COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1259



# SOCIAL SECURITY INVESTMENT IN EQUITIES IN AN ECONOMY WITH SHORT-TERM PRODUCTION AND LAND

Peter Diamond and John Geanakoplos

June 2000

# Social Security Investment in Equities in an Economy with Short-Term Production and Land

Peter Diamond and John Geanakoplos*

June 2000

**Abstract**

This paper explores the general equilibrium impact of social security portfolio diversification into private securities, either through the trust fund or via private accounts. The analysis depends critically on heterogeneity in saving, in production, in assets, and in taxes. Under fairly general assumptions we show that limited diversification increases a neutral social welfare function, increases interest rates, reduces the expected return on short-term equity (and thus the equity premium), decreases safe investment and increases risky investment. However, the effect on aggregate investment, long-term capital values, and the utility of young savers hinges on delicate assumptions about technology. Aggregate investment and long-term asset values often move in the opposite direction. Thus social security diversification might reduce long-term equity value while it increases aggregate investment.

## Introduction

This paper explores the impact of social security portfolio diversification into private securities, either through the trust fund or via private accounts. We evaluate the effect of diversification on prices (of stocks, bonds, and land), on welfare (of young and old, savers and workers) and on investment (in risky production and safe production and in the aggregate). Our analysis is carried out in a general equilibrium, overlapping-generations model.

Most studies of social security diversification have concentrated on the consequences it would have for retirement benefits and the budget viability of the system, ignoring any general equilibrium repercussions (and sometimes even claiming it would have none). By contrast, we analyze the general equilibrium ramifications and show how critically they depend on heterogeneity in saving, in production, in assets, and in taxes.[1]

Among the elderly, Social Security income is distributed very differently than private pension and asset income.[2] For the bottom quintile of the income distribution, 81% of

*The authors are grateful to Saku Aura and Tom Davidoff for research assistance, to Bill Brainard, Alicia Munnell, Jim Poterba and Antonio Rangel for comments, and to the National Science Foundation for research support under grant SBR-9618698. The research reported herein was also supported by a grant from the U.S. Social Security administration (SSA) funded as part of the Retirement Research Consortium. The opinions and conclusions are solely those of the authors and should not be construed as representing the opinions or policy of SSA or any agency of the Federal Government. A previous version of this paper was circulated under the title Social Security Investment in Equities I: Linear Case.

[1]For another analysis of portfolio diversification where general equilibrium effects do matter, see Abel (1999). For examples of claims that make sense in a representative agent model but are not adequate once heterogeneity is recognized, see Financial Economists' Roundtable (1998), and Greenspan (1997).

[2]See Mitchell and Moore (1997), Social Security Administration (1996).

income comes from Social Security, while only 6% is from pensions plus income from assets. For the top quintile, 23% comes from Social Security, while 46% is from pensions and assets — dramatically different percentages. Similarly, there are great differences in saving and investing among current workers. Among all those who were paying social security taxes in 1995, fully 59% held no stock, either directly or through pension plans. Even among those between 45 and 54 years of age, 50% held no stock, directly or indirectly.[3] These differences have important implications for the proposal to invest part of Social Security trust fund reserves in private securities.

We represent this heterogeneity in saving behavior by supposing that there are two types of representative agents, one of which does no saving (except through social security) and the other of which saves and selects a portfolio (and, for simplicity is assumed not to be covered by social security). We refer to the two types of agents as workers and savers.

Our analysis also recognizes the importance of heterogeneity in production. We suppose that there are two short-term technologies, which produce safe and risky output. We also assume there are two long-term technologies, called safe land and risky land, which produce safe and risky output in perpetuity. It is important to distinguish between safe and risky output, because social security diversification will likely change equilibrium prices in such a way as to increase production in one technology sector and reduce it in the other. It is important to consider long-term production, because changes in the equilibrium prices of land will redistribute wealth between generations, whereas changes in the prices of short-term production do not. Wealth redistribution is interesting for its own sake, but also because investment depends on the income of the young savers. Moreover, we shall see that the effect of social security diversification on the value of risky land depends on the elasticity and level of short-term safe and risky production.

Currently the social security trust fund only holds US government bonds. Adding government bonds to the four kinds of production, we are led to consider a model with five assets.

It is important to distinguish income taxes from social security taxes. We shall see that social security diversification is likely to change the rate of interest. A higher interest rate means higher income taxes to pay the higher coupons on government bonds. But the increased income tax burden may fall on households in different proportions than the social security taxes and benefits. This in turn will have feedback effects on the demand for assets and ultimately on the new equilibrium interest rates.

In order to keep the analysis simple, social security is modeled as a combination of a pay-as-you-go system together with a defined contribution system, and not as a defined benefits system. Social security diversification occurs when the asset mix in the defined contribution system is suddenly shifted from bonds toward equities, and then maintained at the higher equity level forever after. For example, if workers were suddenly given discretionary accounts, a number of them who did little or no saving outside of social security would choose to invest part of their discretionary accounts in equities, and then our analysis would apply. More generally, we suppose there is a social security trust fund, which suddenly sells some of its bonds and invests the proceeds in stock, and passes through the difference in net returns between the stocks and bonds to the contemporaneous old.[4] The differences between defined benefit and defined contribution systems as distributors of rate-of-return risk have been explored in OLG models with a single representative agent.[5] This paper is meant to complement those studies. There is

[3]Quoted in Geanakoplos, Mitchell, and Zeldes (1999). See Kennickell, Starr–McCluer, and Sunden (1997) and Ameriks and Zeldes (in progress).

[4]Since we suppose that all workers have the same utility, it makes no difference whether social security accounts are personal or are managed by a trust fund, provided that they choose the same asset mix.

[5]See, for example, Bohn (1997, 1998), Diamond, (1997).

a brief discussion of how to extend the analysis to a defined benefit plan in Section 12.

Our first conclusion, which holds quite generally, is that social security diversification creates the potential for welfare improvements. Diversification from a point of zero exposure to equities raises the sum total of utility in the economy if household utilities are weighted so that in every time period the marginal utility of a dollar for sure is the same for every living saver and every retired worker.[6] The welfare gains come from the superior risk-sharing social security diversification permits when there are workers who do not have savings to invest on their own, and when, in the absence of social security diversification, social security benefits have a low correlation with stock returns.[7] The saving constrained workers, though perhaps more risk averse than the savers *ceteris paribus*, should be more risk tolerant on the margin precisely because they are not exposed to stocks outside of social security. On the other hand, it is also important to bear in mind that in addition to the welfare gains, social security diversification would also likely cause welfare redistributions, so that a Pareto improvement might require additional policy steps.

The transfer of risk from savers to workers following social security diversification leads to further changes in prices and welfare, because savers will adjust their behavior in response to their new situation. The direction of these further changes, however, is ambiguous without further assumptions. We make four assumptions, enough to determine many of the consequences of diversification. However, the effects on long term capital values, on welfare, and on aggregate investment, still remain ambiguous without yet more assumptions on the relative elasticities and levels of production. Rather than making still more assumptions on technology, we work out all the possibilities.

We suppose that the demands by savers for consumption when young, and for safe and risky consumption when old, are normal. (If savers maximize expected utility, normality follows from decreasing absolute risk aversion and increasing relative risk aversion.) We suppose that increases in government bond interest payments raise the payments on government bonds held by social security more than they raise workers' income taxes (and thus raise savers' taxes more than they raise savers' income on the government bonds they hold). We also suppose short-term risky production is along a ray in state space, that is $k$ units of investment in the risky technology produce $g(k)R_s$ units of output in each state $s$. (Short-term safe production $f(k_0)1$ is by definition along a ray.) Finally, we suppose that the output from long-term risky production is independently and identically distributed each period.

In contrast to models where all workers are portfolio unconstrained and technology is homogeneous, we find that social security diversification into equities has substantial and subtle real effects. Under our assumptions, social security diversification would increase risky investment, decrease safe investment, raise interest rates, lower expected returns on short-term risky securities, and reduce the equity premium. Stock market value can go

[6]Since workers do not save, their marginal rates of substitution across time are not proportional to interest rates, and it is impossible to require marginal utilities for young workers to match up as well, for all time periods. But even if they do not match up in the welfare function, the welfare function must be increased by diversification.

[7]Exactly this point was made in Geanakoplos, Mitchell, and Zeldes (1999), who also tried to quantify the welfare gain in a special quadratic example. Here the proposition that there must be welfare gains from a small degree of social security diversification is proved in a formal model, assuming there are workers who do not save and that there is a low correlation between benefits and stock returns. If social security benefits are financed entirely by personal accounts, which exclusively hold bonds, then the assumption of low correlation between social security benefits and stock returns is plausible. If social security benefits depend on a pay-as-you-go system and future real wages are not highly correlated with stock returns, then again the hypothesis of low correlation between social security returns and stock returns is plausible. Another reason to confine the analysis to a defined contributions system is to make clear that the gains to diversification we describe do not rely at all on the risk spreading across generations that could be supported by a defined benefits system. A defined benefits system diversified into equities would create the possibility of still bigger welfare gains.

up or down, as could the welfare of young savers and total investment, depending on the direction of change in land prices and ultimately, on the relative sensitivity and levels of safe and risky production.

It is quite surprising that risky land prices might fall. Indeed the surprise can be put more sharply: if the trust fund begins entirely invested in bonds, and then shifts some of its portfolio from bonds into risky land, the new equilibrium price of risky land may be lower than before (and will be under conditions given in proposition 5). The explanation is that the one period return on risky land is the sum of the risky dividend and next period's capital value of land. Under our assumption that land dividends are independent across time, the capital value is riskless, and therefore its present value will decline with the increase in interest rates. The near term returns from land rise in value, but the future dividends fall in value. In other words, the one-period return on infinitely lived land is a combination of the short-term risky return and the short-term safe return. Diversification makes one go up and the other down.[8] We show that if short-term production is exclusively risky and linear, then total land values must fall when social security diversifies, and if short-term production is exclusively safe, then land values must rise. When there is no short-term production, social security diversification into land also raises aggregate land value. Proposition 5 covers the general case, showing that the sensitivity of production to price changes and the prevailing levels of risky output (relative to safe output) determine the change in land values, given both declining absolute risk aversion and increasing relative risk aversion.

Proponents of social security diversification often say it would help young savers because stocks have traditionally earned a higher return than social security is projected to yield in the future. They have been rightly criticized for sometimes forgetting about the unfunded liability embodied in social security commitments to today's old, and for ignoring the riskiness of stock returns.[9] Naturally our model recognizes both of these considerations, and not surprisingly it shows that the equity premium would fall after diversification. Our analysis also makes clear that the welfare of young savers depends on at least two more considerations. First, their income taxes will rise (to pay the higher interest on government bonds). Second, the assets they buy and sell will change in value. In our model, their purchases of short-term risky assets exactly offset the dividends they receive as owners of firms. The change in price of short-term risky securities therefore does not affect their welfare. Young savers are, however, net buyers of long-term assets like land. Their welfare will therefore be influenced in the opposite direction by changing land prices. Unless long-term capital values go down substantially after diversification, young (and future) savers will be made worse off by social security diversification. On the other hand, today's old savers will be made better off if long-term capital values rise.

The effect of social security diversification on aggregate investment is ambiguous, because risky investment will go up and safe investment will go down. The effect on aggregate investment therefore depends on whether safe or risky production is more elastic. In the extreme case where there is no safe production, aggregate investment must go up, and similarly when all short-term production is safe, aggregate investment must go down. Aggregate investment also depends on young savers' propensity to save. When land prices go up and savers' welfare declines, they will want to save less. When land prices go down, the stimulus to risky production from the change in prices is reinforced by the increased demand by young savers. In general, there might be a trade-off between substitution

[8] Had we assumed that land dividends follow a random walk, increased land values would have become more likely. But then we could not have solved for a steady-state equilibrium. The point anyway is to understand the factors that control the change in long-term capital values, not to settle the empirical question of which factors are strongest.

[9] For assertions that are subject to this criticism, see Forbes (1996) and Moore (1997). This line of criticism is developed in the two papers by Geanakoplos, Mitchell, and Zeldes (1998, 1999).

and income effects. Proposition 5b gives conditions under which the change in aggregate investment is opposite in direction to the change in total land value.

To ease the exposition and also to illustrate the importance of technology in determining the effects of social security diversification, we first analyze a model with linear short-term technology and land.[10] In this model explicit formulas can be obtained. Later we move to a more general model with concave short-term production and land. The comparative statics of equilibrium in the linear model depend on which of the two short-term technologies are actually used in equilibrium. We work out three separate cases, called the bilinear case, the risky linear case, and the safe linear case, depending on whether both linear technologies are used, or just one is used. The results are presented in Propositions 1–3. The risky linear case, arising when there are constant marginal returns to short-term risky investments, but no safe real investment, seems clearly the most realistic among the linear cases, and so we concentrate much of our attention there. In this case social security diversification raises aggregate investment and lowers land values (and therefore stock market value). It also lowers the utility of young savers unless the value of land is sufficiently high relative to the value of government bonds.

The active presence of either linear technology simplifies the analysis because it fixes one equilibrium rate of return. When neither linear technology is in use, we can no longer derive simple formulas for the effects of social security diversification, and we are forced to use an indirect argument. Our results are presented in Proposition 4. Since the indirect argument is applicable more widely, we defer it until after we present our second, more general model with concave short-term production and land. In the concave model we can see in one fell swoop which comparative statics properties hold across equilibrium regimes. We also identify factors that explain whether land values and aggregate investment go up or down after social security diversification. These results appear in Proposition 5, whose proof is given in the Appendix. We conclude the paper by showing how the model could be extended to allow for defined benefits.

The sequence of cases is ordered from simplest to hardest, so the reader can see the pieces of the model introduced in turn. In every case we begin by deriving the welfare implications of social security diversification, conditional on the price changes. We use these conditional welfare effects to derive the price changes by breaking demand into compensated and income effects. We repeatedly use the observation (proved in Aura–Diamond–Geanakoplos, 1999) that our assumption of normal demands and expected utility maximization guarantees that all of the assets and current consumption in our model are Hicksian substitutes.

# 1 Technology

We analyze the equilibrium of a stochastic overlapping-generations economy, where each generation lives for two periods. There is one perishable consumption commodity in each date-event, which can either be eaten or invested using a productive technology. Young consumers have (nonstochastic) endowments, which can be interpreted as earnings from inelastically supplied labor. Consumers and the social security system are described in Sections 2 and 3.

At each date-event there are two short-term investment opportunities which transform the single perishable consumption good into (safe or stochastic amounts of) consumption goods in the next period. In Sections 1–9, we assume a linear, short-term technology to avoid the complications from feedback of investment levels on rates of return to productive

[10] To further ease the exposition, we introduce land only after presenting a preliminary version of the model without land.

investments. The safe investment produces $R_0 k_0$ in the period following an investment of $k_0$, with no durability in the capital, where $R_0 > 1$ is a constant. (Thus we are assuming a positive safe rate of return.) The risky investment produces $Rk$ in the period following an investment of $k$, also with no durability in the capital, where $R > 0$ is a random variable. For convenience, we assume the risky returns to be independently and identically distributed each period.

Equilibrium with the linear technology is described without land in Section 4 and with land in Section 7. We are interested in examining the change in equilibrium caused by social security portfolio diversification from bonds into equity. The qualitative changes in equilibrium depend crucially on which investment technologies are actually used in equilibrium. There are thus four regimes to consider, depending on whether each short-term technology is in use, or is not in use.[11]

In the bilinear regime, examined in Section 5, we assume that both short-term investments are undertaken in equilibrium. This fixes the equilibrium expected returns on safe and risky investments, and makes the analysis simple, since trust fund diversification does not change the rates of return. Nevertheless, there are welfare consequences, which recur in all the other regimes.

In the second, and most plausible linear regime, called the risky linear case, we suppose only the risky investment is undertaken in equilibrium, which leaves the safe return, and thus the interest rate on government bonds, endogenous and liable to be changed by diversification. We examine the risky linear case without land in Section 6, and with land in Section 8.

In Section 7 we add safe and risky land to the model. Safe land produces a fixed amount of the consumption good every period, while risky land produces a random amount of output every period, perfectly correlated with the output of the short-term risky investment. Given the iid assumption for returns on risky investment, the one-period returns from each type of infinitely-lived land turn out to be convex combinations of the safe and risky short term returns. Thus even though there are two land prices as well as the return on government bonds that are liable to be changed by diversification, our most plausible linear model, extended to include land, can still be solved in terms of one endogenous variable.

In the safe linear regime we retain the land, but we suppose only the safe investment is undertaken in equilibrium, which leaves the one-period expected yield from purchase of risky land endogenous, while fixing the return on the safe investment, and so on bonds. Though we regard this model of exclusively safe investments as implausible, its analysis in Section 9 illustrates how some of our comparative statics predictions can be reversed with a different production technology.

In Section 10, we examine the situation with safe and risky land, but where neither short-term investment is undertaken in equilibrium. Then both the interest rate on government bonds and the expected return on risky land are endogenous, and free to vary from social security diversification. As a result we cannot write a simple formula for the effects of social security diversification, though we can still determine its qualitative effects. We defer the comparative statics analysis to the more general case considered in Section 11.

In Section 11 we generalize the short-term production to allow for decreasing returns to investments. Once again there are two endogenous variables. The qualitative effects

[11] When a technology is not in use, we suppose that the marginal utility of beginning to use it is strictly less than the marginal cost of beginning to use it. For completeness, we mention that there are several other regimes, in which a technology is not in use but by coincidence the marginal benefit of beginning to use it is exactly equal to the marginal cost of beginning to use it. Generically, these knife-edge regimes will not be observed, and we ignore them.

of social security diversification then depend critically on the second derivatives of the production functions for safe and risky output, and on the total amount of each type of production. With linear production we were able to derive explicit formulas for the change in equilibrium after social security diversification. The feedback of production on rates of return complicates the analysis, but through an indirect argument, we are still able to determine the signs, if not the sizes, of the changes in prices and the two short-term investments. Moreover, our more limited qualitative analysis applies to every equilibrium regime, so that we do not have to break the analysis into different cases, but can see clearly what qualitative properties are common to all equilibrium regimes.

# 2 Consumers

To bring out most clearly the difference between social security covered workers and wealth holders, we follow an older literature and assume there are workers who do not save and savers who are not covered by social security; that is, two representative agents in each birth cohort.[12] We assume that each worker receives $w$ in the first period of life, with each saver receiving $W$.

We assume no population growth and normalize the population so that there is a unit measure of (identical) savers and a measure of size $n$ of (identical) workers. The representative saver maximizes expected lifetime utility of consumption, taking prices as given. Expected lifetime utility, $V$, is equal to $U_1[C_1] + E\{U_2[C_2]\}$, where $C_1$ is consumption when young, and $C_2$ is consumption when old, and with $U_i$ increasing, concave and twice continuously differentiable. In the model without land, the savers divide exogenous first period wealth, $W$, among consumption and (up to) three tangible assets — government bonds, $B$, and two types of physical capital: $k_0$, which is the safe asset, and $k$, which is the risky asset. In addition, the savers pay income taxes, $T$, in the second period.[13] Thus, we denote expected utility maximization for the representative saver by:

$$\begin{aligned} V &= \max U_1[C_1] + E\{U_2[C_2]\} \\ &\quad \text{s.t. } W = C_1 + B + k_0 + k \\ &\qquad\quad C_2 = (1+r)B + R_0 k_0 + Rk - T, \end{aligned} \tag{1}$$

where the rate of return, $R$, is random, but taxes are not, as of the time of first-period decisions. If the safe real asset is held in equilibrium, then $1+r$ is equal to $R_0$, since the government bond and the safe real asset are perfect substitutes.

Consumer choice can also be viewed in terms of three (composite) consumer goods — first-period consumption and safe and risky second-period consumption, which we denote as $C_1$, $J$ and $K$.[14] It is therefore convenient to imagine that there is a safe financial asset promising one unit of safe consumption and also a risky financial asset promising one unit of risky consumption $R$, so that $J$ and $K$ can be bought directly.[15] With first-period consumption as numeraire, we denote the price of second-period risky consumption as

[12]Feldstein (1985) makes a similar assumption in his classification of agents as rational and myopic. Having savers covered by social security would complicate the notation without changing the analysis.

[13]Taxes are used to pay interest on government bonds. By collecting taxes in the second period of life, they are paid back to the same cohort they are collected from. Collecting taxes in the first period instead would be equivalent to changing the level of government debt outstanding.

[14]Since all trading and production opportunities can be written in terms of these composite commodities, analysis of equilibrium can be done in these terms. Written in this form, the usual properties of compensated demands hold for the vector of consumptions. On the properties of compensated demands in the presence of uncertainty, see Diamond and Yaari (1972) and Fischer (1972). Moreover, analysis can be done in this form without the assumption of expected utility.

[15]When risky real investment is being undertaken, we can interpret the risky financial asset as shares in the output of a risky firm. When there is no real investment being undertaken, then this risky financial

$p_K$. The price of one unit of second-period safe consumption is denoted by $p_J$. When the risky investment is undertaken in equilibrium, $k > 0$, then we must have $p_K = 1$. When safe investment is undertaken in equilibrium, $k_0 > 0$, we must have $p_J = 1/R_0$. Whether or not real safe investment is undertaken in equilibrium, $p_J$ is always equal to $1/(1+r)$, as long as the supply of government bonds to savers is positive. We now restate the consumer choice problem as:

$$\begin{aligned} V &= \max U_1[C_1] + E\{U_2[J + RK]\} = \max V^*(C_1, J, K) \qquad (2) \\ & \quad \text{s.t. } C_1 + p_J J + p_K K = I \\ & \qquad I = W - p_J T. \end{aligned}$$

Demands for all three consumer goods are functions of the prices of second-period safe and risky consumer goods, and of net lifetime wealth. We denote them by $C^*[p_J, p_K, I] = C^*[p_J, p_K, W - p_J T], K^*[p_J, p_K, I] = K^*[p_J, p_K, W - p_J T]$ and $J^*[p_J, p_K, I] = J^*[p_J, p_K, W - p_J T]$. The supply of $C$, $J$, and $K$ and market clearing will be described in Section 4.

We *assume* the function $V^*(C_1, J, K)$ is such that all three of first-period consumption, and safe and risky second-period consumption are normal goods. The normality of the three goods in turn *guarantees* that all three goods are Hicksian substitutes (given the intertemporally additive structure of preferences described in (2)). A sufficient condition for normality of all three goods is that second period utility displays decreasing absolute risk aversion (DARA) and increasing relative risk aversion (IRRA). (For proofs of these assertions, see Aura, Diamond, Geanakoplos, (1999).)[16]

In contrast, we model workers, who also have two-period lifetimes, as nonsavers. Each worker earns a wage, $w$, in the first period (with inelastically supplied labor), pays payroll taxes $t_w$ in the first period, and consumes $w - t_w$. In the second period, workers consume social security benefits, $b$, which may be random, less income taxes $t$. We denote lifetime utility by $v$ and note that it satisfies:

$$v = u_1[c_1] + E\{u_2[c_2]\} = u_1[w - t_w] + E\{u_2[b - t]\}. \qquad (3)$$

We distinguish two sources of taxes since the payroll tax will be used for social security, while the second period income tax will be used to pay part of the interest on the government debt outstanding.

The lack of randomness in income for young workers, $w$, guarantees a lack of randomness in the pay-as-you-go component of the financing of social security benefits for contemporaneous old workers, as we shall see in the next section.

# 3 Government and the Social Security System

It is assumed that each period the government rolls over one-period debt with a value of $G$. The interest payments on this debt are financed by taxes on older workers and older savers, with the principal rolled over to preserve the debt outstanding.

$$T_t + nt_t = Gr_{t-1}, \qquad (4)$$

where taxes collected in period $t$ are used to pay interest at rate $r_{t-1}$ on debt issued in period $t-1$. We assume that taxes are divided in the proportions $a$ and $1-a$, $(0 < a < 1)$

asset is like a contingent futures contract. An investor can acquire the right to future risky consumption by buying the risky financial asset (i.e., the shares or the futures contract) without having to actually undertake any risky investment, provided that he can find somebody who is willing to sell the asset.

[16]For the reader interested in non-expected utility maximization, we must assume that $V^*$ is such that all three goods are normal, *and* that all pairs of goods are Hicksian substitutes.

giving the period and steady-state relations:

$$\begin{aligned} T_t &= aGr_{t-1}; \quad t_t = (1-a)Gr_{t-1}/n; \\ T &= T(r) = aGr = aG(1-p_J)/p_J = T(p_J); \\ t &= t(r) = (1-a)Gr/n = (1-a)G(1-p_J)/np_J = t(p_J). \end{aligned} \tag{5}$$

We use the same symbols $T$ and $t$ whether or not they depend on $r$ or $p_J$; this is not likely to confuse.

We model the social security system as a combination of a trust fund paying benefits out of its earnings and a pay-as-you-go system of taxes and benefits. The social security trust fund holds the value $F$ of government debt, and the value $k$ of risky assets. It thus holds $K^f = \kappa/p_K$ of risky second period consumption (possibly equal to zero at the outset). Denoting the total value of the trust fund by $F_0$, and supposing the trust fund holds only short-term assets, we describe the trust fund budget set in any period by

$$F + \kappa = F + p_K K^f = F_0. \tag{6}$$

It is assumed that all the excess returns (or losses) earned in the trust fund are passed directly to the old workers. Given the need to maintain the trust fund value allocations in bonds and equity, $F$ and $\kappa$, and given constant payroll taxes $t_w$, and given a stationary population, social security benefits satisfy the period and steady-state relations:

$$\begin{aligned} b_t &= t_w + ((1+r_{t-1})F - F + (RK^f_{t-1} - \kappa))/n \\ b &= t_w + (rF + (R-p_K)K^f)/n. \end{aligned} \tag{7}$$

Thus the expected utility of workers, $v$, satisfies the period and steady-state relations:

$$\begin{aligned} v &= u_1[w-t_w] + E\{u_2[t_w - t + ((1+r_{t-1})F - F + (RK^f_{t-1} - \kappa))/n]\} \\ &= u_1[w-t_w] + E\{u_2[t_w - t + (rF + (R-p_K)K^f)/n\} \\ &= u_1[w-t_w] + E\{u_2[t_w - t + (rF_0 + (R-(1+r)p_K)K^f)/n]\}. \end{aligned} \tag{8}$$

We have given (7) and (8) first as accounting identities (defining the benefits for workers), and second as relations that must obtain if there is a steady state. Observe from equation (7) that all the variations in risky asset payoffs held by the trust fund are passed through directly to the current retirees. There is no risk sharing across generations, as there could be in a defined benefits plan, either by spreading return risks across several cohorts or by varying the payroll tax rate. Indeed, we could equally well think of the social security system modeled above as a pay-as-you-go system together with a defined contribution system, perhaps with private accounts. Workers could be allocated part of payroll taxes to invest in equities, while benefits were decreased by one plus the interest rate times the amount shifted to private accounts.

The wage and the payroll tax rate are assumed to be constant over time; the retirement benefits, however are free to vary, and will do so if the rates of return earned on the trust fund holdings vary. Similarly, the second-period income tax will change if the interest rate on government debt changes.

A crucial part of our analysis is that if $K^f = 0$, young workers at time $t-1$ can look forward with certainty to the social security benefits they will receive when they are old at time $t$. The return $r_{t-1}$ they will get from the trust fund bond investment is already locked in. Furthermore, they can perfectly predict the pay-as-you-go portion of their benefits, since, in stationary equilibrium, wages of the young at time $t$ are nonrandom. In reality, of course, future real wages cannot be predicted with certainty. In our judgment, however, they are substantially less random than stock returns. We shall discuss some implications of relaxing the assumption of nonstochastic wages in Section 5.

# 4 Stationary Equilibrium without Land

Stationary equilibrium is defined as the situation in which prices and young savers' consumption and asset holdings are constant through time and across states of nature. All that varies is output, consumption of the old savers and old workers, and social security benefits. Given our assumptions of a single commodity, and stationary and independent productivity shocks, stationary equilibrium will exist.

When savers undertake risky investment, $p_K$ is equal to one. Stationary equilibrium in the model with short-term production, but without land, is then defined by prices and quantities $(r, C_1, C_2, B, k_0, k)$ such that given $r$ and taxes $T$ equal to $T(r)$, the choices $(C_1, C_2, B, k_0, k)$ solve the savers' optimization problem (1), and such that savers' demand for government bonds equals the supply to savers:

$$B = G - F = G - F_0 + \kappa = G - F_0 + p_K K^f \tag{9.E1}$$

$$\begin{aligned} 1 + r &= R_0 \quad \text{if} \quad k_0 > 0 \\ 1 + r &\geq R_0 \quad \text{if} \quad k_0 = 0 \end{aligned} \tag{9.E2}$$

Since the savers are both the demanders and the suppliers of real investment, the investment markets automatically clear if savers solve (1). Moreover, the consumption market automatically clears as well, once the bond market clears, if savers act within their budget sets defined by (1). To check this, we can verify that supply of consumption equals demand,

$$W + nw + (R_0 - 1)k_0 + (R - 1)k + (R - 1)K^f = C_1 + C_2 + nc_1 + nc_2. \tag{9.E3}$$

The reader can verify that after substituting for $C_1$ and $C_2$ from (1), $c_1$ and $c_2$ from (3), taxes from (5), and benefits from (7), (9.E3) reduces to (9.E1).

We can also write all the market clearing conditions in terms of the variables $C^*$, $J^*$, $K^*$ introduced in budget set (2). The budget set (2) separates the consumption and savings decisions of the savers from the production decisions of firms. From now on we interpret $k_0$ to be the safe production chosen by the firms, and we interpret $k$ as the risky production chosen by the firms. Safe consumption market clearing becomes

$$J^*(p_J, p_K, W - p_J T(p_J)) = (G - F_0 + \kappa)/p_J + R_0 k_0 - T(p_J). \tag{10.e1}$$

Savers' demand for safe second period consumption must just meet the supply of safe second period consumption to savers, which is equal to the total principal and interest payments of government bonds, less what is held by the social security system, plus the safe production, less what is owed in taxes. Using the same variables $C^*$, $J^*$, and $K^*$, we can also write market clearing in the risky good market as

$$K^*(p_J, p_K, W - p_J T(p_J)) = k - \kappa/p_K. \tag{10.e2}$$

The supply of risky second period consumption to savers is equal to risky production, less what is held by the social security system. Note that $k$ is now the total input to the risky production technology.

To complete the picture we suppose the production decisions of the firms are taken to maximize profit:

$$\max[p_J R_0 k_0 - k_0] + \max[p_K k - k].$$

This gives

$$\begin{aligned} p_J = 1/(1 + r) &= 1/R_0 \quad \text{if} \quad k_0 > 0 \\ &\leq 1/R_0 \quad \text{if} \quad k > 0 \end{aligned} \tag{10.e3}$$

$$p_k = 1 \quad \text{if} \quad k > 0$$
$$\leq 1 \quad \text{if} \quad k = 0 \qquad (10.e4)$$

Stationary equilibrium is now defined as a vector $(p_J, p_K, k_0, k)$ such that (10.e1)–(10.e4) hold. The reader can check that if $C^*$, $J^*$, $K^*$ satisfy budget set (2), and (10.e1)–(10.e4) hold, then the consumption good market clears as well.

When $k > 0$, equilibrium based on (10.e1)–(10.e4) reduces to the definition of equilibrium given by (9.E1)–(9.E2). From (10.e4), we get $p_K = 1$, and (10.e2) becomes trivial, since $k$ can always be chosen to make (10.e2) hold. Condition (10.e1) is then the same as (9.E1), and (10.e3) is the same as (9.E2).

When $k = 0$, equilibrium (10.e1)–(10.e4) is slightly different from that given by (9.E1)–(9.E2). When $k = 0$, savers do not find it worthwhile to give up consumption when young to invest in risky output when old. In such cases they would typically be willing to sell future contingent goods $R$ at a price $p_K < 1$. It is natural to suppose that in such cases the trust fund would buy contingent promises from the savers at $p_K < 1$, instead of paying 1 by directly investing in the risky technology. Equilibrium (10.e1)–(10.e4) allows for this, by explicitly incorporating a market for future delivery of the risky consumption good, with a price $p_K$ , as described in budget set (2). This possibility becomes of central importance when we introduce land as a second source of risky consumption. Hence from now on we take budget set (2) and (10.e1)–(10.e4) as our definition of equilibrium. The condition defining the savers' holdings of government bonds $B$ given by (9.E1) must still hold, and we continue to use $B$ as a convenient shorthand for the RHS of (9.E1). But equation (9.E1) will not be treated as an independent equation, since it follows from (10.e1)–(10.e4).

We shall restrict attention to economies that have no stationary equilibria in which $k_0 = 0$ and $p_J = 1/R_0$, or in which $k = 0$ and $p_K = 1$. Since these are knife-edge cases, the set of economies we do not analyze is negligible (literally of measure zero). Depending on whether $k_0 > 0$ or $k_0 = 0$, and whether $k = 0$ or $k > 0$, equilibrium can be one of four different types, or regimes. The effect of social security diversification depends crucially on which regime the original equilibrium is in, hence we analyze all four cases separately.[17]

In each case we analyze the effect on equilibrium of a change in trust fund investment in risky assets: $d\kappa = -dF > 0$. Because the wages of both workers and savers are not influenced by past events, the economy achieves stationary equilibrium in a single period after a change in a parameter such as the portfolio allocation of the trust fund. If the unanticipated change comes at some date $t$, then generations born at date $t$ and after will consume as in the new steady state, and generations born at date $t - 2$ and before will consume as in the original steady state. The generation born at date $t - 1$ will consume as if it made consumption and asset choices when young in the original equilibrium, but was then forced to pay taxes and liquidate assets at date $t$ at the new steady state prices. Thus we confine our analysis to stationary equilibria.

# 5 Social Security Diversification in the Bilinear Case

In this section, we assume the economy is such that in equilibrium both physical assets are held, $k_0 > 0$, $k > 0$, with each asset providing constant marginal returns, as described in Section 1. In equilibrium, the interest rate on government bonds must then be equal to the (exogenously fixed) rate of return on the safe asset, and the price of the risky consumption good must be 1, the cost of the risky physical asset.[18] Since prices do

[17] Furthermore, we shall prove that small changes in the trust fund create small changes in equilibrium. Equilibrium before and after social security diversification will therefore be of the same type.

[18] Equilibrium conditions (10.e3) and (10.e4) are thus equalities, fixing $p_J = 1/R_0$ and $p_K = 1$. Conditions (10.e1) and (10.e2) are irrelevant as long as they can be satisfied with $k_0 > 0$ and $k > 0$. As

not change when the trust fund alters its portfolio, savers are left unaffected. (Since the interest rate does not change, second-period taxes do not change, so the budget set of savers is indeed unaffected.) With unchanging prices they demand the same combination of all three consumption goods — first-period, second-period safe and second-period risky consumptions. Thus, if the trust fund sells some bonds to savers, the savers maintain the same lifetime consumption, and also finance the purchase, simply by investing less in the safe technology. The trust fund uses the proceeds of the sale to undertake risky investment. Thus aggregate risky investment goes up, aggregate safe investment goes down and aggregate investment is unchanged. Since the expected return on risky investment must exceed the return on safe investment (for both to be held by risk-averse savers), expected aggregate output goes up. To see this, note that with $p_K = 1$, the savers' first order condition satisfies:

$$U_1'[C_1] = E\{U_2'[C_2](1+r)\} = E\{U_2'[C_2]R\}; \tag{11}$$

(By hypothesis, savers hold both the risky asset and the safe asset, and so must be indifferent at the margin between the two assets). Since $C_2$ and $R$ are perfectly correlated, $U_2'[C_2]$ and $R$ are negatively correlated. Hence the equality of expectations can only hold if $E\{R\} > 1 + r = R_0$.

If the trust fund initially has only a small amount of the risky asset, this policy is a (weak) Pareto gain — savers are not affected and workers gain since the workers are not risk averse for the first bit of investment in risky assets. To see this, consider the change in worker expected utility (noting that interest rates $r$ and therefore taxes $t$ are unaffected) assuming that $\kappa = K^f$ is zero:

$$\begin{aligned} dv/d\kappa &= E\{u_2'[c_2]((R-1-r)/n - dt/d\kappa + F_0(dr/d\kappa)/n)\} \\ &= E\{u_2'[c_2](R-1-r)\}/n \\ &= u_2'[c_2]E\{(R-1-r)\}/n > 0. \end{aligned} \tag{12}$$

The last equality is obtained by noting that second period consumption of workers is certain, hence so is second period marginal utility, so it may be brought outside of the expectation operator. The final inequality follows from the excess expected risky return derived above. Thus we have shown

**Proposition 1** *Suppose both the safe and risky assets are held in stationary equilibrium (in positive quantities). Then increased trust fund investment in risky assets will raise aggregate risky investment, lower aggregate safe investment, leave aggregate investment unchanged and increase expected output. If the trust fund initially held no risky assets, then the diversification will lead to a weak Pareto improvement, increasing the utility of the young workers and leaving the utility of the old workers and all savers unchanged.*

The equity premium is defined as the difference between the expected return on the risky investment and the return on the safe investment, $E\{R\} - (1 + r)$. Since the equity premium must be consistent with the portfolio choice of risk-averse savers (who hold a strictly positive quantity of risky assets by hypothesis), it must be positive in equilibrium. As long as the equity premium is positive, there is an expected utility gain to workers from diversification in a model where they bear no other risk, since then $E\{u_2'[c_2](R-1-r)\} = u_2'[c_2]E\{R-1-r\} > 0$.

The crucial step in this argument in favor of social security diversification is the paradoxical claim that workers are *more* risk tolerant *on the margin* than savers. One might

mentioned earlier in Section 4, equilibrium is equivalently defined in this regime by (10.e1)–(10.e4) or (9.E1)–(9.E2).

suspect the contrary, that savers are more risk tolerant than workers, all else being equal. That is, it may well be that the worker utility $u$ is a concave transformation of the saver utility $U$, thereby displaying more risk aversion at any level of consumption. And workers have lower incomes on average than savers, which also makes them more risk averse, given an assumption of decreasing absolute risk aversion. But all else is not equal. The savers hold the entire risky capital stock of the nation, while the workers hold none (if $K^f$ is zero). Our proof that there are welfare benefits from social security diversification goes through no matter how much more risk averse $u$ is than $U$, and no matter how poor the workers are. It only needed that both $u$ and $U$ are differentiable, and that workers are not exposed to any stock market risk or other risks correlated with stock market risk.

In reality, workers' retirement income is not completely independent of stock returns. Social security benefits are connected by an explicit formula to real wages. Over retirement horizons as long as thirty years, there is considerable covariance between real wages and stock returns. The question then becomes, how big is worker exposure to stocks, how big is the equity premium, and how risk averse are workers? Addressing this question in detail is beyond the scope of this paper. Note, however, that if the workers' and savers' utilities $u$ and $U$ display similar risk aversion, and both display increasing relative risk aversion, then the poorer workers should have a higher fraction of their wealth invested in stocks than the richer savers. Our judgment is that after properly calibrating the stock exposure implicit in aggregate wages, one would come to the conclusion that the average worker is less exposed to stock returns than savers. At the point where the trust fund holds no stock, it seems very likely to us that the average worker would be better off by some investing in equity. The converse would hold only if it would be optimal for such a worker just starting to save, to hold a portfolio with no stocks at all. However, what is best for the average worker may not be best for every worker. Though our model has assumed that all workers are identical, in reality some workers may be far more risk averse, so that for them any additional stock exposure may be bad, preventing social security diversification from being a Pareto gain (Pestieau and Possen, 1999). However, in considering a more general setting with heterogeneous workers, the reader should bear in mind that the lowest income workers would be protected by the safety net (SSI).

By the same logic used in the proof of proposition 1 for our representative worker model, further increases in social security risky asset holdings would also be weak Pareto gains until the optimal portfolio for workers was reached, unless the saver's holdings of the safe real investment reached zero first. In considering the optimal level of social security diversification, we note that since social security benefits become more correlated with stock returns as diversification increases, the welfare benefits to further diversification decline. The proof of Proposition 1 is thus an argument for limited diversification.

The welfare gains from social security diversification described in Proposition 1 sound superficially similar to the popular argument repeatedly seen in the press about the excess return stocks have traditionally earned over bonds. However, for the typical saver, Proposition 1 claims no (ex ante) gain from social security diversification, despite the equity premium. Nor would it if savers were also covered by social security (except for some savers who were 100% in stocks in their portfolio and wanted some of social security to be in stock as well). On the margin, savers should not expect any excess return to stocks in utility terms. For every dollar in the actual social security trust fund that is shifted to equity, the welfare gains described in Proposition 1 only apply to that fraction of each dollar that goes to support the benefits of workers with little financial wealth who do not borrow, and are therefore holding no stocks.[19] If there were no such poor, constrained

[19]Some workers are unable to hold stocks because they have not saved enough. Others do not hold stocks even though they could. Some of the latter may not be optimizing and would also gain from the diversification, while some may not have been willing to bear the cost of learning about stocks and would

workers, as is the case in a representative agent model with only rational savers, a small enough change in trust fund portfolio policy would have no effects at all.[20]

Our judgment that nonsavers are less risk averse on the margin than savers gives rise to the same welfare benefits to diversification in all of our subsequent models. This effect is perhaps controversial. It is not controversial that a substantial fraction of American workers lack the resources to undo in their private portfolios what the social security trust fund might do on their behalf. If the technology is not perfectly elastic, then the presence of these nonsavers will force savers to alter their portfolio holdings after social security diversification. This implies that social security diversification must have real effects, which we investigate in the following models. These real effects, including additional welfare effects from changing asset prices and taxes, do not depend on our assessment of the marginal risk aversion of nonsavers. Indeed, precisely because they do not save, the effect of social security diversification on asset prices and taxes is independent of their (marginal) risk tolerance.

# 6 Social Security Diversification in the Risky Linear Case

It is unrealistic to suppose that technology fixes the returns on safe and risky assets, independent of preferences. We take a step toward realism by moving to economies in which the second type of equilibrium prevails, and the return on government bonds is endogenous. In this regime, $k_0 = 0$ and $k > 0$. (There are no safe investments undertaken in equilibrium at all, but we maintain the existence of an unlimited number of risky investment opportunities yielding the same distribution of returns, some of which are undertaken.) Given the constant marginal returns to risky investments, equilibrium requires $p_K = 1$. The interest rate on government debt is determined by the supply and demand for bonds.[21] We state this market clearance equation first in general terms, from (10.e1), and then, using the value of $p_K$, and the budget constraint in (2), and finally, the equation for taxes, (5), we state it as a single equation in a single variable.

$$\begin{aligned} G - F_0 + \kappa &= p_J J^*(p_J, p_K, W - p_J T(p_J)) + p_J T(p_J); \qquad (13) \\ G - F_0 + K^f &= W - C^*[p_J, 1, W - p_J T(p_J)] - K^*[p_J, 1, W - p_J T(p_J)] \\ &= W - C^*[p_J, 1, W - (1 - p_J)aG] - K^*[p_J, 1, W - (1 - p_J)aG]. \end{aligned}$$

Thus, with $p_K = 1$, we have a single equation to determine the endogenous price of second-period safe consumption. Note that with $G$, $F_0$, and $W$ all fixed, the response of aggregate investment, $K^* + K^f$, to portfolio policy is minus the response of the consumption of savers, $C^*$.

## 6.1 Income Taxes, the Interest Rate, and the Redistribution of Income

In the absence of the perfectly elastic safe technology, a trust fund shift into risky assets will raise the supply of bonds available to the savers, and likely change the market clearing

also gain since they do not have to pay a cost if investment is done centrally. However, workers so risk averse that they should hold no stocks would lose from diversification, as noted above. Some workers may mistakenly be overinvested in stocks and would also lose from trust fund diversification if they do not reduce their stockholdings in response to trust fund investment.

[20] This has been noted by Smetters, (1997) and Bohn (1997, 1998).

[21] Thus condition (10.e3) reduces to $p_J < 1/R_0$, and is irrelevant, and (10.e4) becomes $p_K = 1$, and (10.e2) is automatically satisfied with a suitable choice of $k$. The only nontrivial equilibrium condition is (10.e1). Once again equilibrium is equivalently described by (10.e1)–(10.e4) and (9.E1)–(9.E2).

interest rate. A change in the interest rate has an immediate welfare effect on savers, through their budget set, and on workers, through the payments on the trust fund holdings of bonds. In addition, any change in the interest rate (with no change in gross debt outstanding) requires a change in income taxes to cover interest costs. We present these two welfare implications of changing interest rates before turning to the determination of the change in the interest rate.

From (2), (10.e1), (9.E1), the definition of taxes given in (5), and the envelope theorem, we know that the change in utility to savers from a change in $p_J$ is given by

$$\begin{aligned} \partial V/\partial p_J &= U_1'\{-J^* - d[p_J T(p_J)]/dp_J\} \\ &= U_1'\{-[(G - F_0 + \kappa) - (1 - p_J)aG]/p_J - d[(1 - p_J)aG]/dp_J\} \\ &= U_1'\{-[B - (1 - p_J)aG]/p_J - d[(1 - p_J)aG]/dp_J\} \\ &= -U_1'[B - aG]/p_J. \end{aligned} \tag{14}$$

Hence, if the shares of marginal second-period taxes paid by savers and workers do not match their shares in the holding of government debt (directly by savers and indirectly through social security for workers), a change in the interest rate causes a redistribution of income. Recognizing that $p_J$ is the only endogenous price, it follows from (14) that the response of expected utility to trust fund portfolio diversification satisfies:

$$dV/d\kappa = -\{U_1'[B - aG]/p_J\}\{dp_J/d\kappa\}. \tag{15}$$

We can also look at the welfare effects on savers and workers in units of second period consumption. This is helpful when considering worker and saver welfare together, since a change in interest rate affects workers only in their old age, and since they do not save, there is no connection between their intertemporal marginal rate of substitution and the interest rate. When looking at second period consumption, it is also more convenient to use $r$ rather than $p_J$. From budget set (1) and the assumption that $k_0 = 0$, we have that $B = W - C^* - K^*$, so we can write the expected utility of a cohort of savers as a function of $r$ as:

$$V = U_1[C^*] + E\{U_2[(W - C^* - K^*)(1 + r) + RK^* - T(r)]\} \tag{16}$$

where $r$ is the endogenous interest rate. By the envelope theorem, the changes in $r$ and $T$ have a direct impact on expected utility, but the indirect changes drop out. Thus the change in lifetime expected utility from an increase in trust fund holdings of capital (and so a decrease in trust fund holdings of government debt) satisfies:

$$\begin{aligned} dV/d\kappa &= E\{U_2'\}(B(dr/d\kappa) - dT/d\kappa) \\ &= E\{U_2'\}(B(dr/d\kappa) - aG(dr/d\kappa)) \\ &= E\{U_2'\}(B - aG)(dr/d\kappa). \end{aligned} \tag{17}$$

That is, savers lose from any increase in the government bond interest rate to the extent that their relative holdings of government debt are less than their share of marginal taxes to cover interest costs. If a equals $B/G$, then savers are not affected by a marginal change in trust fund investment policy.[22]

Workers are affected by trust fund investment and by the impact of the interest rate on benefits and taxes. Differentiating expected lifetime utility of workers given by the last line of (8), and setting $p_k = 1$ and $\kappa = K^f$, and then using the definition of taxes from

[22] Actually, if $a = B/G$, then savers obtain a second order benefit from trust fund diversification, assuming $dr/d\kappa$ is not 0.

(5) and the bond clearing equation (9E.1), we have:

$$\begin{aligned} dv/d\kappa &= E\{u_2'((R-1-r)/n - dt/d\kappa + (F_0-\kappa)(dr/d\kappa)/n)\} \quad (18) \\ &= E\{u_2'(R-1-r+(F-(1-a)G)(dr/d\kappa)\}/n \\ &= E\{u_2'(R-1-r)\}/n - E\{u_2'\}(B-aG)(dr/d\kappa)/n. \end{aligned}$$

Diversification affects workers through two channels. As was the case in the bilinear regime, the expected utility of workers increases from bearing some risk if they were bearing none before diversification. In addition, the effect on workers from the change in the interest rate has the opposite sign from its effect on savers, as can be seen by comparing (17) and (18). If $a$ is equal to $B/G$, then this effect is zero and workers only gain from improved risky investment. When $a = B/G$, we have a (weak) Pareto gain, as in the bilinear case

Old savers and old workers at the time diversification is first implemented are not affected, in a model without land. Since (in steady state) all other generations are treated symmetrically, any social welfare function might as well be written in terms of one generation. Denoting the social evaluation of the marginal utility of second-period safe consumption of a worker relative to that of a saver by $m$, the impact on a social welfare function (SWF) equal to the weighted sum of individual expected utilities satisfies:

$$\begin{aligned} dSWF/d\kappa &= m\{ndv/d\kappa\} + 1\{dV/d\kappa\} \quad (19) \\ &= mE\{u_2'(R-r-1)\} - E\{mu_2' - U_2'\}(B-aG)dr/d\kappa. \end{aligned}$$

Thus there is a direct utility gain from improved risk bearing and a redistributive term, which vanishes if $a$ equals $B/G$. If there is an income distribution change from the transfer between savers and workers, its effect depends on the direction of transfer and the sign of $E\{mu_2' - U_2'\}$. In particular, if we choose $m$ so that one unit of second-period safe consumption gives the same marginal social welfare to every old agent, then the redistribution term drops out and total utility is increased by diversification.

Whether $B > aG$ depends on the size of the trust fund and how tax policy responds to increased interest costs. If $F = 0$, then $B = G > aG$, and an increase in interest rates helps savers, because for every extra dollar in interest receipts received, they pay only a $< 1$ dollars extra in taxes. On the other hand, if the trust fund holds all the government bonds, then $B = 0 < aG$, and savers lose from an increase in interest rates.[23]

In reality, the social security trust fund pays benefits to nonsavers and savers. But because of the redistributive nature of the social security benefit rules, nonsavers have a claim on benefits that exceeds their share of income. At the end of 1999, the trust fund was nearly $900 billion, and increasing rapidly. If it is not the case now, surely in a few years $B < aG$ will be more plausible than $B > aG$. Once the trust fund is big relative to the outstanding stock of government bonds, interest rate increases can be expected to help workers and hurt savers.

**Proposition 2a** *Suppose there are no long-lived assets, and no safe real assets held in equilibrium. Suppose there are constant marginal returns to risky investment, and that risky investments are undertaken in equilibrium. Suppose that the share of taxes of savers*

[23] To consider who in reality is a net receiver of government bond interest payments, net of the taxes levied to pay for them, we need to consider which taxes are raised if interest costs are higher. If it is just the income tax increased, then low income people are not taxed at all. However, if the earned income tax credit is altered along with the income tax (violating our assumption that it is taxes on older workers that are adjusted), then the impact is throughout the income distribution. A realistic case would indeed consider tax changes on young workers and savers as well as on old. That would have additional effects, which we are not trying to analyze.

*is at least as large as their share of bond holdings, $B \leq aG$, and that all three demands are normal. Then, trust fund purchases of the risky investment increase the interest rate on government debt, and increase aggregate real investment, though by less than the quantity of the trust fund purchases.*

**Proposition 2b** *Moreover, starting from a trust fund portfolio entirely in bonds, diversification raises the expected utility of workers in every generation (except the original old, who are unaffected). If in addition, $B = aG$, then trust fund diversification does not affect the utility of young savers in every generation (up to first order, increasing it up to second order). If instead, $B < aG$, then diversification (without any other policy changes) lowers the utility of young savers in every generation. Nevertheless, when $B \leq aG$, trust fund purchases of risky investment increase the weighted sum of utility of all workers and savers, weighted so that the marginal social utility of second-period consumption is the same for all.*

Given the truth of Proposition 2a, equations (17), (18), and (19) demonstrate Proposition 2b.

## 6.2 Interest Rate

To prove Proposition 2a, it remains to examine the effect of diversification on the interest rate and on the level of aggregate investment. Suppose we are in equilibrium given by (13). If $K^f$ increases, equation (13) can be brought back into balance by decreasing $C^*[p_J, 1, W-(1-p_J)aG] + K^*[p_J, 1, W-(1-p_J)aG]$. With $B \leq aG$, this can only happen by decreasing $p_J$, that is, by increasing $r$. To see this, differentiate (13), giving:

$$dp_J/d\kappa = -1/(d\{C^*[p_J, 1, W-(1-p_J)aG] + K^*[p_J, 1, W-(1-p_J)aG]\}/dp_J). \quad (20)$$

Letting $I = W-(1-p_J)aG$ be net wealth at the original equilibrium, and letting $V$ be utility at the original equilibrium, and using the standard Slutsky equation[24] relating demand to the sum of compensated demand and income effects, and using the size of the income effect given by (14), we have

$$dp_J/d\kappa = -1/(C_p^c[p_J, 1, V] + K_p^c[p_J, 1, V] + ((C_I^*[p_J, 1, I] + K_I^*[p_J, 1, I])(-[B-aG]/p_J))) < 0, \quad (21)$$

where subscripts $p$ and $I$ refer to partial derivatives with respect to $p_J$ and income $I$, and superscript $c$ means compensated demand. To see that this expression is really less than zero, first note that compensated changes keep expected utility constant and marginal utilities are proportional to prices, implying that $0 = C_p^c + p_K K_p^c + p_J J_p^c = C_p^c + K_p^c + p_J J_p^c$. Since compensated own effects are always negative, $J_p^c < 0$, and it follows that $C_p^c + K_p^c > 0$. Since both $C$ and $K$ are normal goods, and $B - aG \leq 0$, the denominator of (21) is positive.[25] Note that this analysis holds for any $\kappa \geq 0$ consistent with equilibrium in the risky linear regime.

## 6.3 Aggregate Investment

To consider the impact of changing social security portfolio policy on aggregate investment, we need only determine its effect on the consumption of young savers, since the

[24] $dC^*[p_J, 1, W-(1-p_J)aG]/dp_J$ equals $C_p^c + C_I^* dI$, where $dI = dV/U_1'[C_1]$ is the change in income that would give the same utility at the old prices as given by the new prices and the new income.

[25] We note that the intertemporally additive structure of preferences implies that demand for first-period consumption is normal and the assumption of decreasing absolute risk aversion in second-period utility would imply that the demand for risky assets increases with wealth.

consumption of young workers does not change, as noted directly after equation (13). From (13) we know (using Slutsky and the income effect term from (14)) that

$$\begin{aligned} dk/d\kappa &= d\{K^* + K^f\}/d\kappa = -(dC^*[p_J, 1, W - (1-p_J)aG]/dp_J)(dp_J/d\kappa) \quad (22) \\ &= -\{C_p^c[p_J, 1, V] + C_I^*[p_J, 1, I](-[B - aG]/p_J)\}(dp_J/d\kappa) \\ &= \{C_p^c - C_I^*[B - aG]/p_J\}/\{C_p^c + K_p^c - (C_I^* + K_I^*)[B - aG]/p_J\} > 0. \end{aligned}$$

We already saw that the denominator is positive. Also $C_p^c[p_J, 1, V] > 0$, since $J$ and $C$ are Hicksian substitutes if $J$ is normal. (See Aura, Diamond, Geanakoplos (1999).) Furthermore, $C_I^* > 0$, since $C$ is normal. Thus if $B \leq aG$, substitution and income effects go the same way. Hence trust fund diversification lowers $C^*$, thereby raising total risky investment $K^* + K^f$.

Our analysis also shows that $dk/d\kappa = d\{K^* + K^f\}/d\kappa < 1$, when all the goods are normal, for then $K_p^c > 0$ and $K_I^* > 0$, and the denominator of (22) is larger than the numerator.

Complementing these propositions, we note that if an increase in the government bond interest rate redistributes wealth from workers to savers, $B > aG$, then the interest rate may rise or fall.[26]

# 7 Adding Infinitely-Lived Assets

In the model above, no assets last more than one period. Thus a change in the interest rate does not redistribute wealth across generations. To consider intergenerational redistribution, we now add two infinitely-lived assets. A change in social security policy that changes the prices of the long-lived assets the generations trade with each other will redistribute wealth between the old, who own the assets at the time the policy is implemented, and all future generations. We assume fixed quantities of both types of infinitely-lived assets. We will refer to these assets as safe land and risky land. Each unit of safe land provides one unit of consumption, independent of the state of nature, in each period.[27] Each unit of risky land produces the same (realized) output as one unit of the (contemporaneous) risky investment.[28] We denote the supplies of the two assets by $L_0$ and $L$, and their prices by $p_0$ and $p$. Because of the stationary structure of the economy, in stationary equilibrium, these prices are constant over time.

The introduction of new long-term assets raises the interesting question of how their prices are affected by social security diversification. In reality, the stock market is made up of both short-term and long-term investments, and so the effect of social security diversification on stock market prices involves both short-term and long-term asset price changes.

Analogous to (1), we can write the optimization problem of savers as

$$\begin{aligned} V &= \max U_1[C_1] + E\{U_2[C_2]\} \quad (23) \\ &\quad \text{s.t. } W = C_1 + B + k_0 + k + p_0 L_0^* + pL^* \\ &\qquad C_2 = (1+r)B + R_0 k_0 + Rk + L_0^* + p_0 L_0^* + RL^* + pL^* - T. \end{aligned}$$

[26] So far, we have considered two different models with perfectly elastic and perfectly inelastic supplies of safe assets. We could consider an intermediate model with a downward-sloping demand by foreigners for government debt. This would give a change in the equilibrium interest rate that was between the two cases analyzed. In this case, the increase in the interest rate on government debt would involve increased payments abroad as well as transfers from taxpayers to trust fund beneficiaries. Rather than add a foreign sector, we add land.

[27] This might be a fixed number of government consols, the interest on which is financed by taxation on successive generations.

[28] Since the return to risky investment is independent of the level of investment, there is no distinction between land that provides output and land that provides capital input.

Taxes $T(r) = arG$ are defined as before, in equation (5). Assuming that $k > 0$, and that the trust fund holds only one-period government debt and short-term risky investments, stationary equilibrium is defined as a vector $(r, p_0, p, C_1, C_2, k_0, k, B, L_0^*, L^*)$ such that given $(r, p_0, p)$ and $T = T(r)$, the choices $(C_1, C_2, k_0, k, B, L_0^*, L^*)$ maximize (23), and the bond market clears, as given in equation (9.E1), the safe land market clears, $L_0^* = L_0$, and the risky land market clears $L^* = L$.

At first glance, the model with land seems much more complicated, since it now has two more markets, giving two more variables and two more market clearing equations. But as we shall see in a moment, the effects of trust fund diversification in the presence of long-lived assets are modified by the presence of land, but not drastically changed. As before, the analysis depends on which short-term investments are undertaken in the original stationary equilibrium. The case where both are undertaken, $k_0 > 0$ and $k > 0$, is exactly like the case without land since no prices change, and it is not repeated here. We describe the other three cases in detail, starting in Section 8. In the rest of Section 7, we reformulate and simplify the definition of stationary equilibrium with land.

## 7.1 Land and the Dynamic Asset Span

Land (of either type) lasts forever and gives new output forever and is sold each period by the older generation to the younger one. Given a stationary economy, and the fact that land output, though perhaps risky, is independent and identically distributed each period, the price of land (just *after* the realization of output each period) is constant across time, and across realizations of output.[29] The one-period gross return from purchasing land is equal to its dividend that period, plus a constant capital value. The one-period returns on either type of infinitely-lived land are therefore (endogenous) convex combinations of the returns on risky short-term investments and the safe return on government bonds. Thus we can incorporate land into our model without introducing a new risk characteristic.

If a young saver buys one unit of safe land, it costs $p_0$ and yields safe consumption in the second period (from output and resale) of $1 + p_0$. Since this is a perfect substitute for buying $(1 + p_0)/(1 + r)$ units of the government bond, we have

$$p_0 = (1 + p_0)/(1 + r), \tag{24}$$

or

$$p_0 = 1/r = p_J/(1 - p_J). \tag{25}$$

Similarly, by spending $p$ on risky land, the consumer gets the risky dividend that can be purchased at a price of $p_K$, (by investing in the short-lived risky asset) and the ability to sell the asset at price $p$, which has a current value of $p/(1 + r)$. Thus, by arbitrage, we have the equilibrium price of risky land satisfying:

$$p = p_K + p/(1 + r) = p_K + pp_J \tag{26}$$

or

$$p = (1 + r)p_K/r = p_K/(1 - p_J). \tag{27}$$

Thus the prices of both kinds of land are determined by the price of the short-term assets. Both land prices increase with the price of second-period safe consumption; equivalently, land prices decrease when the interest rate rises. The total value of land is

$$P = p_0 L_0 + pL = L_0 p_J/(1 - p_J) + L p_K/(1 - p_J). \tag{28}$$

[29] The iid assumption implies that there is never any "news" about future returns, so land values never change.

Any equilibrium where the trust fund holds $L$ acres of risky land is equivalent to an equilibrium in which the trust fund holds $L$ units of the risky asset and $Lpp_J$ bonds. (The dividends are the same, and by (26) the bond payoffs $pL$ can be used each period to repurchase the same portfolio.) In particular, starting from a portfolio exclusively in short-term assets, a trust fund purchase of $L$ acres of land (obtained by selling bonds) would have precisely the same effect as the purchase of $L$ units of the risky short-term asset (obtained by selling bonds). However, the effect of a further trust fund purchase of land will differ in the two cases because it will change asset prices, giving a different capital gain to the two portfolios. (In other words, after the price change, a different combination of short-term assets would be needed to replicate risky land.) To keep the analysis simple, we shall continue to suppose that the trust fund holds only short-term assets.

### 7.2 Stationary Equilibrium with Land

We can now define equilibrium in terms of the three goods $C^*$, $J^*$, $K^*$, as we did earlier. The presence of land does not change the expected utility maximization problem of savers given in (2). It is just that they have several ways of acquiring the same patterns of second-period consumption. The demand functions, $C^*[p_J, p_K, W - p_J T]$, $J^*[p_J, p_K, W - p_J T]$ and $K^*[p_J, p_K, W - p_J T]$ do not change. We need to recognize that savers acquire safe future consumption by buying bonds, and both types of land. Safe consumption $J^*$ plus taxes for savers must equal the return from holding all the government bonds not held by the trust fund, plus the payoff from safe land, plus the resale value of all land, safe and risky, plus the payoff from short-term safe production. Similarly, risky consumption for savers, $K^*$, is now the demand for the sum of risky short-term investments and the one-period payoffs of risky land, less what is held by the trust fund. Equilibrium profit maximization equations (10.e3) and (10.e4) remain as before, but (10.e1) and (10.e2) must now be modified to include land.

$$J^*(p_J, p_K, W - p_J T(p_J)) + T(p_J) = (G - F_0 + \kappa)/p_J + L_0 + P + R_0 k_0 \qquad (29.\text{e}1^*)$$

$$K^*(p_J, p_K, W - p_J T(p_J)) + \kappa/p_K = L + k \qquad (29.\text{e}2^*)$$

$$P = L_0 p_J/(1 - p_J) + L p_K/(1 - p_J) \qquad (29.\text{e}3^*)$$

# 8 Social Security Diversification in the Risky Linear Case with Land

We suppose the economy with land described in Section 7 is such that in stationary equilibrium there is only risky production, $k_0 = 0$ but $k > 0$. As in Section 6, equilibrium boils down to the market for safe consumption in equation (29.e1*). Using the budget constraint (2) of the savers, the equality $p_K = 1$, the definition of $T(p_J)$, and the formula (28) for the value of total land, we can write the market clearance for bonds as:

$$\begin{aligned} G - F_0 + k &= p_J J^*(p_J, p_K, W - p_J T(p_J)) + p_J T(p_J) - p_J(L_0 + P) \qquad (30) \\ G - F_0 + K^f &= W - C^*[p_J, 1, W - p_J T(p_J)] - K^*[p_J, 1, W - p_J T(p_J)] - p_J(L_0 + P) \\ &= W - C^*[p_J, 1, W - (1-p_J)aG] - K^*[p_J, 1, W - (1-p_J)aG] - p_J(L_0 + L)/(1-p_J). \end{aligned}$$

As before, we have a single equation in a single variable, $p_J$. And, with the same assumptions as before, we will again find that $p_J$ goes down, equivalent to the interest rate going up.

From (29.e2*), aggregate investment in short-lived production, $k$, is trust fund demand for real investment, $K^f$, plus the demand of savers for risky consumption, $K^*$, minus the portion of that demand that is satisfied by purchasing risky land, $L$. Thus, using the second line of (30), and $p_K = 1$, and $P = p_K L + p_J(L_0 + P)$, aggregate investment in risky short-term assets, $K^f + K^* - L$, can be written

$$K^f + K^* - L = p_K(K^f + K^* - L) = W - (G - F_0) - C^*[p_J, p_K, W - p_J T(p_J)] - P. \quad (31)$$

Hence, with $W$, $G$, and $F_0$ all fixed, the response of aggregate investment in short-term assets to social security diversification is minus the sum of the response of consumption of savers, $C^*$, and the change in the value of total land, $P = p_0 L_0 + pL$. With the same assumptions as before, we will again find that aggregate investment goes up.

## 8.1 Expected Utility

In the previous models without land, old savers were not affected by social security diversification. Since no asset lasted longer than their lives, by the time they were old all their assets were paid down or dissolved into consumption goods. But with the introduction of land they have something to sell, whose value might be affected by social security diversification. For example, if land prices go down in value (as the interest rate rises), the old savers at the time of the trust fund diversification lose, *ceteris paribus*. Young savers gain, as do savers in every succeeding generation.

We begin the analysis with old savers at the time of implementation of the policy change:

$$\begin{aligned}
\partial V_{\text{old}}/\partial p_J &= U_2'[dP/dp_J] \quad (32)\\
dV_{\text{old}}/d\kappa &= (\partial V_{\text{old}}/\partial p_J)(dp_J/d\kappa)\\
&= U_2'[dP/dp_J](dp_J/d\kappa).
\end{aligned}$$

Since the new stationary equilibrium is achieved immediately after the trust fund purchases, young savers at the time of the purchases are affected exactly the same way as all future savers, namely

$$\begin{aligned}
\partial V/\partial p_J &= U_1'\{-J^* - d[p_J T(p_J)]/dp_J\} \quad (33)\\
&= U_1'\{-\{[B - (1 - p_J)aG]/p_J + L_0 + P\} - d[(1 - p_J)aG]/dp_J\}\\
&= -U_1'[[B - aG]/p_J + L_0 + P]\\
dV/dk &= (\partial V/\partial p_J)(dp_J/d\kappa)\\
&= \{-U_1'[[B - aG]/p_J + L_0 + P]\}(dp_J/d\kappa).
\end{aligned}$$

The first term in the last line of (33) reflects the within-cohort redistribution between savers and workers as a consequence of different shares in government bonds and in the taxes that finance the interest on the bonds. The second term reflects the across-cohort redistribution from changes in the price of safe consumption which is purchased from the previous generation by buying land. The formulas in (33) appear to depend on the value of land, not the change in the value of land, but this shall be clarified in a moment. Notice that the expected utility of young savers can increase or decrease, depending on the balance of redistributions between savers and workers, and redistributions between old savers and young savers. If the value of all land, $P$, exceeds the total of all government bond promises $G/p_J = G(1 + r)$, then social security diversification must improve the welfare of young savers (assuming $dp_J/dk < 0$), even though it creates a redistribution from young savers to young workers, if $B < aG$. Evidently young savers gain more from old savers than they lose to young workers. A similar conclusion holds if the value of safe land $p_0 L_0$ is greater than the value of government bonds $G$ outstanding, as can be seen from (33), and the identity $p_J(L_0 + P) = p_0(L_0 + L)$.

Equation (18) quantifying the effect of trust fund diversification on workers in the risky linear case without land applies without change in this risky linear case with land. As before, the increased exposure to risky stock and the rise in interest rates make workers better off, assuming $B - aG < 0$ and $dp_J/d\kappa < 0$.

## 8.2 Land Values

With $p_K$ unchanged at a value of one, we can compute the change in total land value by

$$\begin{aligned} dP/dp_J &= L_0[1/(1-p_J) + p_J/(1-p_J)^2] + Lp_K/(1-p_J)^2 \\ &= L_0/(1-p_J)^2 + Lp_K/(1-p_J)^2 \\ &= [L_0(1-p_J+p_J)/(1-p_J) + Lp_K/(1-p_J)]/(1-p_J) \\ &= [L_0 p_J/(1-p_J) + Lp_K/(1-p_J) + L_0]/(1-p_J) \\ &= [P + L_0]/(1-p_J) \\ dP/d\kappa &= (dP/dp_J)(dp_J/d\kappa) \\ &= \{[P + L_0]/(1-p_J)\}(dp_J/d\kappa). \end{aligned} \tag{34}$$

The second line of (34) decomposes the change in the total value of land into the sum of the change in the value of safe land and the change in the value of risky land. It is clear from the formulas that the value of safe land and of risky land each move in the same direction as $p_J$, that is in the opposite direction of the change in interest rates. We shall show in the next subsection that if $B \leq aG$, and demands for all goods are normal, then diversification raises the safe interest rate: $(dp_J/d\kappa) < 0$. Thus trust fund purchases of risky short-term investments (and equivalently trust fund purchases of risky land) reduce the price of risky land (and also the price of safe land). It is a remarkable, and unanticipated, property of the current model that the increase in demand for risky land reduces its price! It is often claimed that if social security bought stocks, it would raise the value of the stock market. This breezy conclusion is seen to be more delicate than it sounds. Since the interest rate increases, it is not so surprising after all to find a tendency for stock prices to decline, for stock prices depend on discounting future returns. When technology fixes the return on short-term risky investments, the interest rate effect is the only one that affects stock prices. We pursue this question of land values further in the next sections.

Using the expression derived in (34) for the change in total land value, we can rewrite the effect on savers' welfare derived in (33) as

$$\begin{aligned} \partial V/\partial p_J &= -U_1'[[B - aG]/p_J + (1-p_J)dP/dp_J] \\ dV/d\kappa &= -U_1'[[B - aG]/p_J + (1-p_J)dP/dp_J](dp_J/d\kappa). \end{aligned} \tag{35}$$

Now it is clear that changing land values does affect young savers. If social security is diversified at time 1, and land prices fall, the young at time 1 do not gain by the whole drop in land prices, since the resale value of the land when they get old also falls.

## 8.3 Social Welfare

The social welfare function discussed in Section 6 simply weighted the utilities of a representative saver and a representative worker from the same generation. Now that different generations receive different treatment, we also need to assign weights to the generations.

In the rest of the paper, we assume social welfare is measured by the weighted sum of all household utilities:

$$SWF = mnv^0_{\text{old}} + V^0_{\text{old}} + \sum_1^\infty \delta(t)[mnv^t + V^t]$$

The superscript $t$ refers to the generation of birth, and we suppose the diversification takes place at time $t = 1$. The weight $m$ is chosen, as before, so that starting from the original equilibrium, an additional dollar gives the same marginal social utility whether it is given to an old saver or an old worker from the same generation. Finally we suppose $\delta(t) = 1/(1 + r)^{t-1}$, where $r$ is the interest rate prevailing in the original equilibrium. With this definition of social welfare, the marginal social utility of a dollar for sure at any time $t$, whether given to any young saver, old saver, or old worker, starting from the original equilibrium allocation, is equal to its market price as of time 1. (The welfare weight on young workers can be arbitrary, since social security diversification does not affect them in our model.) Since the welfare weights are such that redistribution of safe consumption (apart from young workers) has no effect on social welfare, we call SWF a social welfare function with neutral weights.

To calculate the effect of social security diversification on social welfare, the utility gains must be added across all generations. Using (32) and (35), and recognizing that $(1-p_J)$ is equal to $r/(1+r)$, the change in total land value does not affect social welfare. The sum of all savers' utility gains from the fall in land prices, discounted by the equilibrium interest rate, exactly balances the change in utility of the old from the generation in retirement at the time the policy was implemented.

## 8.4 Interest Rate and Proposition 2*

We summarize the positive and normative effects described so far in Proposition 2* (the number 2 refers to the regime, and the * indicates that land is present in the economy):

**Proposition 2a*** *Suppose there is no safe short-term real investment, and that short-term risky investments display constant marginal returns, and are undertaken in equilibrium. Suppose there is both safe and risky land. Suppose also that the share of taxes of savers is at least as large as their share of bond holdings, $B \leq aG$, and that demands for all consumption goods are normal. Then, trust fund purchases of risky investment increase the interest rate on government debt, and increase aggregate investment.*

**Proposition 2b*** *Moreover, the price of safe land falls, as does the price of risky land. The total value of land therefore falls, though $-dP/d\kappa < 1/p_J$.*

**Proposition 2c*** *Moreover, starting from a trust fund invested exclusively in bonds, trust fund purchases of risky investments increase the expected utility of all workers (except the old at the time the policy is implemented, who are unaffected). Old savers at the time the policy is implemented are hurt. If, in addition, $(aG - B) < p_0(L_0 + L)$, then all other savers are helped. If, however, $(aG - B) > p_0(L_0 + L)$, then all other savers are also hurt. In any case, diversification increases a social welfare function with neutral weights.*

Given 2a*, 2c* and the qualitative part of 2b* have already been proved.

It remains to show that trust fund purchases of risky investment increase the interest rate on government debt. Differentiating (31), using the Slutsky equation, using the impact of a price change on $V$ derived in (33), and using the derivative of $P$ from (34), we have:

$$
\begin{aligned}
dp_J/d\kappa &= -1/d(C^* + K^* + P)/dp_J \qquad (36)\\
&= -1/\{C^c_p + K^c_p + (C^*_I + K^*_I)(-(B - aG)/p_J - L_0 - P) + dP/dp_J\}\\
&= -1/\{\{C^c_p + K^c_p\} - \{(C^*_I + K^*_I)(B - aG)/p_J\} + \{-(C^*_I + K^*_I)(L_0 + P) + dP/dp_J\}\}\\
&= -1/\{\{C^c_p + K^c_p\} - \{(C^*_I + K^*_I)(B - aG)/p_J\} + \{(L_0 + P)(1/(1 - p_J) - (C^*_I + K^*_I))\}\}.
\end{aligned}
$$

The first terms in the denominator of the last line of (36) are the compensated demands for first-period consumption and risky second-period consumption with respect to the price of safe second-period consumption and have a positive sum, as noted in the proof of Proposition 2. The next term reflects the redistribution between savers and workers and is positive if savers have a larger share in taxes than in bonds ($B \leq aG$) and have normal demands. The final term reflects the intergenerational redistribution between old savers when the policy is implemented and later cohorts. It is also positive when the demand for safe second-period consumption is normal (which implies that $C^*_I + K^*_I = 1 - p_J J^*_I < 1$) and the price of second period consumption, $p_J$, is between zero and one, that is, the safe interest rate is positive, as we have assumed. Thus $dp_J/d\kappa < 0$, as was the case without land.

From (36) we note that the presence of long-lived assets, $L_0 + L > 0$, decreases the sensitivity of interest rates to trust fund diversification, and thus decreases the size of the interest rate increase.

We can get further information about the size of $dP/d\kappa$ by multiplying out the terms in (36). In the third line of (36), replace $-(C^*_I + K^*_I)(L_0 + P) + dP/dp_J$ with $p_J\{dP/dp_J\} + x$. Using the formula $(1 - p_J)dP/dp_J = (L_0 + P)$ derived in (34), and the fact that $(C^*_I + K^*_I) < 1$, we know that $x > 0$. Now multiplying out the terms in (36), and using the fact that the rest of the terms in the denominator of (36) are positive and the fact that $dp_J/d\kappa < 0$, gives $-p_J dP/d\kappa < 1$, concluding the proof of 2b*. We might interpret the quantitative part of 2b* as follows. Given that a period in this model represents something like 30 years, and that the real interest rate has historically been about 2.3% per annum, a crude estimate of $p_J$ is about $\frac{1}{2}$. A \$500 billion transfer of trust fund assets from bonds into stock, maintained there forever, must lower land prices, but could not lower land prices by more than \$1 trillion.

## 8.5 Aggregate Investment

As noted in (31), the level of investment in short-term production possibilities, $K^f + K^* - L$, is equal to the endowment of young savers less their first-period consumption, less the amount spent on purchasing land, less the unified net debt of the government. Note that $L$ is constant. Hence the change in short-term risky investment is given by the change in the RHS of equation (31). Differentiating (31), using Slutsky with income effect from (33), and the derivative of $P$ from (34), gives

$$
\begin{aligned}
dk/d\kappa &= d\{K^* + K^f\}/d\kappa = -[d(C^* + P)/dp_J][dp_J/d\kappa] \qquad (37)\\
&= -\{C^c_p + (C^*_I)(-(B - aG)/p_J + L_0 + P) + dP/dp_J\}[dp_J/d\kappa]\\
&= -\{\{C^c_p - (C^*_I)(B - aG)/p_J\} + \{-(C^*_I)(L_0 + P) + dP/dp_J\}\}[dp_J/d\kappa]\\
&= -\{\{C^c_p\} - \{C^*_I(B - aG)/p_J\} + \{(L_0 + P)(1/(1 - p_J) - C^*_I)\}\}[dp_J/d\kappa] > 0.
\end{aligned}
$$

The first term $C^c_p$ in the last line is the compensated cross elasticity of first-period consumption with respect to the price of second-period safe consumption. If the demand for riskless second period consumption is normal, then $C^*$ and $J^*$ are Hicksian substitutes and this term is positive (Aura, Diamond, and Geanakoplos, 1999). With normality of demand for first period consumption and redistribution from savers to workers ($B \leq aG$) the second term is positive. The third term is also positive when the demands for second-period safe and risky consumption are normal ($C_I^* < 1$) and the price of second period consumption, $p_J$, is between zero and one, that is, the safe interest rate is positive, as we have assumed. Multiplying by the minus sign in front and by $dp_J/d\kappa < 0$ gives a positive number.

Replacing $[dp_J/d\kappa]$ in the first line of (37) by the first line of (36), we get $dk/d\kappa = [d(C^* + P)/dp_J]/[d(C^* + K^* + P)/dp_J]$. From this we see that whether $dk/d\kappa$ is above or below one depends on the sign of $dK^*/dp_J$, which is the sign of $K_p^c - K_I^*\{[(B - aG)/p_J] + (L_0 + P)\}$. The compensated derivative, $K_p^c$, is positive since the two assets are Hicksian substitutes. Thus, we see that if $-(B - aG) > p_J(L_0 + P)$, then $d\{K^* + K^f\}/d\kappa < 1$. But if $-(B - aG) < p_J(L_0 + P)$, and $K_p^c$ is small (it could be zero if we did not maintain expected utility maximization — see footnote 17), then $d\{K^* + K^f\}/d\kappa$ could be greater than 1. In such a case, the drop in the value of land gives such a big positive income boost to young savers, who are buyers of the land, that they increase their holdings of risky assets $K^*$ even while the competing return on safe assets has gone up. Thus a trust fund purchase of \$500 billion of risky assets could increase real investment by more than \$500 billion.

# 9 Social Security Diversification in the Safe Linear Case with Land

We now turn to economies that display the third equilibrium regime, where there are one-period safe real assets but no one-period risky assets held in equilibrium. Many of our preceding results are now reversed. While not realistic, this case shows the importance of diminished risky investment opportunities. In this extreme case, the interest rate is determined by the return on safe real one-period assets, $p_J = 1/(1 + r) = 1/R_0$, and the price of safe land is technologically determined as well. With no change in the interest rate, taxes do not change. The price of risky consumption, and thus the price of risky land, are determined by market clearance, reflecting both the given interest rate and the evaluation of risky consumption by savers.

Recall that we have assumed the presence of a short-term risky financial asset promising $R$ quite apart from whether there is any real risky production. A seller of this asset could simply deliver out of land dividends, without producing any risky output. Thus even when there is no risky production, we can define social security diversification exactly as before, namely as the sale of bonds and the purchase of short-term risky securities. Recall also that starting from a position with no risky securities, trust fund purchases of short-term risky assets in exchange for bonds has exactly the same effect as the trust fund purchase of risky land in exchange for bonds.

In the absence of risky production, the only source of risky consumption is risky land, and each acre of risky land provides one unit of risky consumption. (Hence $K^*$ can be interpreted as the savers' demand for risky land.) Market clearing in the market for risky consumption, (29.e2*), now reduces to:

$$\begin{aligned} &K^*(p_J, p_K, W - p_J T(p_J)) + \kappa/p_K = L \qquad (38)\\ &K^*(p_J, p_K, W - p_J T(p_J)) + K^f = L \\ &K^*(1/R_0, p_K, W - (1/R_0)aG(R_0 - 1)) + K^f = L. \end{aligned}$$

## 9.1 Expected Utility, the Price of Risky Consumption, the Price of Land, and Proposition 3*

With the interest rate fixed by the return on riskless investments, savers' utility changes only on account of a change in price of risky consumption. We have from the budget set (2), from the envelope theorem, and from (38) that:

$$\begin{aligned} \partial V/\partial p_K &= U_1'\{-K^*\} = -U_1'\{L - K^f\} \\ dV/d\kappa &= -U_1'\{L - K^f\} dp_K/d\kappa. \end{aligned} \quad (39)$$

Thus the expected utility of savers moves in the opposite direction from the price of risky investment. Since the trust fund adds to the demand for risky assets, we should not be surprised to find that $p_K$ rises after social security diversification. The result is that all savers, starting from the young at the time of social security diversification, lose.

This raises an interesting point for the current privatization debate. Many of today's young are clamoring for diversification on the grounds that stocks earn higher returns than bonds. But any rational young saver should already be investing so much of his wealth in stock that he is indifferent on the margin between further investments in stocks and bonds. Thus if prices did not change, the direct effect of social security diversification should be irrelevant to a young saver (even supposing he is covered by social security). However, if the extra demand for risky assets raises $p_K$ (equivalently, if it lowers the expected return savers can get over their lives), then equation (39) shows that it reduces their welfare, provided that the riskless rate does not also change.

We can indeed deduce from differentiating (38), and using the income effect term calculated in (39), that $p_K$ does rise after social security diversification

$$\begin{aligned} dp_K/d\kappa &= -1/\{p_K\{dK^*[p_J, p_K, W - p_J T(p_J)]/dp_K - \kappa/(p_K)^2\}\} \\ &= -1/\{p_K K^c_{p_K} - p_K K^*_I [L - K^f]\} - \kappa/p_K\} > 0. \end{aligned} \quad (40)$$

Since compensated own price effects are always negative, and $K^*$ is a normal good, all the terms in the denominator are negative. Multiplied by the negative sign outside, we get the claimed result. In short, with risky second-period consumption being a normal good, the demand is downward sloping and the price $p_K$ must rise in response to an increase in trust fund holdings of land in order to clear the market.

From the connection between $p_K$ and the price of risky land derived above, we conclude that the price of risky land also rises

$$\begin{aligned} &dp_0/d\kappa = (dp_0/dp_K)(dp_K/d\kappa) = 0 \\ &dp/d\kappa = (dp/dp_K)(dp_K/d\kappa) = [1/(1 - p_J)](dp_K/d\kappa) \\ &dP/d\kappa = L dp/d\kappa = [L/(1 - p_J)](dp_K/d\kappa) = [LR_0/(R_0 - 1)](dp_K/d\kappa). \end{aligned} \quad (41)$$

Putting (40) and (41) together shows that risky land and total land prices rise after diversification, when there is no risky investment undertaken in equilibrium. With a rise in the price of land, the utility of old savers rises when the policy is implemented, since the value of the land they are holding rises. In turn, this lowers the expected utility of young savers and those in future cohorts. Starting from a trust fund invested exclusively in bonds, the expected utility of workers is increased by diversification in the same way as in Proposition 1. We also will show that aggregate investment decreases instead of increasing. We summarize our results in the following proposition:

**Proposition 3a*** *Assuming the presence of safe real one-period investments, but no risky real one-period investments, and that demand for risky second period consumption is normal, then, trust fund purchases of risky investment increase the price of risky consumption, and decrease aggregate real investment.*

**Proposition 3b*** *Moreover, social security diversification raises the price of risky land, leaving the price of riskless land unchanged. The total value of land goes up.*

**Proposition 3c*** *Starting from a trust fund invested exclusively in bonds, trust fund purchases of risky investments increase the expected utility of all workers (except the old at the time the policy is implemented, who are unaffected). Old savers at the time the policy is implemented are also helped. All young and future savers lose utility as a result of the policy. Social welfare, with neutral weights, is increased.*

## 9.2 Aggregate Investment

It remains to prove that aggregate investment declines after trust fund diversification. We begin by examining equilibrium condition (29.e1*). Rearranging (29.e1*), using budget set (2), and market clearance for risky consumption (29.e2*) gives:

$$\begin{aligned} R_0k_0 &= J^*(p_J, p_K, W - p_JT(p_J)) + T(p_J) - (L_0 + P) - R_0(G - F_0 + p_KK^f) \qquad (42)\\ &= R_0(W - C^*(1/R_0, p_K, W - aG(R_0-1)/R_0) - p_KK^*) - (L_0 + P) - R_0(G - F_0 + p_KK^f)\\ &= R_0(W - C^*(1/R_0, p_K, W - aG(R_0-1)/R_0)) - (L_0 + P) - R_0(G - F_0 + p_KL). \end{aligned}$$

Dividing by $R_0$, differentiating with respect to $\kappa$, substituting for the derivative of $P$ from (41), and then using the Slutsky equation together with the income effect derived in (39), we have:

$$\begin{aligned} dk_0/d\kappa &= -(dC^*/dp_K + L/(R_0-1) + L)(dp_K/d\kappa) \qquad (43)\\ &= -(C^c_{p_K} - C^*_I[L - K^f] + LR_0/(R_0-1))(dp_K/d\kappa)\\ &= -(C^c_{p_K} + L[(R_0/(R_0-1)) - C^*_I] + C^*_IK^f)(dp_K/d\kappa) < 0. \end{aligned}$$

To see that the derivative $dk_0/d\kappa$ is negative, note first that since $K$ is normal, $C$ and $K$ are Hicksian substitutes (see Aura, Diamond, and Geanakoplos (1999)), so $dC^c/dp_K > 0$. By normality of $K$ and $J$, $C^*_I < 1$. Since $R_0/(R_0-1) > 1$, the second term is positive. Finally, since $C^*$ is normal and the trust fund holdings of risky consumption are nonnegative, the last term is positive as well. Thus the sum in parenthesis is positive, and since $dp_K/d\kappa > 0$, safe investment declines.

# 10 Social Security Diversification with Land, but No Short-term Production

In the models above, one (or both) of the rates of return was determined by the linear technology. We turn now to economies with steady state equilibria where there is no short-term production at all, implying that we must solve two market-clearing conditions simultaneously in order to determine the two endogenous consumer good prices. This simultaneity problem makes the analysis much harder, so we defer the proofs to the next section and to the Appendix.

When the only output comes from land in fixed supply, there is no response in supply to a change in circumstances. All the adjusting must therefore come through prices. We find that in this case the presumption is that land goes up in value after social security diversification.

As in Section 9, we think of $K^f$ as representing the acquisition of short-term risky returns by the trust fund. As we mentioned earlier, if the trust fund originally holds no risky assets, the effect of small purchases of risky land has the same effect as the purchase of risky consumption.

We begin, as usual, by calculating the effect of social security diversification on savers' welfare. Since both prices $p_J$ and $p_K$ may move in response to a change in $\kappa$, we have a more complicated expression than before, in effect combining the utility effects of regimes 2 and 3, as given in (33) and (39).

$$\begin{aligned} dV/d\kappa &= (\partial V/\partial p_J)(dp_J/d\kappa) + (\partial V/\partial p_K)(dp_K/d\kappa) \qquad (44)\\ &= U_1'\{-J^* - d[p_J T(p_J)]/dp_J\}(dp_J/d\kappa) + U_1'\{-K^*\}(dp_K/d\kappa)\\ &= U_1'\{-[[B-(1-p_J)aG]/p_J + L_0 + P] - d[(1-p_J)aG]/dp_J\}(dp_J/d\kappa)\\ &\quad + U_1'\{-(L-K^f)\}(dp_K/d\kappa)\\ &= -U_1'\{[B-aG]/p_J + L_0 + P\}(dp_J/d\kappa) - U_1'\{L-K^f\}(dp_K/d\kappa)\\ &= -U_1'\{[[B-aG]/p_J](dp_J/d\kappa) + (1-p_J)(dP/d\kappa) - K^f(dp_K/d\kappa)\}. \end{aligned}$$

The last equality follows from differentiating $P = (L_0 + P)p_J + Lp_K$, which gives $(1 - p_J)dP = (L_0 + P)dp_J + Ldp_K$. Observe, from the fourth equality of (44), that the effect of $p_J$ on welfare depends on the sign of $\{[B - aG]/p_J + L_0 + P\}$. From the last line of (44), we conclude that if $K^f = 0$, and $B = aG$, then savers' welfare goes up if and only if the price of land goes down. This is to be expected, since with $K^f = 0$, savers must buy all the land in equilibrium.

Next we indicate that the expected changes do occur in the prices of consumer goods after social security diversifies into risky assets: the price of the good with increased demand goes up, while the price of the good with decreased demand goes down. The proof of Propositions 4a*– 4c* will be given as special cases of the more general model presented in the next section.

**Proposition 4a*** *Suppose there is no short-term production. Suppose that demands are normal and the savers' share of taxes is at least as large as their share of bonds ($B \leq aG$). Then trust fund diversification raises $p_K$ and lowers $p_J$.*

We turn now from the prices of second-period consumption to the prices of land. We have seen in Section 7 that if production is exclusively risky, then total land values go down after social security diversification. On the other hand, we saw in Section 8 that if production is exclusively safe, then total land values go up after diversification. A natural question is what happens when there is no short-term production? From Proposition 4a*, we already know that (when there is no short-term production) diversification lowers the price of safe land, assuming $B \leq aG$, since it raises interest rates. Proposition 4b* asserts that under an additional restriction, social security diversification raises the price of risky land, and of total land. Remarkably, with this restriction, total land value goes up no matter what the proportion of safe land and risky land, even as safe land goes down in value. The new condition is that $[(1-a)G-F_0]/p_J+aG \geq 0$. If the trust fund is sufficiently smaller than the stock $G$ of government bonds, the condition automatically holds. If $\kappa = 0$, this condition can be written equivalently as $(1 - p_J)aG \leq B$ or $aG \leq (1 + r)B$. The conclusion that total land prices rise after diversification also holds without the extra condition if savers display constant absolute risk aversion. Finally, Proposition 4b* asserts that when the trust fund is completely unfunded, starting private accounts in equities or starting a trust fund that borrows to hold equities will raise the total price of land if and only if old savers display increasing relative risk aversion.

**Proposition 4b*** *Suppose that $U_2$ displays decreasing absolute risk aversion and increasing relative risk aversion. Suppose that originally $\kappa = K^f = 0$. Suppose that the savers' share of taxes is greater than their original share of government bonds: $B - aG = [(1-a)G - F_0] \leq 0$. Suppose that in the original equilibrium, $[(1-a)G - F_0]/p_J + aG \geq 0$. Then social security diversification into risky land raises the price of risky land, lowers the price of safe land, and raises the total value of land, provided that $L > 0$. If $[(1-a)G - F_0]/p_J + aG < 0$, but there is constant absolute risk aversion, then the previous conclusions still hold.*

**Corollary 4b*.1** *Consider the very special case where the original trust fund is zero, $F = \kappa = F_0 = 0$, and savers pay all the income taxes for interest, $a = 1$. Then social security diversification into risky land raises the total value of land if old savers display increasing relative risk aversion (no matter what their absolute risk aversion), and lowers the value of land if old savers display decreasing relative risk aversion.*[30]

**Proposition 4c*** *Moreover, under the conditions of Proposition 4b*, social security diversification into risky land improves welfare for old savers and all future workers, and lowers the welfare of all young and future savers.*

Proposition 4c* follows immediately from Propositions 4a* and 4b* and from the expression for savers' welfare derived in (44). Again we defer the proof of Proposition 4b* until the next section.

## 11 Concave Production Technology

In the models analyzed above, short-run technologies were either linear or not worth using. Without altering the assumption that wages are fixed, we now consider a model in which the return to investment decreases with the level of investment. We suppose that the safe technology takes the form $f(k_0)$ and that the risky technology now takes the form $g(k)R$, where $f$ and $g$ are twice differentiable and concave and $R$ is stochastic, as above. We suppose the productive sectors of the economy are owned entirely by the savers.[31] Each saver therefore receives a rent or profit from ownership of technology, in addition to his wage, as income.[32] The model is much more complicated than the linear model considered in Sections 1–9, since there are now two free prices, and it is no longer easy to calculate explicit formulas for the effects of social security diversification. This model includes all the previous models as special cases.

By generalizing the model we can see what general qualitative properties persist across all the equilibrium regimes studied in Sections 1-9. We find in this section that, in general, social security diversification raises the riskless interest rate, and lowers the expected short-term risky return. It decreases safe investment and increases risky investment. Its effect

[30] We also show in the next section that if $U_2$ is quadratic, and $B = aG$, and $\kappa = 0$, then social security diversification raises the total value of land.

[31] That is, each of the unit measure of savers owns access to these technologies in terms of own capital input. Since each saver will invest the same amount, we can do the analysis in terms of aggregates.

[32] This modeling approach differs from that with an externality that could result in the same aggregate output function, but without the separation of returns between the return on capital inputs and the return on ownership of technology. This alternative approach would give a larger return to trust fund investment in capital since there would not be an increase in the return to savers from owning technology.

on total investment could in general go either way. Quite often, aggregate investment will move in the opposite direction from total land value. Social security diversification tends to decrease the value of land if risky investment changes more than safe investment, *ceteris paribus*, and increase the total value of land if safe investment responds more. Social security diversification tends to increase the total value of land if savers display increasing relative risk aversion when old (decreasing the value of land with decreasing relative risk aversion), *ceteris paribus*. Furthermore, social security diversification tends to increase the total value of land if savers' old consumption is safer than the return from all land, and decrease the total value of land if savers' old consumption is riskier than the return from all land.

Before turning to our model, let us note that there are still other variations on the model one might later consider. A further generalization of the model would have been to introduce labor as a nonseparable input to production. If labor were applied at the same time as capital, for example, at planting time, before uncertainty is resolved, there would be little additional complication. But if labor is applied to production after uncertainty is resolved, for example, at harvest time (so that the capital of one generation combines with the labor of the next), then labor income becomes state dependent and there would be no steady state equilibrium (though perhaps a Markov equilibrium). The complications of looking for a nonstationary equilibrium are beyond the scope of this paper.

One could also allow for distinct models of land, depending on whether ownership of land ensures a given level of (possibly stochastic) output each period, or whether the ownership of land provides a given level of capital input to production each period. When the marginal product of capital was given, the two approaches were the same. To limit the length of the paper, we maintain our assumption that the returns from land are independent of capital accumulation.

With our generalization of production, the optimization problem of savers when $k > 0$ now becomes

$$\begin{aligned} V &= \max U_1[C_1] + E\{U_2[C_2]\} \\ &\quad \text{s.t. } W = C_1 + B + k_0 + k + p_0 L_0^* + pL^* \\ &\qquad C_2 = (1+r)B + f(k_0) + g(k)R + L_0^* + p_0 L_0^* + L^* R + pL^* - T(r). \end{aligned} \tag{45}$$

As usual, we find it more convenient to describe equilibrium in terms of budget set (2) and the variables $C^*$, $J^*$, $K^*$, which now requires recognition of the return from owning technologies as part of the definition of income, $I$. We begin with the productive sector, which is assumed to maximize profits, taking as given the prices of the composite commodities of safe and risky second-period consumption. Let

$$\Pi(p_J, p_K) = \max[p_J f(k_0) - k_0] + \max[p_K g(k) - k] \tag{46}$$

Income for the savers is now defined as

$$I(p_J, p_K) = W - p_J T(p_J) + \Pi(p_J, p_K). \tag{47}$$

With this definition of income, we can define savers' demands $C^*$, $J^*$, $K^*$ from budget set (2) as before. Stationary equilibrium is now described by a vector $(p_J, p_K, k, k_0)$ satisfying (46), (47), and (48.e1**)–(48.e5**) below.

$$J^*(p_J, p_K, I(p_J, p_K)) + T(p_J) = (G - F_0 + \kappa)/p_J + L_0 + P + f(k_0) \tag{48.e1**}$$

$$K^*(p_J, p_K, I(p_J, p_K)) + \kappa/p_K = L + g(k) \tag{48.e2**}$$

$$\begin{aligned} p_J = 1/(1+r) &= 1/f'(k_0) \quad \text{if} \quad k_0 > 0 \\ &\leq 1/f'(k_0) \quad \text{if} \quad k_0 = 0 \end{aligned} \tag{48.e3**}$$

$$p_K = 1/g'(k) \quad \text{if} \quad k > 0$$
$$\leq 1/g'(k) \quad \text{if} \quad k = 0 \tag{48.e4**}$$

$$P = L_0 p_J/(1 - p_J) + L p_K/(1 - p_J). \tag{48.e5**}$$

We shall confine our attention to "regular economies", that is economies satisfying two restrictions. The first is that if in any equilibrium, either safe or risky investment is not undertaken, then the corresponding price/marginal product condition in (48.e3**) or (48.e4**) is a strict inequality. The second restriction is that at every equilibrium, if we linearize the five equations (48.e1**)–(48.e5**), and then differentiate with respect to the five variables $(p_J, p_K, k, k_0, P)$, we get an invertible matrix. (If condition (48.e3**) or (48.e4**) is a strict inequality, then we drop the corresponding production input level, fixing it at 0, and also drop the corresponding equation, and look at the remaining $4 \times 4$ or $3 \times 3$ matrix). It can easily be shown that nearly every economy is regular. Thus there is almost no loss of generality in looking only at regular economies.[33]

As in the linear model, there are four equilibrium regimes depending on whether risky or safe investment is undertaken. Nevertheless, since all four of these regimes are consistent with the hypothesis that the economy is regular, we can handle all the cases as part of the same analysis. Our first proposition shows that the effects of social security diversification on short-term prices and investment can be generalized from all four special cases of the linear model to the more general concave model of this section.

**Proposition 5a** *Suppose we have a regular economy with concave short-term production technology and land. Suppose that savers' demand is normal in all three goods, and that the savers' share of taxes is greater than or equal to their share of bonds. Then trust fund diversification (weakly) raises $p_K$ and $k$ and (weakly) lowers $p_J$ and $k_0$. In words, social security diversification raises the riskless interest rate, lowers the expected return on short-term risky securities, increases risky investment, and decreases safe investment.*

Proposition 5a includes Propositions 1, 2a, 2a*, 3a*, and 4a* as special cases. Furthermore, since in the risky linear case and in the safe linear case one of the prices $p_K$ or $p_J$ is fixed by the technology, Proposition 5a and the formula (29.e5**) for land also immediately prove Propositions 2b* and 3b*. Yet the proof of Proposition 5a does not need to make reference to different equilibrium regimes. On the other hand, the proof is indirect, and proceeds by finding a contradiction. The proof is presented in the Appendix.

Proposition 5a is qualitative, so we cannot use it in general to sign the effect of social security diversification on aggregate investment, for that involves comparing the magnitude of the effect of diversification on safe investment and on risky investment. But we can link the change in aggregate investment with the change in total land value. Before stating the next proposition, let us denote the expression for the payoff from all land (including dividends and capital sales) by $\Lambda = L_0 + LR + P$. Suppose WLOG that the units of risky land $L$ and the units for risky production $g$ and $R$ have been chosen so that in the original equilibrium, $p_J = p_K$.

**Proposition 5b** *Suppose we have a regular economy with concave short-term production and land, and suppose that originally $\kappa = 0$. Then the change in value of all land $dP$ after social security diversification is proportional to*

$$\begin{aligned} dP \sim\ & \{PU_1''[C_1] + (1/p_J)E[U_2''[C_2]\Lambda]\}(dk_0 + dk) \\ & - \{[(1-a)G - F_0]/p_J^2\}E[U_2''[C_2]\Lambda]dp_J + (1/p_J)E[U_2''[C_2]\Lambda[1-R]](d\kappa - dk) \end{aligned} \tag{49}$$

[33] That is, if we endow savers with a very small amount $s$ of safe consumption and a very small amount $r$ of risky consumption in their old age, then almost any choice of $s$ and $r$ (precisely, all except for a measure zero set of choices) will give a regular economy.

The term $(dk_0 + dk)$ is the change in aggregate investment. The coefficient multiplying $(dk_0 + dk)$ is negative, since the second derivative of utility is negative. To the extent that the second and third terms can be ignored, (49) indicates that total land value and aggregate investment move in the opposite direction. An increase in aggregate investment tends to be associated with a decrease the value of total land, and a decrease in aggregate investment tends to be associated with an increase the total value of land. From Proposition 5a, we know that if savers' demands are normal, then social security diversification increases risky investment and decreases safe investment. Thus if risky investment responds more to social security diversification then safe investment, total land value tends to go down. The converse also holds if safe investment is more responsive. This is consistent with Propositions 2b* and 3b*, but applies also when there is nonlinear short-term production.

The second term in the expression for $dP$ in Proposition 5b is positive if $B - aG = [(1-a)G - F_0] < 0$ and demands are normal, since it is then minus the product of three negative terms. When demands are normal and $B - aG < 0$, social security diversification reduces the supply of safe consumption for old savers, and this tends to increase their marginal utility of land. If $B = aG$, the second term in (49) is zero.

The sign of the third term in (49) depends on whether old savers' consumption $C_2$ is riskier or safer than the payoff $\Lambda$ to land, and on whether relative risk aversion and absolute risk aversion are increasing or decreasing. Dividing and multiplying by $U_2'[C_2]$, we can write

$$E[U_2''[C_2]\Lambda[1-R]] = E[-(U_2''[C_2]/U_2'[C_2])\Lambda[R-1]U_2'[C_2]]. \tag{50}$$

Notice first that

$$E[[R-1]U_2'[C_2]] = p_K - p_J = 0. \tag{51}$$

Thus the random variable $[R-1]$ has a zero expectation with respect to the measure $m(s) = U_2'[C_2(s)]$. Recall that a random variable always has positive covariance with any (positive) monotonic transformation of itself, with respect to any measure. Notice also that $C_2$ is a positive monotonic transformation of $R$, since consumption is a positive combination of riskless and risky payoffs, and so relative risk aversion $-(U_2''[C_2]/U_2'[C_2])C_2$ is also a positive monotonic transformation of $[R-1]$, if it is increasing in $C_2$. On the other hand, absolute risk aversion $-(U_2''[C_2]/U_2'[C_2])$ is a negative monotonic transformation of $[R-1]$, if it is decreasing in $C_2$.

If $C_2 = \Lambda$, then $-(U_2''[C_2]/U_2'[C_2])\Lambda$ is the relative risk aversion (RRA) of the savers in old age. If in addition savers' old age RRA is constant, then the third expression in (49) is zero. We immediately deduce:

**Corollary 5b.1** *Suppose we have a regular economy with concave short-term production and land, and suppose that savers display constant relative risk aversion in old age. Suppose that originally $\kappa = 0$, $B = aG$, and the levels of short-term production are such that $\Lambda = aC_2$, with $a > 0$. Then the change in value of all land $dP$ after social security diversification is opposite in sign to the change in aggregate investment $dk_0 + dk$.*

If $\Lambda = aC_2$, and RRA is increasing, then the expression (50) represents the covariance of RRA and $[R-1]$, and is positive. If furthermore, $d\kappa > dk$, then the third term in the expression (49) for the sign of $dP$ must be positive. Thus in case $aC_2 = \Lambda$, and $d\kappa > dk$, social security diversification into equity tends to increase total land value if savers display increasing relative risk aversion in old age, and tends to lower land values if savers display decreasing relative risk aversion in old age.

More generally, since $\Lambda$ and $C_2$ are both positive combinations of safe and risky consumption, we can always write $\Lambda = aC_2 - b$, where $a > 0$ and $b$ are constants, and $b > 0$ if and only if $C_2$ is relatively safer than $\Lambda$. We can thus write

$$\begin{aligned} E[-(U_2''[C_2]/U_2'[C_2])\Lambda[R-1]U_2'[C_2]] &= E[-(U_2''[C_2]/U_2'[C_2])aC_2[R-1]U_2'[C_2]] \\ &\quad - E[-(U_2''[C_2]/U_2'[C_2])b[R-1]U_2'[C_2]] \end{aligned} \tag{52}$$

The first term in the RHS of (52) is positive if old savers' RRA is increasing, and negative if RRA is decreasing, as we just saw. If $C_2$ is relatively safer than $\Lambda$, so that $b > 0$, the second term in (52) is positive if old savers absolute risk aversion (ARA), is decreasing, and is negative if ARA is increasing. In case $C_2$ is relatively riskier than $\Lambda$, the situation is reversed.

The proportion of safe consumption in $C_2$ is higher than in $\Lambda$ if and only if,

$$\{[(1-a)G - F_0]/p_J + aG + f(k_0)\}/g(k) > (L_0 + P)/L \tag{53}$$

Putting these arguments together, the following corollaries can be rigorously derived from Proposition 5b, as they are in the Appendix. Corollary 5b.2 gives sufficient conditions for the price of land to rise after diversification.

**Corollary 5b.2** *Suppose that $U_2$ displays decreasing absolute risk aversion and increasing relative risk aversion. Suppose that the savers' share of taxes is greater than their original share of government bonds, $B - aG \leq 0$, and suppose that in the original equilibrium, $\kappa = 0$. Suppose that in the original equilibrium, the proportion of safe consumption in $C_2$ is higher than in $\Lambda$. If social security diversification into equities causes safe investment to change no less than risky investment, then it raises the price of risky land, lowers the price of safe land, and raises the total value of land, if $L > 0$. (Equivalently, if social security diversification lowers the total value of land, then it must increase aggregate investment.) Even if the proportion of safe consumption in $C_2$ is lower than in $\Lambda$, but there is constant absolute risk aversion, the previous conclusions still hold.*

If there is no short-term risky production, then (53) holds automatically as long as $[(1-a)G - F_0]/p_J + aG > 0$. Thus Proposition 4b* follows immediately from Corollary 5b.2, as does Corollary 4.b*.1. The case where there is no short-term production, and $U_2$ is quadratic and $B = aG$, considered in footnote 31, follows immediately from the formula (49).

We can also derive conditions under which total land values fall.

**Corollary 5b.3** *Suppose that $U_2$ displays decreasing absolute risk aversion, and constant or decreasing relative risk aversion. Suppose that the savers' share of taxes is equal to their original share of government bonds, $B - aG = 0$, and suppose that in the original equilibrium, $\kappa = 0$. Suppose that in the original equilibrium, the proportion of risky consumption in $C_2$ is higher than in $\Lambda$. If social security diversification into equities causes risky investment to change more than safe investment, then it lowers the total value of land. (Equivalently, if social security diversification raises the total value of land, then it must decrease aggregate investment.)*

The following proposition also follows immediately from Proposition 5b, and the analysis of utility derived in the proof of 5b in the Appendix.

**Proposition 5c** *Suppose we have a regular economy with concave short-term production technology and land. Suppose that savers' demand is normal in all three goods, and that the savers' share of taxes is greater than or equal to their share of bonds, and suppose that in the original equilibrium, $\kappa = 0$. Then trust fund diversification raises the utility of all young and future workers. If it also raises the price of land, then it helps old savers and hurts all young and future savers. If on the other hand, it lowers the total value of land, then it hurts all old savers and may help or hurt young and future savers. In any case, it increases a social welfare function with neutral weights.*

## 12 Defined Benefits

We have restricted attention so far to a defined contribution social security system for analytical convenience, and to make the point that even there, social security diversification in moderation brings potential welfare gains. We show now that at least for our central risky linear case, we can incorporate a defined benefit structure without changing the comparative statics conclusions we already reached.

In our defined contribution social security system, we supposed that the trust fund maintained a constant value $F$ invested in government bonds, and a constant value $\kappa$ invested in risky securities, distributing any surplus as benefits (or taxes, if the surplus turned negative) to the contemporaneous old. Suppose instead that the fund acted only to maintain $F$, distributing a part of the surplus over $F$ as benefits to the contemporaneous old, and investing the rest in risky equities. Over time the benefits and the size of the trust fund investment in risky equities would change.

Leaving tax rates fixed, the level of benefits would presumably adapt to the level of the trust fund, thereby rising with the return on the portfolio, as does a defined contribution system, but not rising dollar-for-dollar, as the returns got spread over future cohorts. In this way the benefits of a cohort would depend on the realized returns over a longer period of time (than just one period). With a sensible benefit rule, and with a plausible stochastic process for the return on capital, this would raise expected utility for future generations, measured as of the time of implementation of the policy, since a diversified social security system could spread the risk of uncertain returns over many generations. Thus the gain from a diversified portfolio becomes larger with a good policy for determining benefits.

In any "defined benefits system" with risky investments, benefits (and or taxes) must be changed, depending on the returns of the risky investments. The point is to smooth benefits, while recognizing the need to satisfy a nonnegativity constraint should there be a prolonged period of low returns. In the presence of random returns on a nontrivial portion of the trust fund, it is necessary to recognize the probability that the portion of the trust fund invested in risky assets would become negative if both benefits and taxes were unchanged. Thus, every "defined benefit system" must be changed from time to time. The policies that determine such change need to be modeled in order to consider the value of smoothing that comes from defined benefits. In the context of a model with randomness in other aspects of the economic and demographic environment of the social security system, the change in uncertainty from a diversified portfolio would not be such a salient change in the system.[34]

In the risky linear case the variations described above in the trust fund holding of risky securities and in social security benefits have no effect on any equilibrium price. The extra money invested in risky securities is absorbed by an increase in risky production, with no effect on $p_K$. The environment of savers is thus exactly the same as it was in Section 8.

[34] If a defined contribution system is to fulfill its social purpose, it will also need periodic change in response to economic and demographic developments.

The same comparative statics conclusions we reached in the defined contribution system of Section 8 therefore apply without change to the defined benefits system described here, no matter what the precise benefit rule.

# 13 Concluding Remarks

We have considered the implications of changing trust fund portfolio policy away from 100% government debt in a general equilibrium model with four kinds of households (old and young workers, who do no saving, and old and young savers) and four kinds of production (safe and risky, short-term and long-term). We found that a small amount of diversification raises total welfare (suitably defined), though it causes other welfare redistributions among the four household types. Social security diversification changes relative prices, and the composition and amount of real investment. Depending on the structure of marginal taxes, social security diversification raises the interest rate, thereby increasing the payoffs of the social security trust fund and helping workers, but forcing an increase in income taxes that hurts savers. There is almost always a wealth transfer between young savers and old savers, in one direction or the other. If diversification raises the value of land, then old savers gain, but young savers lose on this account, and vice versa. Diversification reduces real safe investment, but increases risky investment and the price of short-term risky consumption. The effect on aggregate real investment and long-term asset prices is ambiguous, but higher aggregate investment tends to lower the price of long-term investments. The common sense conclusion that trust fund diversification would (if it did anything at all) increase real investment and increase stock market value is thus seen to be questionable, as is shown in the risky linear case and the safe linear case. In both cases one or the other of the common sense predictions is reversed.

The model has assumed that the technology is iid. This leaves out the effect of news about future technologies on current asset prices. This would be an interesting extension of the analysis. Presumably this would make asset returns riskier and add to the social value of sharing risks more widely in the economy and so strengthen the case for trust fund investment in equities.

The paper has also assumed that labor is a separable input from capital, and from land. Had we allowed an interaction (say in a constant returns to scale production function depending nonlinearly on every input), changes in investment would have had a feedback effect on wages. This would have created another interesting redistribution between savers and young workers. It also would have had effects on land prices. Depending on how we modeled labor inputs, we might have been forced to consider stochastic wages, which would have obliged us to move to a Markov equilibrium, instead of the steady state we considered here.

There are four points to make relative to the current policy debate.[35] First, contrary to some assertions, the heterogeneity of the population implies that trust fund portfolio choice does have real effects on the economy. Second, while it is appropriate to be concerned about the risk associated with a change in portfolio policy, it seems to us unlikely that workers are so risk averse that a portfolio completely invested in Treasury bonds is optimal. This point is reinforced by the ability of the government to spread risk over successive cohorts since Social Security is a defined benefit system. That is, if a defined benefit system is well-run, there is a stronger case for trust fund investment in private securities than in the models analyzed here which assumed a defined contribution system. Third, the marginal social benefit to diversification declines as the level of diversifica-

[35] For more discussion of Trust Fund investment, see Munnell and Balduzzi (1998) and Panel on Privatization of Social Security (1998).

tion increases (and as workers are exposed to more risk), which puts a limit on the total amount of socially desirable diversification. Diversification increases the value of social security, but there is nothing in our analysis that suggests a reasonable level of diversification alone is sufficient to restore actuarial balance to Social Security. Lastly, the models considered here have substituted equity investment for bond investment, holding constant the level of funding of Social Security. Many proposals for investment in stocks, whether through the trust funds or through individual accounts, use stock investments as a reason to increase or decrease the financing of Social Security (at least in the short run) relative to what might be proposed without such investment. Such a change involves intergenerational redistribution that is not incorporated in the analysis in this paper, though it could be accommodated by our model.[36] Our analysis does apply, however, to proposals that would substitute a portfolio change for cuts in future benefits.

[36] See Smetters (1997).

## References


[1] Abel, Andrew B., 1999, The Social Security Trust Fund, the Riskless Interest Rate, and Capital Accumulation, NBER Working Paper 6991.

[2] Ameriks, John, and Stephen P. Zeldes, in progress, How Do Household Portfolio Shares Vary with Age?

[3] Aura, Saku, Peter Diamond and John Geanakoplos, 1999, Savings in a Two-Period Two-Asset Model, unpublished.

[4] Bohn, Henning, 1997, Risk Sharing in a Stochastic Overlapping Generations Economy, unpublished, University of California at Santa Barbara.

[5] Bohn, Henning, 1998, Social Security Reform and Financial Markets, in Federal Reserve Bank of Boston Conference on Social Security Reform.

[6] Diamond, Peter, 1997, Macroeconomic Aspects of Social Security Reform, Brookings Papers on Economic Activity, 2.

[7] Diamond, Peter and Menahem Yaari, 1972, Implications of the Theory of Rationing for Consumer Choice Under Uncertainty, *American Economic Review*, 62(3): 333–343.

[8] Feldstein, Marty, 1985, The Optimal Level of Social Security Benefits, *Quarterly Journal of Economics*, 100(2): 303–20.

[9] Financial Economists' Roundtable, 1998, Statement on Social Security, unpublished.

[10] Fischer, Stanley, 1972, Assets, Contingent Commodities, and the Slutsky Equations, *Econometrica*, 40(2): 371–385.

[11] Forbes, Steve, 1996, How to Replace Social Security, *The Wall Street Journal*, December 18.

[12] Geanakoplos, John, Olivia S. Mitchell, and Stephen P. Zeldes, 1998, Would a Privatized Social Security System Really Pay a Higher Rate of Return?, in Arnold, Graetz, and Munnell (eds.), *Framing the Social Security Debate*. National Academy of Social Insurance, Brookings Institution Press, Washington, DC.

[13] Geanakoplos, John, Olivia S. Mitchell, and Stephen P. Zeldes, 1999, Social Security Money's Worth, in Mitchell, Myers, and Young (eds.), *Prospects for Social Security Reform*. University of Pennsylvania Press, Philadelphia.

[14] Greenspan, Alan, 1997, Testimony before the Task Force on Social Security of the Committee on the Budget, U.S. Senate, November 20.

[15] Kennickell, Arthur, Martha Starr-McCluer, and Annika E. Sunden, 1997, Family Finances in the U.S.: Recent Evidence from the Survey of Consumer Finances, *Federal Reserve Bulletin*, January.

[16] Mitchell, Olivia S., and James F. Moore, 1997, Retirement Wealth Accumulation and Decumulation: New Developments and Outstanding Opportunities, Working Paper 97-12, The Wharton School, University of Pennsylvania.

[17] Moore, Stephen, 1997, Prepared Testimony on the Future of Social Security for This Generation and the Next, House Ways and Means Committee, Social Security Subcommittee, 105 Cong. 1 Sess., June 24. Washington: Government Printing Office.

[18] Munnell, Alicia H., and Pierluigi Balduzzi, 1998, Investing the Trust Fund in Equities, unpublished, Boston College.

[19] Panel on Privatization of Social Security, National Academy of Social Insurance, 1998, Evaluating Issues in Privatizing Social Security, Washington: National Academy of Social Insurance; available as Diamond (ed.), 1999, *Issues in Privatizing Social Security*, Cambridge: MIT Press.

[20] Pestieau, Pierre, and Uri M. Possen, 1999, Investing Social Security in the Equity Market. Does it Make a Difference? unpublished, University of Liege.

[21] Smetters, Kent, 1997, Investing the Social Security Trust Fund in Equity: An Options Pricing Approach, Technical Paper 1997-1, Congressional Budget Office, Macroeconomic Analysis and Tax Analysis Divisions..

[22] Social Security Administration, 1996, *Annual Statistical Supplement to the Social Security Bulletin.*

# Appendix

**Proof of Proposition 5a** The proof will be by contradiction, rather than by direct computation. Rewriting (48.e1**), after substituting the expression for $T(p_J)$ from (5), we get

$$J^*(p_J, p_K, I(p_J, p_K)) = J \equiv [(1-a)G - F_0 + \kappa]/p_J + L_0 + P + aG + f(k_0). \qquad (54)$$

We can think of $J^*$ as the demand for safe consumption, and $J$ as the supply net of tax. Similarly rewriting (48.e2**) we get

$$K^*(p_J, p_K, I(p_J, p_K)) = K \equiv L + g(k) - \kappa/p_K \qquad (55)$$

Again we think of $K^*$ as the demand for risky consumption, and $K$ as the supply. Multiplying $J^*$ from (54) by $p_J$ and $K^*$ from (55) by $p_K$ and adding, and then using the budget set (2), and the definition of $I$ and $P$ from (46) and (47), and the identity $P = p_J L_0 + p_K L + p_J P$, we get

$$C^*[p_J, p_K, W - (1-p_J)aG] = C \equiv W - [G - F_0] - P - k - k_0 \qquad (56)$$

We think of $C^*$ as the demand for current consumption, and $C$ as the supply.

The strategy of proof consists of differentiating the three supply equations for $J$, $K$, and $C$ derived in (54)–(56), and the equation (48.e5**) defining $P$, with respect to the four variables $d\kappa$, $dp_J$, $dp_K$, and $dP$. From the regularity hypothesis, we know that the change in equilibrium values will indeed be differentiable. Notice that we do not use (48.e3**) or (48.e4**) at all in this part of the proof, so our method applies to every equilibrium regime. We obtain

$$dJ = -\{[(1-a)G - F_0 + \kappa]/p_J^2\}dp_J + [1/p_J]dk + dP + f'(k_0)dk_0 \qquad (57)$$

$$dK = g'(k)dk + [\kappa/p_K^2]dp_K - [1/p_K]d\kappa \qquad (58)$$

$$dC = -dP - dk - dk_0. \qquad (59)$$

Next we calculate the effect on savers' utility from the same changes in the four variables. We get

$$dV/d\kappa = -U_1'\{[[B - aG]/p_J](dp_J/d\kappa) + (1-p_J)(dP/d\kappa) - K^f(dp_K/d\kappa)\}. \qquad (60)$$

This is exactly the same expression we derived in (44) when there was no production. The crucial point is that the firms effectively trade only with the savers. If the savers own the technology, then there is no welfare effect from price changes via production. If the output becomes more valuable, increasing the profit for the firm, then it becomes more expensive for the savers to buy from the firm.

With this important observation, we can rule out three cases by contradiction.

**Case 1** Suppose $dp_J \geq 0$, $dk_0 \geq 0$, $dp_K \geq 0$, $dk \geq 0$. Since we cannot have both $dp_J = 0$ and $dp_K = 0$, in fact at least one price strictly rises. Then from (48.e5**), $dP > 0$. From (59), this implies that

$$dC^* = dC < 0. \qquad (61)$$

Adding $dJ$ and $dK$ from (57) and (58), we have that

$$\begin{aligned} p_J dJ + p_K dK &= -[((1-a)G - F_0 + \kappa)/p_J]dp_J + d\kappa + p_J[dP + f'(k_0)dk_0] \qquad (62) \\ &\quad + p_K g'(k)dk + [\kappa/p_K]dp_K - d\kappa \\ &= -[((1-a)G - F_0 + \kappa)/p_J]dp_J + p_J[dP + f'(k_0)dk_0] + p_K g'(k)dk + [\kappa/p_K]dp_K > 0. \end{aligned}$$

provided that $B - aG = [((1-a)G - F_0 + \kappa)] \leq 0$. Observe next that, given that $dp_J \geq 0$, $dp_K \geq 0$, the compensated change

$$dC^c(p_J, p_K, U) \geq 0, \tag{63}$$

since all three goods are Hicksian substitutes (see Aura, Diamond, Geanakoplos, 1999). To maintain constant utility, the compensated change

$$p_J dJ^c(p_J, p_K, U) + p_K dK^c(p_J, p_K, U) \leq 0. \tag{64}$$

Since by (61), $dC^* < 0$, and by (63), $dC^c(p_J, p_K, U) \geq 0$, it follows, since $C^*$ is normal, that welfare of the savers must have gone down. Since $J^*$ and $K^*$ are also normal, it follows from (64) that

$$p_J dJ^* + p_K dK^* < 0. \tag{65}$$

But (62) and (65) contradict each other, since $dJ = dJ^*$ and $dK = dK^*$.

Before proceeding to case 2, we remark that we could have deduced (64) directly from the fact that savers' utility is additively separable between consumption when young and when old. Similarly (65) holds for the same reason. Thus case 1 does not really need the hypothesis of normality. We shall need this fact when we prove Corollary 5b.3.

**Case 2** Suppose $dp_J \leq 0$, $dk_0 \leq 0$, $dp_K \leq 0$, $dk \leq 0$. We get the same contradiction as in case 1, with all the signs reversed.

**Case 3** Suppose $dp_J \geq 0$, $dk_0 \geq 0$, $dp_K \leq 0$, $dk \leq 0$. Since $J^*$ and $K^*$ are Hicksian substitutes (see Aura, Diamond, and Geanakoplos, (1999)) we must have

$$\begin{aligned} dK^* &\geq (K_I^*)dI, \\ dJ^* &\leq (J_I^*)dI, \end{aligned} \tag{66}$$

where $dI$ is the change in wealth which would cause the same change in welfare (at constant prices) as caused by the price changes. From (57) we deduce (assuming that $B - aG \leq 0$) that

$$dJ^* = dJ \geq dP, \tag{67}$$

and from (58), we know that

$$dK^* = dK = g'(k)dk + [\kappa/p_K^2]dp_K - [1/p_K]d\kappa < [\kappa/p_K^2]dp_K \leq 0. \tag{68}$$

Suppose welfare for the savers went up, or stayed the same, $dI \geq 0$. Then since $K^*$ is a normal good, we know from (66) that $dK^* \geq 0$, a contradiction to (68). Alternatively, suppose that welfare for the savers went down, $dI < 0$, and the total value of land went up or stayed the same, $dP \geq 0$. Then from (66) $dJ^* < 0$. But from (67) we have that $dJ^* = dJ \geq 0$, contradiction. The only remaining possibility is that $dI < 0$ and $dP < 0$. But from $dP < 0$ and the welfare effect described in (60) we deduce that

$$dI = dV/U_1' > K^f dp_K. \tag{69}$$

From the fact that all goods are normal, so that $0 < (K_I^*) < 1/p_K$, and $dp_K \leq 0$, and $dI < 0$, it then follows from (66) and (69) that

$$dK^* \geq (K_I^*)dI > dI/p_K > K^f dp_K/p_K = (\kappa/(p_K)^2)dp_K. \tag{70}$$

But now (68) and (70) are contradictory.

Using profit maximization for the first time, we cannot get any of the mixed cases, such as where $dp_K > 0$ and $dk < 0$, so the theorem follows after eliminating the above three cases. ∎

Now let us turn to the proof of Proposition 5b.

**Proof of proposition 5b** Recall that we denote the payoff from holding all the land by $\Lambda = L_0 + LR + P$. It follows that $d\Lambda = dP$. Furthermore, the marginal utility of land must equal its price, so

$$PU_1'[C_1] = E[U_2'[C_2]\Lambda]. \tag{71}$$

Differentiating (71), and then using $d\Lambda = dP$, we must have

$$\begin{aligned} dPU_1'[C_1] + PU_1''[C_1]dC_1 &= E[U_2''[C_2]dC_2\Lambda] + E[U_2'[C_2]d\Lambda] \\ &= E[U_2''[C_2]dC_2\Lambda] + E[U_2'[C_2]dP]. \end{aligned} \tag{72}$$

We get from equation (56) derived in the proof of Proposition 5a that

$$dC_1 = -dP - dk_0 - dk. \tag{73}$$

Assuming WLOG that $U_1'(C_1) = 1$, we get from (72) and (73) and the marginal utility condition for riskless consumption $p_J = E[U_2'[C_2]1]$ that

$$dP[1 - PU_1''[C_1]] = PU_1''[C_1](dk_0 + dk) + E[U_2''[C_2]dC_2\Lambda] + p_J dP, \tag{74}$$

so

$$dP[1 - p_J - PU_1''[C_1]] = PU_1''[C_1](dk_0 + dk) + E[U_2''[C_2]dC_2\Lambda]. \tag{75}$$

From the equilibrium conditions (54) and (55), also derived in the proof of Proposition 5a, we know that

$$\begin{aligned} C_2 &= J + KR = [(1-a)G - F_0 + \kappa]/p_J + L_0 + P + aG + f(k_0) + [L - \kappa/p_K]R + g(k)R \\ &= [(1-a)G - F_0 + \kappa]/p_J - [\kappa/p_K]R + aG + \Lambda + f(k_0) + g(k)R. \end{aligned} \tag{76}$$

Differentiating this expression, assuming that $\kappa = K^f = 0$, and recalling that $d\Lambda = dP$, gives

$$dC_2 = \{-[(1-a)G - F_0]/p_J^2\}dp_J + [1/p_J]d\kappa - [R/p_K]d\kappa + dP + f'(k_0)dk_0 + g'(k)Rdk. \tag{77}$$

Again without loss of generality we can rescale $R$ so that $p_J = p_K$, giving us

$$dC_2 = \{-[(1-a)G - F_0]/p_J^2\}dp_J + (1/p_J)[1-R]d\kappa + dP + f'(k_0)dk_0 + g'(k)Rdk. \tag{78}$$

It follows from plugging the expression for $dC_2$ from (77) into (75) that

$$\begin{aligned} dP[1 - p_J - PU_1''[C_1]] &= PU_1''[C_1](dk_0 + dk) + E[U_2''[C_2]\{\{-[(1-a)G - F_0]/p_J^2\}dp_J \\ &\quad + (1/p_J)[1-R]d\kappa + dP + f'(k_0)dk_0 + g'(k)Rdk\}\Lambda]. \end{aligned} \tag{79}$$

Moving $dP$ from the RHS to the LHS of (79) gives

$$\begin{aligned} &dP[(1-p_J) - PU_1''[C_1]] - E[U_2''[C_2]\Lambda] \\ &= PU_1''[C_1](dk_0 + dk) + E[U_2''[C_2]\{\{-[(1-a)G - F_0]/p_J^2\}dp_J \\ &\quad + (1/p_J)[1-R]d\kappa + f'(k_0)dk_0 + g'(k)Rdk\}\Lambda]. \end{aligned} \tag{80}$$

From (48.e3**) and (48.e4**), $f'(k_0) = 1/p_J = 1/p_K = g'(k)$ (or else either $dk$ or $dk_0 = 0$). Using the identity $dkR = -dk(1-R) + dk$, we get from (80) that

$$\begin{aligned} &dP[(1-p_J) - PU_1''[C_1]] - E[U_2''[C_2]\Lambda] \\ &= \{PU_1''[C_1] + (1/p_J)E[U_2''[C_2]\Lambda]\}(dk_0 + dk) \\ &\quad + E[U_2''[C_2]\{\{-[(1-a)G - F_0]/p_J^2\}dp_J + (1/p_J)[1-R](d\kappa - dk)\}\Lambda] \\ &= \{PU_1''[C_1] + (1/p_J)E[U_2''[C_2]\Lambda]\}(dk_0 + dk) - \{[(1-a)G - F_0]/p_J^2\}E[U_2''[C_2]\Lambda]dp_J \\ &\quad + (1/p_J)E[U_2''[C_2]\Lambda[1-R]](d\kappa - dk). \end{aligned} \tag{81}$$

The terms multiplying $dP$ on the LHS of the equation are all positive, since $p_J < 1$. ■

**Proof of corollary 5b.2** From increasing relative risk aversion and decreasing absolute risk aversion of $U_2$, one can derive the normality of all the savers demands for first-period consumption, and second-period safe and risky consumption (see Aura, Diamond, and Geanakoplos (1999)). From normality of the demands, Proposition 5a guarantees that $dp_J \leq 0$, $dk_0 \leq 0$, and that $dp_K \geq 0$, $dk \geq 0$. From the hypothesis of the relative sensitivity of risky and safe investment, we deduce that $(dk_0 + dk) \leq 0$. Hence the first term in the expression (49) for the sign of $dP$ in Proposition 5b is positive or zero. The second term likewise, as we already saw from the discussion after Proposition 5b. The hypothesis that the proportion of safe consumption in $C_2$ is higher than in $\Lambda$, together with DARA and IRRA, guarantees that $E[U_2''[C_2]\Lambda[1-R]] \geq 0$, as we also saw in the discussion after Proposition 5b.

To evaluate the third and last term on the RHS of the expression (49) for the sign of $dP$, it only remains to consider $(d\kappa - dk)$. Observe from (48.e2**) that if $dk \geq d\kappa$, then $dK^* \geq 0$, which given that $dp_J \leq 0$ and $dp_K \geq 0$, is possible (since $J^*$ and $K^*$ are Hicksian substitutes) only if welfare has increased for savers or welfare has remained the same and $dp_K = 0$. Since we cannot have both $dp_J = 0$ and $dp_K = 0$, we deduce that either $dp_J \leq 0$ and $dp_K \geq 0$ and welfare has strictly increased for savers, or welfare has remained the same and $dp_J < 0$ and $dp_K = 0$. In either case, $dJ^* > 0$, again because $J^*$ and $K^*$ are Hicksian substitutes and normal goods. Since by hypothesis riskless investment responds more than risky investment, $-dk_0 \geq dk \geq d\kappa > 0$, (57) and (48.e3**) together show that $dJ = dJ^* > 0$ can only occur if $dP > 0$, which proves what we want anyway. So we might as well assume $dk < d\kappa$. This proves that the third and last term in the expression for the sign of $dP$ is positive, thus showing that $dP > 0$.

Since the total value of land increases, and since we know (from the fact that $dp_J \leq 0$) that the total value of safe land must decrease or stay the same, it follows that the value of risky land increased.

Finally, observe that with constant absolute risk aversion it makes no difference if consumption $C_2$ is relatively riskier than land payoffs $\Lambda$. Even if the term $b$ in expression (52) is negative, the very last term in (52) is equal to zero anyway. ■

**Proof of Corollary 5b.3** Since now risky investment responds more than safe investment, by Proposition 5a, $(dk_0 + dk) > 0$. The first term in the expression for the sign of $dP$ in Proposition 5b is thus negative. The second term disappears because $B - aG = 0$. As for the third term, recall how it is broken into two parts in (52). With constant or declining relative risk aversion, the first part is zero or negative. With $b$ negative in (52), on account of the riskiness of $C_2$ relative to the payoffs of land, and decreasing absolute risk aversion, the second part of (52) is negative.

To conclude the proof, we need only verify that we may assume $(d\kappa - dk) \geq 0$. Suppose to the contrary that $dk > d\kappa > 0$. It follows immediately that $dp_K \geq 0$. If in addition, $dp_J > 0$, then the reader can check that exactly the same contradiction as was found in Case 1 of the proof of proposition 5a. (See the remark at the end of Case 1.) So suppose instead that $dp_J \leq 0$. Observe from (58*) that if $dk > d\kappa$, then $dK^* > 0$. Given that $dp_J \leq 0$ and $dp_K \geq 0$, this is possible only if welfare has strictly increased for savers (since $J^*$ and $K^*$ are Hicksian substitutes, even if one of $J^*$ or $K^*$ is not normal, as shown in Aura, Diamond, and Geanakoplos (1999)). From (60), and the fact that $\kappa = 0$, we deduce that $dP < 0$, which is what we wanted to prove anyway. ■