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A STOCHASTIC INFINITE-HORIZON ECONOMY WITH SECURED LENDING, OR UNSECURED LENDING AND BANKRUPTCY

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Abstract: Modeling problems for a monetary economy are discussed and some examples are presented in the context of an infinite-horizon economy with one or two types of traders, who use fiat money to buy a single perishable consumption good. Three instances are considered, all with transactions in fiat money. The first model has no borrowing or lending. The second model permits both borrowing and lending, but all loans are secured. The third model has borrowing and unsecured lending, and takes into account the presence of debtors who are unable to honor their debts and go bankrupt. Borrowing and depositing take place through an outside bank, although in some circumstances a money market could be used instead. Conditions for different forms of lending are discussed. This is a survey of three technical papers, where the mathematical models are developed in detail and the proofs are supplied.

1 Introduction

This is a nontechnical survey of some modeling problems in the theory of money and financial institutions. Several simple examples are presented in order to illustrate the economic distinctions in a dynamic market with fiat money, with or without lending, and with or without active bankruptcy. These examples are part of our current work involving the description and analysis of infinite-horizon exchange economies as strategic market games[1] with many small agents.[2] There are many technical difficulties in the general mathematical analysis, and many open problems. This survey, without delving into the full formal mathematical apparatus needed to make the analysis rigorous, provides a heuristic discussion and illustrates many of the modeling problems.

Three companion papers [Karatzas, Shubik and Sudderth (hereafter, KSS), 1994, 1997; and Geanakoplos, Karatzas, Shubik and Sudderth (hereafter, GKSS), 1997] provide formal proofs of the nature of the stationary wealth distributions and some results on the existence of equilibrium in economies under (i) no lending, (ii) under secured lending, and (iii) under lending with bankruptcy. Where appropriate, we

[1] See Shubik (1973), Shapley (1976), Shapley and Shubik (1977) for expository articles on strategic market games, and Shubik and Whitt (1973) for an infinite horizon game without exogenous uncertainty, Karatzas, Shubik and Sudderth, also Geanakoplos, Karatzas, Shubik and Sudderth (referred to henceforth as KSS, 1994; KSS, 1997; and GKSS, 1997) for an infinite horizon game with exogenous uncertainty.

[2] Technically, a continuum of agents.

indicate conjectures which hold for certain examples, but have not yet been proved in any generality.

Using some extremely simple economic models, our purpose is to illustrate several features of a dynamic economy with fiat money and some forms of loan markets. In particular, we address some of the following questions:

(1) What constitutes fully secured lending?

(2) How is fiat money conserved in an economy with bankruptcy?

(3) How can unpaid debts be discharged?

(4) What are the conditions under which interest rates for lending and saving should or should not be equal?

(5) What is the relationship between the interest rate for saving and the "natural discount factor" (which appears in the utility function defined in Section 4)?

(6) Under what circumstances can a loan market, together with the appropriate default rules, bring about Pareto-optimal equilibrium?

(7) When can a money market substitute for a central bank?

(8) When and why are bounds required on the amounts that an individual can borrow?

(9) What are the information and communication requirements for a viable credit market with secured lending?

(10) Does the introduction of a loan market in an economy with uncertainty improve the welfare of all agents?

(11) What are the conditions required on an outside bank to run a loan market, if the private supply and demand for money are not equal?

(12) How powerful an instrument can monetary policies be, via the control of interest rates for borrowing and saving, or via credit restrictions?

(13) What can be said about the dynamics of the moves of the agents when the economy is out of equilibrium?

(14) What interventions are needed to prevent price crashes or explosions?

(15) What can be said about the velocity of circulation of money?

The general results in our other papers and the examples provided below give answers, or at least some insights, into the answers to these problems.

2 Playable Strategic Market Games

2.1 A change in the paradigm

In economic theory, a contrast is made between statics and dynamics. There are the static general-equilibrium and partial-equilibrium theories and, in contrast, there are a host of oligopoly models, growth theories, innovation models and bargaining theories where some form of dynamics is postulated.

We suggest an intermediate stage between the noninstitutional simplicity of general equilibrium and the explicit behavioralism of macroeconomics or any dynamic model involving motion. This stage contains fully-defined process models studied for their equilibrium properties. Implicit or explicit assumptions are required as to how expectations are formed and how economic agents update their moves in order to describe their laws of motion, even though only equilibrium may be studied.

Before one can discuss behavior carefully, it is necessary to have a full description of the game. All feasible moves by each agent must be specified, along with their payoffs under every conceivable play of the game. This requirement has nothing to do with equilibrium, with rationality or with any assumptions about "how individuals should behave." In terms of game theory it is merely the requirement that all the rules be well-defined. In the terminology of stochastic processes, it is the requirement that the outcome for every element in the state space be well-defined. The easiest way to view this requirement is in terms of an experimental game. If the game is to be played without questions to the referee, the rules must be known to and understood by all, and they must specify clearly what happens under all circumstances.

Another advantage that comes from thinking in terms of a well-defined playable game, is that it helps one to avoid misplaced emphasis on unnecessary details of institutional "realism." Instead of trying to catch all of the features of specific institutions which are the carriers of economic and financial process, one concentrates on minimal institutions, i.e., on those rules which are minimally sufficient to permit the appropriate function to be performed.

The title of this section is "a change in the paradigm." The change proposed here is from equilibrium modeling, which implicitly avoids dynamics and does not fully define the state-space out of equilibrium, to game-theoretic and experimental gaming modeling, which requires a full description of what happens for all moves which occur, no matter how foolish or nonoptimal they may appear to have been *ex post*.

Part of the argument made here is that even the most unconstrained competitive market is surrounded by a host of rules, laws, customs and institutions which limit the action in the game. A glance at the body of rules governing the functioning of the New York Stock Exchange should be sufficient to indicate that the rules are written to let us know what will happen under all circumstances and to constrain behavior in such a manner, that certain forms of panic and disaster which could destroy a market are prevented by the rules. A simple example should help to illustrate this last point. If all trade must be for fiat money, the velocity of trade is bounded, and the authorities do not issue more fiat money, then there is an upper bound to the market price for a given amount of goods being sold for fiat. If individuals know that these bounds exist, then it is reasonable to assume that the expectations formed concerning future prices will also be bounded.[3]

2.2 Rules, institutions, bounds, continuity and compactness

In the associated technical papers we have completely specified several playable games and proved the existence of a stationary wealth distribution in an economy without loans; in an economy with an interest rate formed endogenously by a money-market; and for an economy with an outside bank which lends and accepts deposits, with one or two interest rates set by the bank. Here we discuss the meaning of an equilibrium with a stationary wealth distribution, and discuss the assumptions made to provide sufficient conditions for its existence.

Having made comments about the unsatisfactory aspects of equilibrium theory, it may seem strange that we concentrate on an equilibrium proof. We do so for two reasons. The first is to show that equilibrium results can be viewed as special cases of fully-defined dynamic models. The second is to illustrate the nature of the control conditions needed to constrain the dynamics in order to guarantee the existence of equilibrium. This second purpose can be regarded as an exercise in mathematical institutional economics, in the sense that it is devoted to exploring the conditions, rules, or institutions required to promote "orderly markets."

We suggest that the fussy mathematical details which are sufficient, and in some instances may be necessary, for an existence proof, all have economic meaning and help to cast some light as to how a society needs to supervise its markets. No claim is made that our conditions are completely general. There may be other, possibly better ways to attain the appropriate bounds, compactness and continuity conditions which are needed for the proofs. But *some* such conditions are needed for a completely defined and tractable model which supports the dynamics.

In essence, in order to prove the existence of a fixed point for the mapping which takes old levels of wealth, prices and interest rates into their new levels, we require that these values be bounded (or belong to a compact set). In particular, we would like to make sure that, no matter what strategies are employed by the agents, these bounds are maintained.

Some of the required bounds follow from natural assumptions about the game. For example, natural assumptions about the utility functions of the individuals enable us to show that they will spend enough fiat money to keep the price of the consumption good bounded away from zero (cf. Section 7.2 of GKSS, 1997). To keep the price bounded from above, it is necessary to have a rule providing a bound for the amount that an individual is allowed to borrow before spending. To bound the interest rate in the case of the model with a money market, we consider a central bank that steps into the money market, just before the rate is finally formed, either to add money, or to borrow money in order to keep the rate within "acceptable bounds" (cf. Section 6.3 of KSS, 1997).

[3] If zero goods are offered we assume that the market is not open. If some goods are offered we assume that a minimum unit is traded.

The introduction of what may appear to be a set of *ad hoc* institutional rules designed to bound prices, borrowing and interest rates, is mathematically necessary to establish the appropriate structure for the existence of smoothly functioning competitive strategic markets. At equilibrium, much of the emergency apparatus need not be employed actively, yet it is needed to prevent disaster out of equilibrium. In reality there are often flaws in the safeguards and regulations. The financial disasters which have occurred may be attributed to situations in disequilibrium, where the safeguards (bounding mechanisms) needed to prevent the dynamics from going out of control were not present.

What is the relationship between this "institutionally monitored game" approach and general equilibrium and its variants? General equilibrium is nonstrategic. An attempt to specify the state space for all strategies shows that there are certain (possibly implausible, but nevertheless perfectly feasible) strategies that may lead to items such as a division by zero, which could cause arbitrarily high prices or rates of interest. It does not seem to be possible to define well and fully a strategic game, without at least placing some limits on strategic choice. But the limits required appear to be exactly the sort of rules and customs which have evolved in the construction of financial institutions, complete with rules on specialists, limits on trading, default rules and bankruptcy codes.

In actuality, when the regulatory apparatus is well designed, to a good first approximation the economy may function smoothly and the dynamics remain bounded. But the institutional and administrative design is often difficult and imperfect; thus credit restrictions may be loosened as a result of boom psychology and unrealistically optimistic forecasting, leading to a loosening of the constraints and destroying orderly markets by unbounding credit lines in an exponential manner. Disaster may strike, but, when the disaster is bad enough, play is stopped and the rules of the game change. The general equilibrium formulation, being completely predynamic and abstracting away from the full definition of the strategic freedom permitted, fails to illustrate the need for the extra "rules of the game" needed to carry the dynamics. The strategic market games require this level of detail in order fully to define strategic equilibrium.

3 Modeling Considerations

3.1 Some Needed Explanations in a Theory of Money

There are some relatively mundane aspects of economic life which can easily be ignored in an equilibrium theory concerned with the existence of prices but not with the mechanisms which bring them into being. These are: (1) the presence of fiat money and the nature of the conservation laws concerning its supply in the markets and in the banking system; (2) the existence of the "float," or a transactions need for money; (3) the need for default, bankruptcy and reorganization rules, if lending is permitted, and (4) the nature of interest rates as parameters or control variables or as endogenous variables.[4]

3.2 Preliminary Conditions

There is frequently a trade-off between "realism" and analytical tractability. Economic systems are notoriously multivariate and, except in special cases, little can be said about their dynamics. Our approach here is to try to break down the models into "bite-sized pieces." A listing of the simplifications and the justification for them is given.

1. Number of agents: We utilize a continuum of agents as a reasonable approximation, for many purposes, of a mass anonymous market. The assumption of a continuum of agents not only produces game-theoretic models closely related to general equilibrium models, but also simplifies the strategy sets of individuals and makes the analysis of perfect equilibria more plausible, as the threat structure of the game is kept extremely simple.

2. Number of types of agents: Two agents are of the same type if they have the same preferences and the same distribution of income. They can differ in wealth. As in Lucas (1978), Stokey and Lucas (1989), Bewley (1986) and others, for some purposes we consider one type of agent, i.e., a "representative or typical consumer." For some purposes, however, we consider more than one type of agent.

3. Number of goods traded: Most of our models are with fiat money and one representative or aggregate good. The proof of the existence of equilibrium in KSS (1994) does not depend on the restriction to one type of trader, but the proof of uniqueness of the equilibrium depends on the assumption of one type of good. Uniqueness may be lost for two or more goods.

4. Production and exchange or exchange only: Models with production choices require a considerable complication of the strategy sets which, although highly desirable for many purposes, are not needed for the first investigations which concentrate on the money-market or banking system. The important link between the rate of interest and production can to some extent be illustrated by considering a storable consumable commodity. The act of inventorying may be regarded as a simple form of production.

5. Complete or incomplete markets: We assume the existence of exogenous uncertainty which cannot be perfectly hedged. Furthermore, we impose a structure of spot markets and one-period loan markets.

6. Type of money: We confine our investigation to fiat money. The extra properties of commodity money can be well covered in one- or two-stage models (see, Shubik, 1998). The modification of trading needs caused by the existence of clearinghouses is covered elsewhere (Quint and Shubik, 1995).

[4] A fifth key phenomenon is the *velocity of money*, which is given by the volume of transactions per unit of money per period of time. We delay our study of velocity and avoid the new difficulties it poses, by considering discrete-time models, where velocity is constrained to be between 0 and 1. The relationship between the discrete- and continuous-time models is of importance. But we suspect that the transactions need for money cannot be ade- quately modeled by continuous-time models alone, without imposing some discrete-time aspects of economic life on the continuous-time structure.

7. Existence of credit: The extension of credit, in the context of a money-market, calls for the creation of a least one new financial instrument, the IOU note.[5] The rules concerning its operation must be given explicitly.

8. Secured or unsecured lending: When studying equilibrium conditions it is possible to specify conditions under which no bankruptcy occurs at equilibrium.[6] We split our analysis into two cases, fully secured lending or lending where there may be bankruptcy at equilibrium lending.

9. Bankruptcy and reorganization rules: In a multistage economy, when there is default, rules must be given to cover not only the treatment of bankrupts but also to specify what happens to the creditors, as it is possible that they may not be fully repaid. Even the simplest of rules requires considerable detail. Considerable modeling difficulties are encountered in showing that such rules exist which appear to be reasonable and also lead to a mathematically tractable model.

10. Overlapping generations: The assumption of the existence of individuals with a finite expected life, with overlapping generations, is far more reasonable than assuming infinitely-lived individuals. We believe that our mathematical results can be extended in a straightforward way to encompass these extra features, but defer extending the analysis of overlapping generations to future work.

11. Variation of the money supply: In general, an economy with a variable supply of goods requires that the money supply must be varied, if hoarding or boundary solutions are to be avoided. This requires, in a strategic model of the economy, that one specify how a banking system is in a position to expand or contract the money supply and the supply of credit. For the most part we confine our analysis and examples to economies where the aggregate amount of good for sale each period can be regarded as constant, even though incomes of individual agents are stochastic. This can be done utilizing the construction of Feldman and Giles (1985) as noted in KSS (1994). But even here there may be a need to vary the money supply. A discussion of the general problems involving the variation of the money supply is deferred to a separate investigation.

12. Information conditions: In games with a continuum of agents, without exogenous uncertainty, information conditions do not matter (see Dubey and Shubik, 1981) as a nonatomic individual agent has no strategic power over others. However, if there is randomization in the game, and if some players are informed of the outcome of the randomization and others are not, the nonsymmetric distribution of information could be valuable.

[5] The introduction of a bank which lends and accepts deposits requires two new instruments, its IOU notes given in exchange for deposits and individual IOU notes when it makes loans.

[6] Even though there may be no bankruptcy at equilibrium, a fully strategic model requires the specification of rules to cover default which might occur out of equilibrium.

4 The Market Without Loans

Prior to introducing loans, the first example presented is of an exchange economy utilizing fiat money for all transactions. In the study of a simple exchange economy KSS (1994) we were able to establish general conditions for the existence of a stationary state in an economy, and to provide explicit examples of economies with stationary wealth distributions, which serve to provide some insight into the economic aspects of the analysis.

Example 1: One type of agent, no loans, saturated utility

Suppose that each agent α produces in each period t a random quantity Y_t^α of a consumption commodity, such that $Y_t^\alpha = 3/2$ with probability $\gamma = 1/4$, and $Y_t^\alpha = 1/2$ with probability $1 - \gamma = 3/4$. For convenience, we take the index set for agents to be the unit interval $I = [0, 1]$. Then the total quantity of the commodity Q_t in period t can be written as the integral

$$Q_t = \int_I Y_t^\alpha d\alpha.$$

We assume that Q_t is well-defined and constant from period to period: $Q_t = Q = 3/4$. For each agent α, the variables $Y_1^\alpha, Y_2^\alpha, \ldots$ are assumed to be independent.

Agents are required to offer for sale all of their production,[7] and to use the money they have to buy whatever amount of the good they wish to use for personal consumption. Thus, if an agent α begins period t with S_{t-1}^α units of fiat money, he may bid any amount b_t^α in $[0, S_{t-1}^\alpha]$ for the good.[8] Bids are made simultaneously with each trader knowing only the distribution of his income for the period at the time of the bid. The price p_t of the commodity is formed as the ratio of the total bid to the total production

$$p_t = \frac{\int_I b_t^\alpha d\alpha}{Q},$$

and each trader α receives an amount of the good, $x_t^\alpha = b_t^\alpha / p_t$, in proportion to his bid. At the end of the period each individual α receives his income's worth $p_t Y_t^\alpha$ in fiat money, and enters the next period with wealth

$$S_t^\alpha = S_{t-1}^\alpha - b_t^\alpha + p_t Y_t^\alpha.$$

The market mechanism is shown in Figure 1 below.

[7] The sell-all model is a device which requires that individuals monetize all their assets each period. It is a tax collector's dream as all assets are reevaluated by the market each period. It also enables us to confine our concern to one-commodity models, where fiat money is needed for bidding as there is a payment lag. A simple example of a market of this type is where a modern farmer sends all of his milk production to the market and buys milk for personal consumption.

[8] There is in our opinion a subtle, but important difference between this model and the usual "cash-in-advance" models. The individual must pay cash or credit in order to form price. The process analysis requires that the method for forming prices must be made explicit. The easiest way to do this is a Cournot style price formation mechanism. A Bertrand–Edgeworth style mechanism would lead to a somewhat different process.

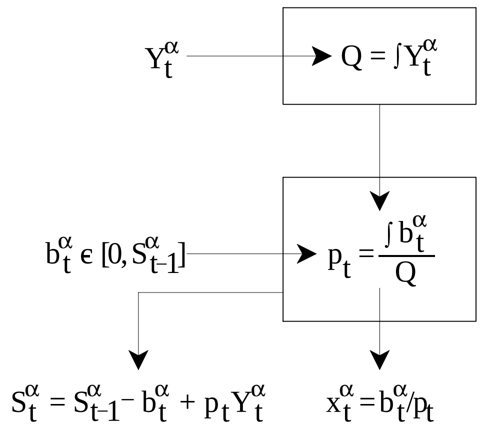

Figure 1. Bids, Consumption, and Income

Suppose now that each agent has the same extremely simple one-period utility function, as in Figure 2, whereby each individual desires up to one unit of the good beyond which utility saturates.

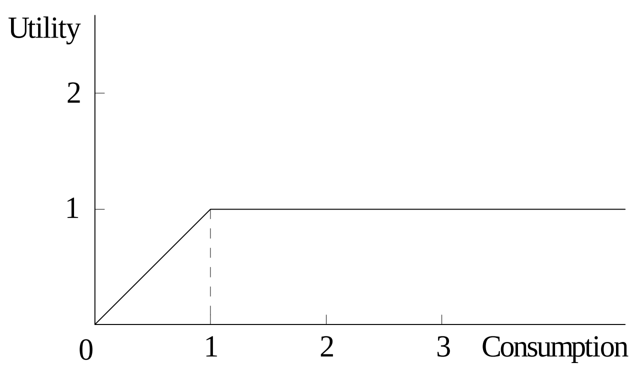

Figure 2. A Simple Utility Function Saturating at One

Every agent α seeks to maximize the expected value of total discounted utility,

$$\sum_{t=1}^{\infty} \beta^t u(x_{t+1}^\alpha),$$

where the discount[9] factor β is in (0,1) and, as shown in Figure 2, the utility function u is given by

$$u(x) = \begin{Bmatrix} x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{Bmatrix}$$

If the prices p_t remain fixed and equal to a constant price p, say $p = 1$, then it is almost obvious that the optimal policy for a trader is to spend up to the saturation

[9] This is sometimes referred to as the "natural discount" where one could interpret the utility function as representing a dynasty of individuals, in which the concern of individual t for the welfare of individual $t+1$ is β. Otherwise $1/\beta$ can be interpreted as expected life-time.

point 1 and, if the trader has more than 1 unit of cash, to save the additional funds for future expenditures. This policy, which is proved to be optimal in (KSS, 1994), can be written in the simple form

$$b = c(s) = \begin{Bmatrix} s, & 0 \leq s \leq 1 \\ 1, & s > 1 \end{Bmatrix}.$$

When an agent α follows this policy, the stochastic process $\{S_t^\alpha\}$ of the agent's successive wealth levels is a Markov chain with stationary distribution concentrated on $\{1/2, 1, 3/2, 2, ...\}$ and transitions as shown in Figure 3.

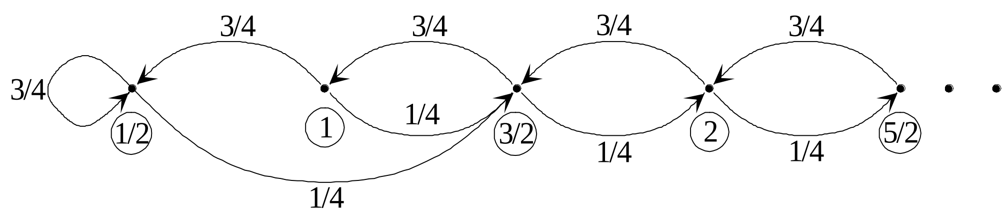

Figure 3. A Markov Chain

For example, at a wealth level of 2, the optimal policy is to spend 1. Then there is a probability of 3/4 that the agent earns 1/2 and begins the next period at 3/2, and a probability of 1/4 that the agent earns 3/2 and moves to 5/2. The stationary distribution for the chain is

$$\mu_{1/2} = \tfrac{1}{2},\ \mu_1 = \tfrac{1}{6},\ \mu_{j/2} = \tfrac{2}{3}\left(\tfrac{1}{3}\right)^{j-2},\ j \geq 3,$$

as illustrated in Figure 4.

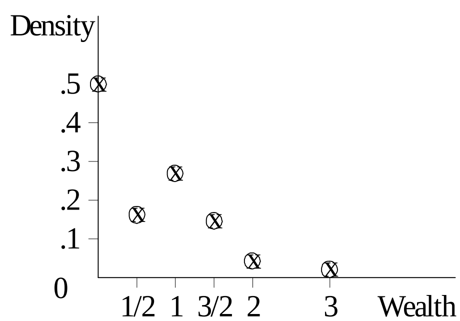

Figure 4. Wealth Distribution

This stationary distribution for the Markov chain of a single agent also corresponds, in equilibrium, to a stationary wealth distribution across agents. That is, although the wealth of each individual follows the Markov chain, the distribution of their aggregated wealth remains fixed at the distribution μ above.

At this equilibrium distribution, one half of the agents have wealth 1/2, all of which they spend, and the other half of the agents have wealth of at least 1 and spend 1. The total bid is therefore

$$B = \tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{2} \times 1 = \tfrac{3}{4}$$

so that the price

$$p = \frac{B}{Q} = \frac{3/4}{3/4} = 1$$

does stay fixed at 1. The total amount of fiat money in the economy is

$$M = \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{6} \cdot 1 + \sum_{j=3}^{\infty} \tfrac{j}{2} \cdot \tfrac{2}{3} \cdot \left(\tfrac{1}{3}\right)^{j-2} = 1.$$

Thus, $M - B = 1/4$ is hoarded every period.

An interpretation of the result is as follows. The economy has a fixed total amount of fiat money, which is used to buy a constant total amount of the perishable good each period, but each individual's income is uncertain even though the total amount of resource in the economy is fixed. Thus if a stationary state exists, there will be a constant price over all time and each individual will appear to face a one-person dynamic program.

In the proof given in KSS (1994), we insist upon a well-defined process model in which the actions of individuals form the market price. The actions are conditioned on the players' expectations. We specify a simplistic model for the formation of expectations, as follows: Each player is treated as though he were composed of two agents. One agent is mathematically sophisticated, but otherwise naive; he is the calculator or program-solver. The other agent is somewhat simplistic; he is the market forecaster. The forecaster uses as his rule, that all prices in the future are going to be the same as the last price observed. He tells this to the calculator, who naively believes him and solves a sophisticated dynamic program based on this simplistic advice. All individuals move based upon the calculations of the programmers using the naive forecasting, and the process continues. It is fairly easy to see that if a stationary Markovian equilibrium exists,[10] and if we start everyone at the equilibrium, then the naive forecasting will be self-fulfilling. There will be hosts of other forecasting methods which would work equally well *in equilibrium*. The real economic problem is how well the forecasting performs *out* of equilibrium; this requires considering the full dynamics. At this time we have no results on the dynamics.

The meaning of the wealth distribution shown in Figure 4 is that if we start all individuals with the initial wealth distribution as indicated, and if all are given the appropriate predicted price, then they will select actions with the following property: after all have acted, and the randomization concerning income has taken place, then this whole wealth distribution will map into itself. The wealth of various individuals

[10] The assumption of a continuum of agents constrains our attention to subgame perfect equilibria as threats by a "small" individual have no influence. This is not true for a strategically active central bank.

will have been moved around, but the overall aggregates will have stayed constant. A difficult open question is what happens when the initial wealth distribution is not the stationary distribution given above.

Example 2: An economy with two types of trader and no loans

Calculating even simple nontrivial examples of infinite-horizon stochastic models with more than one type of trader is difficult. Furthermore, the chances for unique equilibria are not high. There is, however, one easy and economically interesting extension to two types of traders. We add a class of traders who are risk-neutral. One might suspect that they would become the lenders or insurers of the others, when lending is permitted. Here we consider only trade.

We may immediately extend Example 1 above to the situation in which some percentage of the population, say η, is completely risk neutral, and the remaining $1-\eta$ has the utility function of Figure 2. All traders are assumed to have the same income claims to the proceeds obtained from the consumption good being put up for sale.

It is straightforward to check that there is a stationary equilibrium, where the risk-neutral traders spend their wealth each period. and the others follow the strategy $c(\cdot)$ of Example 1, modified only by a different normalization. The wealth distribution of the risk-averse is as before, but for the population as a whole the densities at the wealth levels $.5p$ and $1.5p$ are modified with $1/2$ of the population of the risk-neutral traders added at each point.

In equilibrium, income and expenditure of each group must be constant; thus the *per capita* wealth of the risk-averse traders will be lower than that of the others, even though on the average both types earn and consume the same *per capita* amount.

The difference in risk-aversion is manifested in the hoarding of money by the risk-averse. If this were played as a game where all individuals were started off with the same amount of money, the risk-neutral would gain considerably in any adjustment to equilibrium, by deriving consumption from spending the money which disappears into the hoard of the risk-averse.

5 The Market with Loans from an Outside Bank

5.1 Different types of lending arrangements

There are at least four types of lending structures, all of which coexist today and have manifested themselves at one time or another during history. They are (1) bilateral lending between a specific lender and borrower; (2) aggregated lending and borrowing through a money market; (3) borrowing from an intermediary such as a commercial or inside bank, which may also accept deposits from individuals, or (4) borrowing from and lending to a government or outside bank which controls the supply of fiat money.

Historically it appears that bilateral loans in kind, and then in commodity money, were the oldest form of borrowing and lending. It is possible to set up probabilistic models of borrowing by randomly matched pairs. These may be of interest in economic anthropology in casting some light on the emergence of money markets and banks. But we restrict our concern to economies which have mass anonymous organized markets and an outside bank.

We model first an outside or government bank and, in Section 6 below, a money market, but refrain from considering inside or commercial banks. The reason for this omission is that the model of a commercial bank as a simple profit-maximizer is hard to justify in an economy with incomplete markets, and it is even harder if the bank has a corporate form and is run by fiduciaries. This is a lengthy topic requiring separate treatment covering stockholder and management behavior. At this level of simplicity we therefore leave out both commercial and merchant banks, and concentrate on contrasting behavior in markets with (i) no loans; with (ii) loans via a money market; and with (iii) loans via an outside or central bank.

5.2 The outside or government bank

We now consider an economy closely related to that of Section 3. We introduce an outside bank which is in a position to make loans and to accept deposits. As our examples are extremely simplified, it is possibly helpful to regard them as experimental games which (at least for a finite number of periods and players) could be played in an experimental laboratory. The outside bank may be regarded as being part of the game control, and its policy as set by the referee.

In particular, we consider the outside bank to be a strategic dummy, whose policy is set in advance and known to all (like the Dealer in Blackjack). In our first model we assume that the agents deal only through the bank. Thus, for example, if the bank does not accept deposits, or pays a low interest rate on them, potential lenders do not have the opportunity to locate individual potential borrowers who might be willing to pay more. (If there were n types of traders, we might need to consider different loan interest and deposit rates and credit limits the bank might have for each type of trader.)[11]

When an individual's income is variable and he borrows, it is often feasible that he will find himself in a position where he is unable to repay, unless the lending is secured in a way to prevent default. When default occurs we must have some convention to resolve the situation, so that all parties can proceed. This can be done in many ways;[12] see, for example, Section 5.4 below.

5.3 On secured lending

If the income of an individual is determined by a random variable, and the individual can borrow more than his minimum income, there is a chance that the individual will be unable to pay back the money owed.

[11] In this paper we assume symmetric information conditions hence the different risk profiles are objectively known and provide the basis for "redlining" or placing different constraints and interest rates on different classes of borrowers.

[12] See Shubik (1995, Chs. 11 and 12) for a discussion and historical information.

Even a casual look at the world around us will confirm that there is a limit to the amount that anyone can borrow. When studying equilibrium results in much of economic theory it may be mathematically convenient to assume otherwise. For some purposes the assumption of unbounded availability of borrowing or unbounded short sales may be adequate. But in finance, the lenders are invariably concerned with the probability that they may not be paid back. The axiom of lending in good banking is that it is a function of "character, competence and collateral." In much of the literature in economics, character and competence are rarely modeled.[13] In practical problems of banking and insurance, they are important. Collateral can be split fairly reasonably into two parts. One consists of the physical assets owned by the individual, the other, of the contracts and arrangements the individual has which will generate income in the future. Thus an employment contract indicating that the individual has a five-year commitment at a salary of $500,000 a year from a major corporation is "bankable." Even tenure at a sufficiently financed university has some credit worth at the local bank; it certainly will be taken into account when the individual borrows.

The size of the mortgage market alone, indicates the importance of secured lending.

Here, for simplicity we do not consider the asset structure of the individuals. We consider only their monetary income from the sale of a commodity they produce. A slightly richer model could introduce an asset, such as land, which is the source of this production. It is implicitly present here, but the individuals are not permitted to trade it; hence it need not appear explicitly in the model.[14]

Some lending is based on character and competence. We do not underestimate their importance, but to do justice to them one needs an acceptable theory explaining how these are inferred. Suppose that lending is based on a knowledge of the future claims of each individual to income from the sale of the good they own; if the price at which the borrowers could sell their good were known in advance, the lenders could accurately estimate the ability of a borrower to repay, and they could make sure that the loans are secured. In actual lending practices a "haircut" is usually given to the amount lent, based on the assessed value of the security which is posted. The loan may constitute only a fraction of the current estimated worth of the asset used as security. The difference provides a safety cover for the possibility of changes in price, and other unforeseen contingencies as well as incompetence, comfort and conservatism.

In a fully dynamic system, where prices are being formed after loans have been made, there is, in essence, no such thing as a totally secured loan, unless price bounds are imposed,[15] for the simple reason that one cannot rule out the possibility that the price of whatever is being sold in the future may drop to zero or close enough to zero

[13] Certain aspects of character are implied in the literature on moral hazard.

[14] When land of more than one type is introduced explicitly as a productive asset it is easy to extend the model to include trade in shares of the output thus a dynamic relative to CAPM can be constructed.

[15] But see Shubik and Yao (1990) where if gold is both money and a consumer durable one can construct a totally secured loan as the price of money in terms of itself is always one.

to push the individual into default.

When price plummets, or interest rates become "unreasonable," we may need government intervention to "preserve orderly markets." In our technical papers we note that these apparently "institutional" features of the government, which prevent interest rates or prices from becoming too extreme, provide sufficient conditions for the bounding of certain sets of outcomes.[16]

In the models presented below, we assume that lending is bounded by the minimum amount an individual will have available to pay back, if price has a lower bound. Thus active bankruptcy will always be averted.

Example 3: An outside bank with a zero rate of interest and secured lending

Consider the same economy as in Example 1, with the additional feature of an outside or central bank. Suppose that the bank is willing to accept deposits and to lend any individual up to an amount $k = 1/2$. Assume also that the bank sets the rate of interest $\rho = 0$ for both borrowers and depositors. The bank returns to depositors their deposits at the beginning of each period, and requires borrowers to pay back their loans at the same time.

We may regard this as *secured lending* in the following sense. If the bank has predicted that prices in the future will be $p_t = 1$, then given the incomes of each individual, all will have an income of at least 1/2 and, at least at a rate of interest of $\rho = 0$, will always be able to pay back up to 1/2. If the guess of the bank concerning predicted price is incorrect, and the price turns out to be lower, then some debtors may not earn enough to pay back the loans which were meant to be "secure."[17]

Given that future spot prices are $p_t = 1$ for all t, then it can be shown (cf. KSS, 1997) that an optimal strategy for an individual with wealth s is: If $0 \leq s \leq 1/2$, borrow 1/2 and spend $s + 1/2$; if $1/2 < s < 1$, borrow $1 - s$ and spend 1; if $s \geq 1$, deposit $s - 1$ and spend 1. Thus, exactly as in Example 1, each agent seeks to consume as close as possible to one unit of the good, at which point the utility function saturates. The difference here is that agents with less than one unit of money can now borrow from the bank.

When an agent follows the optimal strategy just described, the stochastic process of the agent's successive wealth becomes a Markov chain as depicted in Figure 3, except that it is now shifted to the left by half a unit. Thus, the stationary distribution for the chain is given by

$$\mu_0 = \tfrac{1}{2},\ \mu_{1/2} = \tfrac{1}{6},\ \mu_{(j+1)/2} = \left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)^j,\ j \geq 1.$$

As in Example 1, this is also the equilibrium distribution of wealth across agents. The total bid is easily calculated to be $B = 3/4$ as before, but now includes bids from agents at 0 and 1/2 that involve borrowing. Thus the equilibrium price is $p = B/Q = (3/4)/(3/4) = 1$, as predicted.

[16] They provide the sufficient conditions for the compactness of the payoffs and enable us to obtain existence proofs.

[17] See Shubik and Yao (1989) for a discussion of secured lending.

The total wealth held by agents prior to borrowing is

$$M = \tfrac{1}{2} \cdot 0 + \tfrac{1}{6} \cdot \tfrac{1}{2} + \sum_{j=1}^{\infty} \left(\tfrac{j+1}{2}\right) \left(\tfrac{2}{3}\right) \left(\tfrac{1}{3}\right)^j = \tfrac{1}{2}.$$

The total *amount borrowed* by agents at 0 and 1/2 is

$$\tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{6} \cdot \tfrac{1}{2} = \tfrac{1}{3}$$

and the *amount deposited* by agents having wealth greater than 1 is

$$\sum_{j=2}^{\infty} \left(\tfrac{j+1}{2} - 1\right) \left(\tfrac{2}{3}\right) \left(\tfrac{1}{3}\right)^j = \tfrac{1}{12}.$$

Notice that the total amount deposited differs from the total amount borrowed, so that the bank must hold an amount R of reserves, in order to meet the demand for loans. The money supply for the society as a whole can be taken to be $R + M = 1/4 + 1/2 = 3/4$, where the bank reserves of $R = 1/4$ together with deposits of $1/12$ are sufficient to provide money for lending.

Because of the saturated utility function in this example, there cannot be an equilibrium with a positive rate of interest paid to depositors. Indeed, for any $\rho > 0$, the interest income of a sufficiently rich individual will exceed his spending of one unit of money and, consequently the wealth of such an individual will increase toward infinity.

The next example has a utility function u that does not saturate and for which the derivative u' is bounded away from zero. This makes it possible to construct an equilibrium with positive and different interest rates for borrowers and depositors.

Observe that in equilibrium the bank must balance its books every period, in the sense that the net interest paid by borrowers to the bank must equal that paid by the bank to depositors. Otherwise, the total wealth held by agents would be growing or shrinking, and, in particular, the distribution of wealth could not remain fixed. In Example 3 with zero interest rates and secured lending, the books trivially balance because the net interest paid by borrowers is zero, and so is that paid to depositors.

Example 4: An outside bank with two different positive interest rates and secured lending

Suppose that each agent α in each period t now produces Y_t^α units of the good where $P[Y_t^\alpha = 1] = 1 - \gamma$ and $P[Y_t^\alpha = 4] = \gamma$, $0 < \gamma < 1$. As before, the total production

$$Q = \int_I Y_t^\alpha d\alpha = 3\gamma + 1$$

is assumed to be the same in every period. Agents are assumed to have the piecewise linear utility function

$$u(x) = \begin{Bmatrix} x, & 0 \le x \le 2\tfrac{1}{2} \\ 2\tfrac{1}{2} + \eta\left(x - 2\tfrac{1}{2}\right), & x > 2\tfrac{1}{2} \end{Bmatrix},$$

where the parameter η is in the interval (0,1). The function u is shown in Figure 5. The bank offers loans to borrowers up to a limit of $k = 1/3$ at the interest rate $\rho_1 = 2$, and it pays depositors at rate $\rho_2 = 1$. The discrepancy between these rates is explained below.

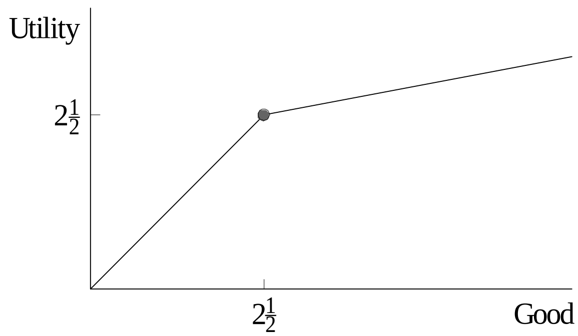

Figure 5

The parameters γ, η, and the discount factor β can be chosen so that the equilibrium price p is 1, the books balance, and the optimal policy for an agent is as follows:

Borrow $\frac{1}{3}$ and spend all for $0 \le s \le 2\frac{1}{6}$;
borrow $2\frac{1}{2} - s$ and spend all $(= 2\frac{1}{2})$ for $2\frac{1}{6} \le s \le 2\frac{1}{2}$
lend $s - 2\frac{1}{2}$ and spend $2\frac{1}{2}$ for $2\frac{1}{2} \le s \le 3$;
lend $\frac{1}{2}$ and spend $s - 2\frac{1}{2}$ for $3 \le s$.

Under this policy, an agent's optimal bid b at wealth s is given by

$$b = c(s) = \begin{Bmatrix} s + \frac{1}{3}, & 0 \le s \le 2\frac{1}{6} \\ 2\frac{1}{2}, & 2\frac{1}{6} \le s \le 3 \\ s - \frac{1}{2}, & 3 \le s \end{Bmatrix}$$

as is shown in Figure 6.

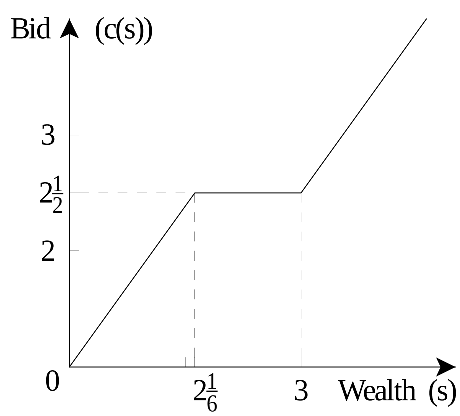

Figure 6

It is easy to check that the wealth of an agent using this policy will, after no more than 2 days, take values in the set {0,2,3,5}. Thus, the stationary distribution of the process is just that of this simple Markov chain shown in Figure 7.

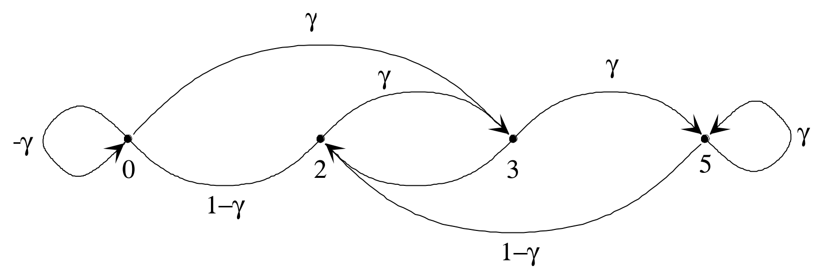

Figure 7

The stationary distribution of this chain is

$$\mu_0 = (1-\gamma)^2,\ \mu_2 = \mu_3 = \gamma(1-\gamma),\ \mu_5 = \gamma^2.$$

Consider now the requirement that the bank balance its books under this equilibrium wealth distribution. Under the proposed optimal policy, agents with wealth 0 and 2 will each borrow 1/3 and pay back $(1+\rho_1) \times \frac{1}{3} = 3\frac{1}{3} = 1$ to the bank. Thus, the aggregate net gain to the bank is $\frac{2}{3}(\mu_0 + \mu_1) = \frac{2}{3}(1-\gamma)$. Agents with wealth 3 or 5 will deposit 1/2 and get back from the bank $(1+\rho_2) \times \frac{1}{2} = 2 \times \frac{1}{2} = 1$. The aggregate net loss to the bank is $\frac{1}{2}(\mu_3 + \mu_5) = \frac{1}{2}\gamma$. The bank will balance its books if and only if $\frac{2}{3}(1-\gamma) = \frac{1}{2}\gamma$; that is, iff $\gamma = \frac{4}{7}$.

Let us fix the parameter γ to be 4/7. It is easy to verify that the price p stays equal to 1 under the stationary distribution. The proof that the policy defined above is optimal is sketched in the appendix.

We are now in a position to start to answer some of the fifteen questions raised in the introduction.

QUESTION 4: *What are the conditions under which lending and deposit rates of interest should or should not be equal?*

OBSERVATION 4: We have constructed an example (Example 4) of an economy with one type of agent and an outside bank with loans and deposits with two different interest rates, one for the borrowers and one for depositors, with no bankruptcy.

In this instance, if we were to consider borrowing and lending through a money market, there would be no equilibrium with an active money market. We believe that the failure of the supply for loans to equal the demand for loans at a single interest rate is due to the "kink," or nondifferentiability, in the utility function. If the utility function were smoothly differentiable, we suspect that the maintenance of two rates would require strict policing by the government bank to prevent arbitrage. However if there are transactions costs associated with the search, contracting and collection procedures in a loan market, then the presence of two rates is consistent with differentiable utility functions. Even with smoothly differentiable utility functions and no saturation of utility, the sufficient conditions for the existence of a single rate which clears the market are somewhat delicate (see KSS, 1997).

The presence of transactions costs is sufficient to justify the introduction of competitive commercial banking, if the banks can profitably provide the matching services at a lower transactions cost than the individuals can in direct lending to each other without intermediaries, or even using a money-market broker.

In the technical companion paper on equilibrium with active bankruptcy (GKSS, 1997) we note that a different reason appears. The two rates can be justified by the requirements for loss reserves for the bank.

QUESTION 5: *When there is a single, nonzero, noninflationary equilibrium deposit and borrowing rate of interest, what is the relationship between the rate of interest and the "natural discount factor" (β in the utility function)?*

OBSERVATION 5: The reason for saving money is to be able to smooth-out consumption, when one is subject to a random income. Saving could be worthwhile even at a zero rate of interest; for instance, if it is guaranteed that a loan will be repaid, a saver is indifferent between lending at a zero rate of interest or hoarding. Any positive rate of interest represents a gain over hoarding, to an individual who wishes to save money for insurance.

If there were no stochastic element in the economy, the equilibrium rate of interest ρ at which the bank's books balance would have to satisfy $1 + \rho = 1/\beta$, where $0 < \beta < 1$ is the "discount factor."[18] When there is a stochastic element, and no bankruptcy at equilibrium, the interest rate which balances the central bank's books and hence guarantees no inflation is more complex. It is determined by the natural discount factor, the full distribution of the income stochastic variable, and the curvature of the utility function. The bank rate must be selected in such a way, that the amounts of borrowing and lending affect each other in equilibrium.

5.4 An outside bank with active bankruptcy

Bankruptcy is an important feature of real economies. Our technical paper (GKSS, 1997) introduces a simple model, in which bankruptcy can and does occur even in equilibrium. The full definition of the rules of our game requires the specification of how the inability to repay is resolved. In a real economy this possibility is handled by laws concerning default, insolvency, and bankruptcy. Rather than get lost in institutional details, we model these rules as simply as possible. Agents who are unable to repay their debts receive a nonmonetary "punishment" in units of utility, but are then forgiven their debts and allowed to continue to play.

To make this "punish and forgive" rule precise, we now assume that each agent's utility function $u(x)$ is defined for negative as well as positive values of x. For $x \geq 0$, $u(x)$ is positive and represents, as it did before, the utility derived from the

[18] Given the simple period-by-period separable utility function we have used, with a constant income each period, there may be no need for active borrowing or lending; but a "shadow rate of interest" may nevertheless exist. If any other rate of interest were announced there would be an opportunity for arbitrage.

consumption of x units of the good. For $x < 0$, $u(x)$ is negative and corresponds to *punishment* incurred for a debt equal to x units of the good, or px units of money when p is the price.

Even though some agents may fail to pay back their loans, the bank must nevertheless balance its books in equilibrium just as before. As the next example illustrates, this can be accomplished if the aggregate repayments of the borrowers are sufficient to make up for those who default.

Example 5: Two interest rates and active bankruptcy

The production variables Y_t^α satisfy $P[Y_t^\alpha = 0] = 1-\gamma$ and $P[Y_t^\alpha = 5] = \gamma$, where the probability parameter γ is in the interval [1/3, 1/2]; thus, the total production is $Q = \int_I Y_t^\alpha d\alpha = 5\gamma$. Notice that an agent, who borrows and spends, will with probability $1-\gamma$ receive no income. Thus, secured lending is impossible even in equilibrium. Agents have the utility function

$$u(x) = \begin{Bmatrix} x, & -\infty < x \le 1 \\ 1 + \eta(x-1), & x \ge 1 \end{Bmatrix},$$

where $0 < \eta < 1$. The bank lends to borrowers up to a limit of $k = 1$ at the interest rate $\rho_1 = 1/\gamma - 1$and pays depositors at rate $\rho_2 = 0$.

As is shown in Appendix B of GKSS (1997), the parameters η and β can be chosen so that the equilibrium price p is 1 and each agent's optimal policy is as follows:

$$\begin{cases} \text{Borrow } 1-s \text{ and spend 1 for } 0 \le s \le 1; \\ \text{save } s-1 \text{ and spend 1 for } 1 < s \le 2; \\ \text{save 1 and spend } s-1 \text{ for } s > 2. \end{cases}$$

For $s < 0$, an agent is punished $u(s) = s$ and then plays the policy above from 0.

After at most two steps, the wealth of an agent playing this policy will lie in the set $\{-1/\gamma, 0, 1, 5 - 1/\gamma, 5, 6\}$. A bankrupt agent at the negative position $-1/\gamma$ is punished by the same amount $u(-1/\gamma) = -1/\gamma$ and then immediately moves to 0 before making a bid. After this move, the chain is as pictured in Figure 8 below. For example, an agent at $s = 0$ borrows 1, spends it, and owes the bank $(1+\rho_1)\times 1 = 1/\gamma$. With probability $\bar{\gamma} = 1-\gamma$, the agent receives no income, goes bankrupt at $-1/\gamma$, is punished, and is then set back at $s = 0$. With probability γ, the agent receives an income of 5 units of money, pays the debt of $1/\gamma$, and moves to $5 - 1/\gamma$.

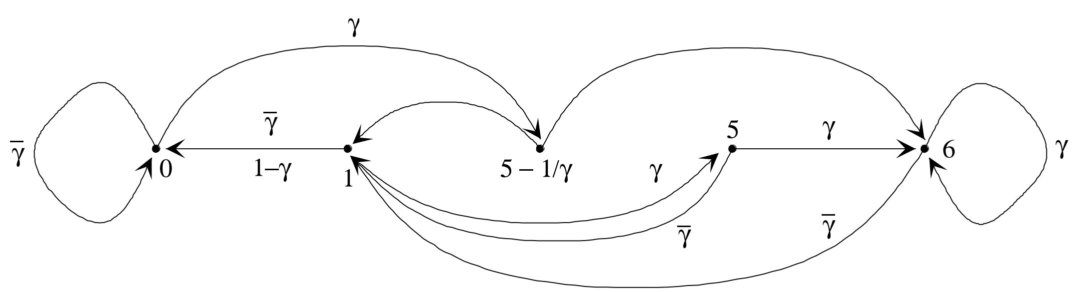

Figure 8

The stationary distribution for the chain is

$$\mu_0 = (1-\gamma)^2,\ \mu_1 = \gamma(1-\gamma),\ \mu_{5-1/\gamma} = \gamma(1-\gamma)^2,\ \mu_5 = \gamma^2(1-\gamma),\ \mu_6 = \gamma^2,$$

the total bid is

$$B = \mu_0 \cdot 1 + \mu_1 \cdot 1 + \mu_{5-1/\gamma} \cdot \left(5 - \tfrac{1}{\gamma} - 1\right) + \mu_5 \cdot (5-1) + \mu_6 \cdot (6-1) = 5\gamma.$$

Thus the price $p = B/Q = (5\gamma)/(5\gamma) = 1$ is in agreement with our assumption.

Observe also that the bank does balance its books. Indeed, the bank has no net loss to its depositors at positions $5 - 1/\gamma$, 5, and 6, because $\rho_2 = 0$. The borrowers at position $s = 0$ each borrow 1; the proportion γ of borrowers who get an income of 5 will each pay back $(1+\rho_1) \times 1 = 1/\gamma$, and the total amount paid back will be $\gamma \times 1/\gamma = 1$.

6 A Money Market for Loans

Let us assume now that there is no outside bank but that, instead, agents bid IOU notes for loans or offer money for lending in a money market. An interest rate is now formed endogenously by dividing the aggregate bid of borrowers for loans by the aggregate funds offered for lending. After borrowing or lending in the money market, agents bid for the commodity and its price is formed just as before.

For the money market model to be in equilibrium with a stationary wealth distribution, *two* balance equations must be satisfied. First, the amount of money borrowed must be equal to the amount lent, and, for the interest rate to be well-defined, these quantities cannot equal zero. Secondly, the amount paid back to depositors must equal that paid back by borrowers. The second condition follows from the first in the case of secured lending. [See KSS (1995) and GKSS (1997) for a detailed explanation.]

The next two examples illustrate first a money market with secured lending and then with active bankruptcy.

Example 6: A money market with secured lending

Assume that every agent has the utility function of Example 4 as shown in Figure 5, and that the production variables, represented by Y, satisfy $P[Y=1] = 1/2 = P[Y=4]$. We want to exhibit a money-market equilibrium with interest rate $\rho = 1$, price $p = 1$, and bound on loans $k = 1/2$. If these parameters are fixed, then, for certain values of η and β, an optimal bid for an agent with wealth s is

$$b = c(s) = \begin{Bmatrix} s + \frac{1}{2}, & 0 \le s \le 2 \\ 2\frac{1}{2}, & 2 \le s \le 3 \\ s - \frac{1}{2}, & s \ge 3 \end{Bmatrix},$$

as is shown in an appendix to KSS (1997). Observe that under this policy, an agent with less than $2\frac{1}{2}$ units of money borrows the maximum or just enough to reach $2\frac{1}{2}$, and an agent with more than $2\frac{1}{2}$ units will lend the excess up to a maximum of $1/2$.

An agent who uses this policy will reach a wealth s in 0, 2, 3, 5 and then follow the Markov chain of Figure 7 with the parameter $\gamma = 1/2$. The stationary distribution is $\mu_0 = \mu_2 = \mu_3 = \mu_5 = 1/4$. All of the agents at 0 and 2 are borrowing 1/2 and paying back 1; those at 3 and 5 are lending 1/2 and getting back 1. Thus, the books balance. It is easy to check that the price is $p = 1$, under the given policy and wealth distribution.

The example works also with a bank in place of the money market.

Example 7: A money market with active bankruptcy

Consider the economy of Example 5 but with a money market rather than a bank. Because many borrowers go bankrupt and fail to pay their debts, the *ex ante* interest rate $\rho_1 = \frac{1}{\gamma} - 1$ charged to borrowers differs from the ex post rate $\rho_2 = 0$ paid to lenders. The same equilibrium as in Example 5 will work for a money-market, if the total amount of money borrowed equals the total amount lent. Under the policy and wealth distribution from Example 5, the total amount borrowed is $\mu_0 \times 1 = (1-\gamma)^2$, and the total lent is $\mu_{5-1\gamma} \times 1 + \mu_5 \times 1 + \mu_6 \times 1\gamma$. For equilibrium, we need that these amounts be equal; this leads to the equation $(1-\gamma)^2 = \gamma$, only one root of which, namely, $\gamma = (3-\sqrt{5})/2 = .38$, is a probability.

This example suggests, what seems to be true more generally, that *a money-market equilibrium is more delicate than an equilibrium with a central bank.*

Question 7: *When can a money market or a central bank be substitutes?*

Observation 7: A central bank can always replace a money market by declaring an exogenous interest rate (or rates) equal to the endogenous rate determined by the money market. A money market can substitute for a bank only if lending and borrowing are in balance.

Question 8: *When and why are bounds required on the amounts that an individual can borrow?*

Observation 8: The credit-granting standards and the default and reorganization rules of a society are, in essence, a public good which facilitates the decentralization of risk-taking in investment. A loan market can be killed (on the private lender side) by having no penalty against default, unless default is bounded (see Observation 9); it is killed (on the borrower side) by having so high a penalty that borrowing is discouraged.

Without some limit on the generation of IOU notes, bidding could be unbounded and the game ill-defined. An easy and reasonably natural way to handle this problem is to introduce explicitly limits on the ability of individual debtors to generate IOU notes. But this “fix” (which may be needed to keep strategy sets compact) must have an interpretation in terms of the information requirements in a market with borrowing. This is the subject of Question 9 and is dealt with below.

QUESTION 9: *What are the information requirements on credit restrictions and on the selection of criteria for secured lending and bankruptcy rules?*

OBSERVATION 9: There is a basic difficulty in the running of loan markets, where the loans are not immediately secured by assets which are always worth more than the loan. This difficulty is manifested in the expense and complexity of credit evaluation procedures, information requirements, and collection costs.

Anyone who has been involved with real-world loan markets is aware of substantial expenses in legal, accounting, auditing, administrative and collection costs. In this investigation we do not deal directly with these features, but concentrate on the representation of the overall role of evaluation. In essence, whenever there is a large homogenous group of agents who wish to borrow, but have no assets for security, the lenders cannot expect to be paid back if the borrowers are lent more than the present value of their expected income. But this at least requires that the size of their expected income be known. Ideally, for tight control in a loan market with n types of borrowers, and leaving aside administrative costs, the lenders need to know at least the expected income of each type, the probability that an individual who has been lent a specific amount will go bankrupt, and the force of the default penalty on his behavior. For secured lending, the lender needs to be able to evaluate accurately the future worth and condition of current assets and future income of individuals. In fact, many loans do involve in-depth, detailed "hand-tailored," one-on-one arrangements. In a mass-market the luxury of much personalized information can hardly be afforded.

By explicitly introducing the appropriate parameters in terms of bounds on borrowing and estimates of default, together with appropriate penalties, we are able to construct a well-defined playable game. But the need for this detail highlights the observation that the key to successful financing lies in the credit evaluation and policing procedures. These at least appear as logical necessities in the strategic market game models.

QUESTION 10: *In an economy with uncertainty, does the introduction of a loan market improve the welfare of the society as a whole?*

OBSERVATION 10: It is natural to guess that the introduction of an active loan-market will improve welfare for the society as a whole. In these simple one-dimensional economies with only one type of agent, we can take the integral of utilities as a measure of total welfare. However, the following example shows that a loan-market can *decrease* total welfare in equilibrium.

Example 8: A loan market that decreases total welfare

Consider an economy with the piecewise-linear utility function of Example 4 and with production variables, represented by Y, such that $P[Y = 1] = 1/2 = P[Y = 4]$. Suppose first that there is a loan market (either through an outside bank, or a money market) with secured lending, a bound on loans of $k = 1/2$, interest rates $\rho_1 = \rho_2 = 1$, and price $p = 1$ in equilibrium. As is shown in the appendix to KSS (1997), the optimal policy for an individual is

$$b = c(s) = \begin{Bmatrix} s + \frac{1}{2}, & 0 \le s \le 2 \\ 2\frac{1}{2}, & 2 \le s \le 3 \\ s - \frac{1}{2}, & s \ge 3 \end{Bmatrix}$$

for certain values of the parameters β and η; for example, $\beta = 1/4$ and $\eta = 1/11$. The stationary wealth distribution æ for the corresponding Markov chain is the uniform distribution on the four-point set $\{0,\ 2,\ 3,\ 5\}$. Therefore, the total welfare in equilibrium is

$$\begin{aligned} \int u(c(s))\mu(ds) &= \tfrac{1}{4}[u(c(0)) + u(c(2)) + u(c(3)) + u(c(5))] \\ &= \tfrac{1}{4}\left[u\left(\tfrac{1}{2}\right) + u\left(2\tfrac{1}{2}\right) + u\left(2\tfrac{1}{2}\right) + u\left(4\tfrac{1}{2}\right)\right] \\ &= 2 + \tfrac{\eta}{2}. \end{aligned}$$

For the same economy without the possibility of borrowing or lending, the optimal policy can be shown to be

$$\tilde{c}(s) = \begin{Bmatrix} s, & 0 \le s \le 2\frac{1}{2} \\ 2\frac{1}{2}, & 2\frac{1}{2} \le s \le 4 \\ s - 1\frac{1}{2}, & s \ge 4 \end{Bmatrix}$$

for certain values of β and η again including $\beta = 1/4$ and $\eta = 1/11$. This time the stationary wealth distribution $\tilde{\mu}$ is uniform on the four-point set $\{1, 2\frac{1}{2}, 4, 5\frac{1}{2}\}$, and total welfare in equilibrium is

$$\begin{aligned} \int u(\tilde{c}(s))\tilde{\mu}(ds) &= \tfrac{1}{4}\left[u(\tilde{c}(1)) + u\left(\tilde{c}\left(2\tfrac{1}{2}\right)\right) + u(\tilde{c}(4)) + u\left(\tilde{c}\left(5\tfrac{1}{2}\right)\right)\right] \\ &= \tfrac{1}{4}\left[u(1) + u\left(2\tfrac{1}{2}\right) + u\left(2\tfrac{1}{2}\right) + u(4)\right] \\ &= \tfrac{17+3\eta}{8} \\ &= 2 + \tfrac{\eta}{2} \text{ for } \eta < 1. \end{aligned}$$

On the other hand, it is not difficult to give examples where a loan market improves total welfare.

7 Observations on a System with Fiat Money and Loans

We consider the remaining questions raised in Section 1.

QUESTION 1: *What constitutes fully secured lending?*

OBSERVATION 1: Secured lending can only be well-defined if the price system is bounded, and the loan repayment is *always* feasible.

QUESTION 2: *How is fiat money conserved in an economy with bankruptcy?*

OBSERVATION 2: When bankruptcy takes place, IOU notes or debt instruments are destroyed. The specific amount of fiat money in the system is conserved. But as noted below, its distribution may be influenced by the bankruptcy settlement.

QUESTION 3: *How can unpaid debts be discharged?*

OBSERVATION 3: A bankruptcy and reorganization procedure is designed specifically to wipe out the IOU notes and settle matters by the redistribution of the fiat money and other assets in the economy.

QUESTION 6: *Under what circumstances can a loan market, together with appropriate default rules bring about Pareto optimality?*

OBSERVATION 6: Although we have not done so in our current model, we could consider debtors who could hide their ability to repay. Thus, an individual could cheat by stating that he has no income, when, in fact he does. Ideally, if an all-seeing outside bank, which is fully informed of all states of the system and forgives unfortunate debtors, can recognize and punish those who cheat, it will be able to convert the stochastic income of each individual into its mean value. By concavity of the utility function the mean value consumption will be Pareto optimal and dominate the stochastic consumption. It can do this by charging an appropriately high interest rate and limiting the amount that any individual can borrow to his expected income. Repayments will equal borrowing, the effective rate of interest will be zero, and each individual will borrow up to his limit set by the bank. His income will always be less than or equal to the amount he owes. As he cannot cheat he pays back whatever he can; thus, in equilibrium, at the start of each period his net wealth is zero. He borrows to the limit once more. This amount has been set by the "all-seeing bank" to equal to his expected income. Each individual borrows and spends a nonrandom amount each period, but repays a random amount equal to whatever he earns. He is forgiven any shortfall.

In actuality, such a costlessly run, all-seeing bank does not exist. The information conditions required for such a bank are enormous. It would have to know the details of every random variable influencing each individual, and the realizations of every random variable, in order to distinguish between the honest poor and those concealing income.

QUESTION 11: *What are the conditions required on an outside bank to run a loan market, if the private supply and demand for money will not be equal?*

OBSERVATION 11: The outside bank can meet excess demand for loans by lending from its own reserves as in Example 3. It could likewise absorb excess supply. As Example 7 illustrates, a money market is not as flexible.

QUESTION 12: *How powerful an instrument can monetary policy be via the control of the loan and deposit rates and credit restrictions?*

Observation 12: A precise reply cannot be given to this question without a sensitivity analysis over a specified domain; intuitively it appears, however, that the importance of credit restrictions as a way of controlling the dynamics of expectations is critical.

Question 13: *What can be said about the dynamics when the economy is out of equilibrium?*

Observation 13: The approach adopted here may be regarded as a necessary preliminary to the development of dynamics. In order to study equilibrium we were required fully to specify the laws of motion, but the existence proof of equilibrium, though hopeful, does not provide a proof of convergence from a position of disequilibrium. In particular, it is easy to construct many equations of motion based on different expectations, all of them consistent with the same equilibrium but with different dynamics. The classical "cobweb" example can be modified to show this.

Question 14: *What interventions are needed to prevent price crashes or explosions?*

Observation 14: The conditions discussed in Section 2.2 indicate the importance of bounding prices and interest rates. Thus it appears that items such as "orderly markets," and government monetary intervention when interest rates are "too high or low," are not merely institutional facts, but ways in which the state-space of the game is kept compact, and emerge as conditions necessary for the existence of equilibrium. In particular, the debate on whether certain financial processes could have unbounded variances appears to hinge on both some specific empirical facts concerning individual and mass investment behavior, and on delicate mathematical conditions involving division by zero or unboundedness of strategy sets. These considerations lead us to conjecture that for smoothly functioning markets near equilibrium, much of conventional finance theory with continuous processes may serve as a good approximation. But during panics or under stress the institutions and their rules may not be sufficient. These rules include conditions governing specialists, limiting or extending special credits, and limiting trade.

If the dynamics goes too far out of control, the reaction of the society is to close markets and reorganize the institutions. However, even when the system stays within politically acceptable bounds, there may be equilibrium conditions which support jumps in stock prices. In this survey we have not modeled the stock market explicitly. This remains to be done, but the observations of Mandelbrot (1963, 1966, 1967) and the models of Arthur (1994) and Bak and Paczuski (1995) suggest several approaches to modeling the stock market as an infinite horizon strategic market game.

Question 15: *What can be said about the velocity of money?*

Observation 15: In the models presented here the concept of fiat money is well defined. It could be regarded as a physical entity such as dollar bills. By specifying that there be only one trading session per period, the velocity is at most one per period. Using the type of fully defined game model suggested here, it would be possible to run experimental games where, by permitting several trading sessions per period, one could experimentally observe the velocity of fiat money. But in an actual economy two difficulties appear. As there are many "near monies" or money substitutes such as checks, private banknotes, credit cards, book transfers, etc...what one wishes to define as money used in transactions becomes complex. Furthermore, even simple observations on how many times a week a 25 cent piece changes hands are not easy to make.

8 APPENDIX: An Optimality Proof

In several of our examples, we have claimed that a certain policy is optimal for an individual agent when the economy is in equilibrium. Here we illustrate some general techniques by sketching a proof of optimality for the policy of Example 4. As we noted above, proofs for most of the other examples can be found in the three companion papers.

Suppose that the economy of Example 4 is in equilibrium with the fixed parameters $k = 1/3$, $\rho_1 = 2$, $\rho_2 = 1$, $p = 1$, $\gamma = 4/7$, η, and β where the last two parameters remain to be chosen. Consider the situation of an individual agent with initial wealth $S_0 = s$. The agent can choose to bid any amount of money b in the interval $[0, s + 1/3]$. Since the price p is 1, the bid b buys b units of the good and the agent receives $u(b)$ units of utility. If $0 \le b \le s$, the agent deposits $s - b$ in the bank and begins the next period with wealth

$$S_1 = (1 + \rho_2)(s - b) + Y = 2(s - b) + Y$$

where, by assumption, $P[Y = 1] = 3/7$ and $P[Y = 4] = 4/7$. If $s < b \le s + 1/3$, the agent borrows $b - s$ from the bank and moves to

$$S_1 = -(1 + \rho_2)(b - s) + Y = 3(s - b) + Y.$$

The two formulas for S_1 can be written in the form

$$S_1 = g(s - b) + Y,$$

where

$$g(x) = \begin{Bmatrix} 3x, & x \le 0 \\ 2x, & x > 0 \end{Bmatrix}.$$

After the first move, the game continues from the new position but with discounted future returns.

This game for a single agent is essentially a discounted dynamic programming problem in the sense of Blackwell (1965). (A technical difference is that Blackwell assumes a bounded return function in each period.) Let $V(s)$ be the supremum over all possible policies of the expected discounted return of an agent starting at s. Then V satisfies the Bellman equation:

$$V(s) = \max_{0 \le b \le s+1/3} [u(b) + \beta E(V(g(s - b) + Y))].$$

Next let $Q(s)$ be the return starting from s if the agent uses the policy defined in Example 4. To show that the policy is optimal amounts to proving that $Q(s) = V(s)$ for all s. However, by a standard result of dynamic programming, it suffices to show that Q satisfies the Bellman equation. Equivalently, we need to show that for each s, the function

$$\psi(b) = \psi_s(b) = u(b) + EQ(g(s-b)+Y) \tag{1}$$

attains its maximum on the interval $0 \le b \le s + 1/3$ at $b = c(s)$.

First, we need to calculate Q and its derivative. The key to the calculation is the functional equation

$$Q(s) = u(c(s)) + \beta EQ(g(s-b)+Y).$$

If we substitute in the right-hand side using the definition of $c(s)$, and the distribution of Y, and the definition of g above, then this functional equation can be written more explicitly as

$$Q(s) = \left\{ \begin{array}{ll} s + \frac{1}{3} + \beta\left[\frac{3}{7}Q(0) + \frac{4}{7}Q(3)\right], & 0 \le s \le 2\frac{1}{6} \\ 2\frac{1}{2} + \beta\left[\frac{3}{7}Q\left(3s - 6\frac{1}{2}\right) + \frac{4}{7}Q\left(3s - 3\frac{1}{2}\right)\right], & 2\frac{1}{6} \le s \le 2\frac{1}{2} \\ 2\frac{1}{2} + \beta\left[\frac{3}{7}Q(2s-4) + \frac{4}{7}Q(2s-1)\right], & 2\frac{1}{2} \le s \le 3 \\ 2\frac{1}{2} + \eta(s-3) + \beta\left[\frac{3}{7}Q(2) + \frac{4}{7}Q(5)\right], & s \ge 3 \end{array} \right\}.$$

The first and fourth lines can be substituted in the second and third to get

$$Q(s) = \left\{ \begin{array}{ll} 2\frac{1}{2} + \frac{3}{7}\beta\left\{3s - 6\frac{1}{2} + \frac{1}{3} + \beta\left[\frac{3}{7}Q(0) + \frac{4}{7}Q(3)\right]\right\} & \\ \quad +\frac{4}{7}\beta\left\{2\frac{1}{2} + \eta\left(3s - \frac{6}{2}\right) + \beta\left[\frac{3}{7}Q(2) + \frac{4}{7}Q(5)\right]\right\}, & 2\frac{1}{6} \le s \le 2\frac{1}{2} \\ 2\frac{1}{2} + \frac{3}{7}\beta\left\{2s - \frac{1}{4} + \frac{1}{3} + \beta\left[\frac{3}{7}Q(0) + \frac{4}{7}Q(3)\right]\right\} & \\ \quad +\frac{4}{7}\beta\left\{2\frac{1}{2} + \eta(2s-4) + \beta\left[\frac{3}{7}Q(2) + \frac{4}{7}Q(5)\right]\right\}, & 2\frac{1}{2} \le s \le 3 \end{array} \right\}.$$

Differentiate to get

$$Q'(s) = \left\{ \begin{array}{ll} 1, & 0 < s < 2\frac{1}{6} \\ \frac{9}{7}\beta + \frac{12}{7}\eta\beta, & 2\frac{1}{6} < s < 2\frac{1}{2} \\ \frac{6}{7}\beta + \frac{8}{7}\eta\beta, & 2\frac{1}{2} < s < 3 \\ \eta, & s > 3 \end{array} \right\}.$$

The right- and left-derivatives can be calculated at the endpoints by continuity from the right and left, respectively. For Q to be concave, we need

$$1 \ge \frac{9}{7}\beta + \frac{12}{7}\eta\beta \ge \frac{6}{7}\beta + \frac{8}{7}\eta\beta \ge \eta. \tag{2}$$

The middle inequality holds automatically; we *assume* the other two inequalities hold also.

We can now check the Bellman equation for Q by showing that the function $\psi = \psi_s$ from (1) attains its maximum at $b = c(s)$. Consider four cases.

Case 1: $0 \le s \le 2\frac{1}{6}$

Since $c(s) = s + 1/3$, we need $\psi'(b) \ge 0$ for $b \le s + 1/3$. (We are somewhat careless about endpoints; this doesn't matter because of continuity.)

If $0 \le b \le s$, then

$$\begin{aligned}\psi(b) &= b + \beta\left[\tfrac{3}{7}Q(2(s-b)+1) + \tfrac{4}{7}Q(2(s-b)+4)\right],\\ \psi'(b) &= 1 - \tfrac{6}{7}\beta Q'(2(s-b)+1) - \tfrac{8}{7}\beta Q'(2(s-b)+4)\\ &\ge 1 - \tfrac{14}{7}\beta\\ &\ge 0 \text{ if } \beta \le \tfrac{1}{2}.\end{aligned}$$

If $s \le b \le s + \frac{1}{3}$, then

$$\begin{aligned}\psi(b) &= b + \beta\left[\tfrac{3}{7}Q(3(s-b)+1) + \tfrac{4}{7}Q(3(s-b)+1)\right],\\ \psi'(b) &= 1 - \tfrac{9}{7}\beta Q'(3(s-b)+1) - \tfrac{12}{7}\beta Q'(3(s-b)+4)\\ &\ge 1 - 3\beta\\ &\ge 0 \text{ if } \beta \le \tfrac{1}{3}.\end{aligned}$$

Case 2: $2\frac{1}{6} \le s \le 2\frac{1}{2}$

Here $c(s) = 2\frac{1}{2}$. So we need $\psi'(b) \ge 0$ for $b \le 2\frac{1}{2}$ and $\psi'(b) \le 0$ for $b \ge 2\frac{1}{2}$.

If $0 \le b \le s$, then exactly as in Case 1

$$\psi'(b) \ge 1 - \tfrac{14}{7}\beta \ge 0 \text{ if } \beta \le \tfrac{1}{2}.$$

If $s \le b \le 2\frac{1}{2}$, then

$$\psi(b) = b + \beta\left[\tfrac{3}{7}Q(3(s-b)+1) + \tfrac{4}{7}Q(3(s-b)+4)\right].$$

Now

$$3(s-b)+1 \le 1 \text{ and } 3(s-b)+4 \ge 3\left(-\tfrac{1}{3}\right)+4 = 3,$$

and thus

$$\begin{aligned}\psi'(b) &= 1 - \tfrac{9}{7}\beta - \tfrac{12}{7}\beta\eta\\ &\ge 0\end{aligned}$$

by (2).

If $2\frac{1}{2} \le b \le s + \frac{1}{3}$, then

$$\psi(b) = 2\tfrac{1}{2} + \eta\left(b - 2\tfrac{1}{2}\right) + \beta\left[\tfrac{3}{7}Q(3(s-b)+1) + \tfrac{4}{7}Q(3(s-b)+4)\right].$$

Also

$$3(s-b)+1 \le 1 \text{ and } 3(s-b)+4 \ge 3\left(-\tfrac{1}{3}\right)+4 = 3,$$

and thus

$$\begin{aligned}\psi'(b) &= \eta - \tfrac{9}{7}\beta - \tfrac{12}{7}\beta\eta\\ &\le 0\end{aligned}$$

by (2).

The remaining two cases, namely $2\frac{1}{2} \leq s \leq 3$ and $s \geq 3$, can be treated analogously.

In the last case, an additional condition

$$\eta = \tfrac{6}{7}\beta + \tfrac{8}{7}\beta\eta$$

must be imposed. All of our conditions are satisfied, for example, when $\beta = 1/3$ and $\eta = 6/13$.

Remarks

1. This example gives us a stationary equilibrium with two different interest rates. However, the optimal policy is not unique. It should be possible, but perhaps painful, to get an example with more uniqueness.

The example gives some intuition as to why it is difficult to prove an existence theorem for interest rates which balance the books. Here we start with interest rates which give a simple Markov chain. The chain varies nicely as a function of the other parameters. If we fix the other parameters and vary the interest rates, the behavior of the chain is more complicated.

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