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On the Ranking of Bilateral Bargaining Opponents*

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Abstract: We fix the status quo (Q) and one of the bilateral bargaining agents to examine how shifting the opponent's ideal point (type) away from Q in a unidimensional space affects the Nash and Kalai-Smorodinsky bargaining solutions when opponents differ only in their ideal points. The results are similar for both solutions. As anticipated, the bargainer whose ideal point is farthest from Q prefers a opponent whose ideal is closest to her own. A similar intuitive ranking emerges for the player closest to Q when opponent's preferences exhibit increasing absolute risk aversion. However, if the opponent's preferences exhibit decreasing absolute risk aversion (DARA), the player closest to Q prefers a more extreme opponent. This unintuitive result arises for opponents with DARA preferences because the farther their ideal point is from Q, the easier they are to satisfy. 130

Keywords: Game Theory, Nash bargaining problems, bargaining solutions, rankings.

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1 Introduction

We examine bilateral bargaining situations where principals delegate bargaining to agents (e.g., voters to governments in international negotiations or to central and sub-national authorities in intergovernmental negotiations; workers to unions in wage negotiations). With heterogenous agents, the outcome of bargaining depends on the bargaining pair and on their preferences over the set of alternatives. Choosing a delegate requires ranking the agreements reached between different pairs. Though not concerned with delegation, we study bargaining outcomes in a unidimensional space.

In Gallego and Scoones (2005), voters elect one of three parties to represent them in intergovernmental negotiations. The elected State formateur engages in intergovernmental Nash (1950) bargaining over policy with its Federal counterpart. Agreements depend on the identity and risk "attitude" of the formateurs. Voters rank the anticipated agreements. If parties have quadratic utility functions, voters rank agreements and party's ideal policies identically. However, if one party is more risk averse than the other, agreements may not follow the party's ranking. We show that policies and party's ranking may differ when the agents' type and status quo matter. We extend and bring insights to their findings in a general bargaining framework.

A bilateral bargaining problem is defined by a set of feasible utility payoffs (S) including the disagreement point (D) that prevails if negotiations fail over a set of alternatives (X). Several unique solutions to the bargaining problem have been proposed in the literature[1]. In this paper, we concentrate on the commonly used Nash (NS) and Kalai-Smorodinsky (KS) solutions.

[1] For an excellent discussion on bargaining solutions see Thomson (1994).

We study the effect that varying the opponent's type has on the bargaining solutions. We assume agents have concave utility functions and opponent's type is defined by its ideal point but maintain the remaining characteristics of the opponent's utility function constant. Opponents' utilities are then just perfect translations of each other. We have then a family of problems indexed by pairs of negotiators. We also assume the existence of a status quo or fallback position completely outside the agents' control. This seems natural if the status quo is the outcome of previous negotiations arrived at by perhaps different agents. We cast our model in a complete information riskless framework to isolate the effect of types and the status quo on the different solutions.

We assume bargainers' utility functions are single-peaked[2] over a unidimensional space with L and R representing the two agents. To rank agreements between different pairs, we fix one agent and shift the opponent's ideal point by $\alpha \geqslant 0$ units, α identifies the opponent's type. We fix R (L) and shift L (R) to the right. To avoid repetition, some cases are reflections of those we consider, we assume the status quo Q is to the *left* of R's ideal point. Single-peakedness and agents' types bring out the role of the status quo (relative to the ideal points) in these solutions[3].

Varying opponent's type affects the components of the bargaining problem: the set of feasible payoffs, the concavity of the opponent's utility at a given point and the disagreement outcome. The set of alternatives over which the players agree to negotiate $A \subseteq X$ depends on the pair involved, may expand or shrink as α increases but remains anchored for one of the

[2]This is a common assumption in political economy models.

[3]Under single peakedness the status quo may constrain the set of feasible alternatives.

players. The change in A and the rightward translation of the opponent's utility affect the set of feasible payoffs S in ways that "add" and "subtract"[4] from it. S and S_α may not be contained within one another. If preferences do not exhibit constant risk-aversion throughout X, the translation changes the concavity of the opponent's utility function. As the opponent's ideal point shifts away from Q, her disagreement point and bargaining position worsen (Thomson 1987). Even under the assumption of perfect translations, the simultaneous interaction of these changes render ranking agreements difficult.

We find that the NS and KS may *not* increase in opponent's type. The ranking depends on whether the opponent is the one closest or farthest from Q. Given a pair of utility functions, the results are similar for both solution concepts. In the following summary, Q lies to the left of L's ideal point.

When we fix R and shift L, R prefers a less extreme opponent, i.e., whose ideal point is closer to her own. However, when we fix L and shift R, whether L prefers R or the shift to R_α depends on the absolute risk aversion (ARA) of R's utility function u_R (a property connected to its concavity). When u_R exhibits IARA (increasing ARA), L prefers R. When u_R has DARA (decreasing ARA), L prefers R_α. The "unintuitive" DARA result in a sense contrasts (the models differ) with Kihlstrom *et al.'s* (1981) finding that, for a fixed set of alternatives[5], an agent prefers a *more* risk averse opponent who is easier to satisfy under both solution concepts (Köbberling and Peters 2003).

In our model, fallback positions matter and affect bargaining outcomes.

[4] Though related, adding and substracting from S in our model differs from similar concepts developed by Myerson and Thomson (1980). Whereas in their model, the original equilibrium solution remains an element of the Pareto set; in ours it may not.

[5] In our model the set of feasible alternatives may change. Kannai (1977) first observed the results of Kihlstrom *et al.* (1981) and Roth (1977).

Since both solutions (NS and KS) move closer to R's ideal point when L_α's ideal point moves to the right, this result is in the spirit of Thomson (1987) where both solutions hurt the player whose fallback position worsens for a fixed S[6]. In contrast with Thomson, though R_α's disagreement point worsens in α, R_α's bargaining position improves if u_R has IARA.

In the applied bargaining literature, preferences are given certain mathematical representations. Since quadratic utilities exhibit IARA, the ranking of solutions and opponent's ideal points coincide[7]. By contrast, logarithmic utilities exhibit DARA and L ranks opponents opposite to their ideal points.

We find that the counter-intuitive ranking of opponents in the DARA case is due to the opponent with ideal point farther from Q being easier satisfy. More importantly, we show also that even under IARA preferences, a minor departure from IARA near the solution is enough to "upset" the ranking of agreements and the preferences over opponents' types (see Example 2 below).

2 The Model

We assume two bargainers have different preferences over a unidimensional set of alternatives X. Their unimodal or single-peaked utilities have different ideal points located somewhere in the interior of X. Let L and R represent the two agents. The given status quo is outside the agents' control[8].

L and R have utility functions, u_L and u_R respectively, defined on the

[6] Others also examine the effect of disagreement outcomes on bargaining solutions (see e.g., Wakker 1987, Bossert 1994 and Bossert and Peters 2002).

[7] This supports the assumption made in multiparty (> 2) models where policy is modelled as a convex combination of the ideal points of parties involved in negotiations.

[8] In political economy models, voters and parties have different ideal policies to capture differences in preferences over policies. The status quo may represent agreements reached in previous legislative, international trade, or intergovernmental policy negotiations.

compact interval $x \in X = [a, b]$. For arbitrary unimodal utility functions, let $\widehat{L} \equiv \arg\max u_L(x)$ and $\widehat{R} \equiv \arg\max u_R(x)$ be the ideal points of L and R with $\widehat{L} < \widehat{R}$. (See example with quadratic utilities in Figure 1.)

If L and R simultaneously demand the payoff pair (u, v) (i.e. L demands u and R demands v where u and v are real numbers) for which $u \leq u_L(x)$ and $v \leq u_R(x)$ for some $x \in X$, they both receive their demands. Otherwise (i.e. if, for all $x \in X$, either $u > u_L(x)$ or $v > u_R(x)$), they receive the default payoffs $(u_L(x_0), u_R(x_0))$ for some fixed status quo $x_0 \in X$. To avoid repetition, some cases are reflections of those we consider, we assume $x_0 < \widehat{R}$.

In our Nash (1950) bargaining problem, the *feasible set* S is

$$\{(u, v) \mid u_L(x_0) \leq u \leq u_L(x) \text{ and } u_R(x_0) \leq v \leq u_R(x) \text{ for some } x \in [a, b]\}.$$

Given concave utility functions, it can be shown S is a compact convex subset of $\mathbf{R}^2$ (as in Figure 2 below[9]). The *disagreement point* is $D = (u_L(x_0), u_R(x_0))$. Every other point in S weakly dominates D (i.e. neither agent is worse off at this other point and at least one is better off). The pair (S, D) constitute the *bargaining problem* (where S and D depend on x_0).

The *Pareto optimum set* $[PO(S)]$ is then given by

$$PO(S) = \{(u, v) \in S \mid \text{ either } u' < u \text{ or } v' < v \text{ for all other } (u', v') \in S\}$$

and the subset of alternatives $A \equiv \{x \in X \mid (u(x), v(x)) \in PO(S)\}$ is the *bargaining set*. Both the NS and KS solutions belong to $PO(S)$ and correspond to (possibly different) points $x^* \in A$.

[9]Figure 2(a) shows the feasible set S for the quadratic utility functions of Example 1. Figure 2(b) shows the feasible set when S is bound on the right by the parametric curve $\{(u_L(x), u_R(x)) | x \in [a, b], u_L(x) \geq u_L(x_0), u_R(x) \geq u_R(x_0)\}$. The NS and the KS are unaffected by which feasible set is taken. We take the latter form of S in our figures.

When the status quo lies between the ideal points of the two bargainers, $x_0 \in [\widehat{L}, \widehat{R}]$, the bargaining solution is $x^* = x_0$ regardless of the solution concept used, since the bargaining set is $A = \{x_0\}$.

From now on assume $x_0 < \widehat{L}$. For $x \in [x_0, \widehat{L}]$, L's and R's preferences are aligned, i.e., both prefer agreements to the right of x. Thus, agreements acceptable to both cannot lie to the left of $\widehat{L}$ (i.e. $A \subset [\widehat{L}, b]$). Let $\overline{x_0} > \widehat{L}$ be the agreement that keeps L indifferent to the status quo, i.e., $u_L(\overline{x_0}) = u_L(x_0)$[10]. Whether $\overline{x_0}$ constrains the bargaining set depends on the location of R's ideal point relative to $\overline{x_0}$. When $\overline{x_0} \in (\widehat{L}, \widehat{R}]$, since L rejects any proposal to the right of $\overline{x_0}$, $A = [\widehat{L}, \overline{x_0}]$. However, when $\widehat{R} < \overline{x_0}$, for $x \in [\widehat{R}, \overline{x_0}]$, both prefer agreements to the left of $\overline{x_0}$, $A = [\widehat{L}, \widehat{R}]$ in this case.

We now illustrate the NS and KS solutions using quadratic utilities.

Example 1 *(Figure 1) Suppose u_L and u_R are given by*

$$\begin{array}{l} u_L(x) = -(x-1)^2 + 1 \\ u_R(x) = -(x-2)^2 + 1 \end{array}. \tag{1}$$

Figure 1 about here

Clearly, the agents want to agree on a payoff pair $(u_L(x), u_R(x))$ for some x between the vertices of these parabolas (i.e. for some $x \in [1, 2]$) since every other point (u, v) in the feasible set is dominated by a point of this form. When $x_0 \in [1, 2]$, the agents agree on x_0, the only Pareto optimum point. When L wants to increase his payoff by demanding something higher than $u_L(x_0)$ (i.e., $x < x_0$), R opts for x_0 to avoid a decrease in her payoff.

When $x_0 < 1$ (or $x_0 > 2$), both increase their payoff if they agree on $(u_L(x), u_R(x))$ for some $x \in A = [1, 2]$. Figure 2 plots the boundary of the

[10] If there is no such $\overline{x_0}$ (i.e. $u_L(x) > u_L(x_0)$ for all $x > \widehat{L}$ that are in X), we take $\overline{x_0}$ as the right-hand endpoint of X (i.e. $\overline{x_0} = b$).

feasible set S when $x_0 = 0$ and $\overline{x_0} = 2$. Every point in the interior of S has another point on the boundary in the first quadrant that dominates it. This boundary represents the Pareto optimum set.

Figure 2 about here

The Nash solution (NS) in this example is the unique point on the Pareto set corresponding to the x^* that maximizes the Nash product (see Section 3)

$$\max_{x \in A} \; [u_L(x) - u_L(x_0)] \; [u_R(x) - u_R(x_0)].$$

A straight forward calculation with $x_0 = 0$ yields $x^* = (9 - \sqrt{17})/4 \cong 1.219\,2$.

The Kalai-Smorodinsky solution (KS) in this example is the unique point corresponding to x^* on the Pareto set for which (see Section 4)

$$\frac{u_L(x) - u_L(x_0)}{u_L(\widehat{L}) - u_L(x_0)} = \frac{u_R(x) - u_R(x_0)}{u_R(\widehat{R}) - u_R(x_0)},$$

i.e., x^* satisfies $4u_L(x) = u_R(x) + 3$ where $x \in A$. When $x_0 = 0$, $x^* = 4/3$.

In Figure 2, the solutions are the intersection of the relevant level curve (NS) or the ray from the disagreement point to the maximum utilities (KS) with the boundary of the feasible set S.

As is well known, an agent's risk attitude, associated with the concavity of his utility function for given ideal points, can affect each bargaining solution (Kannai 1977, Kihlstrom *et al.* 1981, Roth 1979). If in Example 1, u_L is fixed and u_R maintains her ideal point but increases in concavity (e.g. $u_R(x_0) = -c(x-2)^2 + 1$ for some $c > 1$), the NS moves closer to L's ideal point. We are not interested how changes in a player's attitude to risk for given ideal points effects the solutions; rather, we consider what happens when the ideal points change through horizontal translations of the utility functions for a given status quo. We state the problem in technical terms.

Problem: Suppose $u_{\cdot}(x)$ (where $u_{\cdot}$ is either u_L or u_R) is a concave utility function defined on $x \in X = [a, b]$ with its maximum in the interior[11]. Let x_0 be the fixed status quo for some $x_0 \in [a, \widehat{L}]$. If $u_{\cdot,\alpha}(x) = u_{\cdot}(x-\alpha)$ for some $\alpha > 0$ (i.e. a horizontal translation to the right by α units), describe how the solution depends on α. Does the bargaining solution $x(\alpha)$ of a horizontal shift of α units increase as α increases?

We rephrase in terms of ranking opponents. Suppose L can choose between two R opponents whose utility functions are perfect translations of each other with different ideal points. The best for L is that R's ideal point coincides with his own to agree on their common ideal point. However, if this is not possible, it seems intuitive that the agent whose ideal point is closest to L's should provide L with the better payoff for any bargaining solution. We expect the ranking of agreements and opponents' ideal points to coincide. Sections 3 and 4 rank the NS and KS solutions respectively.

3 The Nash Solution

In his seminal paper, Nash (1950) shows the existence of a unique solution (NS) to his bargaining problem satisfying the following axioms: Pareto optimality, symmetry (invariance under all permutations of agents), contraction independence (independence of irrelevant alternatives), and scale invariance

[11] In fact, we assume that the domain of these utility functions contains a larger interval than X so that their horizontal translations that we consider are still defined for all $x \in X$.

(invariance to positive affine transformations). We now define the Nash solution to our bargaining problem (S, D) in formal terms.

Let $NS(A)$ be the pair of equilibrium payoffs of the NS and let $x^* \in A$ be the agreement associated with $NS(A)$. The NS corresponds to the x that maximizes the Nash product $[NP(x)]$

$$NP(x) \equiv [u_L(x) - u_L(x_0)]\ [u_R(x) - u_R(x_0)].$$

Though our problem differs from Nash's (1950) (symmetry and scale invariance no longer hold and disagreement outcomes matter), there is a single NS (i.e. $NS(A)$ is well defined) since the feasible set S is compact and convex for any u_L and u_R. Assuming these utility functions are sufficiently smooth (i.e., continuous and differentiable), the NS can be found by setting the derivative of the Nash product to zero. $x^* \in A$ is the unique solution to

$$\frac{u_R'(x)}{u_L'(x)} = -\frac{u_R(x) - u_R(x_0)}{u_L(x) - u_L(x_0)}$$

and corresponds to the point where the level curve $NP(x) = C$ for some constant $C > 0$ meets the feasible set S at a point of tangency (Figure 2).

3.1 Ranking R's opponents

We fix R and vary L's ideal point, so that $u_{L,\alpha}(x) = u_L(x - \alpha)$ for some $\alpha > 0$. The $x(\alpha) = x$ associated with the NS in this case satisfies

$$\frac{u_R'(x)}{u_L'(x-\alpha)} = -\frac{u_R(x) - u_R(x_0)}{u_L(x-\alpha) - u_L(x_0-\alpha)}.$$

Given the *rightward* shift of $A_{L,\alpha}$, we anticipate R prefers opponents' with ideal points closer to her own. As the following result confirms, R orders the $x(\alpha)$ as she orders L's ideal points (i.e. $x(\alpha) > x(\widehat{\alpha})$ iff $\alpha > \widehat{\alpha} \geq 0$).

Theorem 1 *Suppose u_L is a unimodal utility function with* $\arg\max u_L = \widehat{L}$ *and the status quo is at $x_0 < \widehat{L}$. Let $u_{L,\alpha}$ be the horizontal translation of u_L to the right by α units (i.e. $u_{L,\alpha}(x) = u_L(x - \alpha)$). Suppose u_R is an increasing utility function for all $x \in [x_0, \widehat{L} + \alpha]$. Then the agreement $x(\alpha)$ associated with the NS between L_α and R satisfies $\frac{dx(\alpha)}{d\alpha} > 0$. Thus, if u_R is unimodal and $\widehat{R} > \widehat{L} + \alpha$, R prefers a less extreme opponent, i.e., an opponent whose ideal point is closer to her own.*

The analytic proof of this theorem (and all others) can be found in the Appendix. To gain a more conceptual understanding of this result, we study how translating u_L affects the components of the bargaining problem; namely,
(i) the set of feasible alternatives $A_{L,\alpha}$,
(ii) the disagreement outcomes $D_{L,\alpha}$ and
(iii) the set of feasible payoffs $S_{L,\alpha}$
and how these affect the Pareto set $PO_{L,\alpha}$ and the Nash product curves.

An increase in α causes a rightward shift in $A_{L,\alpha}$. Had nothing else changed, this can only benefit R, i.e., $\frac{dx(\alpha)}{d\alpha} \geqslant 0$ (Thomsom and Myerson 1980). Moreover, L_α's disagreement outcome worsens as α increases. Had nothing else changed, L_α's bargaining position worsens relative to R's (Thomson 1987). Finally, had nothing else changed, at high levels of $x \in A_{L,\alpha}$, the upward shift of L_α's utility relative to L's makes it easier for R to satisfy L_α rather than L (Thompson and Myerson 1980). Though each of these effects on its *own* does not guarantee that $\frac{dx(\alpha)}{d\alpha} > 0$, the following qualitative description of their separate effects on the feasible set and the Nash product curves does. We illustrate the effects for the quadratic utility functions of Example 1 but they are similar for all unimodal utilities.

We first study how increasing α affects the feasible set ignoring its effect on the Nash product curves. From Figure 3, the Pareto set $PO(S_{L,\alpha})$ dominates $PO(S)$ over some range that includes the agreement reached between L and R. Thus, an increase in α can only benefit R, i.e., $\frac{dx(\alpha)}{d\alpha} \geqslant 0$.

Figure 3 about here

Now, we analyze the effect of increasing α on the Nash product curves ignoring its effect on the feasible set. Given x_0 and the upward shift of L_α's utility, the trade-off in the Nash product curves $NP(A_{L,\alpha})$ moves in R's favor (the thin and solid equilibrium Nash level curves in Figure 3(a)). So that changes that affect the implementation of the Nash solution also favor R.

The simultaneous changes in the feasible set and in the Nash product curves *reinforce* one another to R's benefit making it easier to satisfy L_α rather than L at *high* $x \in A_{L,\alpha}$. R gets a higher payoff when facing L_α rather than L (solid versus dashed horizontal lines intersecting the u_R axis in Figure 3(a)) and prefers an opponent whose ideal point is closer to her own.

3.2 Ranking L's opponents

Fix L and vary R, so that $u_{R,\alpha}(x) = u_R(x - \alpha)$ for some $\alpha > 0$. The NS $x(\alpha) = x$ in this case satisfies

$$\frac{u_R'(x-\alpha)}{u_L'(x)} = -\frac{u_R(x-\alpha) - u_R(x_0-\alpha)}{u_L(x) - u_L(x_0)}.$$

An increase in α causes $A_{R,\alpha}$ to expand away from $\widehat{L}$ up to $\widehat{R}_\alpha = \overline{x_0}$. Since $x^* \in A_{R,\alpha}$, had nothing else changed, this suggests R_α can only benefit, i.e., $\frac{dx(\alpha)}{d\alpha} \geqslant 0$ (Thomsom and Myerson 1980). Moreover, R_α's disagreement outcome worsens as α increases. Had nothing else changed, R_α's bargaining position worsens relative to L's, suggesting $\frac{dx(\alpha)}{d\alpha} \leqslant 0$ (Thomson 1987). Finally, had nothing else changed, at low levels of $x \in A_{R,\alpha}$, the downward shift of R_α's utility relative to R's makes it easier for L to satisfy R_α rather than R, suggesting $\frac{dx(\alpha)}{d\alpha} \leqslant 0$ (Thompson and Myerson 1980). However, their simultaneous net effect on the bargaining outcome is unclear, i.e., $\frac{dx(\alpha)}{d\alpha} \gtreqless 0$, making the analysis more complex than in Section 3.1.

Figure 4 about here

Theorem 2 *Suppose u_L is a unimodal utility function with* $\arg\max u_L = \widehat{L}$ *and the status quo is $x_0 < \widehat{L}$. Suppose u_R is an increasing utility function for all $x \in [x_0, \widehat{L}]$. Let $u_{R,\alpha}$ be the horizontal translation of u_R to the right by α units (i.e. $u_{R,\alpha}(x) = u_R(x - \alpha)$). If u_R has increasing absolute risk aversion, the $x(\alpha)$ associated with the NS between L and R_α satisfies $\frac{dx(\alpha)}{d\alpha} > 0$. If u_R is unimodal, L prefers an opponent whose ideal is closer to his own. If u_R has decreasing absolute risk aversion, the $x(\alpha)$ associated with the NS between L and R_α satisfies $\frac{dx(\alpha)}{d\alpha} < 0$, L prefers a more extreme opponent.*

Our main contribution here is to show that the concavity of u_R (i.e., R's absolute risk aversion) affects the ranking of opponents. Theorem 2 shows that R_α's bargaining position vis à vis L, i.e., whether $\frac{dx(\alpha)}{d\alpha} \gtreqless 0$, depends on the simultaneous effect these factors have on R_α's utility. Thus, the concavity of R_α's utility, i.e., R_α's strength of preference (Peters 1992)[12], is a major

[12]Peters (1992) defines a *strength of preference relation* as follows. For a player facing four choices $\{a, b, c, d\} \in A$, let the binary relation $\succsim^*$ be a complete transitive binary relation on $A \times A$. If $(a, b) \succsim^* (c, d)$, then the player prefers the change from b to a to the change from d to c, i.e., for utility function u, $u(a) - u(b) > u(c) - u(d)$. Peters proves that for two players the utility function of the player with the weaker strength of preference relation is a concave transformation of the other player's utility.

determinant of $\frac{dx(\alpha)}{d\alpha}$. To show this we use the Arrow-Pratt coefficient of absolute risk aversion (ARA) as it measures changes in concavity that are invariant to positive linear transformations even in riskless environments such as ours (Mas-Colell *et al.* 1995). R_α's coefficient of ARA is

$$ARA_{R,\alpha} \equiv -\frac{u''_{R,\alpha}}{u'_{R,\alpha}}$$

Since we allow R_α's ideal point to shift as far as the upper bound of $X = [a, b]$, we examine situations where R's ARA uniformly increases or decreases over the relevant range. By definition, u_R exhibits increasing (decreasing) ARA (respectively IARA and DARA) when $\frac{d}{dx}\left(-\frac{u''_R}{u'_R}\right) > (<)0$.

The above theory can be applied to common utility functions. For instance, suppose u_L and u_R are quadratic as in Example 1. Since

$$\frac{d}{dx}\left(-\frac{u''}{u'}\right) = \frac{(u'')^2 - u'''u'}{(u')^2},$$

and $u''' = 0$, quadratic utility functions exhibit IARA and so both L and R prefer an opponent whose ideal point is closer to their own. On the other hand, the utility function $u_R(x) \equiv \ln x$ used in Figure 5 (where the set of alternatives X can be taken as $[a, 3]$ where a is positive and close to 0) satisfies

$$\frac{d}{dx}\left(-\frac{u''}{u'}\right) = -\frac{1}{x^2} < 0.$$

That is, $u_{R,\alpha}$ exhibits DARA and so $\frac{dx(\alpha)}{d\alpha} < 0$ by Theorem 2.

Figure 5 about here

DARA (Figure 5(b)). Since R_α is a perfect translation of R's utility, R's utility is a concave transformation of R_α's (Mas-Colell *et al.* 1995). Thus,

R_α's utility increases faster than R's for $x \in A_{R,\alpha}$, i.e., R_α has lower strength of preference, is less tough in negotiations, than R. L can more easily satisfy R_α than R at *low* levels of x since, to avoid the breakdown of negotiations, R_α accepts a bigger compromise than R. The trade-off of the feasible set and the Nash product curves improves in L's favor increasing L's payoff. L prefers an opponent whose ideal point is farther from its own.

IARA (Figure 4(a)). In this case, R_α is tougher in negotiations than R (i.e. R_α has stronger strength of preference). L gives up more when bargaining with R_α than with R, i.e., $\frac{dx(\alpha)}{d\alpha} > 0$ and prefers R to R_α.

As stated in the Introduction, only the DARA result contrasts in a sense (the models differ) with that of Kihlstrom *et al.* (1981) where for a given set of alternatives a player prefers a more risk averse opponent. In our framework, we no longer work in an ordinal domain but work instead in one where strength of preferences matters and interpersonal comparisons are possible, i.e., Nash's (1950) scale invariance axiom no longer holds (Thomson 1994).

Remark. Our use of ARA should not be interpreted as asserting players are risk averse[13] in our model. We use the Arrow-Pratt coefficient of ARA because it measures changes in the concavity of R_α's utility relative to R's. Changes in concavity are associated with R_α's strength of preferences relative to R's (Peters 1992) and not to changes in R_α's risk aversion since there is no risk in our model. The ARA coefficient reflects the responsiveness of R_α's utility to the combined effect of changes in the components of the bargaining problem and in the Nash product curves. It is unfortunate that the ARA

[13] Köbberling and Peters (2003) distinguish between utility and probabilitic risk aversion. Volij and Winter (2002) distinguish attitude towards wealth from attitude towards risk.

measure has been associated with a player's attitude towards risk even in riskless environments rather than the player's strength of preference. In our context, a player's strength of preference decreases as the concavity of player's utility function increases, i.e., as the coefficient of ARA increases.

The proofs of Theorems 1 and 2 in the Appendix rely heavily on the assumption that the coefficient of ARA does not change sign on the interval $[x_0, x(\alpha)]$ (i.e. the utility function is either uniformly IARA or uniformly DARA). We now ask what happens if ARA changes sign somewhere in the set of alternatives. For instance, from the Appendix, we see that $x(\alpha) = x$ decreases in α when R's utility function is translated if and only if

$$u'_L(x)\,[u'_R(x) - u'_R(x_0)] + [u_L(x) - u_L(x_0)]\,u''_R(x) > 0. \tag{2}$$

Thus, if utilities are risk neutral (or close to being risk neutral) near $x(\alpha)$, then $u''_R(x) = 0$ and (2) is satisfied. In particular, a small change in a utility function that exhibits IARA uniformly to one that is risk neutral near the NS will change the sign of $\frac{d}{d\alpha}(x(\alpha))$. This is illustrated in the following example where, for clarity, we make the interval of risk neutrality fairly large[14].

Example 2 *For mathematical convenience, we change $u_L(x)$ in Example* 1 *to $u_L(x) = -x^2$ and take $u_{R,\alpha}(x) = u_L(x-\alpha)$. Let the status quo $x_0 = -7$. If $\alpha = 3$, the NS is $x(3) = 1$. We make both players risk neutral near this NS by taking secants to the downward parabola for $u_L(x)$ in the intervals $x \in [-3,-1]$ and $x \in [0,2]$. Thus, redefine $u_L(x)$ as*

$$u_L(x) = \begin{cases} 4x+3 & \textit{if } -3 \le x \le -1 \\ -2x & \textit{if } 0 \le x \le 2 \\ -x^2 & \textit{otherwise} \end{cases}$$

[14]If u_R is risk neutral for all alternatives in X, then $x(\alpha)$ is constant (i.e. $\frac{d}{d\alpha}(x(\alpha)) = 0$).

and we keep $u_{R,\alpha}(x) = u_L(x-\alpha)$.

For α *close to* 3, *the NS is close to* 1. *Near* $x = 1$, *we maximize* $f_\alpha(x) \equiv [u_L(x) - u_L(-7)][u_R(x) - u_R(-7)]$. *For* $\alpha = 3$, $f_3(x) = [-2x+49][4(x-3)+3+100] = -8x^2+14x+4459$ *which has a maximum when* $-16x+14=0$. *The new NS is* $x = 7/8$. *As* α *varies slightly from* 3, *we maximize*

$$f_\alpha(x) = [-2x+49]\left[4(x-\alpha)+3+(-7-\alpha)^2\right] = (-2x+49)\left[4x+52+10\alpha+\alpha^2\right].$$

Now $\frac{df_\alpha(x)}{dx} = -16x+92-20\alpha-2\alpha^2 = 0$ *when* $x^*(\alpha) = \frac{46-10\alpha-\alpha^2}{8}$. *Clearly, as* α *increases (i.e.* $u_R(x)$ *is translated farther to the right),* $x(\alpha)$ *decreases (i.e. moves to the preferred solution of player L).*

Remark. The above discussion using $u_R(x) = \ln x$ as in Figure 5 illustrates Theorem 2 must be applied with care in the DARA case since the utility function that is always DARA must be increasing and so cannot be unimodal. From the proof in the Appendix, clearly we only need DARA uniformly on the interval $[x_0, x(\alpha)]$ to conclude $\frac{dx(\alpha)}{d\alpha} < 0$. Thus, we can modify $u_R(x)$ for $x > x(\alpha)$ so that it becomes unimodal outside this interval without affecting the statement of Theorem 2.

Care must also be taken when applying the above theory to functions that are not always differentiable. For instance, Euclidean preferences ($u(x) = -|x-\widehat{x}|$) common in political economy models, exhibit constant ARA everywhere except at $\widehat{x}$. If $u_R(x) = -\left|x - \widehat{R}\right|$, then the NS between L and R_α may equal $\widehat{R}_\alpha$ (i.e., $\frac{dx(\alpha)}{d\alpha} = \frac{d\widehat{R}_\alpha}{d\alpha} > 0$). On the other hand, for α sufficiently large, $x(\alpha) < \widehat{R}_\alpha$ and then $x(\alpha)$ is constant (i.e., $\frac{dx(\alpha)}{d\alpha} = 0$) as in the risk neutral situation[15].

[15] In Theorem 1 when L_α has Euclidean preferences, for small values of α, $x(\alpha) > \widehat{L}_\alpha$

4 The Kalai-Smorodinsky Solution

Kalai and Smorodinsky (1975) replace Nash's contraction independence axiom with individual monotonicity, requiring that in equilibrium a player benefits from any expansion of his feasible alternatives/payoffs. The KS selects the Pareto optimum point in S at which the utility gains for each agent from the disagreement point D are *proportional* to their maximum possible gains in utilities, i.e., proportional to the difference between the maximum utilities achievable within the feasible set S and their disagreement outcome. We now formally define the KS solution to our bargaining problem (S, D).

Let $KS(A)$ be the pair of equilibrium payoffs of the KS and let x^* be the agreement associated with $KS(A)$. Let $u_L(\widetilde{L}) = \max\{u_L(x) \mid (u_L(x), u_R(x)) \in S\}$ and $u_R(\widetilde{R}) = \max\{u_R(x) \mid (u_L(x), u_R(x)) \in S\}$ be respectively the *maximum* utilities L and R can achieve within S. Note that $\widetilde{L} \geq \widehat{L}$ with equality when $\widehat{L} \in A$. Similarly, $\widetilde{R} \leq \widehat{R}$. Then, $x^* \in A$ satisfies

$$\frac{u_R(x^*) - u_R(x_0)}{u_L(x^*) - u_L(x_0)} = \frac{u_R(\widetilde{R}) - u_R(x_0)}{u_L(\widetilde{L}) - u_L(x_0)}. \tag{3}$$

The components of the KS solution refer to the left and right hand side (LHS and RHS) of (3). The RHS, $s(x_0) = \frac{u_R(\widetilde{R}) - u_R(x_0)}{u_L(\widetilde{L}) - u_L(x_0)}$, represents the agreement on how utility gains from the disagreement point must be *shared*. The KS is the $x \in A$ that equates the maximum gain in utilities achievable within the feasible set S (LHS) to $s(x_0)$. Geometrically, $KS(A)$ corresponds to the point where the ray from the disagreement point $D = (u_L(x_0), u_R(x_0))$ to the maximum utilities $M = (u_L(\widetilde{L}), u_R(\widetilde{R}))$ intersects the feasible set S (Figure 2). $s(x_0)$ determines the shareable gains over the disagreement point (the

and then $x(\alpha)$ is constant. Once α is sufficiently large, $x(\alpha) = \widehat{L}_\alpha$ and increases in α.

DM ray) and the feasible set determines *where* along the ray the solution is located. The KS is then the *proportion* of the maximum possible gain in utilities to which the players agree (the point along the DM ray).

In Sections 4.1 and 4.2, Theorems 3 and 4 show that the theoretical NS results of Section 3 (i.e. Theorems 1 and 2) extend to the KS as well.

4.1 Ranking R's opponents

Fix R and vary L, $u_{L,\alpha}(x) = u_L(x - \alpha)$ for some $\alpha > 0$. Define $\overline{x_0(\alpha)}$ as the solution of $u_{L,\alpha}(x) = u_{L,\alpha}(x_0)$ for $x > \widehat{L}_\alpha \equiv \widehat{L} + \alpha$[16]. Although $\widehat{R}$ and x_0 remain fixed, the KS depends on whether $\widehat{R} < \overline{x_0(\alpha)}$ or $\widehat{R} > \overline{x_0(\alpha)}$ as this determines the bargaining set, i.e., whether $A_{L,\alpha} = [\widehat{L}_\alpha, \widehat{R}]$ or $A_{L,\alpha} = [\widehat{L}_\alpha, \overline{x_0(\alpha)}]$. When $\widehat{R} < \overline{x_0(\alpha)}$, $\{\widehat{L}_\alpha, \widehat{R}\} \subset A_{L,\alpha}$ and the KS $x(\alpha) = x$ satisfies

$$\frac{u_R(x) - u_R(x_0)}{u_L(x-\alpha) - u_L(x_0)} = \frac{u_R(\widehat{R}) - u_R(x_0)}{u_L(\widehat{L}) - u_L(x_0 - \alpha)}.$$

When $\widehat{R} > \overline{x_0(\alpha)}$, we know $\widehat{R} \notin A_{L,\alpha}$, so that the KS $x(\alpha) = x$ satisfies

$$\frac{u_R(x) - u_R(x_0)}{u_L(x-\alpha) - u_L(x_0)} = \frac{u_R(\overline{x_0(\alpha)}) - u_R(x_0)}{u_L(\widehat{L}) - u_L(x_0 - \alpha)}$$

since the maximum utility R can achieve in $S_{L,\alpha}$ is $u_R(\overline{x_0(\alpha)})$.

Regardless of whether $\widehat{R}$ is or is not in $A_{L,\alpha}$, given the rightward shift in $A_{L,\alpha}$ we anticipate that R's ranking of opponents monotonically increases in α, i.e. the KS, $x(\alpha)$, strictly increases in α.

Theorem 3 *Under the assumptions of Theorem 1, the agreement $x(\alpha)$ corresponding to the KS bargaining solution between L_α and R satisfies* $\frac{dx(\alpha)}{d\alpha} > 0$.

[16] Again, if no such $\overline{x_0(\alpha)}$ exists, we set $\overline{x_0(\alpha)} = b$.

While the statements of Theorem 1 and 3 are the same with NS replaced by KS, given differences in the two solution concepts the intuition leading to them also differs. The intuition for Theorem 3 is similar regardless of whether $\widehat{R} < \overline{x_0(\alpha)}$ or $\widehat{R} > \overline{x_0(\alpha)}$, so we only discuss that pertaining to $\widehat{R} < \overline{x_0(\alpha)}$. As in the NS, an increase in α affects the feasible set and the share of the maximum possible gains in utilities (the implementation of the KS solution).

We examine how increasing α affects the feasible set disregarding changes in the agreed upon share. We know that, for $\alpha > 0$, there is a range where $PO(S_{L,\alpha})$ lies above and to the right of $PO(S)$. The trade-off at the Pareto frontier shifts in R's favor when facing L_α rather than L (Section 3.1). As in the NS case, these changes favor R but are not sufficient to guarantee that $\frac{dx(\alpha)}{d\alpha} > 0$. However, there are other changes that do.

We now study how increasing α affects $s_{L,\alpha}(x_0)$, R and L_α's agreed upon share of the maximum gains in utilities, ignoring its effect on the feasible set. By assumption, the maximum utilities are such that $(u_{L,\alpha}(\widehat{L}_\alpha), u_R(\widehat{R})) = (u_L(\widehat{L}), u_R(\widehat{R}))$. Under complete information, R understands that, although L_α's fallback position worsens relative to L's, L_α's maximum gain in utility increases, $u_{L,\alpha}(\widehat{L}_\alpha) - u_{L,\alpha}(x_0) > u_L(\widehat{L}) - u_L(x_0)$. To avoid the breakdown of negotiations L_α agrees to R's demand of a bigger share. For $\alpha > 0$, the slope of $D_{L,\alpha}M$ ray flattens relative to that of DM, i.e., $s_{L,\alpha}(x_0) < s(x_0)$ (Figure 3(b)). Had nothing else changed, the redistribution of the share in R_α's favor assure R_α a greater payoff within the feasible set S.

The combination of changes to the feasible set and improvements in R's share leads to an *additional* effect that also favors R. Since L_α and R understand that over some range $PO(S_{L,\alpha})$ strictly dominates $PO(S)$, they agree to take a higher proportion of L_α's greater maximum utility gain (relative to L's). The KS then picks the point on $PO(S_{L,\alpha})$ that is proportional to L_α's now higher maximum gain in utility. In Figure 3(b), the flatter $D_{L,\alpha}M$ ray intersects the Pareto set at a point where $x(\alpha) > x^*$, i.e., $\frac{dx(\alpha)}{d\alpha} > 0$. Thus, R's payoff increases in α.

4.2 Ranking L's opponents

Fix L and vary R, so that $u_{R,\alpha}(x) = u_R(x-\alpha)$ for some $\alpha > 0$. We take into account that beyond a certain α the set of feasible alternatives changes. For α small enough such that $\widehat{R}_\alpha < \overline{x_0}$, we have that $u_L(\widehat{R}_\alpha) = u_L(\widehat{R}+\alpha) > u_L(\overline{x_0})$ and the bargaining set is $A_{R,\alpha} = [\widehat{L}, \widehat{R}_\alpha]$. The KS solution, $x(\alpha) = x$, satisfies

$$\frac{u_R(x-\alpha) - u_R(x_0-\alpha)}{u_L(x) - u_L(x_0)} = \frac{u_R(\widehat{R}) - u_R(x_0-\alpha)}{u_L(\widehat{L}) - u_L(x_0)}. \tag{4}$$

For large enough α, $\widehat{R}_\alpha > \overline{x_0}$, so that $u_L(\widehat{R}_\alpha) = u_L(\widehat{R}+\alpha) < u_L(\overline{x_0}) = u_L(x_0)$. The set of alternatives is $A_{R,\alpha} = [\widehat{L}, \overline{x_0}]$. The KS $x(\alpha) = x$ satisfies

$$\frac{u_R(x-\alpha) - u_R(x_0-\alpha)}{u_L(x) - u_L(x_0)} = \frac{u_R(\overline{x_0}) - u_R(x_0-\alpha)}{u_L(\widehat{L}) - u_L(x_0)}. \tag{5}$$

The relative location of $\widehat{R}_\alpha$ and $\overline{x_0}$ determines which equation, (4) or (5), is relevant for our analysis. Theorem 4 shows that the qualitative behavior of $\frac{dx(\alpha)}{d\alpha}$ is independent of their location. However, α affects the components of the bargaining problem and the share of the maximum possible gain in utilities (the implementation of the KS) through R_α's strength of preferences (ARA). As in the NS, the KS depends on R's ARA.

Theorem 4 *Suppose the assumptions of Theorem 2 hold. If u_R has IARA, the KS solution $x(\alpha)$ between L and R_α satisfies $\frac{dx(\alpha)}{d\alpha} > 0$. L prefers a less* *extreme opponent. If u_R has DARA, the KS solution $x(\alpha)$ between L and R_α satisfies $\frac{dx(\alpha)}{d\alpha} < 0$. L prefers a more extreme opponent.*

The statements of Theorems 2 and 4 are the same with NS replaced by KS. Since the intuition behind Theorem 4 is similar when $\widehat{R}_\alpha > \overline{x_0}$, we assume that $\widehat{R}_\alpha < \overline{x_0}$. Note that $A_{R,\alpha}$ expands as α increases until $\widehat{R}_\alpha = \overline{x_0}$.

We study how increasing α affects the agreed upon share $s_{R,\alpha}(x_0)$ ignoring changes in the feasible set. Under complete information, L understands that R_α's maximum gain in utility increases as R_α's fallback position worsens. L demands a bigger share as "compensation" for bargaining with a R_α rather than R. Thus, $s_{R,\alpha}(x_0) < s(x_0)$ (a steepening of the $D_{R,\alpha}M$ ray in Figures 4(b) and 5(c)). Within the feasible set S, L_α's bigger share ensures L_α's payoff increases relative to L's. This suggests $\frac{dx(\alpha)}{d\alpha} > 0$.

Now consider the effect of increasing α on the feasible set ignoring its effect on $s_{R,\alpha}(x_0)$. Within the relevant range, the downward shift in R_α's utility relative to R's shifts $PO(S_{R,\alpha})$ over some range in and below $PO(S)$. If we maintain the share constant at $s(x_0)$, the shrinkage of the Pareto set forces the agents to agree to a smaller proportion of the maximum gain in utilities, and the payoff of both L and R_α fall, so that $\frac{dx(\alpha)}{d\alpha} \gtreqless 0$.

The combined effect of changes to the agreed share and in the feasible set is unclear, i.e., $\frac{dx(\alpha)}{d\alpha} \gtreqless 0$. One of our major contributions is to show that the improvement in L's bargaining position depends again on R_α's strength of preference (toughness in negotiations), i.e., on u_R's ARA.

DARA (Figure 5(c))**.** Since some payoffs in S are no longer available to L and R_α, a redistribution of share in L's favor is not enough. Both players must also agree to take a smaller proportion of their maximum gains in utilities (i.e., a lower point along the $D_{R,\alpha}M$ ray). Moreover, under DARA, R's utility is a concave transformation of R_α's (Mas-Colell *et al.* 1995) implying R_α has lower strength of preferences than R. Since R_α is less tough in negotiations, the trade-off at the Pareto frontier moves in L's favor. Thus, the KS moves in L's favor, $\frac{dx(\alpha)}{d\alpha} < 0$ and L prefers R_α to R.

IARA (Figure 4(b)). When u_R exhibits IARA, R_α is tougher than R in negotiations so that L gives up more when bargaining with R_α rather than R and $\frac{dx(\alpha)}{d\alpha} > 0$. L then prefers R to R_α.

5 Conclusions

We examine the Nash and the Kalai-Smorodinsky solutions in a bilateral bargaining model when an agent faces an opponent who may be one of an infinite type when the status quo is outside the agents' control. To isolate the effect of types and the status quo in a model where the entire utility function matters, we make opponents identical in every respect except their ideal points. When the status quo lies between the players' ideal points, the degenerate bargaining problem leaves the status quo in place. Otherwise, the ranking of opponents depends on the agent doing the ranking. Surprisingly, the results are similar for both solution concepts.

Assuming (as in the text) $x_0 < \widehat{L}$, our model provides the following predictions. R prefers an opponent whose ideal is closest to her own since the simultaneous changes to the Pareto set and the implementation of the solution concepts benefit R as α increases. A similar ranking emerges for L when u_R exhibits IARA. For these two cases, the ranking seems intuitive. However, our main message is that, if u_R exhibits DARA, L prefers an opponent whose ideal point is farthest from his own. Though initially unintuitive, the result is rather intuitive. L prefers R_α to R since R_α is less tough in negotiation. In general, L prefers the less tough opponent. Finally, we find that any minor departure of u_R from IARA near the solution is enough to upset the ranking.

The Nash and the Kalai-Smorodinsky solutions to our problem do not exhibit sensitivity to risk. Peters (1992, p. 1018) states "in a model without lotteries, it is incorrect to explain theoretical results by referring to the risk attitude(s) of the decision makers." When we introduce types we leave behind the world of ordinal preferences and move into one where strength of preference matter. In a riskless bargaining environment, strength of preference is related to the concavity of the utility function and not to the player's aversion to risk and a major determinant of a player's bargaining power. The Arrow-Pratt coefficient of ARA measures changes in the concavity of utility function, one of the driving forces behind our results. As α increases, the concavity of R_α's utility function changes at a given $x \in A_{R,\alpha}$, affecting R_α's strength of preferences and her bargaining power. Our results emphasize that bargaining outcomes depend on the location of the players' ideal points relative to the status quo and on the players' toughness in negotiation.

Our results complement Gallego and Scoones' (2005) results on the ranking of intergovernmental Nash bargaining agreements between the Federal and the three State formateurs. When parties have quadratic utilities (IARA), they find as do we that agreements follow the ranking of the party's ideal policies. They find that if the right-most party is more risk averse than the others, agreements may not follow the party's ranking. We show this reverse ranking arises when the right-most party's utility exhibits DARA.

6 Appendix

Proof of Theorem 1. Let $A \subset X$ be the bargaining set described in Section 2[17]. The NS is the x that maximizes the Nash product

$$\max_{x \in A} \; NP(x) = [u_L(x) - u_L(x_0)] \; [u_R(x) - u_R(x_0)].$$

The NS between L_α and R is $x(\alpha) \in A_{L,\alpha}$ and the unique solution of

$$F(x, \alpha) = 0$$

where $F(x, \alpha) \equiv u'_{L,\alpha}(x) \; [u_R(x) - u_R(x_0)] + [u_{L,\alpha}(x) - u_{L,\alpha}(x_0)] \; u'_R(x)$. By implicit differentiation,

$$\begin{aligned} \frac{dx(\alpha)}{d\alpha} &= -\frac{\partial F(x,\alpha)/\partial \alpha}{\partial F(x,\alpha)/\partial x} \\ &= -\frac{-u''_L(x-\alpha)[u_R(x) - u_R(x_0)] + [-u'_L(x-\alpha) + u'_L(x_0-\alpha)]u'_R(x)}{u''_L(x-\alpha)[u_R(x) - u_R(x_0)] + 2u'_L(x-\alpha)u'_R(x) + [u_L(x-\alpha) - u_L(x_0-\alpha)]u''_R(x)}. \end{aligned}$$

Now, $\frac{dx(\alpha)}{d\alpha} > 0$ since the denominator is negative for $x \in A_{L,\alpha}$ and

$u''_L(x-\alpha)[u_R(x)-u_R(x_0)]+[u'_L(x-\alpha)-u'_L(x_0-\alpha)]u'_R(x) < 0.$ ∎

Proof of Theorem 2. For L and R_α, the NS $x(\alpha)$ is the solution of

$$G(x, \alpha) = 0$$

[17] That is, A is a compact interval with left endpoint $\widehat{L}$. Notice that

$$\frac{d}{dx}[(u_L(x) - u_L(x_0))\,(u_R(x) - u_R(x_0))] = u'_L(x)(u_R(x) - u_R(x_0)) + (u_L(x) - u_L(x_0))u'_R(x)$$

is positive at $x = \widehat{L}$ and negative at the right endpoint (since either $u_L(x) - u_L(x_0) = 0$ or $u'_R(x) = 0$ there). Since $[u_L(\widehat{L}) - u_L(x_0)][u_R(\widehat{L}) - u_R(x_0)] > 0$ and $\frac{d^2}{dx^2}[(u_L(x) - u_L(x_0))(u_R(x) - u_R(x_0))] < 0$, the NS corresponds to a unique x^* in the interior of A.

where $G(x, \alpha) \equiv u'_L(x)[u_{R,\alpha}(x) - u_{R,\alpha}(x_0)] + [u_L(x) - u_L(x_0)]u'_{R,\alpha}(x)$. Now

$$\begin{aligned}\frac{dx(\alpha)}{d\alpha} &= -\frac{\partial G(x,\alpha)/\partial\alpha}{\partial G(x,\alpha)/\partial x} \\ &= -\frac{u'_L(x)[-u'_R(x-\alpha) + u'_R(x_0-\alpha)] - [u_L(x) - u_L(x_0)]u''_R(x-\alpha)}{u''_L(x)[u_R(x-\alpha) - u_R(x_0-\alpha)] + [u_L(x) - u_L(x_0)]u''_R(x-\alpha) + 2u'_L(x)u'_R(x-\alpha)}.\end{aligned}$$

The denominator is again negative but now the numerator,

$$u'_L(x)[u'_R(x-\alpha) - u'_R(x_0-\alpha)] + [u_L(x) - u_L(x_0)]u''_R(x-\alpha) \gtreqless 0.$$

By substituting $G(x, \alpha) = 0$, this numerator is (with $x(\alpha) = x$)

$$\begin{aligned} & u'_L(x)[u'_R(x-\alpha) - u'_R(x_0-\alpha)] - u'_L(x)u''_R(x-\alpha)\frac{u_R(x-\alpha) - u_R(x_0-\alpha)}{u'_R(x_0-\alpha)} \\ = \;& u'_L(x)\left[u_R(x-\alpha) - u_R(x_0-\alpha)\right]\left[\frac{u'_R(x-\alpha) - u'_R(x_0-\alpha)}{u_R(x-\alpha) - u_R(x_0-\alpha)} - \frac{u''_R(x-\alpha)}{u'_R(x_0-\alpha)}\right] < 0\end{aligned}$$

if and only if

$$\frac{u''_R(x-\alpha)}{u'_R(x_0-\alpha)} < \frac{u'_R(x-\alpha) - u'_R(x_0-\alpha)}{u_R(x-\alpha) - u_R(x_0-\alpha)}.$$

Since $u'_R \neq 0$ for all $x \in A_{R,\alpha}$, by Cauchy's Mean Value Theorem, $\frac{u'_R(x-\alpha)-u'_R(x_0-\alpha)}{u_R(x-\alpha)-u_R(x_0-\alpha)} = \frac{u''_R(\xi-\alpha)}{u'_R(\xi-\alpha)}$ for some $x_0-\alpha < \xi-\alpha < x(\alpha)-\alpha$. If u_R is IARA (i.e. $\frac{d}{dx}\left(-\frac{u''_R}{u'_R}\right) > 0$), $\frac{u''_R(\xi-\alpha)}{u'_R(\xi-\alpha)} > \frac{u''_R(x-\alpha)}{u'_R(x_0-\alpha)}$ and so $\frac{dx(\alpha)}{d\alpha} > 0$. If u_R is DARA, then $\frac{dx(\alpha)}{d\alpha} < 0$[18]. ■

Proof of Theorem 3. We consider whether $\widehat{R} \in A_{L,\alpha}$ (case a) or $\widehat{R} \notin A_{L,\alpha}$ (case b). Typically, (b) holds for small α, (a) for large α.

(a) Here $u_{L,\alpha}(\widehat{R}) > u_{L,\alpha}(x_0)$. The KS $x(\alpha) = x$ between L_α and R satisfies

$$\frac{u_R(x) - u_R(x_0)}{u_{L,\alpha}(x) - u_{L,\alpha}(x_0)} = \frac{u_R(\widehat{R}) - u_R(x_0)}{u_L(\widehat{L}) - u_{L,\alpha}(x_0)}.$$

[18]Note that u_R only needs to be uniformly IARA (or uniformly DARA) over $[x_0, x(\alpha)]$ (and not over all of X) to apply this method of proof. In fact, if $\widehat{R}_\alpha \in [x_0, x(\alpha)]$, then u_R cannot be uniformly DARA since $\frac{d}{dx}\left(-\frac{u''_R}{u'_R}\right) \cong \left(-\frac{u''_R}{u'_R}\right)^2 > 0$ near $\widehat{R}_\alpha$.

Since $\{\widehat{L}_\alpha, \widehat{R}\} \in A_{L,\alpha}$, the KS solution $x(\alpha)$ also satisfies

$$F(x,\alpha) = 0$$

where $F(x,\alpha) \equiv \left[u_R(\widehat{R}) - u_R(x_0)\right][u_L(x-\alpha) - u_L(x_0-\alpha)]$
$-[u_R(x) - u_R(x_0)]\left[u_L(\widehat{L}) - u_L(x_0-\alpha)\right]$. By implicit differentiation,

$$\begin{aligned}\frac{dx(\alpha)}{d\alpha} &= -\frac{\partial F(x,\alpha)/\partial\alpha}{\partial F(x,\alpha)/\partial x} \\ &= -\frac{\left[u_R(\widehat{R}) - u_R(x_0)\right][-u_L'(x-\alpha) + u_L'(x_0-\alpha)] - [u_R(x) - u_R(x_0)][u_L'(x_0-\alpha)]}{\left[u_R(\widehat{R}) - u_R(x_0)\right][u_L'(x-\alpha) - u_L'(x_0-\alpha)] - u_R'(x)\left[u_L(\widehat{L}) - u_L(x_0-\alpha)\right]}.\end{aligned}$$

Since the denominator is negative, $\frac{dx(\alpha)}{d\alpha} > 0$ if and only if

$$\left[u_R(\widehat{R}) - u_R(x_0)\right][-u_L'(x-\alpha) + u_L'(x_0-\alpha)] - [u_R(x) - u_R(x_0)]\,u_L'(x_0-\alpha) > 0.$$

Substituting $F(x,\alpha) = 0$ and $u_L'(\widehat{L}) = 0$, this is true if and only if

$$\begin{aligned}&\frac{\left[u_L(\widehat{L}) - u_L(x_0-\alpha)\right]\left[u_R(\widehat{R}) - u_R(x_0)\right]}{u_L(x-\alpha) - u_L(x_0-\alpha)}[u_L'(x-\alpha) - u_L'(x_0-\alpha)] \\ < \; & [u_R(x) - u_R(x_0)]\left[u_L'(\widehat{L}) - u_L'(x_0-\alpha)\right]\end{aligned}$$

if and only if

$$\frac{u_L'(x-\alpha) - u_L'(x_0-\alpha)}{u_L(x-\alpha) - u_L(x_0-\alpha)} < \frac{u_L'(\widehat{L}) - u_L'(x_0-\alpha)}{u_L(\widehat{L}) - u_L(x_0-\alpha)}.$$

Let $h(x) \equiv \frac{u_L'(x) - u_L'(x_0)}{u_L(x) - u_L(x_0)}$. Then

$$h'(x) \equiv \frac{u_L''(x)[u_L(x) - u_L(x_0)] - [u_L'(x) - u_L'(x_0)]\,u_L'(x)}{[u_L(x) - u_L(x_0)]^2} < 0.$$

Since $x(\alpha) > \widehat{L}$, $h(x(\alpha)) < h(\widehat{L})$. The numerator is negative and so $\frac{dx(\alpha)}{d\alpha} > 0$.

(b) For α small, $u_{L,\alpha}(\widehat{R}) = u_L(\widehat{R}-\alpha) < u_L(x_0-\alpha) = u_{L,\alpha}(x_0)$. Let $\overline{x_0(\alpha)}$ be such that $u_{L,\alpha}(x_0) = u_{L,\alpha}(\overline{x_0(\alpha)})$ with $\widehat{L} < \overline{x_0(\alpha)}$. The KS satisfies

$$\frac{u_R(x)-u_R(x_0)}{u_L(x-\alpha)-u_L(x_0-\alpha)} = \frac{u_R(\overline{x_0(\alpha)})-u_R(x_0)}{u_L(\widehat{L})-u_L(x_0-\alpha)}.$$

Thus, the $x(\alpha)$ between L_α and R is the solution of

$$F(x,\alpha)=0$$

where $F(x,\alpha) \equiv \left[u_R(\overline{x_0(\alpha)})-u_R(x_0)\right]\left[u_L(x-\alpha)-u_L(x_0-\alpha)\right]$ $-\left[u_R(x)-u_R(x_0)\right]\left[u_L(\widehat{L})-u_L(x_0-\alpha)\right]$. By implicit differentiation,

$$\begin{aligned}
\frac{dx(\alpha)}{d\alpha} &= -\frac{\partial F(x,\alpha)/\partial\alpha}{\partial F(x,\alpha)/\partial x} \\
&= -\frac{\left[u_R(\overline{x_0(\alpha)})-u_R(x_0)\right]\left[-u_L'(x-\alpha)+u_L'(x_0-\alpha)\right]-\left[u_R(x)-u_R(x_0)\right]u_L'(x_0-\alpha)}{\left[u_R(\overline{x_0(\alpha)})-u_R(x_0)\right]\,u_L'(x-\alpha)-u_R'(x)\,\left[u_L(\widehat{L})-u_L(x_0-\alpha)\right]} \\
&\quad -\frac{u_R'(\overline{x_0(\alpha)})\frac{d\overline{x_0(\alpha)}}{d\alpha}\left[u_L(x-\alpha)-u_L(x_0-\alpha)\right]}{\left[u_R(\overline{x_0(\alpha)})-u_R(x_0)\right]\,u_L'(x-\alpha)-u_R'(x)\,\left[u_L(\widehat{L})-u_L(x_0-\alpha)\right]}.
\end{aligned}$$

Since the denominator is negative, $\frac{dx(\alpha)}{d\alpha} > 0$ if and only if

$$\begin{aligned}
&\left[u_R(\overline{x_0(\alpha)})-u_R(x_0)\right]\left[-u_L'(x-\alpha)+u_L'(x_0-\alpha)\right]-\left[u_R(x)-u_R(x_0)\right]u_L'(x_0-\alpha) \\
&+u_R'(\overline{x_0(\alpha)})\frac{d\overline{x_0(\alpha)}}{d\alpha}\left[u_L(x-\alpha)-u_L(x_0-\alpha)\right] > 0. \qquad \text{(A.1)}
\end{aligned}$$

Substituting $F(x,\alpha)=0$ and $u_L'(\widehat{L})=0$, we get the first term of (A.1) is

$$\begin{aligned}
&\left[u_R(\overline{x_0(\alpha)})-u_R(x_0)\right]\left[-u_L'(x-\alpha)+u_L'(x_0-\alpha)\right]-\left[u_R(x)-u_R(x_0)\right]u_L'(x_0-\alpha) \\
= &-\frac{\left[u_L(\widehat{L})-u_L(x_0-\alpha)\right]\left[u_R(x)-u_R(x_0)\right]}{u_L(x-\alpha)-u_L(x_0-\alpha)}\left[u_L'(x-\alpha)-u_L'(x_0-\alpha)\right] \\
&+\left[u_R(x)-u_R(x_0)\right]\left[u_L'(\widehat{L})-u_L'(x_0-\alpha)\right] \\
= &\left[u_R(x)-u_R(x_0)\right]\left[u_L(\widehat{L})-u_L(x_0-\alpha)\right]\left[\frac{u_L'(\widehat{L})-u_L'(x_0-\alpha)}{u_L(\widehat{L})-u_L(x_0-\alpha)}-\frac{u_L'(x-\alpha)-u_L'(x_0-\alpha)}{u_L(x-\alpha)-u_L(x_0-\alpha)}\right].
\end{aligned}$$

Let $h(z) \equiv \frac{u_L'(z-\alpha)-u_L'(x_0-\alpha)}{u_L(z-\alpha)-u_L(x_0-\alpha)}$. Then, for $z > \widehat{L}+\alpha$, we have

$$h'(z) \equiv \frac{u_L''(z-\alpha)\left[u_L(z-\alpha)-u_L(x_0-\alpha)\right]-\left[u_L'(z-\alpha)-u_L'(x_0-\alpha)\right]u_L'(z-\alpha)}{\left[u_L(z-\alpha)-u_L(x_0-\alpha)\right]^2} < 0.$$

Since $x(\alpha) > \widehat{L}+\alpha$, the first term is positive. The second term of (A.1)

$$u_R'(\overline{x_0(\alpha)})\frac{d\overline{x_0(\alpha)}}{d\alpha}\left[u_L(x-\alpha)-u_L(x_0-\alpha)\right]$$

is positive if $\frac{d\overline{x_0(\alpha)}}{d\alpha} > 0$. This follows since u_L is unimodal and concave. Alternatively, $\overline{x_0(\alpha)}$ satisfies $H(\overline{x_0(\alpha)},\alpha)=0$ where $H(x,\alpha) \equiv u_L(x_0-\alpha) - u_L(x-\alpha)$. Thus,

$$\frac{d\overline{x_0(\alpha)}}{d\alpha} = -\frac{\partial H(\overline{x_0(\alpha)},\alpha)/\partial\alpha}{\partial H(\overline{x_0(\alpha)},\alpha)/\partial\left(\overline{x_0(\alpha)}\right)} = \frac{u_L'(x_0-\alpha)-u_L'(\overline{x_0(\alpha)}-\alpha)}{-u_L'(\overline{x_0(\alpha)}-\alpha)} > 0. \quad \blacksquare$$

Proof of Theorem 4. As in Theorem 3, there are two cases for the KS solution, except that case (a) is now for small α and case (b) for large α.

(a) For α small, $u_L(\widehat{R}+\alpha) > u_L(x_0)$. The $x(\alpha)$ between L and R_α satisfies

$$G(x,\alpha) \equiv 0$$

where $G(x,\alpha) \equiv \left[u_R(\widehat{R})-u_R(x_0-\alpha)\right]\left[u_L(x)-u_L(x_0)\right]$
$-\left[u_R(x-\alpha)-u_R(x_0-\alpha)\right]\left[u_L(\widehat{L})-u_L(x_0)\right]$. By implicit differentiation,

$$\begin{aligned}\frac{dx(\alpha)}{d\alpha} &= -\frac{\partial G(x,\alpha)/\partial\alpha}{\partial G(x,\alpha)/\partial x} \qquad \text{(A.2)}\\ &= -\frac{u_R'(x_0-\alpha)\left[u_L(x)-u_L(x_0)\right]+\left[u_R'(x-\alpha)-u_R'(x_0-\alpha)\right]\left[u_L(\widehat{L})-u_L(x_0)\right]}{\left[u_R(\widehat{R})-u_R(x_0-\alpha)\right]u_L'(x)-u_R'(x-\alpha)\left[u_L(\widehat{L})-u_L(x_0)\right]}.\end{aligned}$$

The denominator is negative. On substituting $G(x,\alpha)=0$ and $u_R'(\widehat{R})=0$, the numerator of (A.2) becomes

$$\begin{aligned}
& -u_R'(x_0-\alpha)\frac{[u_R(x-\alpha)-u_R(x_0-\alpha)]\left[u_L(\widehat{L})-u_L(x_0)\right]}{u_R(\widehat{R})-u_R(x_0-\alpha)} \\
& -[u_R'(x-\alpha)-u_R'(x_0-\alpha)]\left[u_L(\widehat{L})-u_L(x_0)\right] \\
= \; & [u_R(x-\alpha)-u_R(x_0-\alpha)]\left[u_L(\widehat{L})-u_L(x_0)\right] \\
& \times\left[\frac{u_R'(\widehat{R})-u_R'(x_0-\alpha)}{u_R(\widehat{R})-u_R(x_0-\alpha)}-\frac{u_R'(x-\alpha)-u_R'(x_0-\alpha)}{u_R(x-\alpha)-u_R(x_0-\alpha)}\right].
\end{aligned}$$

Let $k(z)\equiv\frac{u_R'(z-\alpha)-u_R'(x_0-\alpha)}{u_R(z-\alpha)-u_R(x_0-\alpha)}$. Then

$$\begin{aligned}
k'(z) \; & \equiv \; \frac{u_R''(z-\alpha)\left[u_R(z-\alpha)-u_R(x_0-\alpha)\right]-\left[u_R'(z-\alpha)-u_R'(x_0-\alpha)\right]u_R'(z-\alpha)}{\left[u_R(z-\alpha)-u_R(x_0)\right]^2} \\
& = \left[\frac{u_R''(z-\alpha)}{u_R'(z-\alpha)}-\frac{u_R'(z-\alpha)-u_R'(x_0-\alpha)}{u_R(z-\alpha)-u_R(x_0)}\right]\frac{u_R'(z-\alpha)}{u_R(z-\alpha)-u_R(x_0)} \\
& = \left[\frac{u_R''(z-\alpha)}{u_R'(z-\alpha)}-\frac{u_R''(\xi-\alpha)}{u_R'(\xi-\alpha)}\right]\frac{u_R'(z-\alpha)}{u_R(z-\alpha)-u_R(x_0)}
\end{aligned}$$

for some $x_0-\alpha<\xi-\alpha<z-\alpha$. As in the proof of Theorem 2 above, $\frac{dx(\alpha)}{d\alpha}>0$ if u_R is IARA and $\frac{dx(\alpha)}{d\alpha}<0$ if u_R is DARA.

(b) For large α, $u_L(\widehat{R}+\alpha)<u_L(x_0)$. There is a unique $\widehat{L}<\overline{x_0}<\widehat{R}$ so that $u_L(x_0)=u_L(\overline{x_0})$. The $x(\alpha)=x$ between L and R_α satisfies

$$G(x,\alpha)\equiv 0$$

where $G(x,\alpha)\equiv[u_R(\overline{x_0}-\alpha)-u_R(x_0-\alpha)]\left[u_L(x)-u_L(x_0)\right]$
$-\left[u_R(x-\alpha)-u_R(x_0-\alpha)\right]\left[u_L(\widehat{L})-u_L(x_0)\right]$.

By implicit differentiation, $\frac{dx(\alpha)}{d\alpha}=-\frac{\partial G(x,\alpha)/\partial\alpha}{\partial G(x,\alpha)/\partial x}$ is given by

$$-\frac{\left[-u_R'(\overline{x_0}-\alpha)+u_R'(x_0-\alpha)\right]\left[u_L(x)-u_L(x_0)\right]+\left[u_R'(x-\alpha)-u_R'(x_0-\alpha)\right]\left[u_L(\widehat{L})-u_L(x_0)\right]}{\left[u_R(\overline{x_0}-\alpha)-u_R(x_0-\alpha)\right]u_L'(x)-u_R'(x-\alpha)\left[u_L(\widehat{L})-u_L(x_0)\right]}.$$

The denominator is negative. On substituting $G(x,\alpha)=0$, the numerator is

$$\begin{aligned}
& [u_R'(\overline{x_0}-\alpha)-u_R'(x_0-\alpha)]\,\frac{[u_R(x-\alpha)-u_R(x_0-\alpha)]\left[u_L(\widehat{L})-u_L(x_0)\right]}{u_R(\overline{x_0}-\alpha)-u_R(x_0-\alpha)} \\
& -[u_R'(x-\alpha)-u_R'(x_0-\alpha)]\left[u_L(\widehat{L})-u_L(x_0)\right] \\
= \; & [u_R(x-\alpha)-u_R(x_0-\alpha)]\left[u_L(\widehat{L})-u_L(x_0)\right] \\
& \times\left[\frac{u_R'(\overline{x_0}-\alpha)-u_R'(x_0-\alpha)}{u_R(\overline{x_0}-\alpha)-u_R(x_0-\alpha)}-\frac{u_R'(x-\alpha)-u_R'(x_0-\alpha)}{u_R(x-\alpha)-u_R(x_0-\alpha)}\right].
\end{aligned}$$

From part (a) where $k(z)\equiv\frac{u_R'(z-\alpha)-u_R'(x_0-\alpha)}{u_R(z-\alpha)-u_R(x_0-\alpha)}$, $k'(z)=\left[\frac{u_R''(z-\alpha)}{u_R'(z-\alpha)}-\frac{u_R''(\xi-\alpha)}{u_R'(\xi-\alpha)}\right]\frac{u_R'(z-\alpha)}{u_R(z-\alpha)-u_R(x_0)}$ for some $x_0-\alpha<\xi-\alpha<z-\alpha$. As in the proof of Theorem 2 above, $\frac{dx(\alpha)}{d\alpha}>0$ if u_R is IARA and $\frac{dx(\alpha)}{d\alpha}<0$ if u_R is DARA. ∎

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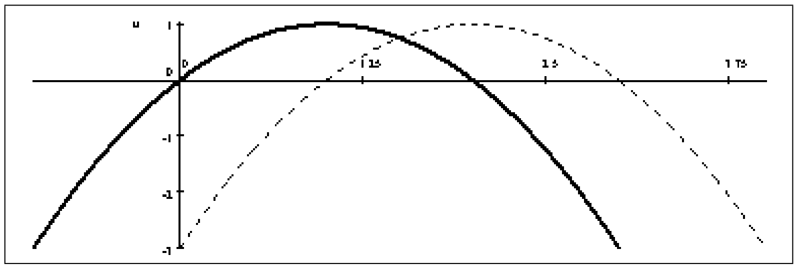
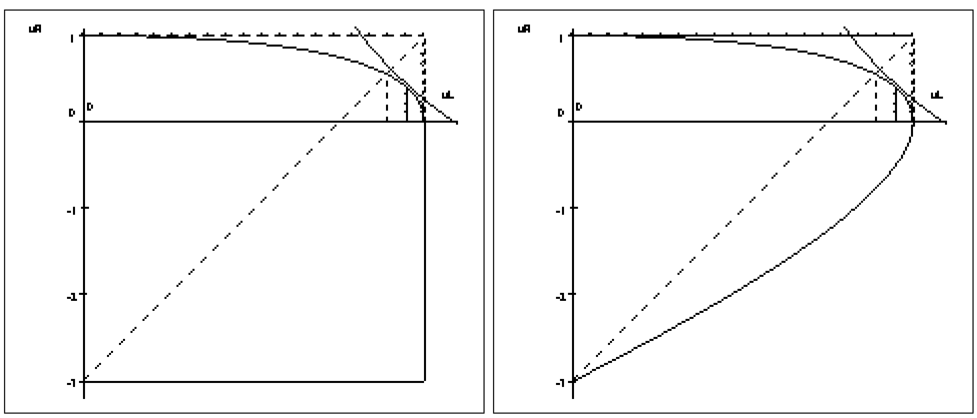
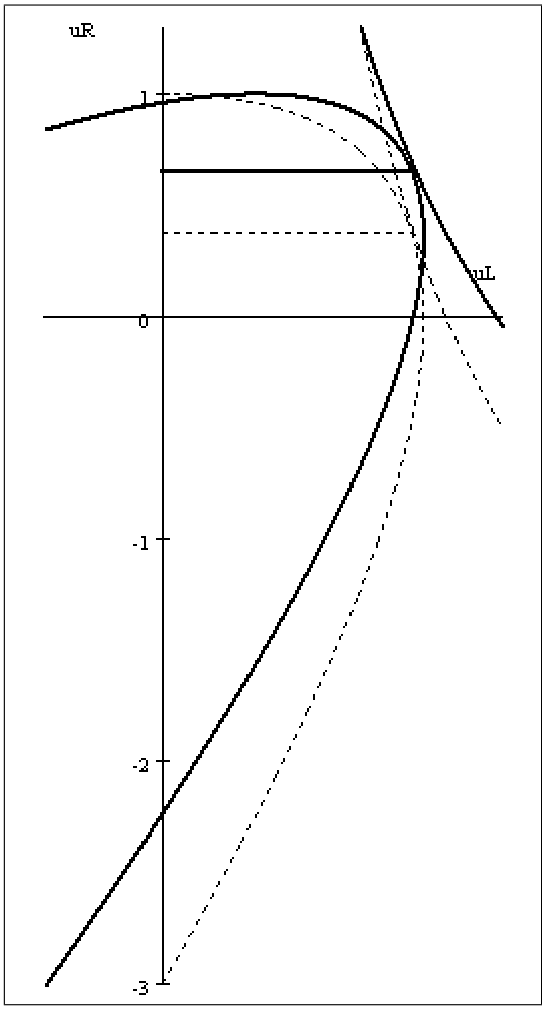


Figure 1. The utility functions of Example 1. The solid left parabola is $u_L(x)$ and the dashed right parabola is $u_R(x)$.

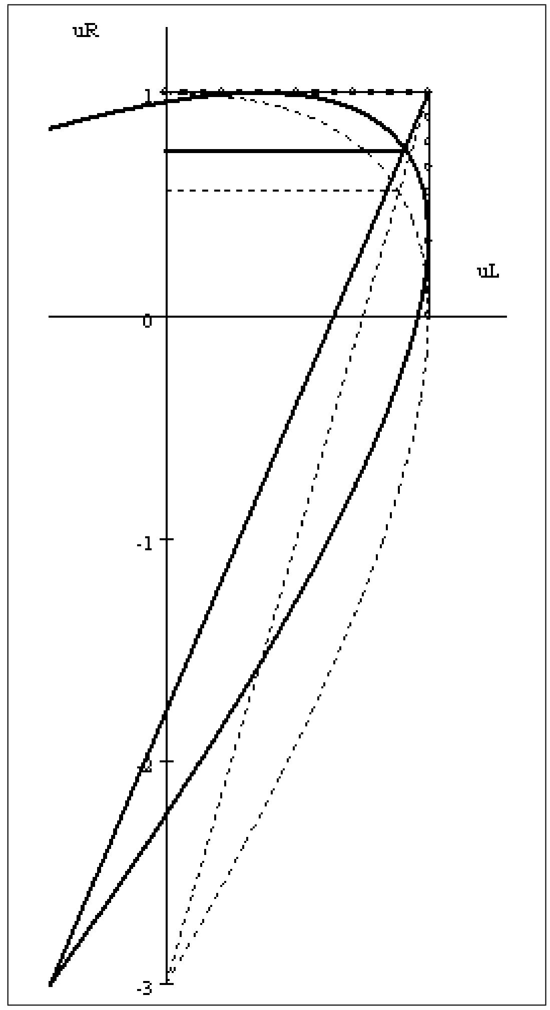


(a) (b)

Figure 2. The feasible set S for Example 1 with status quo $x_0 = 0$ is the region bounded by the vertical axis and the thick curve. The disagreement point is $D = (0, -3)$. Diagram (a) shows S according to the definition whereas (b) replaces the right-hand boundary with the parametric curve $\{(u_L(x), u_R(x)) \mid x \in [a, b], u_L(x) \geqslant u_L(x_0), u_R(x) \geqslant u_R(x_0)\}$. The Pareto optimum set is the portion of S that lies in the first quadrant. Also shown are the NS and KS for Example 1. The NS is the point of intersection of this thick curve with the solid curve (indicated by the vertical dotted line and corresponds to $x^* = 1.2192$) and KS is the point of intersection the thick curve with the solid line (indicated by the vertical dashed line and corresponds to $x^* = 4/3$).

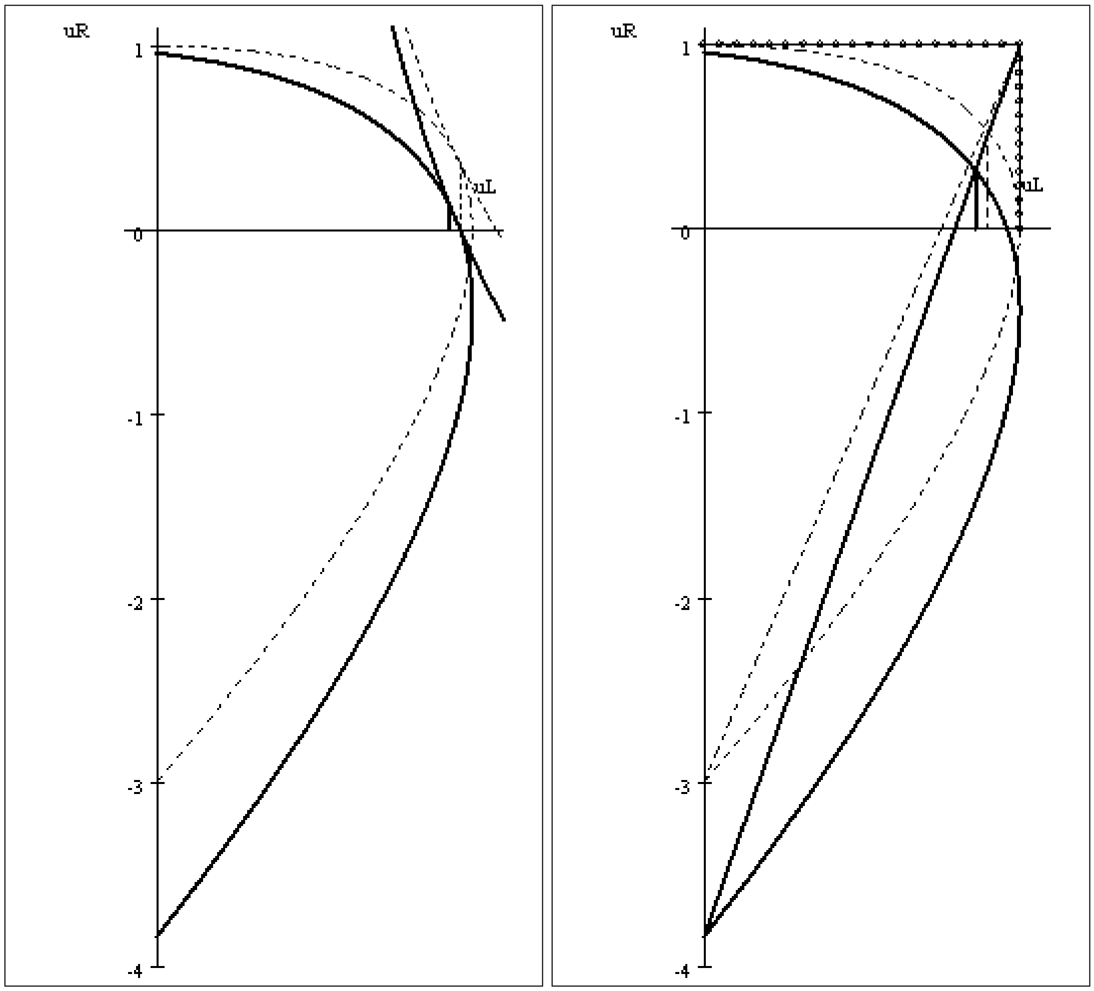


(b) NS solution



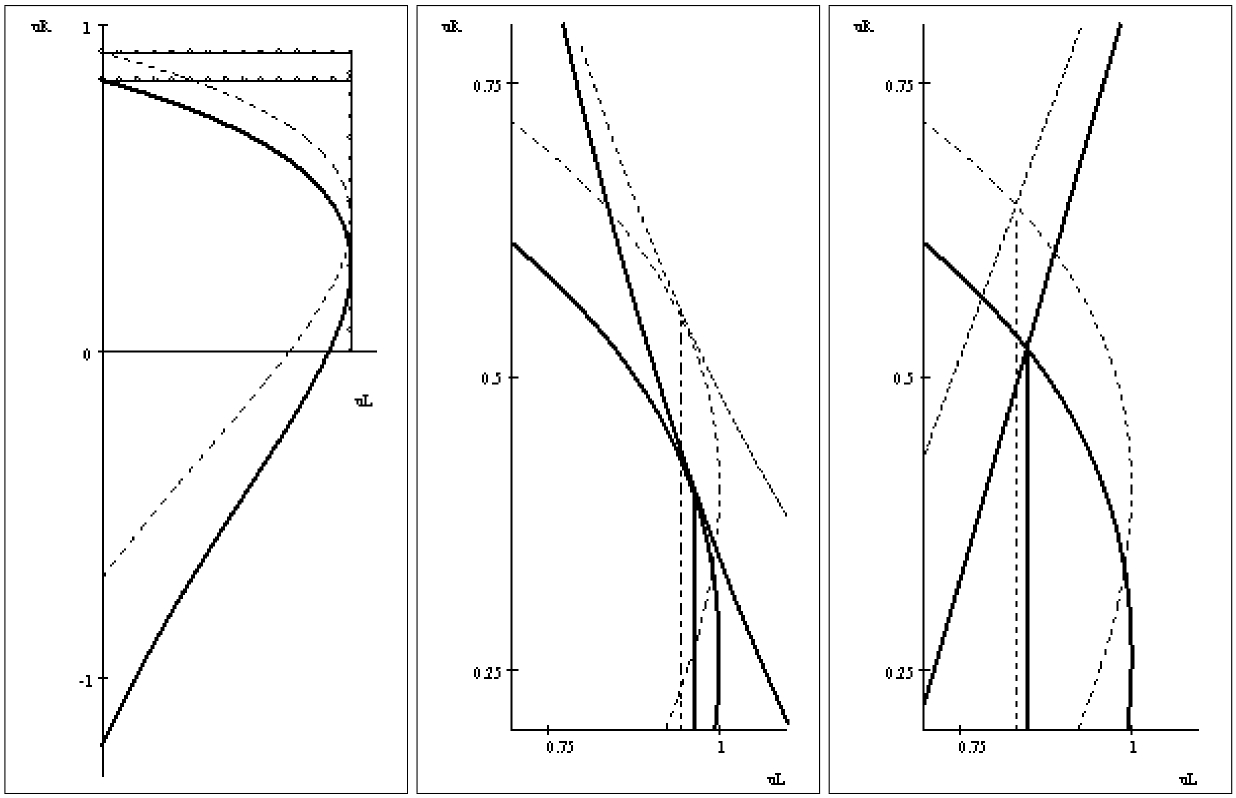
(c) KS solution

Figure 3. The boundary of S for the quadratic utility functions of Example 1 with $x_0 = 0$ when $\alpha = 0$ (the thick dashed curve corresponding to Figure 2) and for *L's utility function shifted by* $\alpha = 0.2$ (the thick solid curve). $D = (0, -3)$ when $\alpha = 0$ and $D_{L,\alpha} = (-0.44, -3)$ when $\alpha = 0.2$, so that $\overline{x_0(\alpha)} > \widehat{R}_\alpha$. The solutions ((a) NS and (b) KS) are the intersections of these thick curves with the dashed ($\alpha = 0$) and solid ($\alpha = 0.2$) *horizontal* lines. $x(\alpha)$ increases in α since $u_R(x)$ is an increasing function near $x(\alpha)$.



(a) NS solution (b) KS solution

Figure 4. The boundary of S for the quadratic utility functions of Example 1 with $x_0 = 0$ when $\alpha = 0$ (the thick dashed curve corresponding to Figure 2) and for *R's utility function shifted by* $\alpha = 0.2$ (the thick solid curve). $D = (0, -3)$ when $\alpha = 0$ and $D_{R,\alpha} = (0, -3.84)$ when $\alpha = 0.2$, so that $\overline{x_0} = 2 < \widehat{R}_\alpha$. The solutions ((a) NS and (b) KS) are the intersections of these thick curves with the dashed ($\alpha = 0$) and solid ($\alpha = 0.2$) *vertical* lines. $x(\alpha)$ decreases in α since $u_L(x)$ is a decreasing function near $x(\alpha)$.



(a) The feasible set
(b) NS solution
(c) KS solution

Figure 5. The feasible (thick dashed curve) set for the utility function $u_L(x) = -(x-1.5)^2 + 1$ and $u_R(x) = \ln x$ with status quo $x_0 = 0.5$. The disagreement point is $D = (0, -\ln 2)$ when $\alpha = 0$. The other feasible set (thick solid curve) is *for* u_R *shifted by* $\alpha = 0.2$. Now $\overline{x_0(\alpha)} < \widehat{R}_\alpha$ (in fact, $\widehat{R}_\alpha = \infty$ since u_R is increasing). How the NS and KS change with α is unclear in (a) so the relevant region is blown up in (b) and (c). In diagrams (b) and (c) the *vertical line* shifts right as α increases, i.e., $x(\alpha)$ decreases in α.