, P is the predecessor relation, and S is the successor relation on H. Next, let us introduce an other relation on H, called *maximal initial subhistory*, denoted by p. This is defined as: $h'ph$ if and only if when $h = (a^k)_{k=1}^{K}$, $h' = (a^k)_{k=1}^{K-1}$. We also write $h' = p(h)$. The inverse of this relation is denoted by s, and hsh' if and only if $h'ph$, and we may write $h \in s(h')$ accordingly. In graph-theoretical terms, p is the immediate predecessor relation, and s is the immediate successor relation. Finally, we will make use of a further relation, called the *subhistory* relation, denoted by Q. The definition of this invokes the fact that histories can be identified as sequences of actions. We say that $Q(h) = (a_k)_{k=1}^{L}$ is a subhistory of $h = (a^k)_{k=1}^{K}$, if two conditions are met. First, each $a^{k'}$ which is part of $(a_k)_{k=1}^{L}$ has to designate the same action a_i as some $a^{k''}$ which is part of $h = (a^k)_{k=1}^{K}$. Second, if two action tokens $a^{k'}$ and $a^{k'''}$ are part of $Q(h)$, and they correspond to $a^{k''}$ and $a^{k''''}$ in h, respectively: then $a^{k'}$ and $a^{k'''}$ preserve the same order in $Q(h)$ as $a^{k''}$ and $a^{k''''}$ had in the sequence h.

The various relations defined above should be extended for the sake of the coming analysis to the set of information sets. Due to the nature of the object $\langle H, R, \mathbf{I}\rangle$, there are several legitimate extensions. The following two

issue is fairly different from the present one.

are adopted. For two information sets I and I', I' precedes I, that is $I'PI$ if and only if $\exists h' \in I'$ and $\exists h' \in I'$ such that $h'Ph$. We can write $I' \in P(I)$, and the inverse relation S is naturally defined. Similarly, for two information sets I and I', I' immediately precedes I, that is $I'pI$ if and only if $\exists h' \in I'$ and $\exists h \in I$ such that $h'ph$. We can write $I' \in p(I)$and, again, the inverse relation s is naturally defined. The employment of the same letter for denoting these relations between information sets as those between histories is justified by the fact that we recognize only one extension.

For the remaining case of predecessor relations between histories and information sets, note that histories can be viewed as singleton information sets.

18. A second set of auxiliary concepts involves the idea of experience, introduced by *Osborne* and *Rubinstein*[26]. The *experience of actions* of the decision maker at history $h \ni D$ is denoted by $V(h)$. It is defined as that sequence $(a^l)_{l=1}^{L}$ which is a subhistory of $h = (a^k)_{k=1}^{K}$, and is such that $\forall a^{l'}$ which is part of $(a^l)_{l=1}^{L}$, $\exists h' \in P(h) \cap D$ such that the action a_i corresponding to $a^{l'}$ is in $A(h')$. This amounts to saying that $V(h) = (a^l)_{l=1}^{L}$ is that subsequence of h which is constituted by actions made previously by the decision maker, as opposed to chance. Similarly, $W(h) = (\alpha^m)_{m=1}^{M}$ is the *chance experience* at $h \in D$. Here $(\alpha^m)_{m=1}^{M}$ is a subhistory of $(a^k)_{k=1}^{K}$, and for $\forall a^{m'}$ part of $(\alpha^m)_{m=1}^{M}$, $\exists c \in P(h) \cap C$ such that $(\alpha^{m'}) \in A(c)$. Thus this is the subsequence of h made up of the chance moves in it. This latter concept will not be employed in the current section, but some use will be made of it in the next one.

The most important concept in this cluster is the *experience* of the decision maker at h, denoted by $X(h)$. It is defined as the sequence $((I^{l-1}, a^l)_{l=1}^{L}, I^L))$. This sequence has the following properties. The elements a^l are just the elements of $V(h)$. And the elements I^l are the elements of $Y(h)$, the sequence making up the *experience of information sets*. This sequence is defined as follows. For $l < L$, I^l is such that if a^{l+1} is in (I^l, a^{l+1}) which is part of $X(h)$, and further if $a^{l+1} \in A(h')$ for some $h' \in P(h) \cap D$: then $h' \in I^l$. Finally, I^L is the information set which contains h.

19. Recall that an extensive form decision problem is a tuplet $\Delta = \langle H, R, f_c, \mathbf{I}, u \rangle$ and that H may stand for a finite tree or for a finite set of histories. Note that for our purposes, the extensive form $\Delta = \langle H, R, \mathbf{I} \rangle$ can suitably represent a given decision problem.

It is useful to introduce than a third group of auxiliary concepts which refer to *subproblems* of an extensive form Δ. The first among these are the history-induced (h-induced, or c-induced) subproblems, denoted by Δ^h or Δ^{c}[27]. In Δ^h, the set of histories H^h consist of h and $\forall h' \in H$ such that $h' \in S(h)$. The player assignment function R^h is the projection of R on H^h. Similarly, the information partition $\boldsymbol{I}^h$ is the projection of $\boldsymbol{I}$ on H^h. Formally, $\boldsymbol{I}^h = (I \in \mathbf{I} | \mathrm{I} \cap H^h \neq 0$. There is further a partition $\mathbf{I}_s^h$ of immediate successors of h, a projection of $\mathbf{I}$ on the set H_s^h for which it is true that $\forall h' \in H_s^h$, $h' \in s(h)$. And c-induced subproblems are analogously derived. The second kind of subproblem is that of the information set induced (I-induced) subproblem, denoted by Δ^I, which is defined, with a

[26] Osborne–Rubinstein, *A Course in Game Theory*, p. 203. See also Piccione–Rubinstein, "On the Interpretation of Decision Problems with Imperfect Recall", pp. 9-10.

slight abuse of notation, as $\cup_{h\in I}\ \Delta^h$. For a more precise definition one would have to first define the union operation on subproblems. Finally, we have the action induced (a_i-induced) subproblems, denoted by Δ^{a_i} This consists of action tokens corresponding to a_i and $\cup_{\{h|\ \exists h':a_i \in A(h),\ (h',a_i = h\}}\Delta^h$. An additional, but related concept is *containment*. Here consider some Δ^h. Then if for some $I \in \boldsymbol{I}^h$ and for $\forall h' \in I$ we have $h' \in \Delta^h$, we say that I is contained in that Δ^h. There are analogous concepts of containment for Δ^I and Δ^a.

Finally, let us define the length of a history h as $\mathtt{l}(h) = |\mathrm{K}|$ whenever $h = (a^k)_{k=1}^{K}$. One can then also define multi-staged information sets I, for which $\forall h, h'$ such that $h \in I$ and $h' \in I$, we have $\mathtt{l}(h) = \mathtt{l}(h')$.

20. Next we identify classes of extensive form decision problems. All these classes are related to properties of the extensive form $\Delta = \langle H, R, \mathbf{I}\rangle$.

DEFINITION 1: *An extensive form decision problem features* perfect information}, *if each information set in* Δ *is singleton*.

DEFINITION 2: *An extensive form decision problem features* perfect recall *if for* $\forall I \in \boldsymbol{I}$ *and* $\forall h, h' \in I$, *we have* $X(h) = X(h')$. *Otherwise it features* imperfect recall.

DEFINITION 3: *An extensive form decision problem features* perfect recall of information sets, *if for* $\forall h, h', I$ *such that* $h \in I$, *and* $h' \in I$, *we have* $Y(h) = Y(h')$.

DEFINITION 4: *An extensive form decision problem is* multi-staged, *if each of its information sets are multi-staged.*

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[27] Note that the symbol Δ is used both for denoting extensive forms and subproblems, and thus is employed for the reference to somewhat dissimilar mathematical objects.

DEFINITION 5: *An extensive form decision problem features* cross-branch relevance, *if there exists* $I \in \Delta$ *such that* $\exists I' \in \Delta^{\mathrm{I}}$ *which is not contained* in Δ^{I} *or if there exists* $c \in \Delta$ *such that* $\exists I' \in \Delta^{\mathrm{c}}$ which is not contained in Δ^{c}.

DEFINITION 6: *An extensive form decision problem features* absent-mindedness, if $\exists I \in \boldsymbol{I}$ *and* $\exists h, h' \in I$, *such that* $h \in S(h')$.

DEFINITION 7: *An extensive form decision problem features* precedence reversal, if $\exists I, I' \in \boldsymbol{I}$, *such that* $I \in S(I')$ *and* $I' \in S(I)$.

Many of these concepts are adapted from earlier works and I retained the original name for them. The concept of perfect information decision problem is standard. The current definition of perfect recall is the same as in *Osborne* and *Rubinstein*[28]. Perfect recall of information sets and absent-mindedness are from *Piccione* and *Rubinstein*[29] (see *Figures 4* and *5* for examples of each of them.) Multi-staged problems are named by *Battigalli.*[30]. Instances of cross-branch relevance and precedence reversal are shown on *Figures 6* and *7*. Note that the concept of precedence reversal is connected to the standard concept of "crossing information sets", but the example on *Figure 7* shows that the current name may be more accurate.

21. Consider the extensive form represented on *Figure 8*, which could even claim right to belong to a separate class, to be defined as:

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[28] *Cf.* Osborne–Rubinstein, *A Course in Game Theory*, p. 203.

[29] *Cf.* Piccione–Rubinstein, "On the Interpretation of Decision Problems with Imperfect Recall", pp. 9–10.

[30] As it was pointed out to me by *Pierpaolo Battigalli*, the requirement of multi-stagedness is part of the original formalization of games in von *Neumann–Morgenstern*, "The Theory of Games and Economic Behavior", see for example pp. 77–79.

DEFINITION 8: *An extensive form decision problem features* imperfect recall of chance moves *if* $\exists I, h, h' \in$ *and* $c, c' \in C$, *such that* $h \in s(c)$, $h' \in S(c')$, *and* $c' \in S(c)$.

Further, the configuration on *Figure 8* can be more precisely captured by the following definition:

DEFINITION 9: *An extensive form decision problem features* unmitigated imperfect recall of chance moves *if* $\exists I, h, h' \in I$, *and* $c, c' \in C$, *such that* $h \in s(c)$, $h' \in s(c')$, *and* $c' \in s(c)$.

Note that by *Definition 2* above, the problem on *Figure 8* features perfect recall. But allowing this possibility would make an analysis of optimal solutions to imperfect recall problems exceedingly more complicated and ambiguous.

But it is not clear at all that we should allow for this possibility. Notice that on *Figure 8* the chance vertex *c'* is an immediate successor of the other chance vertex *c*. This may be viewed as an illegitimate configuration, since any set of chance moves which are connected by the immediate precedence relation could be collapsed into one, on the strength of the consideration that only the outcome of the whole series of connected chance moves is relevant for the decision maker. According to this view, the physical problem represented in the Ex Ante state is already a model of the forthcoming decision situation, and multiple chance moves are appropriately compressed. Therefore, in this essay, I do not allow for the possibility of a chance vertex being an immediate successor of an other chance vertex.

6. A Characterization of the Relationship between Imperfect Recall Extensive Forms

22. This section is devoted to a characterization of how the various sorts of extensive forms featuring imperfect recall relate to each other. This exercise is useful for two reasons. First, it fosters the understanding of the basic patterns of imperfect recall and, second, it supports the analyses of specific decision problems where the decision maker has to face these extensive forms. Indeed, its results are crucial inputs to examinations of how the best solution to a given imperfect recall problem depends on the characteristics of the underlying extensive form[31]. In turn, the current classification of extensive forms (§20), and the characterization of their relationships can be fully justified only by results on the class-dependent properties of optimal extended strategies. No attempt at such a justification can be carried out on this occasion.

The characterization will be presented in form of a series of statements. While the proofs of them vary significantly in difficulty and nature, I preferred to call them each a 'lemma'.

We start by recognizing that the whole set of extensive form decision problems can be divided without residuals to multi-staged and non-multi-staged problems. Given this first division, we approach the task of characterization by first situating perfect recall problems in the now divided field.

LEMMA 1: *Each perfect recall problem is multi-staged, but there are multi-staged problems which do not feature perfect recall.*

Proof: For the first part of the statement, note that, by hypothesis, there are no chance vertices in the problem which would be immediate successors of each other. So, trivially, $W(h) = W(h')$ for each I and each $h, h' \in I$. By perfect recall, it is also true that $V(h) = V(h')$. But then $l(h) = l(h')$.

For the second part, suppose for a contradiction that there exists no such problem. But then F*igure 4* provides a counterexample.

Next we situate problems with absent-mindedness in this first division.

LEMMA 2: *No problem with absent-mindedness is multi-staged, but there exist problems which are neither multi-staged nor feature absent-mindedness.*

Proof: As to the first part, note that by definition, if a problem features absent-mindedness, $\exists I$ and $\exists h, h' \in I$, such that $h' \in S(h)$. Now consider an action a_i which is part of h', but not of h. One such action is the one for which it is true that $(h,a_i) \in H$, and it may be the only such action. But then $l(h) \neq l(h')$

As to the second part, suppose for a contradiction that there exists no such problem. But then *Figure 6* provides a counterexample.

The characterization proceeds by inquiring about the place of problems with cross-branch relevance in the division. The following four observations give a first description of the relationship of these problems to some of the other kinds:

[31] In subsection 5.2 of "*On Optimal Solutions to Decision Problems with Imperfect Recall*" (Essay 2), I give an explicit characterization of how the best solution to one class of problems depends on the structure of its extensive form.

LEMMA 3: *There exist problems with cross-branch relevance which are multi-staged, and there also exist problems with cross-branch relevance which are not multi-staged.*

Proof: For both parts of the statement, suppose for a contradiction that there exist no such problems. But then *Figures 1* and *6* provide counter-examples.

LEMMA 4: *Problems with perfect recall may or may not feature cross-branch* relevance.

Proof: Suppose for a contradiction that the statement was not true. But then consider *Figures 14* and *20* for counterexamples.

LEMMA 5: *For problems with cross-branch relevance which are not multi-staged, some of feature absent-mindedness and others do not.*

Proof: For both parts of the statement, suppose for a contradiction that there exist no such problems. But then *Figures 9* and *6* provide counter-examples.

LEMMA 6: *There exist problems with absent-mindedness which do not feature cross-branch relevance.*

Proof: Suppose for a contradiction that there exists no such problem. But then *Figure 5* provides a counterexample.

The next step in the characterization attains a placement of problems with precedence-reversal in this field, by means of the following three lemmas:

LEMMA 7: *No problem with precedence-reversal is multi-staged.*

Proof: By definition, if a problem features precedence reversal, then there exist information sets I and I' for which $I \in S(I')$ and $I' \in S(I)$. Then we can show that it cannot be the case that both are multi-staged.

Suppose then that I is multi-staged, that is that if $h, h' \in I$ then $l(h) = l(h')$. Without loss of generality, suppose also that $\exists h'' \in I'$ such that $h \in S(h'')$ and $\exists h''' \in I'$ such that $h''' \in S(h)$. But then $h''' \in S(h'')$, thus by the reasoning in the proof of *Lemma 2*, I' is not multi-staged. Therefore the problem itself cannot be multi-staged either.

LEMMA 8: *There is no problem with precedence-reversal which would not feature cross-branch relevance at the same time.*

Proof: Note first that by the above *Lemma 7,* a problem with precedence-reversal cannot be multi-staged.

Now suppose for a contradiction that there is a problem with precedence-reversal which does not feature cross-branch relevance. By definition, there are two information sets, I and I', in the problem for which it is true $I \in S(I')$ and also $I' \in S(I)$.

Consider first the case when one of these information sets is singleton, without loss of generality this could be I', and suppose its only element is h''. Then if $h, h' \in I$, it has to be the case that $h'' \in S(h)$ and $h' \in S(h'')$. But then h is not in $\Delta^{I'}$ so I is not contained in $\Delta^{I'}$. Thus the problem features cross-branch relevance.

Consider next the case when neither I nor I' is singleton. Suppose then that $h, h' \in I$ and $h'', h''' \in I'$. Without loss of generality, we have $h' \in S(h'')$ and $h''' \in S(h)$. Then for the subproblem $\Delta^{I'}$ it is true that h is not contained in it. So for I' there is a successor information set, I, which is not contained in $\Delta^{I'}$. Therefore the problem features cross-branch relevance. This finishes the *proof*.

LEMMA 9: *There exist problems with precedence reversal which feature cross-branch relevance, but not absent-mindedness; and others which* *feature both cross-branch relevance and absent-mindedness. On the other hand, there exist problems which do not feature precedence reversal, but still: feature cross-branch relevance, but not absent-mindedness; feature absent-mindedness but not cross-branch relevance; feature both cross-branch relevance and absent-mindedness.*

Proof: For all five parts of the statement, suppose for a contradiction that there exist no such problems. But then *Figures 15, 7, 6, 5,* and *9*, respectively, provide counterexamples.

The final class of problems which we have to place are those with perfect recall of information sets. The five lemmas below describe their location:

LEMMA 10: *There is no problem with perfect recall of information sets which would feature precedence reversal.*

Proof: Suppose for a contradiction that there is a problem which has perfect recall of information sets and also precedence reversal. Then for each $\hat{I}$ of the problem and each, h, $h' \in \hat{I}$, we have $Y(h) = Y(h')$ by definition. By precedence-reversal, there are also information sets I and I' for which it is true that $I \in S(I')$ and $I' \in S(I)$.

Without loss of generality, suppose also that $h \in S(h'')$ and $h''' \in S(h)$, where $h, h' \in I$ and $h'', h''' \in I'$. Then I' is part of $Y(h)$. So we have two possibilities. Either h' is not in $S(h'')$, in which case $Y(h) \neq Y(h')$. Or $h' \in S(h'')$, and therefore I is part of $Y(h''')$. From that follows that $Y(h'') \neq Y(h''')$. Then the problem cannot feature precedence-reversal.

LEMMA 11: *Each perfect recall problem has perfect recall of information sets, but there are problems with perfect recall of information sets which do not have perfect recall.*

Proof: For the first part, note that in a perfect recall problem, for each information set I and each $h,\ h' \in I,\ X(h) = X(h')$, trivially. Then by definition of the sequences X and Y, this implies that $Y(h) = Y(h')$.

For the second part, suppose for a contradiction that there exists no such problem. But then *Figure 4* provides a counterexample.

LEMMA 12: *Each multi-staged problem which does not feature cross-branch relevance is one with perfect recall of information sets.*

Proof: The proof proceeds by exclusion of possibilities. By hypothesis, the problem cannot feature cross-branch relevance. By the first part of *Lemma 2,* it cannot be one with absent-mindedness. Then by *Lemma 7,* it cannot feature precedence reversal either. Then we next note that by the first part of *Lemma 1,* by *Lemma 6,* and the first part of *Lemma 11*, each perfect recall problem is multi-staged, and features perfect recall of information sets but not cross-branch relevance. But there are no more categories in this characterization.

LEMMA 13: *No problem with perfect recall of information sets features absent-mindedness.*

Proof: Suppose for a contradiction that there is such a problem. Then since it has absent-mindedness, there is an information set I in it, for which $h,\ h' \in I$ and $h \in S(h')$. By the definition of experience of information sets, I is part of $Y(h)$, but by the same definition, I is not part of $Y(h')$. Thus $Y(h) \neq Y(h')$, and thus the problem cannot be that of perfect recall of information sets.

Then the extensive forms shown on *Figure 13* cannot belong to a problem with perfect recall of information sets.

LEMMA 14: *There are problems with perfect recall of information sets which are multi-staged, have no perfect recall and feature cross-branch relevance. There are problems with perfect recall of information sets which are not multi-staged and feature cross-branch relevance.*

Proof: For each of the four parts, suppose for a contradiction that there exist no such problems. But then consider *Figures 11* and *12*, which provide counterexamples.

In the last round, we establish the autonomy of problems with absent-mindedness and cross-branch relevance, respectively.

LEMMA 15: *There exist problems featuring absent-mindedness which have neither cross-branch relevance nor precedence-reversal.*

Proof: Suppose for a contradiction that there exists no such problem. But then *Figure 5* provides a counterexample.

LEMMA 16: *There exist problems featuring cross-branch relevance which are multi-staged but have neither perfect recall nor perfect recall of information sets; and there exist problems featuring cross-branch relevance which are not multi-staged, and have neither perfect recall of information sets, nor absent-mindedness, nor precedence-reversal.*

Proof: For both parts of the statement, suppose for a contradiction that there exist no such problems. But then *Figure 1* and *6* provide counterexamples.

The above characterization can be conveniently represented by a Venn-diagram. Also, note that due to the particular definition which was given to cross-branch relevance, not each noteworthy category is recognized by this characterization. The reasons for this and arguments for possible remedies are not reported here.

IV.

7. The Concept of Strategy

23. So may imperfect recall decision problems be solved? Given his desires and beliefs, and the rules of the problem, the decision maker forms reasons to act in certain ways while he is in the Ex Ante state. Since the First Story holds, he can see through the whole problem in that state. So he can discern a best course of action, a course of action which is best for him in the context of the given problem. (Of course, if there is a chance move in the problem the best course could be contingent on how certain chance moves are resolved.) Then he commences with the implementation of this best course by undertaking his first action.

Now, in an extensive form decision problem, new reasons may appear after some events. If these conflict with earlier reasons, we should say that an instance of time inconsistency arises. (See *Figure 4* where at I_2, the decision maker would want to do something else then what the best course of action perceived in the Ex Ante state demands.) If, as we have assumed, preferences are stable throughout the problem, these new reasons cannot have a conative origin. This is because the decision maker knows everything about the problem in the Ex Ante state and he can rank the feasible sequences of actions according to his preferences, and nothing what happens later on will affect this ranking. Due to the stipulations of the First Story, the reasons for acting in a certain way will never lose authority at later points in time in the problem. In the Ex Ante state, by the First Story, the emergence of any new reasons could have been foreseen, and all contingencies are discerned, so nothing which emerges in the course of the problem can overrule the authority of the Ex Ante reasons. From this it follows that the new reasons can only have an epistemic ground, and have to be due to losing epistemic resources.[32] Thus implementing the course of action which is best from the Ex Ante point of view, possibly taking into account inferior interim reasons, becomes the main task for the decision maker in an imperfect recall problem.

24. While discussing extensive form decision problems, the analysis of how the best course can be implemented is usually organized around the concept of strategy. We can say that a strategy renders a certain action to each information set. Thus the basic, standard concept of strategy, called here *simple strategy* represents it as a function from information sets to actions available at those histories which make up the given information set. Denote this function by, $\sigma : \mathbf{I} \to A,$ where the set A in the range of σ meets the appropriate requirements for restriction (§14). Denote the set of all feasible strategies by Σ, and exclude mixed strategies from consideration. It renders an action to each information set in the problem, irrespective of whether a particular information set can be reached while implementing the strategy or not. Since it is assumed here that the decision maker cannot commit a mistake in carrying out a desired action after a

[32] The argument above sanctions attempts to translate time inconsistency due to changing preferences into time inconsistency due to the lack of intertemporal common knowledge, as in *John Geanakoplos* (1989): "Game Theory without Partitions, and Applications to Speculation and Consensus", mimeo., Yale University. *Cf.* Ferreira-Gilboa-Maschler, ``Credible Equilibria with Utilities Changing during the Play".

history, a less inclusive notion of strategy involves only those information sets in the domain of σ which could be reached in the implementation of the strategy. Denote this set $\mathbf{I}^* \subset \mathbf{I}$. Of course, the derivation of $\mathbf{I}^*$ requires the comparison of each $\sigma \in \Sigma$. Note that this concept is superimposed on the description of the problem since it is not part of the tripartite decomposition. And it is subject to interpretational difficulties.

Some of these difficulties have been discussed by *Ariel Rubinstein*[33] in the context of game theory. The concept of strategy, he writes, can be approached in two ways. First, it could mean a plan for a course of actions (possibly contingent on resolution of uncertainty). This plan does not specify what should be done in contingencies which cannot arise if the plan is actually followed. But a strategy could also mean a complete specification of what to do under all possible circumstances. This more inclusive concept of strategy may arise from taking into account what leads to the articulation of the best course of actions. This articulation has to involve a testing of all possible courses, at the end of which the best one appears and such a testing has to consider all the possible states the decision maker could be in. Thus a plan is the outcome of a deliberation, and leads to an eventual course of actions. Even if it is assumed that the decision maker can make no mistakes while carrying out the plan, the discrepancy of the two approaches is significant in the context of game theory since one's own course of action depends on beliefs on what is the best course of action the other player has in mind. If some hypothesis about the plan of the other player is contradicted, which is an unexpected event in just this sense, an adjustment has to be made which necessarily

[33] Rubinstein, "Comments on the Interpretation of Game Theory", pp. 910–912.

involves making conceptions about the whole range of hypothetical reasoning on the part of the other player.

In a decision problem, no such unexpected events can arise if mistakes are impossible. If the decision problem is characterized by perfect recall, not only are there no unexpected events, there cannot be new reasons to act, either. If there is imperfect recall, still no unexpected events can arise in the above sense, but as we have seen new reasons for action may arise due to the loss of epistemic resources. Still, we should not lose sight of the earlier conclusion that the decision maker would like to implement the best course of action as it appears in the Ex Ante state, and the reason for the adoption of this course has unrivalled authority. We considered strategies as a focal means for implementation, but have not yet reached the full understanding of how strategies can matter in a decision problem.

25. So we may ask again, what can a strategy be? In case of perfect recall, again, the discerning of the best course of action is followed by an initial action, trusting that the same course will be sustained later on while committing further actions along the optimal path. The trust is well-grounded, since no new reasons for acting otherwise will appear, thus no enforcement of that course is necessary. It is not important to remember the strategy later on, it will be readily regenerated by new reflection on the problem during the execution of the previously discerned best course.

Now, in the case of imperfect recall we know that new reasons may appear. Then, because of the authority of the Ex Ante reasons we may think of the decision maker as complying with the strategy formed in the Ex Ante state, even if an independent reflection on what to do next would prompt otherwise. This presupposes that the strategy is kept in mind and the decision maker consults it before any further action is taken. So in an imperfect recall problem the strategy may coordinate the actions in the problem, and further it communicates the best reasons for acting in a certain way. These are two new aspects of what a strategy is in a decision problem which do not appear in the standard formulation.

The following fiction may further enhance the legitimacy of making explicit these two aspects. Think of the decision maker as an aggregate of temporal/modal selves, one self for each conceivable sequence of events after which a decision has to be made. Then we can imagine the Ex Ante state as a convention of these selves which deliberates on what each of them should do when it is his turn to act. Due to the privileged status of the Ex Ante state and the fact that there is ample reason to suppose that that state is capable of "integrating the personality" of the decision maker[34], these selves should opt for what is best for all of them. This convention thus brings about a strategy, and each self is instructed to follow this strategy because this assures that the best course of actions is pursued.

26. At this point, a further difficulty arises in the interpretation of strategies. As we have seen, a fundamental part of the specification of the beliefs in decision and game theory involves the concept of information sets. And information sets are collections of histories regarded as indistinguishable after a certain history. Because of this indistinguishability, in each of these information sets the same set of actions must be available to decision makers, otherwise this one aspect could reveal the actual history the decision maker is at. Strategies, in turn, have to refer to information sets. Now, it could be the case that it would be advantageous to change the strategy within the problem at some point and hoping that later selves will follow suit (see the example on *Figure 1,* discussed in §6). However,

[34] *Cf.* Arrow, *Social Choices and Individual Values*, p. 2.

according to the standard account, since later selves act according to the one strategy constructed in the Ex Ante convention, it is impossible to take advantage of the discernment of the new best course.

One of the characteristic themes of this essay is that it urges the recognition of an extension of the concept of the simple strategy, which allows the updating of a current strategy during the problem. This, a new object, is called *extended strategy*, and it is defined by the function $\theta : \mathbf{I} \times \Sigma \rightarrow A \times \Sigma$, where $\theta \in \Theta$ denotes the generic element. Here the same restriction applies to A as above. Extended strategies are formed in the Ex Ante state and prescribe two operations for each information set. The first operation is the carrying out of instructions according to the strategy regarded as valid, the second operation is contingent on the information set and may call for the specification of a new valid strategy. This new strategy is then passed on to the next information set in which a decision is to be made and which is to be reached next, given the action just committed. There this strategy will be regarded as valid. Finally, there is an initial strategy (which may be denoted as σ_0) which is determined in the Ex Ante state.[35] So changing strategies articulated in the Ex Ante state is sanctioned even in this extension, but Ex Ante considerations may prescribe extended strategies which call for the updating of valid strategies at certain information sets.

The difference between the two definitions can be brought out by a reference to a well-known explication of what strategies are in a perfect

[35] This concept appeared first in my (1994) "An Analysis of Decision Problems in Time: The Concept of the Plan and Imperfect Recall", under the label "plan". *Joseph Halpern* defined later a concept which is virtually the same as the one described here independently from me, in his "On Ambiguities of the Interpretation of Game Trees".

recall context. [36] According to this fiction, strategies are pocket books, each page of which refers to an information set. Then on each such page of a given book there is an action inscribed, the one which ought to be committed at the information set in question. (The interpretation for behavioral strategies employs a straightforward generalization, it imagines a probability distribution for each page.) Then the choice of a strategy amounts to the choice of such a book. Now I propose that this imagery can be retained for the case of extended strategies. Just picture a same sort of book which not only contains a prescription for what to do at a given information set, but also may prescribe the switching to some other book (in an other pocket) if that is necessary. Thus the decision maker starts out with a given book which he may later on throw away and continue to act from an other one.

In the light of the discussion above, we can ascertain that it is indeed in the case of imperfect recall when a decision maker truly needs a strategy, which cam guide him during the course of the problem. In case of perfect recall, strategies are superfluous since can be always reproduced. Their formulation serves only theoretical convenience.

27. An analysis in the framework admitting extended strategies could be conducted in two steps. First, consider the following reduced account of a decision problem. I suggest that temporal/modal selves within the problem could be deprived of epistemic resources, without abandoning the basic structure of the problem. In this reduced account, selves are only capable of recognizing the information set they are in, and carrying out

[36] See Kuhn, "Extensive Games and the Problem of Information", p. 200. *Cf.* Luce–Raiffa, *Games and Decisions*, p. 159.

instructions inscribed in the strategy. In an analysis of imperfect recall, assuming less rather than more about beliefs can only be enlightening.

If the privilege of having epistemic states will be withdrawn from the decision maker while out of the Ex Ante state, the strategy employed by him can be characterized as part of a description of a finite automaton solving the problem. This possibility is explored at some length in the next point. However, here I would like invoke an outcome of that exploration, the capability of treating extended strategies as algorithms for solving the extensive form decision problem. It would then be possible to present an analysis of how to solve imperfect recall problems in terms of the language of finite automata, in a sense this would be the most accurate presentation.

In a second step, one could relax the assumption that decision makers have no epistemic life within the problem, and seek the corresponding definition of extended strategies in this case. Given the argument in §§ 24–26, the analysis would turn out to be completely analogous to the one in the first case. However, the consideration of this two-step procedure allows us to finish the specification of how beliefs are modelled here. Earlier, we characterized fully the beliefs in the Ex Ante state. Then, by focusing on interim beliefs concerning the location within the physical problem (and excluding certain wild cases of forgetting) the attention was shifted towards the specification of what is known at individual information sets. Since we have just ascertained that nothing crucial is lost if we strip temporal/modal selves of much of their epistemic resources within the problem, this settles what is assumed of their beliefs there: capability of identifying information sets and of following the instructions of the strategy book. No talk of incoherent knowledge structures can have a force here. This, finally, also fixes the interpretation of beliefs concerning what the strategy is. There is no problem with strategy recall here; which is just quite alright, since it seems to be a grave challenge to model beliefs about strategies.

28. It may be of interest to ask the question: how should we proceed if we wanted to represent decision makers by abstract automata? The natural model for this representation is the object of Moore-automaton which has been already introduced into the study of bounded rationality phenomena[37]. A Moore-machine for a single decision maker, may be described by the quadruple $\langle Q, q^0, f, \tau\rangle$ Here Q denotes the possible states of the machine, q^0 its initial state, the function $f : Q \rightarrow A$ the output function rendering states into actions, and $\tau : Q \times A \rightarrow Q$ the transition function which renders a new state to the previous state and the induced action. It is not difficult to see that if we examine a decision problem with perfect information, then simple strategies correspond to the output function, and if further a stop rule is imputed into the description, it is convenient to interpret the states as histories. Note also that the decision tree itself is already inscribed into the description of the machine then. If there are chance histories in the problem and also possibly non-singleton information sets, then the interpretation becomes somewhat more involved; and some additional work has to be done.

Without giving then a full translation of imperfect recall decision problems and extended strategies into the formalism of Moore-machines, let me only indicate here how some pivotal components of our problem could be expressed in this framework. Define then $F^*: D \times \mathbf{I} \rightarrow \Sigma$ as the function which assigns a strategy to each decision history already reached.

[37] See the references in *footnote 4*, and Osborne–Rubinstein: *A Course in Game Theory*, pp. 140-143, 164-168.

Then the appropriate output function for an analysis of decision problems which admits extended strategies could be, $f : D \times \mathbf{I} \times \Sigma \rightarrow A \times \boldsymbol{F}$ whose value is $f\,(d, I, \sigma) = (\sigma\,(\mathrm{I}), F^{*}\,(d, I))$: the action to be taken at the history according to the valid strategy σ and the new (updated) strategy. This is then the representation of extended strategies. Finally, the transition function should be, $\tau : D \times \boldsymbol{I} \times A \times \boldsymbol{F} \rightarrow D \times I \times \boldsymbol{F}$ whose value is $\tau\,(d, I, a, \sigma) = (d, I, F^{*}\,(d, I))$: which prescribes the new state. Note that there is the same sort of redundancy between the output function and transition function here as appears at the simple example of a Moore-automaton representing the solution of perfect information problems by means of simple strategies, shown in the previous paragraph.

8. Solutions of Imperfect Recall Decision Problems by means of Extended Strategies

29. *In lieu* of a complete characterization of optimal extended strategies for special classes of imperfect recall decision problems[38], I offer below several examples of how extended strategies can solve certain, simple but focal, problems of forgetting.

The first example is represented on *Figure 5*. This is the simplest problem featuring absent-mindedness, and the one which generated most

[38] For an analysis of the existence of optimal extended strategies for a large class of imperfect recall problems, see "On Optimal Solutions to Decision Problems with Imperfect Recall", Essay 2.

of the quandaries reported in the paper of *Piccione* and *Rubinstein*[39]. Its associated perfect recall problem is shown on *Figure 5.* Clearly, the task is to reach the terminal history where utility 9 is in the offering. The implementation tree Γ_E^* is shown on *Figure 5.* Here the optimal extended strategy calls for an initial strategy σ_0 which renders action L to the only information set I. It also calls for updating the first strategy to σ_1 at I, so that $\sigma_1 (I) = \{R\}$. In this way the utility 9 will be reached for sure.

Figure 1 represents the second example. This is a multi-stage problem without absent-mindedness and precedence reversal, albeit it is not of perfect recall of information sets. If the initial chance move is $\{l\}$, the decision maker wants to reach gains 6; if it is $\{r\}$, he wants to get to where utility 4 is given to him. One optimal extended strategy prescribes the initial strategy σ_0 as follows: $\sigma_0 (I_1) = \sigma_0 (I_2) = \{D\}$, $\sigma_0 (I_3) = \{L\}$. If the problem reaches I_{1}, no updating is necessary. But if it reaches I_2, then there a new strategy σ_1 should be made valid, for which: $\sigma_1 (I_1) = \sigma_1 (\mathrm{I}_2) =$ $\{D\}$,

$\sigma_1 (\mathrm{I}_3) = \{R\}$. And there is an other optimal extended strategy symmetrical to the previous one.

The third example is illustrated on *Figure 15*. This is a problem with precedence reversal, but without absent-mindedness. It is apparent that the decision maker wants to reap 8 under all circumstances. There is only one optimal extended strategy for this problem, with the following features. The initial strategy σ_0 should be: $\sigma_1 (\mathrm{I}_1) = \{r\}$, $\sigma_0 (\mathrm{I}_2) = \{L\}$. And no matter what, at either I_1 or I_2, this should be updated to have a new valid strategy which says: $\sigma_1 (\mathrm{I}_1) = \{l\}$, $\sigma_1 (\mathrm{I}_2) = \{R\}$. In this way, getting 8 is assured.

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[39] *Piccione–Rubinstein*, "On the Interpretation of Decision Problems with Imperfect

The final example is very simple (see *Figure 4*). This problem is of perfect recall of information sets. Here the initial strategy should prescribe $\{l\}$ for the first information set and $\{L\}$ for the second; and that is it. There is no need for updating.

30. Note, however, that the treatment by the current analysis of this last problem is symptomatic of its inability to identify sharply what constitutes time inconsistency due to epistemic reasons in a decision problem[40].
But we can say this much. If there is a need for updating in an optimal extended strategy then surely there is an instance of time inconsistency there. If there arises no such need, then we have to fall back on the following comparison. Find out how much the optimal extended strategy can achieve and then how much the optimal simple strategy can achieve in a given problem. If there is a discrepancy, then there has to be time inconsistency again. We may even say that forgetting is relevant then. But this comparison falls short of a formal characterization of when epistemically driven time inconsistency arises.

9. The Concept of Strategy (cont.)

31. Below, I will consider some possible objections against the analysis offered in this essay. I propose boldly that the set of stipulations termed as the First Story should not be disputed, since that is effectively an anchor for the discussion of decision problems. Challenging that story may lead to a premature stalemate. Then counterarguments can only be levered against

Recall", pp. 6–7, *et passim*

my interpretation of the concept of strategy. To repeat, I argued that in the context of imperfect recall decision problems strategies ought to be viewed as vehicles of coordination and even communication.

It is the latter role which raises the counterobjection that since the rules and the specification of the beliefs of the decision maker in the extensive form completely describe the information available to him concerning the problem, it is problematic to endow strategies with communicative roles. First, as I have argued at length above, it is not immediately clear what the information presented to the decision maker is in a decision problem. The basic construct of information sets itself has two parts, a formal and a conceptual. The formal construes information sets merely as an information partition on decision vertices. The conceptual part interprets this partition as sets of histories which are indistinguishable for the decision maker. Now, we know that the standard interpretation of the concept of information sets originating in the work of *Kuhn,* itself an attempt at organizing the discussion of von *Neumann* and *Morgenstern,* is intimately related to cases of perfect recall and the standard conception of strategy. Further, this formulation is not controversial in case of perfect recall problems. But it is just not true that information sets completely describe the information available to the decision maker. If the First Story holds, the decision maker can construct a tree in the Ex Ante problem, and therefore can identify each edge of this tree. Later on, while being at a given information set, he can only discern classes of actions, the ability to identity of the edges is gone. This is not a central point, but indicates the difficulty in the statement that the extensive form fully describes the

[40] *Piccione–Rubinstein*, "On the Interpretation of Decision Problems with Imperfect Recall", pp. 5, 16–19.

information the decision maker has for the problem. It is indeed a special task to specify the full range of beliefs of the decision maker.

Second, the concept of the strategy is not an integral part of a decision problem[41], it is rather superimposed on it. It ought to refer to the means with which the decision maker can implement the best course of actions identified in the Ex Ante state. The fundamental, standard notion of strategy as a function from information sets to feasible actions at the information sets is entrenched because nothing more is needed in case of perfect recall problems, the study of which is rarely transcended. If strategies are vehicles of implementation, then they may serve functions which become crucial only in the case of imperfect recall or in instances when complexity matters. It is important to recall here the introduction of the fiction of the randomization device for the implementation of mixed strategies. Nothing in the formulation of a decision problem or a game mentions such an entity, still it is regarded as a legitimate notion to employ. Thus there is nothing sacrosanct in the traditional notion of strategy, however, one has to be careful while extending the standard concept in making explicit the presuppositions behind the extension. This is just what the earlier discussion tried to accomplish.

32. Maybe the most pertinent criticism[42] which can be raised against the particular extension proposed in this essay is that it allows under

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[41] The original development of the concept in the book of von *Neumann* and *Morgenstern* would deserve special attention. Here I would only like to remind of the fact that they arrived to the now standard definition as the outcome of a sustained engagement with many difficulties of formalization. Compare von Neumann–Morgenstern, *Theory of Games and Economic Behavior*: "Each player selects his strategy – i.e. the *general principles governing his choices* – freely", p.49 (italics are mine) with pp. 79–81 and 84.

[42] Conversations with *Pierpaolo Battigalli* were most useful for the formulation of the arguments below.

circumstances that, in effect, two different actions are assigned to the same information set. To see this, consider again the example on *Figure 1* when the optimal strategy is updated after a particular chance move. An other way to express the criticism is to state that the information set in which two different actions maybe enacted under two different resolution of the chance move is effectively refined by the extended strategy. But note that even in the case of perfect recall games, in an analogous sense, it could be the case that information sets are effectively refined (as in *Figure 16*). So the refinement produced by the use of extended strategies is merely an outcome of the implementation of the best course of action.

A variation of the previous argument maintains that extended strategies eliminate imperfect recall problems by the "hidden" communication via the strategy updating. That is, the problem of imperfect recall is rather assumed away by the extension. The case when there are no epistemic states during the problem shows that this is not the case, the temporal self at the information set in question does not get to know what happened before, he only executes the instructions inscribed in the strategy regarded by him as valid. Nor is the extended strategy a "hidden" counting device, counting is not an issue here.

33. The very idea that there could be communication among the selves can be regarded as flawed as well. It is indeed true that no communication is necessary in the special case of perfect recall, the strategy can be regenerated by later selves. We saw that even when there is no updating in the extended strategy, remembering the optimal strategy formed in the Ex Ante state helps to execute the right action at information set jeopardized by imperfect recall (*Figure 4*). It is through the strategy that the self at the

Ex Ante state can communicate with the later ones. Then when there is updating, it can be said that selves having informational advantage over some later ones communicate through the updated strategy. This is a crucial point. Recall the fictitious story of the selves convening before the execution of the best course would have started in the Ex Ante state (§ 25). They can be seen as agreeing on an optimal strategy, and adhering to it later on. Some may argue that updating is a deviation of a subcoalition of the selves to a new strategy, and this deviation is coordinated via the communication of what the new strategy is. Now the new objection is that such a coalitional deviation is not allowed in an extensive form decision problem. I think this objection involves an implicit adoption of the perspective of game theory on coalitional deviations. But it is not the case, unless we have specified this among the strictures on the beliefs of the decision maker, that we should view the decision maker as a collection of temporal selves playing a strategic game (of coordination). And then no a priori restriction can be imposed on the feasibility of coalitional deviation (but one can talk about sequential rationality then). This conclusion is really grounded in the overarching authority of the reasons formed in the Ex Ante State, which is itself a reflection of the closure of the world in any Ex Ante state, for the sake of formal analyses (see § 39).

But there are cases when it is not possible to assume that there could be a communication of the deviation, and this is the case when the decision maker is made up of different persons having the same objective, this is the case of teams. Here a team playing Bridge is the classical example (but the pair of arbitrators in the dispute settlement system depicted in the first point also exemplify this kind of a situation, see also §36). It is possible, indeed, it is in the spirit of Bridge, that players 'form a strategy' before the game starts, but they are certainly not allowed to communicate with each other over and above what is feasible in the game. So for a team problem with separate persons involved, for whom certain means of communication are precluded, or for the case when it is not known what strategies or when they cannot be remembered, 'updating strategies' may not be a possibility. For a decision maker who can be regarded as a single person who preserves his identity, and for whom there are no {\em a priori} limits to remember strategies, updating is available. Again, this is because a strategy is not part of the description of the extensive form decision problem, and therefore the extent to which strategies can be recalled or an updating can be communicated is a matter of further modelling choice.

34. The previous discussion underscores two auxiliary themes of this essay. First, it argues for a separation of the issue of strategy recall from the rest of the problem and therewith also allows the examination of the complexity of the strategy in isolation from the other issues in the problem[43]. If one studies a problem with absent-mindedness (see *Figure 5* again), different solutions appear as different assumptions are made about the complexity of the strategy the player can carry out. If indeed he can use the pocket system, he can be sure that he finds the right exit. If he can employ only *one* strategy constructed in the Ex Ante state, he may do worse. Finally, if he cannot recall any strategy we can model him as a succession of selves playing a coordination game, and start really to think about what sort of analysis is the proper one then[44].

[43] This is one of the conclusions in *Halpern*, "On the Ambiguities in the Interpretation of Game Trees", as well.

[44] Analyses of situations when strategies change within the problem appear in game and decision theory elsewhere. In one theory of renegotiation in repeated games, players may renegotiate their equilibrium strategies after some history. See, for example, David Pearce (1987): "Renegotiation-Proof Equilibria: Collective Rationality and

The second theme is the possibility to draw a demarcation line between team problems and imperfect recall decision problems, maintaining that team problems do not allow updating strategies during the problem. Modelling Bridge appears then as a separate project[45].

10. Changes in Preferences

35. This section discusses the possibility that the desires of the decision maker may change during the problem. This change is represented here in a restricted way: through the assignment of a different utility function for each self identified by a decision history. This can be suitably achieved by positing a set of functions $u_h : Z \rightarrow \Re$ where u_h is the payoff function of the decision maker after history h.

The issue of preference changes becomes significant only if one is entitled to view the new desires as carrying some authority for the decision maker, and therefore standing as reasons during the problem. I will not attempt to give a full account of how to analyze these cases. While maintaining that the Ex Ante intentions still must be powerful in influencing the outcome of these conflicts of desires and thus reasons among the selves, I would only like to give an illustration of how preference changes might interact with imperfect recall. More specifically,

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Intertemporal Cooperation", mimeo., Yale University. See also the work of Geir Asheim (1991): "Individual and Collective Time Inconsistency", mimeo., Norwegian School of Economics, on individual and collective time inconsistency, where after each history, decision makers submit a new strategy to the temporal selves acting after them.

[45] Compare Binmore, *Fun and Games*, pp. 458–459 with pp. 573–602.

I would like to point out some difficulties in the analysis of situations where both forgetting and changing preferences are present.

As a start, let us examine an example of a classical instance of preference change and time inconsistency. Consider the decision problem depicted on *Figure 17*. At each terminal history, the numbers on the top indicate the payoffs of the Ex Ante self, the ones at the bottom show the payoffs of the later self. Suppose that at the first information set the decision maker has to make a decision about whether he should call his friend some puzzles concerning the methodology of the social sciences or not. If he does not, he will gain a payoff of 1. If he does, he has to make a further decision between interrupting the conversation after four hours and writing his important treatise on functionalism, or continuing the debate until midnight. Before calling the friend, it is clear that a discussion first and then stopping it for the sake of writing produces a gain of 2. Arguing until midnight, however, gives a meager gain of 0. It is also known that in the middle of the debate, after four hours of talk, the decision maker will see the gains differently. He will think that continuing the debate will bring him closer to perfection in writing his treatise (giving him 2), while abandoning the talk generates a self-image of himself as someone lacking in perseverance in thinking things through (giving him 0). What should the decision maker do at the first information set? Instead of trying to answer this question, let me note that an important feature of this example, as it is set up and as usually these examples are set up, is that the first self is really assumed to release control over the situation after the Ex Ante state and also that his reasons have some independent authority over those of the later selves. The examples below suggest that one way to retain an advantage over the later self might be the exploitation of the epistemic superiority enjoyed in the Ex Ante state.

36. So let us essay first an example which makes it clear that even without a conflict among different selves, forgetting can be beneficial for the decision maker. Consider the next example, which is a transcription of a model of *Abhijit Banerjee*[46]. An individual has two opportunities to invest. Both times, the cost of the investment is either 0 with probability *q*, or *c* with probability 1 — *q*, and the realizations of these costs are independent across time. The return is a with probability *p*, and *b* with probability 1— *p* and the exact value is decided once and for all at the beginning of the problem. Also, stipulate that $a > c > b > 0$. It is assumed that at the first time he has a chance to invest, the individual can observe the value of the return and make an investment decision accordingly. On the second occasion, the individual cannot remember what the returns were, only the investment decision he has made. Now if the constellation of the parameters is such that $\frac{p}{p+(1-p)q} >, \frac{c}{a-b}$, then the decision maker will invest if he remembers he had invested the first time, even if his costs are high at the second time. Now if it is also true of the parameters that $\frac{c}{a-b} > p$, then the decision maker with high cost would not invest if he would happen not to know whether he had invested before or not. So he would be better off if he could not remember whether he had invested or not.

In the next example, shown on *Figure 18*, when a conflict of desires among the selves is "reintroduced", forgetting is beneficial again. If the decision maker knew that he is at node d_4, he would want to do $\{l\}$: which runs against the Ex Ante self's preference of doing {*r*}. Given the probabilities for the chance moves, however, the decision maker while at d_4, will prefer to do {*r*}, since that gives him a larger expected payoff. Thus the decision maker is better off with imperfect recall.

Is there a sense in which we can allow the Ex Ante self to induce a later self, who is in jeopardy of being under the control of some desire lacking authority for the Ex Ante self, to forget? That is, can we allow the Ex Ante self to bring about the very problem on *Figure 18*? In a paper on self-deception, *Mark Johnston* suggests[47] the advantages of techniques which can cause retroactive forgetting, and thus are quite practical for self-deceptive purposes. Can the consideration of such a technique be justified in our decision theoretic model? While I believe such a technique should be listed in the description of the extensive form, I wish to note also that this question is somewhat homologous to the idea of passing through dangerous phases in the course of a decision problem by means of the employment of extended strategies. One fascinating way to deal with this issue would be an explanation in terms of "evolutionary game theory"[48], the ability of a decision maker to forget at the node d_4 is selected by adaptive mechanisms in this perspective, or even survival is assured by forgetting successfully.

37. The example on *Figure 19* can demonstrate how different assumptions about intertemporal common knowledge of the selves might affect the analysis of decision problems. In this example, the second self does not remember his first action.

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[46] Abhijit Banerjee (1991): "The Economics of Rumors", mimeo., Princeton University

[47] See Mark Johnston (1988): "Self-Deception and the Nature of the Mind", in: *Perspectives on Self-Deception*, edited by B. McLaughlin and A. O. Rorty, Berkeley: University of California Press, pp. 63–91.

Consider first the case when the self in the second information set has a good basis to form a knowledge about how much his earlier self knew. He thinks, correctly, that the first self knows that he will later forget, that is he will not be able to distinguish among certain histories, and that his preferences will be also changed. This epistemic scenario would result in a somewhat banal Nash-equilibrium (1,1) among the selves, a famously Pareto-inferior outcome (here in the intrapersonal sense).

In the second case, the second self thinks that the first self does not know that he will forget. Then he will think that the first self played $\{L\}$, and will play himself $\{r\}$ accordingly. Knowing this, the first self will play $\{R\}$, so this breakdown of intertemporal common knowledge results in achieving (3,0), an improvement over the previous case. Again, can the first self induce the second to forget about something that had been known by the first self?

In the last case, the latter self thinks that the first self does not know that desires will change. This will make him think that $\{R\}$ was played, and then this self will play $\{l\}$, which then induces the equilibrium (1,1) again.

No doubt one could continue with exhibiting even more cases highlighting scenarios when the deflation of the knowledge of one of the selves about some entity in the problem could turn out to be beneficial from the Ex Ante point of view. So in this context an apparently new dimension of the analysis of imperfect recall decision problems surfaced as well: how do assumptions, and furthermore theories, about intertemporal common knowledge among selves engage with the analysis?

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[48] *Cf.* Johnston: "Self-Deception and the Nature of the Mind", pp. 88–89.

11. Further Remarks

38. The presentation of the First Story was a self-conscious attempt at fixing the frame of all the decision problems analyzed here, its main function in the discussion can be seen as the creation of a situation in which decision makers are confronted with the problem in a time outside the temporal framework of the problem. It also assures that they face a problem with clearcut boundaries. Then we could suitably talk about deliberation and reason-giving in this Archimedean time, and contrast this with the merely algorithmic execution of the outcome of the deliberation process, the best course of actions. It is indeed a most trivial fact about decision making that deliberation, intention formation and the concomitant plans for actions occur periodically, and are followed by phases of plan execution. Consider chess. Here any acquaintance with how that game is usually played suggests that players sometimes spend a long time thinking and forming a strategy, then this "slow" phase is followed by a rapid exchange of moves when no new deliberation seems to be made. Clearly, in the case of chess, these cycles of deliberation are due to the complexity of the game.

The concept of bounded rationality refers to instances when a decision maker lacks perspicacity concerning the structure of a problem. Each of these instances could in principle induce deliberational cycles. And imperfect recall decision problems could also produce them. Then we can view the concept of extended strategy from a different angle. Suppose that we chase the decision maker out of the Ex Ante state, assuming that there is no point in which he can survey the whole problem. Then the analysis of imperfect recall decisions problems with the help of extended strategies could be useful in identifying those points of the problem when the decision maker needs to deliberate again.

Closely connected issues abound, they include the timing of reason-giving[49], rationalization, justification. There are points in time, in any somewhat complex intertemporal decision problem, when the whole decision problem is regimented again. A new perspective is constructed which issues a demand for new reasons and plans[50]. This may involve commitment to new epistemic stances which are not even grounded in experience[51]. The study of imperfect recall problems could identify some of these triggers.

39. It has been forcefully argued by *Binmore*[52] that many of the interpretational problems of formal decision and game theory are due to the fact that in order to be able to formalize a decision problem for an outside observer, the world in which the problem appears is effectively closed by the analysis. There is a reference here to a distinction between small worlds and large worlds made by *Savage*[53], where decision theory is claimed to apply only in the former case. The formalization of the situation generates a standard for success for the decision maker himself, who, if he knows as much as the modeler or Analyst, should be able to form a

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[49] The article of Eldar Shafir, Itamar Simonson, and Amos Tversky (1993):"Reason-Based Choice", *Cognition* 49, 1: 11–36, offers ample empirical illustration

[50] See many arguments of Michael Bratman, for example, his (1992): "Planning and the Stability of Intention", *Minds and Machines* 2, 1: 1–16.

[51] See Bastian van Fraassen (1984): "Belief and the Will", *Journal of Philosophy* 81, 5: 235–256.

[52] See Kenneth Binmore (1991): "De-Bayesing Game Theory", mimeo., University of Michigan; see also Stephen Morris and Hyun Song Shin (1992): "Noisy Bayesian Updating and the Value of Information", mimeo., University of Pennsylvania and Oxford University, or the very different analysis in Michael Bacharach (1991): "Variable Universe Games", mimeo., Oxford University.

[53] Leonard Savage ([1954], 1972): *The Foundations of Statistics*, New York: Dover, pp. 8–10.

strategy which only needs the adherence to an algorithm for its implementation. Bayesian updating is one such algorithm and allied calculus[54].

This is just like in the case of forgetting, where the world sometimes reopens and a new regimentation has to close it again. Again, the implicit hierarchy built into the concept of extended strategy could separate the Ex Ante, formalized point of view of the Analyst from the *in medias res* condition of the decision maker.

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[54] The obvious connections to the so-called Harsányi doctrine, which does not allow the presence of non-regimented events in a problem, should be closely examined. See Harsányi, "Games with Incomplete Information Played by 'Bayesian' Players"