UNIQUENESS OF EQUILIBRIUM IN THE MULTI-COUNTRY RICARDO MODEL

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Uniqueness of Equilibrium in the Multi-Country Ricardo Model

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Abstract

We present two arguments, one based on index theory, demonstrating that the multi-country Ricardo model has a unique competitive equilibrium if the aggregate demand functions exhibit gross substitutability. The result is somewhat surprising because the assumption of gross substitutability is sufficient for uniqueness in a model of exchange but not, in general, when production is included in the model.

Keywords: Ricardo model, Gross substitutes, Uniqueness

JEL Classification: D51, F11

It is well known that the competitive equilibrium is unique in a pure exchange economy when the market excess demand function satisfies the assumption of gross substitutability. However, if we introduce an arbitrary constant returns to scale technology, a unique equilibrium is ensured only if the market excess demand function satisfies the weak axiom of revealed preference. Since gross substitutability does not imply the weak axiom we can construct examples using an activity analysis model of production in which there are several equilibria even though the market demand functions display gross substitutability. The first such example can be found in Kehoe [4].

There are few results on how the assumption of the weak axiom may be relaxed, and uniqueness still prevails, as we impose conditions on the technology. One notable example is the case where there is only one primary factor of production and each productive activity produces a single good, using other produced goods as inputs in addition to the primary factor. In this case, the nonsubstitution theorem implies that the technology alone uniquely determines the equilibrium price. In the present paper we consider the Ricardo model in which there are many primary factors, the labor in each country, but in which each good is produced using labor alone. We demonstrate that the assumption of gross substitutability on the market excess demand is sufficient to guarantee the uniqueness of the competitive equilibrium for the multi-country

Ricardo model. Two distinct proofs are discussed. The first is based on the induced properties of the excess demand for labor and requires that the gross substitutability condition hold everywhere. The second applies index theory directly to the market demand function for goods and requires only that the gross substitute condition be satisfied at the equilibrium price.

To emphasize the application of our results to standard models of international trade, our analysis supposes a fixed supply of labor. In Wilson [5], both proofs of the uniqueness of equilibrium are extended to include models that incorporate a variable supply of labor, so long as the excess demand for labor and goods satisfies the gross substitute assumption. That paper also extends the proof of Hildenbrand and Kirman [2] and provides a direct proof of the existence of equilibrium in a Ricardo model with gross substitutes that does not appeal to a fixed point theorem.

1 Uniqueness in the Model of Exchange

All of the arguments in the paper are essentially more complicated versions of simple and well known arguments for pure exchange economies. For this reason, we first review how they work in a model of pure exchange.

For any positive integer k, let R^k denote k-dimensional Euclidean space, and $R^k_{++} \equiv \left\{x \in R^k : x_i > 0 \text{ for all } i.\right\}$ denote the interior of the nonnegative orthant. Let $S^k_{++} = \left\{x \in R^k_{++} : \sum_{i=1}^k x_i = 1\right\}$ denote the interior of the unit simplex of dimension $k-1$. For any $x, y \in R^k$, we define $x \geq y$ to mean that $x_i \geq y_i$ for all $i = 1, ..., k$, and $x > y$ to meant that $x \geq y$ and $x_i > y_i$ for some i.

Definition 1 *A function $f : R^n_{++} \to R^n$ is a* market excess demand function *if it is homogeneous of degree zero in prices and satisfies the Walras Law:*

$$f(\pi) = f(\lambda\pi) \quad \text{for all } \pi \in R^n_{++} \text{ and } \lambda > 0.$$
$$\sum_{i=1}^{n} \pi_i f_i(\pi) = 0.$$

Definition 2 *The market excess demand satisfies* gross substitutability *if at all prices $\pi \in R^n_{++}$ and each good i*

$$\frac{\partial f_i(p, w)}{\partial p_k} > 0 \quad \text{for all } k \neq i.$$

Notice that the market excess demand function and therefore the gross substitutability condition are defined only when all prices are strictly positive. (The gross substitute condition is inconsistent with homogeneity of prices if one or more prices are zero.)

To guarantee the existence of an equilibrium with strictly positive prices (and to guarantee that this is only equilibria were we to allow for zero prices), we impose the following boundary condition. Using homogeneity of the demand function, we may normalize prices so that $\pi \in S^n_{++}$.

Definition 3 *A market excess demand function satisfies the* boundary condition *if for any sequence of price vectors* $\{\pi^t\} \in S^n_{++}$, *we have*

$$\min[\pi^t_1, ..., \pi^t_n] \to 0 \text{ implies } \max[f_1(\pi^t), ..., f_n(\pi^t)] \to \infty.$$

An elementary example satisfying all of these conditions arises when each household has a strictly positive endowment of each good and a Cobb-Douglas utility function. (Since the gross substitute condition is linear it is clearly satisfied when the individual excess demand functions are aggregated to obtain the market excess demands.)

The standard argument for uniqueness of equilibrium in a model of exchange with gross substitutes is extraordinarily simple.

Theorem 4 *Suppose the market excess demand function satisfies the gross substitutability. Then the equilibrium* $\pi \in R^n_{++}$ *is unique up to a scalar multiple.*

Proof. Suppose that π and π^* are two equilibrium price vectors that are not proportional. Then we may use the assumption of homogeniety to normalize π^* so that there is a nonempty proper subset of goods I for which

$$\pi_i = \pi^*_i \text{ if } i \in I$$
$$\pi_i < \pi^*_i \text{ if } i \notin I.$$

But then for any good $i \in I$, the gross substitutability assumption implies

$$0 = f_i(\pi) > f_i(\pi^*) = 0$$

which is a contradiction. ■

1.1 General Equilibrium with Production

We describe the production side of the economy by an activity analysis matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1k} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nk} \end{bmatrix}.$$

Each column of A represents a feasible production plan, with negative entries referring to inputs into production and positive entries to outputs. The activities can be used simultaneously at arbitrary non-negative activity levels $x = (x_1, \dots, x_k)$ so that the production possibility set available to the economy as a whole is given by

$$Y = \{y = Ax \text{ for } x \geq 0\}.$$

Free disposal of commodities is described by the presence of n columns in A which form the negative of a unit matrix.

A competitive equilibrium is given by a price and activity level pair (π, x) such that

- $f(\pi) = Ax$ and
- $\pi A \leq 0$ with equality for column j if $x_j > 0$.

In order to guarantee the existence of an equilibrium the following assumption is typically made:

Assumption: There exists a non-zero price vector $\pi \geq 0$ such that $\pi A \leq 0$.

Under this assumption and the ones previously made about the market excess demand functions it is straightforward to demonstrate the existence of a competitive equilibrium.

1.2 Uniqueness in the Ricardo Model

Let there be m countries and n goods with the output in country j of good i for a single unit of that country's labor given by $a_{ij} > 0$. Suppose that each country j has a fixed endowment of labor L_j and let $f_i^j(p, w_j)$ denote the demand for good i in country j, given the price vector $p \in R^n_{++}$ and the county's wage w_j. Let $f_i(p, w) \equiv \sum_j f_i^j(p, w)$ represent the aggregate demand for good i. If we suppose that each $f^j = (f_1^j, ..., f_n^j)$ is homogeneous of degree 0, and satisfies the budget constraint $\sum p_i f_i^j(p, w_j) = w_j L_j$, then f will also be homogeneous of degree 0 in prices and wages and satisfy the Walras Law

$$\sum_{i=1}^{n} p_i f_i(p, w) \equiv \sum_{j=1}^{m} w_j L_j.$$

To ensure that each good is produced by some country in equilibrium we assume that the market demand is strictly positive at all prices.

Definition 5 *The market aggregate demand f satisfies* gross substitutability *if at all prices $(p, w) \in R^n_{++} \times R^m_{++}$ and each good i and country j*

$$\frac{\partial f_i(p, w)}{\partial p_k} \geq 0 \quad \textit{for all } k \neq i$$
$$\frac{\partial f_i(p, w)}{\partial w_j} > 0.$$

If good i is produced in country j in the equilibrium with wage rate w, then

$$p_i a_{ij} = w_j \text{ and } p_i a_{ik} \leq w_k \text{ for all other countries } k.$$

It follows that

$$p_i = \min[w_k/a_{ik}]$$

so that the equilibrium wage vector w uniquely determines the equilibrium price vector p. Because of this fact about the Ricardo model, we will occasionally find it convenient to refer to an equilibrium in terms of wages alone.

Theorem 6 *Suppose the market demand function satisfies gross substitutability, then the equilibrium wage vector w is unique up to a scalar multiple.*

Suppose w and w^* are both equilibrium wage vectors that are not proportional. Then we may normalize w^* so that there is a nonempty proper subset of countries J for which

$$\begin{aligned} w_j &= w_j^* \text{ if } j \in J \\ w_j &> w_j^* \text{ if } j \notin J. \end{aligned}$$

We have the following simple observation about the two equilibria which is valid for all Ricardo models, regardless of assumptions on the market demand functions.

Observation: A good that is produced by a country $j \in J$ in the equilibrium with wages w^* will be produced only by the countries in J in the equilibrium with wages w.

Proof of Theorem 6 We see from this observation that there are some goods that are produced exclusively by the counties in J in the equilibrium with wages w. Let us define I to be the set of these goods, i.e.,

$$I \equiv \{i : p_i < w_k/a_{ik} \text{ for all } k \notin J\}.$$

Also let I^* be the set of goods for which the countries in J are least cost producers under wages w^*:

$$I^* \equiv \left\{i : p_i^* = \min_{j \in J} w_j^*/a_{ij}\right\}.$$

Some of the goods in I^* may be produced by countries not in J in the equilibrium with wages w^*.

The observation tells us that

$$I^* \subseteq I.$$

We have

$$p_i = p_i^* \text{ for } i \in I^*$$

and

$$\text{if } i \notin I^* \text{ then } p_i^* < \min_{j \in J} w_j^*/a_{ij} = \min_{j \in J} w_j/a_{ij} = p_i.$$

Since goods in I are produced only by countries in J at wage vector w it must be true at equilibrium that the cost of purchasing the world demand for the goods in I is less than or equal to the wages received by the counties in J under w :

$$\sum_{i \in I} p_i f_i(p, w) \leq \sum_{j \in J} w_j L_j.$$

Moreover since the countries in J produce only goods in I^* at wage vector w^*, it follows that the value of the world demand for the goods in I^* is greater than or equal to the wages received by the countries in J under w^* :

$$\sum_{j \in J} w_j^* L_j \leq \sum_{i \in I^*} p_i^* f_i(p^*, w^*).$$

But since $w_j = w_j^*$ for $j \in J$ we have

$$\sum_{i \in I} p_i f_i(p, w) \leq \sum_{j \in J} w_j L_j = \sum_{j \in J} w_j^* L_j \leq \sum_{i \in I^*} p_i^* f_i(p^*, w^*).$$

This inequality is valid for all Ricardo models regardless of the assumptions made about the market excess demand functions. If we now make the assumption of gross substitutability then

$$f_i(p, w) > f_i(p^*, w^*) \text{ for all } i \in I^*$$

and we obtain the inequalities

$$\sum_{i \in I} p_i f_i(p, w) \geq \sum_{i \in I^*} p_i f_i(p, w) = \sum_{i \in I^*} p_i^* f_i(p, w) > \sum_{i \in I^*} p_i^* f_i(p^*, w^*).$$

The contradiction between the last pair of inequalities completes the proof of uniqueness of the competitive equilibrium under gross substitutability.

2 Index Theory

Index theory is a sophisticated method of analysis used to study the solutions of systems of non-linear equations

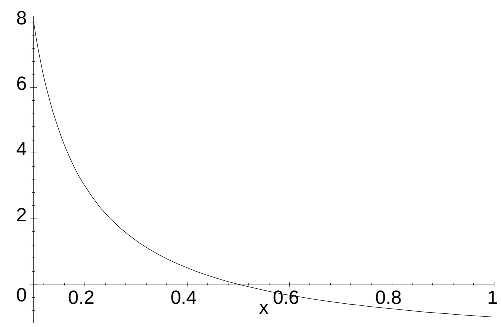
$$g_i(x_1, \ldots, x_n) = 0, \text{ for } i = 1, \ldots, n.$$

Under mild assumptions on the problem, we can associate an index, ± 1, with each solution of the system of equations depending on the local behavior of the functions at that point. The main theorem of index theory states that the sum of the indices over the entire set of solutions is equal to $+1$. This global result permits us to assert the uniqueness of the solution to the system of equations on the basis of local behavior; for example if each solution has an index of $+1$ then there can be only one solution.

We shall provide a simple illustration of the main theorem in the very special case of an exchange economy with 2 goods. A thorough and accessible presentation of index theory may be found in the volume by Garcia and Zangwill [1].

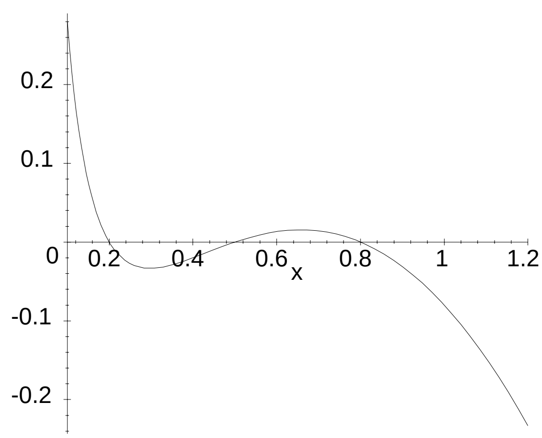
2.1 Exchange Economies with Two Goods

Consider a pure exchange economy with two goods. Since the boundary condition implies that set of equilibrium prices are strictly positive, we may normalize the second price to be unity and consider the excess demand for the first good as a function of its own price. Let $f_1(\pi)$ denote the excess demand for good 1, given the price vector $(\pi, 1)$. Our boundary condition and the Walras Law implies that $\lim_{\pi \to 0} f_1(\pi) = \infty$ and $\lim_{\pi \to \infty} \pi f_1(\pi) = -\infty$. Figure 1 illustrates a market excess demand function for good 1 with a single equilibrium at $\pi = 1/2$.



Notice that at the equilibrium price the market excess demand for good 1 crosses the π axis from above so we must have $f_1'(\pi) < 0$ at this equilibrium.

The next figure illustrates a case with three equilibria.



At the first and third equilibria we have, as before,

$$f_1'(\pi) < 0$$

but at the middle equilibrium we have the reverse inequality

$$f_1'(\pi) > 0$$

More generally, suppose there are n equilibria,

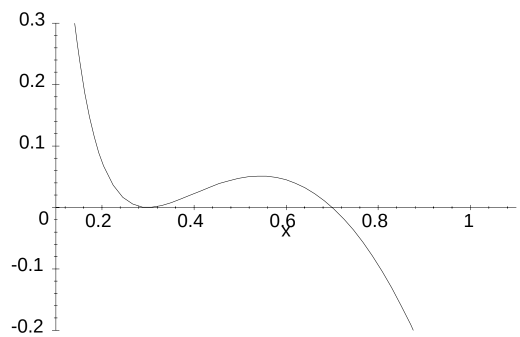
$$\pi^1 < \pi^2 < \cdots < \pi^n.$$

Then if the excess demand is continuous and satisfies the boundary condition and the demand function is never tangent to the π axis, then the first crossing of f_1 must be from above, the second from below, and ultimately the last crossing must be from above. There must, therefore, be one more crossing of f_1 from above than from below.

Our argument is valid for any excess demand that does not have a *degenerate* equilibrium π at which the excess demand function is tangent to the π axis. At a degenerate equilibrium π, where

$$\frac{df_1(\pi)}{d\pi} = 0,$$

the number of equilibria changes dramatically with small perturbations in the demand function, as illustrated in figure 3.



We therefore exclude this nongeneric class of economies from our analysis, by restricting our attention to *regular* economies, in which none of the equilibria are degenerate.

For any nonzero number x, let sign$[x]$ be $+1$ if $x > 0$ and -1 if $x < 0$. Then, for a regular economy we define

$$\text{Index}(\pi) = \text{sign}[-f'(\pi)]$$

for each equilibrium π. If $\left\{\pi^1, ..., \pi^k\right\}$ is the set of equilibrium prices our analysis implies that

$$\sum_{1=1}^{k} \text{Index}(\pi^j) = 1$$

It follows that a necessary and sufficient condition that there be a single equilibrium is that each equilibrium have an index of $+1$.

2.2 The Index for a Model of Exchange with n Goods

This illustrative result can be extended to an excess demand function with n goods. We use the notation

$$f_{ij} = \partial f_i / \partial \pi_j.$$

The assumption that the excess demands are homogeneous of degree zero implies that

$$\sum \pi_j f_{ij} \equiv 0$$

so that the Jacobian matrix

$$J(f) = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{n,n} \end{bmatrix}$$

is singular. In order to define the index of an equilibrium we examine an arbitrary principal minor of the Jacobian, say,

$$D(f) = \begin{bmatrix} f_{11} & \cdots & f_{1,n-1} \\ \vdots & \ddots & \vdots \\ f_{n-1,1} & \cdots & f_{n-1,n-1} \end{bmatrix}$$

Definition 7 *An excess demand function f is regular if $f(\pi) = 0$ implies $D(f)$ is non singular.*

Definition 8 *The index associated with an equilibrium price vector is defined to be*

$$\text{sign}[\det(-D(f))]$$

We have the following major result from index theory that links the indices of all of the equilibria of a model of exchange.

Theorem 9 *(Index Theorem) If $\pi^1, \pi^2, \ldots, \pi^k$ are the competitive equilibrium of a* regular *exchange economy that satisfies the boundary condition, then*

$$\sum_{j=1}^{k} \text{Index}(\pi^j) = 1.$$

2.3 Uniqueness in a Model of Exchange with Gross Substitutes

To demonstrate uniqueness of the equilibrium in an n-good pure exchange economy with gross substitutes we simply show that the index of each equilibrium is positive, i.e., that

$$\det \begin{bmatrix} f_{11} & \cdots & f_{1,n-1} \\ \vdots & \ddots & \vdots \\ f_{n-1,1} & \cdots & f_{n-1,n-1} \end{bmatrix} < 0.$$

The assumption of gross substitutability implies that the matrix

$$-\begin{bmatrix} f_{11} & \cdots & f_{1,n-1} \\ \vdots & \ddots & \vdots \\ f_{n-1,1} & \cdots & f_{n-1,n-1} \end{bmatrix}$$

has positive entries on the main diagonal and negative entries elsewhere, so that it is a Leontief matrix. But it is also a *productive* Leontief matrix since homogeneity of prices implies

$$\sum_{j=1}^{n-1} \pi_j(-f_{ij}) = \pi_n f_{i,n} > 0 \text{ for } i = 1, \ldots, n-1.$$

The classical result that a productive Leontief matrix has a positive determinant implies that the index is positive, and therefore, the equilibrium is unique.

2.4 Index Theory with Production

We shall now discuss the index theorem for an equilibrium model in which production is described by an activity analysis matrix. The first presentation of index theory for this model appears in Kehoe's Ph.D. thesis (1979) and in the subsequent paper [3] in Econometrica.

As before let the market excess demand functions be $f_i(\pi)$, and let A be the activity analysis matrix. The equilibrium is given by a price vector π and a set of activity levels $x \geq 0$, such that

- $f_i(\pi) = Ax$ and
- $\pi A \leq 0$ with equality for those activities that are used at a positive level.

Let S be the subset of activities used at a positive level in the particular equilibrium in question and let s be the number of these activities. The equilibrium conditions yield the following set of $n+s$ equations in $n+s$ variables.

$$\begin{aligned} f_i(\pi) - \sum_{j \in S} a_{ij} x_j &= 0 \\ \sum_i \pi_i a_{ij} &= 0 \text{ for } j \in S \end{aligned}$$

as well as inequalities stating that the remaining activities make a non-positive profit. The Jacobian of the system of equations is the $n+s \times n+s$ matrix

$$J = \begin{bmatrix} F & -A_S \\ A_S^T & 0 \end{bmatrix}$$

where F is the $n \times n$ matrix of derivatives of the excess demand functions

$$F = \begin{bmatrix} f_{11} & \cdots & f_{1j} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{j1} & \cdots & f_{jj} & \cdots & f_{jn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nj} & \cdots & f_{nn} \end{bmatrix}$$

and A_S represents the subset of activities used in this equilibrium.

As in the model of exchange this Jacobian J is singular. In order to have a well defined index for this equilibrium we need to assume that the problem is *non-degenerate* in the sense that the rank of J is $n + s - 1$. We then calculate the sign of the determinant of the principal minor obtained by striking out the jth row and column of the matrix $-J$, where j is one of the first n rows (and columns). If the determinant is positive the index is $+1$; if the determinant is negative the index is -1.

We then have the important, general theorem that if the model is non-degenerate, then the sum of the indices over all of the equilibria is $+1$. It follows that the equilibrium is unique if every equilibrium has an index of $+1$.

2.5 Index Theory and the Ricardo Model

In the Ricardo model the price vector $\pi = (p, w)$ has two components: p the goods prices and w the wage rates. We assume that there are n goods and m countries. The activity analysis matrix takes the form

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & 0 & \cdots & a_{1m} & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n1} & \cdots & 0 & \cdots & a_{nm} \\ -1 & \cdots & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & -1 & \cdots & -1 \end{bmatrix},$$

with A that part of the activity analysis matrix involving outputs and C the rows referring to the m countries.

With general market demand functions the Jacobian of demand with respect to prices p and wages w is given by

$$\begin{bmatrix} \partial f_1/\partial p_1 & \cdots & \partial f_1/\partial p_n & \partial f_1/\partial w_1 & \cdots & \partial f_1/\partial w_m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial f_n/\partial p_1 & \cdots & \partial f_n/\partial p_n & \partial f_n/\partial w_1 & \cdots & \partial f_n/\partial w_m \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The zeros in the last m rows arise because there is no demand for liesure.

Since the market demand functions are homogeneous of degree zero, we have, for each good i

$$\sum_j p_j \partial f_i/\partial p_j + \sum_j w_j \partial f_i/\partial w_j = 0.$$

As before we assume that at equilibrium the s activities in the set S are used. Then the matrix used to calculate the index of the equilibrium is of size $n+m+s$. It has the form

$$\begin{bmatrix} -F & -L & -A_S \\ 0 & 0 & -C_S \\ A_S^T & C_S^T & 0 \end{bmatrix}$$

where

$$f_{ij} = \partial f_i/\partial p_j,$$

L is a matrix of size $n \times m$ with entries $l_{ij} = \partial f_i/\partial w_j$ and

$$\begin{bmatrix} A_S \\ C_S \end{bmatrix}$$

is the set of activities used at equilibrium. The index of the equilibrium is the sign of the determinant of a principal minor of this matrix obtained by striking out the jth row and column, where $1 \leq j \leq n$. We assume that the model is non-degenerate so that the index is well defined.

With more detail the index matrix is given by

$$\begin{bmatrix} -\partial f_1/\partial p_1 & \cdots & -\partial f_1/\partial p_i & \cdots & -\partial f_1/\partial p_n & -l_{11} & \cdots & -l_{1j} & \cdots & -l_{1m} & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ -\partial f_i/\partial p_1 & \cdots & -\partial f_i/\partial p_i & \cdots & -\partial f_i/\partial p_n & -l_{i1} & \cdots & -l_{ij} & \cdots & -l_{im} & \cdots & -a_{ij} & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ -\partial f_n/\partial p_1 & \cdots & -\partial f_n/\partial p_i & \cdots & -\partial f_n/\partial p_n & -l_{n1} & \cdots & -l_{nj} & \cdots & -l_{nm} & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{ij} & \cdots & 0 & 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}.$$

where column $m+n+i$ and row $m+n+i$ indicate that good i is produced in country j. The index associated with the equilibrium is $+1$ if the determinant of the principal minor obtained by striking out the row and column associated with a particular good is positive; the index is -1 if the sign is negative.

We simplify the index matrix, while retaining the sign of all principal minors, by multiplying columns, and rows, $1, \ldots n, n+1, \ldots, n+m$ by

$$p_1, \ldots p_n, w_1, \ldots, w_m$$

and using the fact that if good i is produced in country j then $p_i a_{ij} = w_j$.

After this simplification the index matrix becomes

$$\begin{bmatrix} v_{11} & \cdots & v_{1i} & \ldots & v_{1n} & -e_{11} & \cdots & -e_{1j} & \cdots & -e_{1m} & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ v_{i1} & \cdots & v_{ii} & \ldots & v_{in} & -e_{i1} & \cdots & -e_{ij} & \cdots & -e_{im} & \cdots & -w_j & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ v_{n1} & \cdots & v_{ni} & \ldots & v_{nn} & -e_{n1} & \cdots & -e_{nj} & \cdots & -e_{nm} & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & \ldots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \ldots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & w_j & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \ldots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & w_j & \ldots & 0 & 0 & \cdots & -w_j & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{bmatrix},$$

with

$$
\begin{aligned}
e_{ij} &= p_i w_j \partial f_i / \partial w_j \\
v_{ij} &= -p_i p_j \partial f_i / \partial p_j \text{ and} \\
\sum_j v_{ij} &= \sum_j e_{ij}.
\end{aligned}
$$

And finally we divide columns $n+m+i$ of the activity analysis matrix, and the rows of its transpose, by the corresponding wage rate obtaining the following matrix, which is sufficently important to warrant a formal definition:

Definition 10 *We define the index matrix I to be*

$$
\begin{bmatrix}
v_{11} & \cdots & v_{1i} & \cdots & v_{1n} & -e_{11} & \cdots & -e_{1j} & \cdots & -e_{1m} & \cdots & 0 & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
v_{i1} & \cdots & v_{ii} & \cdots & v_{in} & -e_{i1} & \cdots & -e_{ij} & \cdots & -e_{im} & \cdots & -1 & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
v_{n1} & \cdots & v_{ni} & \cdots & v_{nn} & -e_{n1} & \cdots & -e_{nj} & \cdots & -e_{nm} & \cdots & 0 & \cdots \\
0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots
\end{bmatrix}
$$

The columns to the right of the E matrix depend on the activities in use at the equilibrium. Each such column has two non-zero entries: a -1 in a row corresponding to a good being produced and a $+1$ in a row corresponding to a country which produces that good. The negative transpose of these columns appears below the block of zeros.

The assumption of gross substitutability

$$
\begin{aligned}
\partial f_i / \partial p_j &> 0, \text{ for } i \neq j \text{ and} \\
\partial f_i / \partial w_j &> 0.
\end{aligned}
$$

implies that $e_{ij} > 0$ and that V is a Leontief matrix. V is a *productive* matrix since

$$
\sum_j v_{ij} = \sum_j e_{ij} > 0.
$$

We have the following theorem implying uniqueness of the equilibrium:

Theorem 11 *Under the assumption of Gross Substitutability, the determinant of the principal minor of I obtained by striking out its jth row and column, where $1 \leq j \leq n$, is* ***positive*** *and the index of the equilibrium is therefore* $+1$.

The basic idea of the proof is to observe that the determinant of the principal minor obtained by striking out the jth row and column of I is *precisely* the derivative of $\det I$ with respect to the jth diagonal entry v_{jj}. In order to work with these derivatives we generalize the matrix I by allowing its diagonal entries to vary.

Let $I(\xi)$ be the matrix

$$\begin{bmatrix} \xi_1 & \cdots & v_{1i} & \cdots & v_{1n} & -e_{11} & \cdots & -e_{1j} & \cdots & -e_{1m} & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ v_{i1} & \cdots & \xi_i & \cdots & v_{in} & -e_{i1} & \cdots & -e_{ij} & \cdots & -e_{im} & \cdots & -1 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ v_{n1} & \cdots & v_{ni} & \cdots & \xi_n & -e_{n1} & \cdots & -e_{nj} & \cdots & -e_{nm} & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{bmatrix},$$

with

$$\xi_i \geq v_{ii}.$$

According to our previous discussion the index associated with the equilibrium is equal to

$$\text{sign}[\partial \det I(\xi)/\partial \xi_j] \text{ when } \xi_i = v_{ii}, i = 1, ..., n$$

The proof of our theorem will be complete if we can show the stronger statement: that $I(\xi)$ is increasing in each of variables ξ_i when $\xi_i \geq v_{ii}$.

It is useful to observe that the equilibrium has associated with it a bi-partite graph G with $m + n$ vertices, one for each good and one for each country. There is an edge connecting good i and country j if in this equilibrium good i is produced in country j. The graph may very well be disconnected; if it is we partition G into connected subgraphs $G_1, \ldots, G_p$. For example if the model involves 4 goods and 3 countries and the particular equilibrium uses the following activities

$$\begin{bmatrix} A_S \\ C_S \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 & 0 \\ 0 & 0 & a_{31} & 0 & 0 \\ 0 & 0 & 0 & a_{42} & a_{43} \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

then goods $1, 2, 3$ are produced in country 1 alone and good 4 is produced in countries 2 and 3. The graph has two components (goods $1, 2, 3$ and country 1) and (good 4 and countries $2, 3$).

Now let us demonstrate, by induction on the size of the Ricardo model, that

$$\partial \det I(\xi)/\partial \xi_j > 0 \text{ when } \xi_i \geq v_{ii}.$$

This result implies that the function

$$\det I(\xi_1, \ldots, \xi_n)$$

is increasing in each variable when $\xi_i \geq v_{ii}$. Since $\det I(\xi_1, \ldots, \xi_n)$ is zero when $\xi_i = v_{ii}$ for all i it must be positive for $\xi_i > v_{ii}$.

In our induction argument we will use the fact that the corresponding matrix is strictly positive for Ricardo models - satisfying the assumption of Gross Substitutability - with a smaller number of goods and countries.

Let us take $j = 1$ since the other arguments are identical. The derivative

$$\frac{\partial I(\xi)}{\partial \xi_1}$$

is the determinant of the principal minor obtained by striking out the first row and column of $I(\xi)$. Columns $n+m+1, \ldots, n+m+s$ of $I(\xi)$ each contain two non-zero entries indicating that a particular good is produced in a particular country. Those columns describing the several countries producing good 1 will be replaced, in the principal minor, by columns containing a single entry of $+1$ in the row of that country; the columns involving goods other than good 1 will contain both $+1$ and -1. And similarly for the rows.

For example if $I(\xi)$ has the form

$$\begin{bmatrix}
\xi_1 & \dots & v_{1i} & \dots & v_{1n} & -e_{11} & \dots & -e_{1j} & \dots & -e_{1m} & \dots & -1 & 0 & \dots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
v_{i1} & \dots & \xi_i & \dots & v_{in} & -e_{i1} & \dots & -e_{ij} & \dots & -e_{im} & \dots & \vdots & -1 & \dots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
v_{n1} & \dots & v_{ni} & \dots & \xi_n & -e_{n1} & \dots & -e_{nj} & \dots & -e_{nm} & \dots & 0 & 0 & \dots \\
0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & 0 & \dots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 & 1 & \dots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & 0 & \dots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & \dots & \dots & \dots & 0 & 0 & \dots & -1 & \dots & 0 & \dots & 0 & 0 & \dots \\
0 & \dots & 1 & \dots & 0 & 0 & \dots & -1 & \dots & 0 & \dots & 0 & 0 & \dots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots
\end{bmatrix}$$

then columns and rows $k, k+1$ indicate that good 1 and good i are both produced in country j. After striking out column and row 1 the minor is given by

$$\begin{bmatrix}
\ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & \xi_i & \cdots & v_{in} & -e_{i1} & \cdots & -e_{ij} & \cdots & -e_{im} & \cdots & \vdots & -1 & \cdots \\
\ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & v_{ni} & \cdots & \xi_n & -e_{n1} & \cdots & -e_{nj} & \cdots & -e_{nm} & \cdots & 0 & 0 & \cdots \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots \\
\ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots \\
\ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots \\
\ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & 0 & 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 & 0 & \cdots \\
\cdots & 1 & \cdots & 0 & 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 & 0 & \cdots \\
\ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

But now column k contains a single non-zero entry 1 in row j and row k contains a single non-zero entry -1 in column j. If we expand the minor by these two entries in turn we obtain the smaller minor

$$\begin{bmatrix} \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \cdots & \xi_i & \cdots & v_{in} & -e_{i1} & \cdots & \cdots & -e_{im} & \cdots & -1 & \cdots \\ \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \cdots & v_{ni} & \cdots & \xi_n & -e_{n1} & \cdots & \cdots & -e_{nm} & \cdots & 0 & \cdots \\ \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots \\ \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots \\ \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \cdots & 1 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots \\ \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}.$$

But now the column and row corresponding to any other good produced in country j will also have a single ± 1 in it and can also be removed without changing the value of the determinant of the minor obtained by striking out the first row and column. We have the following crucial observation.

Observation: The determinant of the minor obtained by striking out the rows and columns of $I(\xi)$ for any particular good will be unchanged if we then strike out the rows and columns for **all other goods which are produced in any country that produces that particular good.**

At this point we consider two cases, each with its own example.

Case 1. The subgraph of goods and countries containing good 1 is not the entire graph. In this case after repeated application of the observation we arrive at the determinant $I(\xi)$ for the corresponding solution to the Ricardo model in which all of the goods and countries in the subgraph containing good 1 are discarded. This is a smaller problem - which also satisfies the assumption of Gross Substitutability - and by induction the corresponding determinant is positive. It follows that

$$\partial \det I(\xi)/\partial \xi_i > 0 \text{ when } \xi_i > v_{ii}.$$

As an example of this case consider the model with 4 goods and 3 countries in which goods $1, 2, 3$ are produced in the first country and good 4 is produced in the second and third countries. The activity analysis matrix is

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 & 0 \\ 0 & 0 & a_{31} & 0 & 0 \\ 0 & 0 & 0 & a_{42} & a_{43} \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

and the graph has two components, (goods $1, 2, 3$ and country 1) and (good 4 and countries $2, 3$)

The index matrix $I(\xi)$ is

$$\begin{bmatrix} \xi_1 & 0 & 0 & 0 & -e_{11} & -e_{12} & -e_{13} & -1 & 0 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 & -e_{21} & -e_{22} & -e_{23} & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \xi_3 & 0 & -e_{31} & -e_{32} & -e_{33} & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \xi_4 & -e_{41} & -e_{42} & -e_{43} & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The derivative of the determinant of $I(\xi)$ with respect to ξ_1 is

$$\det \begin{bmatrix} \xi_2 & 0 & 0 & -e_{21} & -e_{22} & -e_{23} & 0 & -1 & 0 & 0 & 0 \\ 0 & \xi_3 & 0 & -e_{31} & -e_{32} & -e_{33} & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \xi_4 & -e_{41} & -e_{42} & -e_{43} & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

After systematically striking out the rows and columns in the component of the graph containing good 1 we arrive at the determinant

$$\det \begin{bmatrix} \xi_4 & -e_{42} & -e_{43} & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

for a Ricardo model with a single good - good 4 - and countries $2, 3$. Of course,

$$\xi_4 > e_{42} + e_{43}$$

Case 2. The subgraph containing good 1 is the entire graph. In this case repeated application of the observation will lead us to a point when we strike the rows and columns for the last good. This results in the determinant

$$\det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

whose determinant is clearly +1.

An example of this case arises in a model with 4 commodities and 3 countries in which goods $1, 2, 3$ are produced in country 1, goods $2, 4$ are produced in country 2 and good 4 is produced in country 3. The matrix of activities used in equilibrium is

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{42} & a_{43} \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

In this example, the graph of goods and countries is completely connected.

The index calculation is based on the matrix

$$\begin{bmatrix} \xi_1 & 0 & 0 & 0 & -e_{11} & -e_{12} & -e_{13} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 & -e_{21} & -e_{22} & -e_{23} & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & \xi_3 & 0 & -e_{31} & -e_{32} & -e_{33} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_4 & -e_{41} & -e_{42} & -e_{43} & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The derivative of the determinant of $I(\xi)$ with respect to ξ_1 is

$$\det \begin{bmatrix} \xi_2 & 0 & 0 & -e_{21} & -e_{22} & -e_{23} & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & \xi_3 & 0 & -e_{31} & -e_{32} & -e_{33} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \xi_4 & -e_{41} & -e_{42} & -e_{43} & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we strike out rows and columns corresponding to country 1, then goods 2 and 3, then country 2 and good 4, we obtain the determinant

$$\det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} > 0.$$

This concludes the proof of Theorem 11 and demonstrates the uniqueness of the competitive equilibrium.

Wilson [5] provides an alternative argument for this result.

References

[1] C. B. Garcia and W. I. Zangwill, *Pathways to Solutions, Fixed Points and Equilibria*, Prentice-Hall, 1981.

[2] Hildenbrand, Werner and Alan. P. Kirman, *Equilibrium Analysis*, North Holland, Amsterdam, 1988, Chapter 3.

[3] Kehoe, Timothy, "An Index Theorem for General Equilibrium Models with Production," *Econometrica*, Vol. 48, No. 5. (Jul., 1980), pp. 1211–1232.

[4] Kehoe, Timothy, "Multiplicity of Equilibria and Comparative Statics," *The Quarterly Journal of Economics*, Vol. 100, No. 1. (Feb., 1985), pp. 119–147.

[5] Wilson, Charles, "Notes on Ricardian Models with Gross Substitutes," www.wilsonc.econ.nyu.edu, unpublished, 2003.