Michael J. Dueker

Michael J. Dueker is an economist at the Federal Reserve Bank of St. Louis. Richard I. Jako provided research assistance.

Hypothesis Testing with Near-Unit Roots: The Case of Long-Run Purchasing-Power Parity

THE HYPOTHESIS THAT the purchasing power of a given currency, like the dollar, will be equal across countries has strong appeal. If the hypothesis is true, then inflation and exchange rate movements will be such that a given currency will, over time, lose equal amounts of its purchasing power in all countries. The sequence of events by which deviations from purchasing-power parity would be eliminated can best be illustrated by example: If the dollar could purchase more goods in other countries than in the United States, then U.S. consumers would purchase more goods from abroad, which would raise the demand for foreign currencies relative to the dollar and lead to a depreciation of the dollar and eventual equalization of the dollar's purchasing power across countries. Despite the intuitive appeal of such arguments for long-run purchasing-power parity, statistical tests have been mixed. This paper argues that previous test results have conflicted because tests of purchasing-power parity have relatively low power under both the null hypothesis that it holds, and the null that it fails. Hence this paper contains tests of both null hypotheses and shows that frequently neither is rejected for monthly data from five major industrialized countries. This result serves as a caution against testing only one null hypothesis, finding that the null hypothesis cannot be rejected for a broad set of countries and concluding that there is robust evidence for or against the theory of long-run purchasing-power parity.

The theory of long-run purchasing-power parity (PPP) implies that a currency's purchasing power is equal across countries in long-run equilibrium, but does not specify how long deviations from this equilibrium can last. Large and persistent departures from PPP in the last 20 years, however, have cast doubt on the validity of PPP. As we will discuss later, there is a literature which tests whether long-run PPP holds, that is, whether departures from PPP are transitory. This article aims to reconcile some of the disparate results from previous studies by using a long-memory model, which can do more than classify deviations from PPP as temporary or permanent: it can provide specific measures of their persistence. Such measures are useful because large, persistent differences in a currency's purchasing power across countries can greatly affect trade flows and the allocation of resources.

Empirically, long-run PPP holds if the real exchange rate, which equals the nominal exchange rate multiplied by the ratio of the domestic and foreign price levels, is mean-reverting. This article will conform with the majority of the empirical PPP studies by using consumer price indexes

to calculate the real exchange rate. If price indexes measured the prices of identical baskets of goods across countries, absolute PPP would hold if $P^* = S \times P$, where P is the domestic price of the goods basket, P^* is the foreign price and S is the exchange rate in terms of units of foreign currency per unit of domestic currency. Because consumer price indexes do not measure the cost of identical baskets of goods across countries, however, relative PPP modifies the relationship to account for the ratio of the values of the two distinct baskets of goods: $P^* = kSP$, where k is the ratio of the value of the foreign basket to the domestic basket. The domestic country's real exchange rate with the foreign country is then $1/k$ and equals SP/P^*.[1]

The conventional approach to testing for long-run purchasing-power parity consists of testing for a unit root in the real exchange rate: Long-run PPP holds if the real exchange rate is mean-reverting but not if it has a unit root. Previous tests have shown little power to reject whichever of the two null hypotheses is employed. Tests whose null hypothesis is that the real exchange rate contains a unit root generally fail to reject, whereas tests whose null is that long-run PPP holds also often fail to reject. These disparate findings are reconciled, however, if there is long memory in the real exchange rate, which enables both acceptance and rejection of long-run PPP at conventional significance levels.[2]

This paper employs long-memory models to obtain estimates of the orders of integration of real exchange rates on a continuous scale. The advantage of estimating the order of integration on a continuous scale is that we can confirm that long-memory time series behavior in real exchange rates is a possible source of the discrepancies between previous tests of long-run

PPP. The finding of long memory in real exchange rates also allows us to judge whether the real exchange rate reverts to its mean within an economically meaningful time frame.

WHY PURCHASING-POWER PARITY MIGHT NOT HOLD

Before discussing statistical tests of PPP, it is worth repeating Engel's (1992) list of possible reasons for the empirical failure of PPP:

1. Barriers to trade such as tariffs and transportation costs.
2. Different consumption preferences across countries.
3. The presence of non-traded goods in price indexes.
4. Prices which are sticky in terms of the currency in which the good is consumed.

Barriers to trade, such as tariffs, are an obvious reason why the same goods do not sell at the same price throughout the world. Different consumption preferences, on the other hand, would lead consumers in each country to choose different baskets of goods. Because price indexes are constructed for baskets of goods designed to represent a particular country's consumption, an apparent failure of PPP could be due to different rates of price inflation across two country's distinctive baskets of consumption goods, rather than different prices for the same goods across countries. When included in price indexes, non-traded goods can also muddle the interpretation of the real exchange rate, because non-traded goods can be idiosyncratic and are thus not directly comparable across countries. Nevertheless consumer price indexes will be used in this paper, despite the presence of non-tradeables, because wholesale price indexes can fail to reflect the underlying rate of inflation accurately.[3] The fourth source of failure, sticky

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[1]Summers and Heston (1991) tabulate the costs of nearly-identical baskets of goods across countries, rather than use existing price indexes. They define the PPP nominal exchange rate to be P^*/P and use this implied exchange rate, rather than the market exchange rate, to make cross-country comparisons. The Summers and Heston measures of the price levels could take the place of commonly used consumer price indexes when testing long-run PPP, as could wholesale price indexes. The Summers and Heston data, however, are only available on an annual basis and include data extrapolated between five year data collection periods. The analysis in this paper will be limited to the use of consumer price indexes to facilitate comparison with previous studies.

[2]Long memory, as will be discussed later, means that the order of integration of a time series process is greater

than zero. If the order of integration is greater than 0.5, the series is not covariance stationary and if the order of integration is greater than one, the series does not have a mean. Long memory is not the same as an autoregressive near-unit root, because a series with a near-unit autoregressive root is still integrated of order zero, and is not considered a long-memory process.

[3]For example the wholesale price index for Japan suggests that Japan has had deflation on average from 1980 to the present, whereas the GDP deflator and CPI show moderate inflation.

prices, can best be explained by an example: Japanese autos sold in Japan and also exported to the United States have sticky prices in yen when sold in Japan and sticky prices in dollars when sold in the United States. Any exchange rate fluctuations would cause the yen (or dollar) price of the same model of car to differ across the Pacific. Thus autos might contribute to the failure of PPP in the true sense: the same good being sold at different prices (net of taxes) across countries.

PREVIOUS TESTS OF PURCHASING-POWER PARITY

Tests of PPP in the literature can be classified according to many criteria. In this brief review of a large literature, three features will receive attention: 20th century annual data vs. post-1973 monthly data; the use of consumer price indexes versus wholesale price indexes in the calculation of the real exchange rate; and whether or not price levels are assumed to be measured with error.[4] The aim of this review is to illustrate the lack of consensus that has emerged from studies of long-run PPP and identify which modeling choices might have influenced the outcomes of those tests.

Coughlin and Koedijk (1990) conduct unit-root tests on real exchange rates using post-1973 monthly data and consumer prices and find that the unit-root null hypothesis cannot be rejected. They also examine whether the real exchange rates are cointegrated with factors thought to determine the real exchange rate. Cheung and Lai (1993b) use post-1973 monthly data on consumer prices and allow for measurement error in prices. They use a Johansen (1991) likelihood ratio test for cointegrating vectors, in which long-run PPP is the null hypothesis, and generally fail to reject long-run PPP.[5] Thus the studies of Coughlin and Koedijk (1990) and Cheung and Lai (1993b) illustrate the importance of the null hypothesis in testing long-run PPP. Edison and Fisher (1991) and Fisher and Park (1991) represent another pair of studies that differ in the null hypothesis employed and the general conclusions about long-run PPP. Cheung and

Lai (1993a) study 20th century annual data and find evidence that the real exchange rate has long memory, but not a unit root for most countries studied. Pippenger (1993) uses wholesale price indexes and finds evidence that PPP between Switzerland and various countries appears to hold in the long run.

Overall a lack of consensus emerges from empirical tests of long-run PPP. The choice of null hypothesis is one source of discrepancy, and it appears that results from a long-memory model can reconcile the apparently conflicting results of several previous studies of long-run PPP, which differ primarily in their choice of null hypothesis. Consequently, tests of each null hypothesis will be highlighted in the estimation results below.

TESTING PURCHASING-POWER PARITY

Much research on whether PPP holds in the long run consists of performing a unit-root test on the real exchange rate. It is well known, however, that unit-root tests, especially those having a unit root as the null hypothesis, like Dickey-Fuller, have little power against long-memory alternatives. Such unit-root testing consists of classifying economic variables as either integrated of order zero [I(0)] or one [I(1)].[6] In contrast, we use a parametric long-memory model in which data series, like the real exchange rate, are modeled as integrated of order d, denoted I(d), where d does not have to be an integer. Any series that is integrated of order $d < 1$ will return eventually to its mean (or its deterministic trend), so shocks to the real exchange rate are not permanent if the real exchange rate is integrated of order $d < 1$.

This paper also provides information about the sources of PPP failure by examining the components of the real exchange rate. The ratio of the price levels may have a higher order of fractional integration than the nominal exchange rate, or vice versa. If r is the real exchange rate, s is the nominal exchange rate, p is the domestic price level, and p* is the foreign price level (all in natural logs), then $r = s + (p-p^*)$. If s is I(d1), (p-p*) is I(d2), then r will generally be in-

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tegrated of order max{d1,d2}. If r is I(b) where b < max{d1,d2}, then the nominal exchange rate and the price ratio (p-p*) are fractionally cointegrated, that is, they share the same stochastic trend to some extent.[7] The real exchange rate does not have a unit root if b < 1.

We can also examine the point estimates of d1 and d2 and see which component appears to have the strongest trend. This comparison answers critics of flexible exchange rates who argue that floating rates actually have caused the real exchange rate to be less stable than it would have been under fixed nominal exchange rates.[8] If d1 > d2, then shocks to the nominal exchange rate are more persistent than shocks to the relative price levels. The latter is somewhat curious, because proponents of the switch to a flexible exchange rate regime envisioned flexible exchange rates as sources of real exchange rate stability in a world in which countries might have persistent differences in inflation rates. Yet if shocks to the nominal exchange rate are more persistent than shocks to the relative price levels, then the nominal exchange rate has persistence above what is potentially useful in reducing the variance of the real exchange rate. In the empirical results that follow, the possibility of excess persistence in the nominal exchange rate will be examined.

BACKGROUND ON LONG-MEMORY MODELS

For many time series, autoregressive moving-average (ARMA) models serve as a parsimonious way to summarize the autocovariance structure of the data. One limitation of such models is that ARMA processes are integrated of order zero, and the autocovariances die off relatively quickly, even when a root in the autoregressive polynomial is near one. Thus ARMA models can be called short-memory models, because a shock affects the level of the series for a relatively short time.

Long-memory models, in contrast, are suitable for data that have slowly decaying coefficients

in their moving-average representations. The fractional ARMA model can serve as a long-memory model, yet it adds only one parameter to a standard ARMA model. To illustrate, we begin with a simple ARMA(1,1) applied to the first difference of a data series y, where L is the lag operator, ε is a mean-zero disturbance, ρ is the AR coefficient, and θ is the MA coefficient:

$$(1) \quad (1 - \rho L)(1 - L)y = (1 + \theta L)\varepsilon$$

A fractional ARMA model is simply an ARMA model applied to fractionally differenced data:

$$(2) \quad (1 - \rho L)(1 - L)^d y = (1 + \theta L)\varepsilon$$

The fractional differencing operator is evaluated by taking a Taylor series expansion around $L = 0$:[9]

$$(3) \quad (1 - L)^d = 1 - dL + \frac{d(d-1)L^2}{2} - \frac{d(d-1)(d-2)L^3}{3!} + \ldots$$

Two characteristics of fractionally integrated data are worth noting. First, a series that is integrated of order d (I(d)) with d < 1 reverts to its mean (or at least to its deterministic trend). Second, if d < .5, the series is covariance stationary. At first glance, it might seem counter-intuitive that a mean-reverting series can fail to be covariance stationary. With long memory, however, the departures from the mean can be sufficiently persistent that the variance of the series is infinite.

Furthermore, two commonly assumed data-generating processes fit within the subset of fractional integration: trend and difference stationarity. Fractional integration offers a bridge between the controversial assignment of a data series as either trend or difference stationary, so that questions about stationarity assumptions

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[7]For reasons outlined below, the restriction that the coefficients on s and (p-p*) equal one is relaxed, so that the order of integration of a general linear (cointegrating) combination of s, p and p* is assumed to be the order of integration of the real exchange rate. The concept of cointegration has been generalized [Granger (1986)] to include cases in which series have stochastic trends that only partially offset each other. This is called fractional cointegration. Originally, cointegration meant that a partic-

ular linear combination of two strongly trending series was I(0).

[8]For example, Aliber (1993) notes that "the U.S. dollar appreciated from 1979 to 1985 even though the U.S. inflation rate was higher than the inflation rates in Germany and Japan."

[9]The concept of fractional differencing was developed by Granger and Joyeux (1980) and Hosking (1981).

may be avoided. For example, if y is trend stationary, we might model y as

(4) $y_t = \mu t + \varepsilon_t$

In differences, equation (4) looks like

(5) $(1 - L)y_t = \mu + (1 - L)\varepsilon_t$

If y is difference stationary, then

(6) $(1 - L)y_t = \mu + \varepsilon_t$

Now suppose that y is fractionally integrated of order d. The first differences of y are then equal to

(7) $(1 - L)y_t = \tilde{\mu} + (1 - L)^{1-d}\varepsilon_t$

where $\tilde{\mu}$ is proportional to μ. Clearly trend stationarity (d = 0) and difference stationarity (d = 1) are bridged by fractional integration, which allows for intermediate cases. An intuitive way to understand why fractional integration is an intermediate case between trend and difference stationarity is to interpret each shock in a difference-stationary process to be a permanent shift away from any previous trend; a shock in a trend-stationary process is a short-lasting shift from the trend; a shock in a fractionally integrated process is a long-lasting shift from the trend. This paper uses estimates of the order of fractional integration to discriminate, if possible, between long-lasting shifts from the mean real exchange rate and permanent shifts in the real exchange rate (the unit-root case).

ESTIMATES OF LONG MEMORY IN REAL EXCHANGE RATES

The data used in this article consist of 234 monthly observations of the nominal exchange rates and consumer price indexes for the United States, Great Britain, Germany and Japan from June 1973 to November 1992.[10] Thus six bilateral relationships will be examined.

Fractional ARMA models are used to estimate the orders of integration of the nominal exchange rates, the ratios of the price levels and

the real exchange rates. The general form of the fractional ARMA model is

(8) $A(L)(1 - L)^d y_t = B(L)\varepsilon_t$

where y is I(d) and ε is assumed to have zero mean, no serial correlation and variance σ^2. A(L) is an autoregressive polynomial of order p and B(L) is a moving-average polynomial of order q:

(9) $A(L) = 1 - \rho_1 L - \rho_2 L^2 - \ldots - \rho_p L^p$

(10) $B(L) = 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q$

Estimation was carried out using the Fox and Taqqu (1986) frequency-domain estimator of fractional ARMA models. The estimator is based on an approximation to the likelihood. Dahlhaus (1988, 1989) has analyzed the Fox and Taqqu estimator and has shown that it shares the same asymptotic efficiency as exact maximum-likelihood estimation. Further details regarding the estimator appear in the appendix.

Before presenting estimation results, we must discuss how the real exchange rate was calculated. Any mismeasurement of the price levels can lead to spurious changes in the mean of the real exchange rate and bias tests toward rejection of long-run PPP. To minimize the possibility of spurious rejections of long-run PPP, the real exchange rate was calculated by estimating a fractionally cointegrating relationship between the nominal exchange rate (s), the domestic price level (p) and the foreign price level (p*):[11]

(11) $s_t = \alpha_0 - \alpha_1 p_t + \alpha_2 p^*_t + \varepsilon_t$

The residuals from equation (11) were then treated as the real exchange rate for unit-root testing with the fractional ARMA model. Cheung and Lai (1993b) and Pippenger (1993) also estimate a general cointegrating relationship, rather than impose $\alpha_1 = \alpha_2 = 1$. They both argue that, because of measurement error in price indexes and unequal weights attached to the same good in different indexes, it is undesirable to impose unit coefficients on the price indexes when studying whether the real exchange rate is mean-reverting. The Phillips and Hansen (1990)

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[10]Koedijk and Schotman (1989) find that the real exchange rates between 15 industrialized countries are fairly well spanned by the real exchange rates between the United States, Japan, Germany and Great Britain.

[11]Cheung and Lai (1993a) discuss the asymptotic theory behind estimating regressions where the residuals are fractionally integrated.

Table 1
Estimated Orders of Integration

Countries	Real exchange rate (b)	t-statistic for 1−b	Nominal exchange rate (d1)	Price ratio (d2)
United States– United Kingdom	1.06 (.039)	−1.54	1.12 (.031)	1.60 (.043)
United States– Germany	.903 (.067)	1.45	1.17 (.028)	1.21 (.081)
United States– Japan	1.24 (.067)	−3.58	1.22 (.070)	1.62 (.080)
United Kingdom– Germany	.872 (.047)	2.72	1.01 (.052)	1.57 (.079)
United Kingdom– Japan	1.19 (.034)	−5.59	1.15 (.039)	1.12 (.039)
Germany– Japan	1.13 (.207)	−.628	1.19 (.142)	1.50 (.091)

NOTE: Standard errors are in parentheses.

method of estimating cointegrating relationships was used for equation (11). This method accounts for simultaneity in the determination of left- and right-hand side variables. Cheung and Lai (1993b) suggest using estimation procedures that take into account interactions between left- and right-hand side variables. In fact, the Phillips-Hansen estimates of α_1 and α_2 indicate that they should not be restricted to equal one. For example, for the nominal exchange rate between Britain and the United States, the estimates of α_1 and α_2 are 1.45 and 1.22, respectively.

The next point of focus is the null hypothesis to be tested: PPP holds as the null hypothesis if the hypothesis that $b < 1$, where b is the order of integration of the real exchange rate, is not rejected; the alternative null hypothesis that PPP fails is not rejected if the null that $b \geq 1$ is not rejected. Using the fractional ARMA model, it is easy to test both null hypotheses and show how the results depend on the choice of the null.

Table 1 contains the main results on the estimated orders of integration of the relevant series. Simple t-tests can be used to test for long-run PPP by dividing one minus the estimated order of integration of the real exchange rate by its standard error. Doing this, we see that the null hypothesis that $b < 1$, where b is

the order of integration of the real exchange rate, is rejected for only two of the six pairs: United States/Japan and Britain/Japan. These are the significantly negative t-statistics in table 1. Thus long-run PPP is not rejected as a null hypothesis in four of six cases. If we reverse the null, however, we can reject the null that $b \geq 1$ for only one pair: Britain/Germany.[12] This is the significantly positive t-statistic in table 1. The results for United States/Germany are borderline with a t-statistic of 1.45, but this is not significant at the usual 5 percent level of significance in a one-tailed t-test, where the critical value is 1.658.

Overall, the orders of integration of real exchange rates are often close enough to one that neither null hypothesis is rejected. This explains some discrepancies between past tests of long-run PPP. Coughlin and Koedijk (1990) used Dickey-Fuller unit-root tests on real exchange rates and could not reject the null that long-run PPP fails to hold. Cheung and Lai (1993b), on the other hand, tested the null that long-run PPP holds and did not find many rejections of long-run PPP. The results from the long-memory model reconcile these findings.

The estimates from the long-memory models do more than give unit root tests, however, by

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[12]Orders of integration greater than one for data in logs imply that the growth rates of the series have long memory.

providing an estimate of the order of integration of the real exchange rate on a continuous scale. Whichever null is used, one result is clear: Even if long-run PPP holds, it is very slow in developing. Assuming that the order of integration of the real exchange rate is 0.9, 73 percent of a shock is still present after 12 months; 68 percent after 24 months; 65 percent after 36 months; and 63 percent after 48 months. As a practical matter, it seems fair to conclude that PPP does not hold within a time horizon that is economically relevant. Uncovering this type of information is the chief advantage of estimating the order of integration on a continuous scale. With other unit-root testing methods, we are forced to view a series as being either I(0) or I(1). Such a polar characterization may not provide practical information about the persistence of the shocks.

Table 1 also provides information about the persistence of shocks to the nominal exchange rate relative to shocks to the ratio of the price levels. In five of six cases, point estimates suggest that the nominal exchange rate has a lower order of integration than the price ratio.[13] For Britain/Japan the point estimate of the order of integration is 1.15 for the nominal exchange rate vs. 1.12 for the price ratio, but this difference does not appear to be statistically significant.[14] Thus the conjecture that nominal exchange rates in the post-Bretton Woods era have shown excess persistence appears to be false. In general, greater persistence in the nominal rate would be needed to offset the persistence in the price ratios for the real exchange rate to be rendered I(0). This is because no I(0) linear combination can exist, for example, between a series that is I(.8) and one that is I(.2). The series that is I(.2) does not have enough of a trend with which to offset the relatively strong trend in the variable that is I(.8).

Another finding from table 1 is that inflation differentials are fractionally cointegrated in some cases. For example, the estimates indicate that $(p_{US} - p_{JPN})$ is I(1.62) and $(p_{US} - p_{UK})$ is I(1.60), but the difference $(p_{UK} - p_{JPN})$ is only I(1.12). This means that the inflation differentials between the United States and Britain and between the

United States and Japan appear much more persistent than the inflation differential between Britain and Japan. In other words, inflation rates in Britain and Japan come closer to sharing a common trend with each other than with inflation in the United States.

Tables 2 through 4 report the fractional ARMA parameter estimates fully only for the bilateral relationships for the United States for the sake of brevity. The key result in these tables is that in fractional ARMA models the fractional differencing parameter can capture the long-run behavior of the data, freeing AR parameters to match the short-run dynamics. If, on the other hand, an ARMA model instead of a fractional ARMA model were fit to the data, the autoregressive polynomial would be forced to have a near-unit root.

In table 2 several AR parameters are negative, and the largest equals 0.53 in the fractional ARMA model of the ratio of the price levels between the United States and Britain. In table 3 both estimated AR parameters for the nominal exchange rate between the United States and Germany are negative, implying that all positive dependence in the exchange rate beyond the first lag is due to the large positive value of the fractional-differencing parameter. The largest root in an AR polynomial in table 3 is 0.37, which is far from the unit circle, in the real exchange rate between the United States and Germany. Estimates of the model of the real exchange rate between the United States and Japan, found in table 4, also show two negative AR coefficients. In fact all of the AR polynomials in table 4 have roots with real parts that are very far from the unit circle. They are $-.05$, $-.03$, and $.08$, respectively, for the real exchange rate, the nominal exchange rate and the price ratio. With the inclusion of the fractional differencing parameter, the AR parameters can take values which allow the fractional ARMA model to capture both long-run dependence and short-run dynamics in the data. The shaded insert and figures 1 through 3 provide a visual check of the match between the covariance structure of the data and that implied by the estimated fractional ARMA model.

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[13]An order of integration above one for variables in logs means that the growth rates display long memory. In the case of the price ratio, the corresponding growth rate is the inflation differential across the two countries. For the nominal exchange rate, it is the rate of exchange rate appreciation.

[14]A formal test would require joint estimates of the two fractional ARMA models, however.

Table 2
Fractional ARMA Models: United States-United Kingdom

	Real Exchange Rate		
	Parameter	Parameter value	Standard error
Order of integration	b	1.06	.039
AR	ρ_1	−.228	.096
	ρ_2	.012	.063
MA	θ_1	.534	.084
	Nominal Exchange Rate		
Order of integration	d_1	1.12	.031
AR	ρ_1	−.161	.100
	ρ_2	.002	.045
MA	θ_1	.452	.090
	Ratio of Price Levels		
Order of integration	d_2	1.60	.043
AR	ρ_1	.529	.051
	ρ_2	−.089	.043
MA	θ_1	−.809	.027

Table 3
Fractional ARMA Models: United States-Germany

	Real Exchange Rate		
	Parameter	Parameter value	Standard error
Order of integration	b	.903	.067
AR	ρ_1	.508	.132
	ρ_2	−.050	.060
MA	θ_1	−.176	.140
	Nominal Exchange Rate		
Order of integration	d_1	1.17	.028
AR	ρ_1	−.332	.099
	ρ_2	−.017	.039
MA	θ_1	.463	.105
	Ratio of Price Levels		
Order of integration	d_2	1.21	.081
AR	ρ_1	.326	.162
	ρ_2	.044	.071
MA	θ_1	−.127	.134

Table 4
Fractional ARMA Models: United States–Japan

	Parameter	Parameter value	Standard error
Real Exchange Rate			
Order of integration	b	1.24	.067
AR	ρ_1	−.105	.126
	ρ_2	−.191	.066
MA	θ_1	−.224	.172
Nominal Exchange Rate			
Order of integration	d_1	1.22	.070
AR	ρ_1	−.065	.113
	ρ_2	−.174	.073
MA	θ_1	.226	.150
Ratio of Price Levels			
Order of integration	d_2	1.62	.080
AR	ρ_1	.159	.108
	ρ_2	−.288	.058
MA	θ_1	−.749	.019

A Visual Check of the Results

Once a fractional ARMA model has been estimated, a visual diagnostic check of the adequacy of the specification and estimates can be obtained by looking at a plot of the periodogram of the data alongside a plot of the spectral density implied by the specified fractional ARMA model. If the model fits the data well, the two graphs will have the same general turning points. The area under the spectral density equals the variance. Thus the height of the spectral density at any point indicates how much of the variance is due to cycles of that frequency. In this way, the spectral density summarizes the autocovariance structure. Figures 1 through 3 provide a look at the three data series for the United States/Germany: the real exchange rate, the nominal exchange rate and the price ratio. The figures show that the estimated fractional ARMA models roughly envelope the smoothed periodograms. Figure 1 shows why the outcome of the unit root test on the U.S./Germany real exchange rate is borderline: It is unclear whether the upturn in the smoothed periodogram at very low frequencies is significant or whether the point estimate of a negative order of integration for the differenced real exchange rate is correct.[1] One other thing to note is that without ARMA parameters the model would be able to fit only series that have globally concave (or globally convex) spectral densities.[2] With ARMA parameters, however, figure 1 illustrates that a fractional ARMA can generate turning points in the spectral density. The best-fitting globally concave spectral density would obviously provide a much less satisfactory fit of the U.S./Germany real exchange rate than the one with turning points shown in figure 1.

[1] In theory the distinction is clear: A series with a negative order of fractional integration will have a spectral density value of zero at frequency zero; a series with a positive order of fractional integration will have a spectral density value of infinity at frequency zero. For figures 2 and 3, it is clear from the periodogram that the series have positive orders of integration. This is not clear for the real exchange rate in figure 1.

[2] Without ARMA parameters, the fractional ARMA model is called the fractional noise model. Its spectral density is globally concave if the fractional differencing parameter from equation (2) is negative; it is globally convex if the fractional differencing parameter is positive.

Figure 1

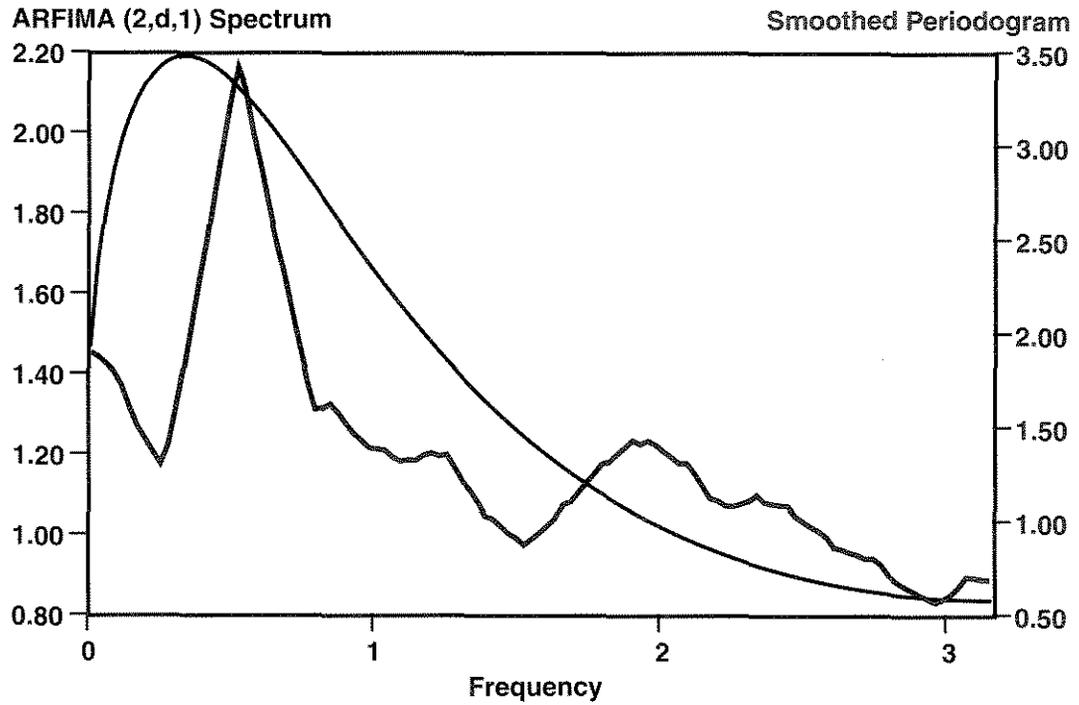**Spectrum of Differenced Real Exchange Rate: U.S./Germany**

Figure 2

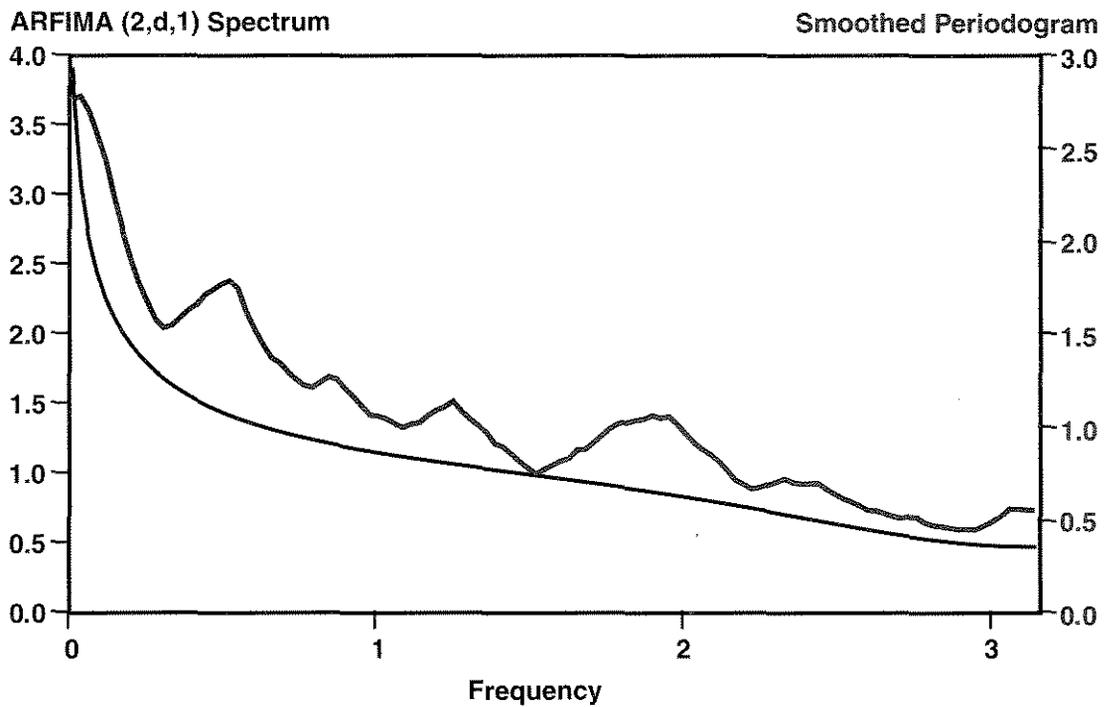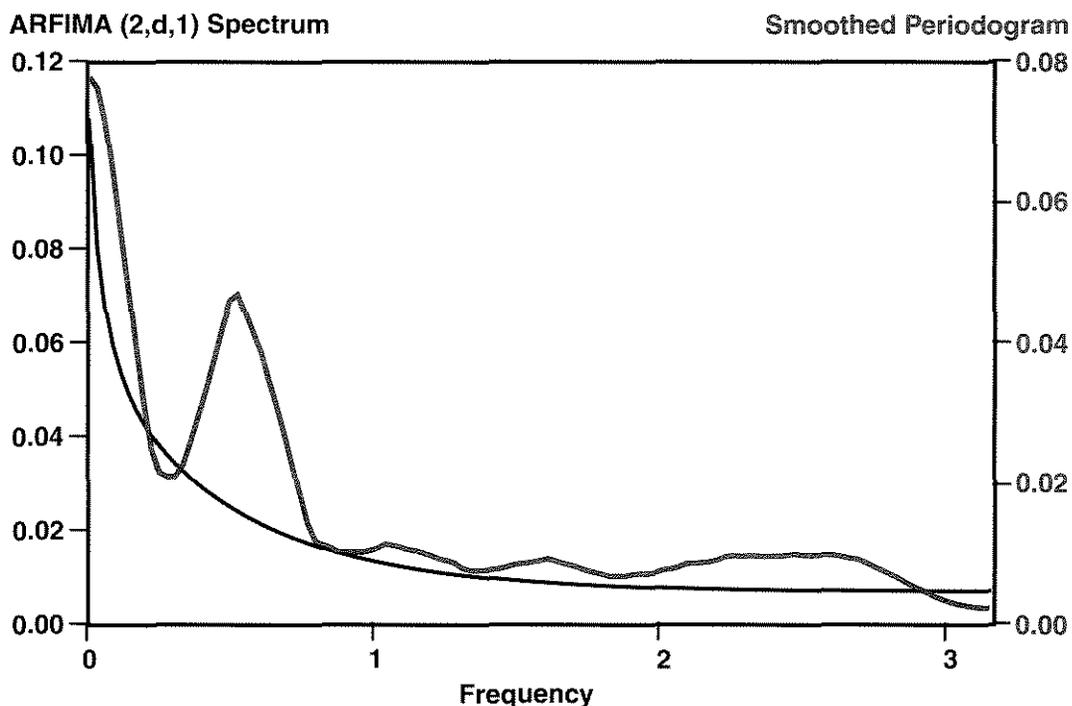**Spectrum of Differenced Nominal Exchange Rate: U.S./Germany**

Figure 3
Spectrum of Differenced Price Level Ratio: U.S./Germany

ARFIMA (2,d,1) Spectrum   Smoothed Periodogram

Frequency

CONCLUSIONS

This article illustrates the key role played by the null hypothesis in testing for unit roots in real exchange rates. If b is the order of integration of the real exchange rate, then the null that $b \geq 1$ is difficult to reject, in which case one would presume that long-run purchasing-power parity does not hold. When the null is that $b < 1$, we also find few rejections, so long-run PPP apparently holds. When this type of ambiguity appears, it is helpful to estimate the order of integration on a continuous scale. The fractional ARMA models presented here do this and the standard errors on b for the six real exchange rates studied show that even if $b < 1$, it is not far enough away from one to make a strong case that purchasing-power parity is empirically relevant.

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Appendix:
The Fox and Taqqu Estimator

Dahlhaus (1988) discusses why the Fox and Taqqu (1986) frequency domain estimator is an approximate maximum-likelihood estimator, sharing the same optimality properties as exact maximum-likelihood estimation. The Fox and Taqqu estimator is derived from the following minimization problem:

$$(12)\ min \sum_{\lambda_k} [ln(\sigma^2 f(\lambda_k;\theta)) + \frac{I(\lambda_k)}{\sigma^2 f(\lambda_k;\theta)}]$$

where $I(\lambda_k)$ is the vector of periodogram ordinates of the data and $\sigma^2 f(\lambda_k)$ is the spectral density function implied by the parameterized model. For the fractional ARMA model in equation (8), the spectral density equals

$$(13)\ f(\lambda) = \frac{|B(e^{i\lambda})|^2}{|A(e^{i\lambda})|^2}(1-e^{i\lambda})^{-d}(1-e^{-i\lambda})^{-d},$$

where A and B are polynomials defined in equations (9) and (10). The objective function is minimized with respect to θ and σ^2. An intuitive description of the objective function is that one wants to choose parameters that will make the spectral density function implied by the model look like the periodogram of the data.