OPTIMAL INVENTORY POLICIES WHEN SALES ARE DISCRETIONARY

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Optimal Inventory Policies when Sales are Discretionary*

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Abstract

Inventory models customarily assume that demand is fully satisfied if sufficient stock is available. We analyze the form of the optimal inventory policy if the inventory manager can choose to meet a fraction of the demand. Under classical conditions we show that the optimal policy is again of the (S, s) form.

The analysis makes use of a novel property of $K - \textit{concave}$ functions.

Keywords: Optimal Inventory Policies, K-Concavity, Discretionary

1 The Model

It is customary in inventory theory to assume that demand is fully satisfied if sufficient stock is available. I shall consider a variation of the classical model in which the inventory manager may elect to meet an fraction of the

*Two colleagues at Yale, George Hall and John Rust, have established a close relationship with a Connecticut company whose primary business activity is the purchase, storage and eventual sale of a variety of steel products to local manufacturers in northeastern United States. Rust and Hall were kind enough to invite me to visit the company and discuss their procedures for inventory management, thereby reintroducing me to a research topic that I had left almost forty years ago. These discussions suggested a variation of the classical inventory model that forms the subject of the current paper. I would also like to thank Professor Guillermo Gallego for his careful reading of this paper and suggestions for improvements.

demand, if the sequence of costs and revenues make such a choice profitable. Under classical conditions we show that the optimal policy is again of the (S, s) form.

The decision not to meet the demand of a customer may be warranted if the current sales price is sufficiently low compared to the cost of restocking the item in the next period, so that it is profitable to refuse a sale, even allowing for a potential loss in good will. Such a situation might arise if the cost, sales and demand parameters vary substantially over time, possibly in a stochastic fashion. In order to keep the notation simple, however, I shall begin the analysis under the assumption that the costs and demand distributions are constant over time, and examine the more general case in the final section of the paper.

Consider an inventory model with a finite number of time periods and with instantaneous delivery of orders. The inventory at the beginning of the period is x and the stock is raised to y units at a cost of

$$c(y-x) = \begin{cases} 0; & \text{if } y = x \\ K + c\cdot(y-x); & \text{if } y > x \end{cases}.$$

The demand during the period is a random variable ξ with density function $\phi(\xi)$. The special feature of this model is that the manager may choose to sell any quantity q with $0 \leq q \leq \min[y, \xi]$ at the market price p. Sales that are not made at the end of the period are permanently lost. The discount factor is α.

Let $f_n(x)$ be the maximum of the discounted expected profit with an initial stock level of x and with n periods remaining. The cost of ordering $y - x$ units is $c(y - x)$ and if q units are sold, revenue is pq and the level of inventory at the beginning of period $n - 1$ is $y - q$. Since the sales level is decided after the demand is realized, the dynamic programming equation is given by

$$\begin{aligned} f_n(x) &= \max_{y \geq x}[-c(y-x) + \int_0^\infty \max_{0 \leq q \leq \min[y,\xi]} \{pq + \alpha f_{n-1}(y-q)\}\phi(\xi)d\xi] \qquad (1) \\ &= \max_{y \geq x}[-c(y-x) + py + \int_0^\infty \max_{0 \leq q \leq \min[y,\xi]} \{p(q-y) + \alpha f_{n-1}(y-q)\}\phi(\xi)d\xi] \end{aligned}$$

We shall demonstrate that the optimal policy in each period is an (S, s) policy.

In the customary treatment of inventory theory the random demand ξ is fully satisfied - to the extent that stock is available - and the integral

$$\int_0^\infty \max_{0 \leq q \leq \min[y,\xi]} \{pq + \alpha f_{n-1}(y-q)\}\phi(\xi)d\xi$$

is replaced by

$$p\int_0^\infty \min[y,\xi]\phi(\xi)d\xi + \alpha f_{n-1}(0)\int_y^\infty \phi(\xi)d\xi + \alpha\int_0^y f_{n-1}(y-\xi)\phi(\xi)d\xi,$$

if unsatisfied demand is lost, and by

$$p\int_0^\infty \min[y,\xi]\phi(\xi)d\xi + \alpha\int_0^\infty f_{n-1}(y-\xi)\phi(\xi)d\xi,$$

if unsatisfied demand is backlogged. The new possibility of discretionary sales requires some modifications of the familiar argument.

In my paper of some 40 years ago [4] I dealt only with the case in which demand is backlogged. In addition sales did not generate revenue directly; there was instead a shortage cost associated with unmet demand. The objective was to minimize the discounted stream of expected purchase, shortage and storage costs, rather than maximizing the stream of profits as in the current paper.

The explicit introduction of revenues and the change in objective appear in the paper by Hall and Rust [3], about which I shall say more later. They consider only the case of lost sales and include a curious condition, which in the current simple context is equivalent to assuming that $p > \alpha c$, i.e., that it is profitable to make a sale at the end of a period and pay the marginal cost of reordering that stock at the beginning of the next period. When I first came across this additional condition, I assumed that it was superfluous and could be discarded without compromising the proof that the optimal ordering rule was an (S, s) policy. But I was in error; without this additional condition the optimal policy may be more complex.

It seemed odd to me that the difficulty in verifying the optimality of an (S, s) policy arose precisely in the case in which an economic calculation would lead one to decline the potential sale. And so I naturally wondered whether presenting the inventory manager with the option of discretionary sales would restore the optimality of these simple policies without any additional conditions. The major conclusion of the paper is that this conjecture is correct.

2 A Preliminary Observation

Let us make a preliminary observation about the maximization that appears inside the integral in the fundamental dynamic programming equation (1). Let $d > 0$ be fixed and $f(y)$ be a continuous function defined on $[0, \infty)$ with a finite number of local maxima. Define

$$g(y) = \max_{0 \leq q \leq \min[y,d]} f(y-q).$$

For each y, the value of q that maximizes $f(y-q)$ subject to the constraint $0 \leq q \leq \min[y, d]$ need not be unique; to be specific we shall select $q(y)$ to be the maximizer that yields the *smallest* value of $y - q$. We have the following result:

Lemma 1 *The function $y - q(y)$ is monotonically increasing in y.*

Proof. The lemma follows from a simple revealed preference argument. Let

$$\begin{aligned} q &= q(y) \\ q' &= q(y') \end{aligned}$$

with $y' > y$. We wish to show that $y' - q' \geq y - q$. Assume to the contrary that $y' - q' < y - q$.

Then

$$0 \leq q \leq y - (y' - q') \leq \begin{cases} q' \leq d \text{ and} \\ y \end{cases}$$

so that $y - (y' - q')$ satisfies the constraints associated with y. Since it was not selected it must be true that

$$f(y' - q') \leq f(y - q)$$

with $y' - q' \geq y - q$ if we have equality.

But on the other hand

$$0 \leq q \leq y' - (y - q) \leq \begin{cases} q' \leq d \text{ and} \\ y' \end{cases}$$

so that $y' - (y - q)$ satisfies the constraints associated with y'. Again it was not selected and therefore

$$f(y - q) \leq f(y' - q')$$

with $y' - q' \leq y - q$ if we have equality. But these two inequalities imply that we do have equality and therefore $y' - q' = y - q$, contradicting the assumption that $y' - q' < y - q$. ■

3 $K - concavity$

A function $g(x)$, defined for $x \geq 0$, is $K - concave$ if

$$g(y) \leq K + g(x) + (y - x)\frac{g(x) - g(x-b)}{b}$$

for all $y \geq x > x - b \geq 0$. We shall demonstrate recursively that the functions $f_n(x)$ are $K - concave$.

The customary arguments for the optimality of (S, s) policies make use of the following elementary properties of $K - concave$ functions:

- If $f(x)$ is $K - concave$, then it is also $K' - concave$ for $K' \geq K$.
- If $f_1(x), \ldots, f_n(x)$ are all $K - concave$ then

  $$\sum_1^n p_i f_i(x)$$

  is also $K - concave$ if $p_i \geq 0, \sum_1^n p_i = 1$. Alternatively if $f(x, \xi)$ is $K - concave$ in x for each ξ then so is

  $$\int_0^\infty f(x, \xi)\phi(\xi)d\xi$$

  for probability density functions $\phi(\xi)$.

Our extension of the classical inventory model requires an additional property of $K - concave$ functions that I had not been aware of previously.

Property: Let $f(y)$ be $K - concave$ and for fixed $d \geq 0$, define

$$g(y) = \max_{0 \leq q \leq \min[y,d]} f(y - q).$$

Then $g(y)$ is also $K - concave$.[1]

[1] In a private communication, Gallego [2] presented an argument for a very similar result: if f is $K - concave$, then for fixed $d > 0$, the function

$$g(y) = \max_{y \leq x \leq y+d} f(x)$$

is also $K - concave$.

Proof. We shall demonstrate that

$$g(y) \leq K + g(x) + (y - x)\frac{g(x) - g(x - b)}{b}$$

for $y \geq x > x - b \geq 0$. Let $q_1 = q(y)$ and $q_2 = q(x - b)$ so that

$$\begin{aligned} g(y) &= f(y - q_1) \text{ and} \\ g(x - b) &= f(x - b - q_2). \end{aligned}$$

$>$From the previous lemma we have $y - q_1 \geq x - b - q_2$. We can assume that

$$y - q_1 > x - b - q_2$$

since otherwise $g(y) = g(x) = g(x - b)$ and the inequality is trivially correct.

We have

$$\begin{aligned} g(y) &= f(y - q_1) \\ &\leq f(z) + \frac{y - q_1 - z}{z - x + b + q_2}[f(z) - f(x - b - q_2)] + K \end{aligned}$$

for any z with

$$y - q_1 \geq z > x - b - q_2.$$

If we select

$$z = \frac{b}{(y - x + b)}(y - q_1) + \frac{y - x}{(y - x + b)}(x - b - q_2)$$

then

$$\frac{y - q_1 - z}{z - x + b + q_2} = \frac{y - x}{b}$$

so that

$$g(y) \leq f(z) + (y - x)[\frac{f(z) - g(x - b)}{b}] + K.$$

To complete the proof we need only show that $0 \leq x - z \leq \min[d, x]$ so that $x - z$ is a feasible choice for x and therefore $g(x) \geq f(z)$.

It is easy to see that

$$x - z = \frac{b}{(y - x + b)}q_1 + \frac{y - x}{(y - x + b)}q_2$$

so that $0 \leq x - z \leq d$. But since $z \geq 0$, we also have $x - z \leq x$. ■

4 The Recursive Argument

Our proof of the optimality of (S, s) policies follows standard lines: we demonstrate recursively that the value functions $f_n(x)$ are $K - concave$, and as a by-product we deduce the form of the optimal policy. I am presenting this completely routine argument in rather tedious detail, because I would like to take this opportunity to improve upon the casual treatment in my early paper [4], in which various functions were assumed to be differentiable, when they may not have been.

Assuming that $f_{n-1}(x)$ is $K - concave$, we see that

$$\max_{0 \le q \le \min[y,\xi]} \{p(q-y) + \alpha f_{n-1}(y-q)\}$$

is also $K - concave$ for each ξ. It follows that

$$g_n(y) = py + \int_0^\infty \max_{0 \le q \le \min[y,\xi]} \{p(q-y) + \alpha f_{n-1}(y-q)\} \phi(\xi) d\xi$$

is $K - concave$ as is $g_n(y) - cy$.

If the stock level at the beginning of period n is x it is profitable to order to $y \ge x$ if

$$\begin{aligned} g_n(y) - K - c \cdot (y - x) &> g_n(x) \text{ or} \\ g_n(y) - cy - K &> g_n(x) - cx, \end{aligned}$$

and if we do order from x it is to that level $y \ge x$ that maximizes

$$g_n(y) - cy.$$

Define S_n to be a global maximizer of $g_n(y) - cy$, say the smallest such maximizer if there are several. If the stock level $x < S_n$ we purchase to S_n only if

$$g_n(x) - cx < g_n(S_n) - cS_n - K.$$

- The first part of the argument for the form of the optimal policy is to show that if we order from a particular $x < S_n$ then we also order from all $x - b < x$. For suppose to the contrary that $x - b < x < S_n$ and

$$\begin{aligned} g_n(x-b) - c \cdot (x-b) &> g_n(S_n) - cS_n - K \\ g_n(x) - cx &< g_n(S_n) - cS_n - K \end{aligned}$$

so that we order to S_n from x but not from $x - b$. But these inequalities are inconsistent with the fact that $g_n(y) - cy$ is $K - concave$ since the definition of $K - concavity$ and the inequalities imply that

$$\begin{aligned} g_n(S_n) - cS_n &\leq K + g_n(x) - cx \\ &\quad + (S_n - x)\frac{g_n(x) - cx - (g_n(x-b) - c \cdot (x-b))}{b} \\ &< K + g_n(x) - cx \end{aligned}$$

a contradiction. If we define s_n to be the unique $x < S_n$ (if there is such an x) with

$$g_n(x) - cx = g_n(S_n) - cS_n - K,$$

then for $x < S_n$ an order is placed if and only if $x < s_n$.

- To complete the argument for the form of the optimal policy in period n, it is necessary to show that no ordering takes place from a stock level $x > S_n$. But if $y > x > S_n$ then

$$\begin{aligned} g_n(y) - cy &\leq K + g_n(x) - cx + (y - x)\frac{g_n(x) - cx - (g_n(S_n) - cS_n)}{x - S_n} \\ &\leq K + g_n(x) - cx \end{aligned}$$

  since $g_n(S_n) - cS_n \geq g_n(x) - cx$ for all x. This completes the demonstration that the optimal policy in period n is the (S_n, s_n) policy.

Finally, to finish the recursion, we must show that $f_n(x)$ is also $K - concave$. But

$$f_n(x) = \begin{cases} g_n(x) & \text{for } x \geq s_n, \\ g_n(s_n) - c \cdot (s_n - x) & \text{for } x < s_n. \end{cases}$$

In order to demonstrate that

$$f_n(y) \leq K + f_n(x) + (y - x)\frac{f_n(x) - f_n(x-b)}{b}$$

for $y \geq x > x - b$, we consider three cases:

- $x \leq s_n$. In this case

$$\begin{aligned} \frac{f_n(x) - f_n(x-b)}{b} &= c \text{ and} \\ f_n(x) &= f_n(s_n) + c \cdot (x - s_n) \end{aligned}$$

so that the inequality to be verified is

$$g_n(y) - cy \leq g_n(s_n) - cs_n + K$$

which is surely correct since

$$g_n(y) - cy \leq g_n(S_n) - cS_n = g_n(s_n) - cs_n - K.$$

- $x > s_n \geq x - b$. In this case

$$f_n(x-b) = f_n(s_n) + c \cdot (x - b - s_n).$$

We must consider two subcases.

  - If

$$\begin{aligned} f_n(x) - cx &\geq f_n(s_n) - cs_n \\ &= f_n(S_n) - cS_n - K \\ &\geq f_n(y) - cy - K \end{aligned}$$

then

$$\begin{aligned} f_n(y) &\leq K + f_n(x) + c \cdot (y - x) \\ &\leq K + f_n(x) + (y - x)\frac{f_n(x) - f_n(x-b)}{b} \end{aligned}$$

because

$$\begin{aligned} \frac{f_n(x) - f_n(x-b)}{b} &= \frac{f_n(x) - (f_n(s_n) + c \cdot (x - b - s_n))}{b} \\ &\geq c. \end{aligned}$$

  - On the other hand if

$$f_n(x) - cx \leq f_n(s_n) - cs_n$$

then

$$
\begin{aligned}
(b+s_n-x)f_n(x) &\le (b+s_n-x)f_n(s_n)+(b+s_n-x)c\cdot(x-s_n) \\
&= bf_n(s_n)-(x-s_n)[f_n(s_n)-c\cdot(s_n-x+b)] \\
&= bf_n(s_n)-(x-s_n)f_n(x-b)
\end{aligned}
$$

so that

$$
\frac{f_n(x)-f_n(s_n)}{x-s_n} \le \frac{f_n(x)-f_n(x-b)}{b}.
$$

Since $f_n = g_n$ for the arguments y, x, s_n we see that

$$
\begin{aligned}
f_n(y) &\le K+f_n(x)+(y-x)\frac{f_n(x)-f_n(s_n)}{x-s_n} \\
&\le K+f_n(x)+(y-x)\frac{f_n(x)-f_n(x-b)}{b}.
\end{aligned}
$$

- In the final case $x-b \ge s_n$ and the desired conclusion follows from the fact that $f_n = g_n$ for the three arguments $y, x, x-b$.

5 Extensions of the Basic Model

The decision not to meet demand fully may be economically sound if there is sufficient variation, possibly stochastic, in the costs, revenues and demand distributions of the model. The argument for the optimality of (S, s) policies carries through virtually unchanged if the set-up costs K_n, the marginal costs c_n, the selling prices p_n and the demand densities $\phi_n(\xi)$ are deterministic functions of time, as long as the set-up costs satisfy the relationship

$$
K_n \ge \alpha K_{n-1}
$$

for all n. If this relation is satisfied the value function $f_n(x)$ will be $K_n -$ *concave*; if it is not satisfied then it is easy to produce examples in which $f_n(x)$ is not $K-$*concave* and the optimal ordering policy in period n is more complex.[2]

A further extension, which has been examined by several authors recently [5], [1], [3] is to assume that these parameters evolve according to an underlying Markov process. The first paper assumes that unsatisfied demand is

[2] If demands are fully satisfied to the extent that stock is available, and excess demand is lost, then the conditions $p_n \ge \alpha c_{n-1}$ are also necessary.

backlogged and the second that it is lost; both papers work with a Markov process with a finite number of states. The paper by Hall and Rust involves profit maximization, has a continuous state process, and assumes that unsatisfied demand is lost. This configuration of assumptions requires the authors to impose an additional condition essentially stating that it is economically sound for all demand to be met.

If the discrete process is in state i at the beginning of a period, then the cost and demand parameters of that period are given by, say,

$$\begin{array}{l} K_i, c_i, p_i \text{ and} \\ \phi_i(\xi) \end{array}$$

and the state of the system at the beginning of the subsequent period is described by a Markov transition matrix

$$P = [P_{i,j}] .$$

To specify the model fully some assumptions are required about the timing of purchase and sales decisions and the inventory manager's knowledge of tomorrow's state. For concreteness, I shall assume that the state of the system next period is revealed at the beginning at that time and that ordering and sales decisions are made prior to this knowledge.

The value function $f_n(i,x)$ will depend not only on the number of periods remaining, and the current stock level, but also on the state of the system i occurring at the beginning of period n. The recursive relationship connecting these value functions is given by

$$\begin{aligned} f_n(i,x) &= \max_{y\geq x}[-c_i(y-x) + \int_0^\infty \max_{0\leq q\leq \min[y,\xi]} \{p_i q + \alpha \sum_j P_{i,j} f_{n-1}(j, y-q)\}\phi_i(\xi)d\xi] \\ &= \max_{y\geq x}[-c_i(y-x) + p_i y \\ &\quad + \int_0^\infty \max_{0\leq q\leq \min[y,\xi]} \{p_i(q-y) + \alpha \sum_j P_{i,j} f_{n-1}(j, y-q)\}\phi_i(\xi)d\xi]. \end{aligned}$$

It is straight-forward to argue that if the functions $f_{n-1}(j,x)$ are $K_{n-1,j}-$ *concave* then $f_n(i,x)$ will be $K_{n,i}-$ *concave* if the condition

$$K_{n,i} \geq \alpha \sum_j P_{i,j} K_{n-1,j}$$

holds. The optimal policies will therefore be of the (S,s) form if this condition is satisfied for all n and i.

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