LONG RUN COVARIANCE MATRICES FOR FRACTIONALLY INTEGRATED PROCESSES

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Long Run Covariance Matrices for Fractionally Integrated Processes[1]

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Abstract

An asymptotic expansion is given for the autocovariance matrix of a vector of stationary long-memory processes with memory parameters $d \in [0, 1/2)$. The theory is then applied to deliver formulae for the long run covariance matrices of multivariate time series with long memory.

Keywords: Asymptotic expansion, Autocovariance function, Fourier integral, Long memory, Long run variance, Spectral density.

JEL Classification: C22

1. Motivation

Stationary long memory processes have extensive applications in economics and finance, particularly with regard to modeling financial variables like volatility and trading volume. The autocovariances of such processes decay according to a power law and the spectra are undefined at the origin. Correspondingly, conventional formulae and estimation procedures for long run variance matrices that apply under weak dependence are no longer relevant under long range dependence. Nonetheless, some modified versions of these (typically infinite dimensional) quantities do exist and are useful in the development of asymptotics involving long memory time series, for instance, in the estimation of fractionally cointegrated systems (Kim and Phillips, 1999; Robinson and Hualde, 2003; Velasco, 2003; Davidson, 2004;

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Davidson and Hashimzade, 2007). Henry and Zaffaroni (2003) provide a recent survey of the many applications of fractional integration and long-range dependence in macroeconomics and finance.

This note shows how to define long run covariance matrices for general multivariate fractionally integrated processes and we focus here on cases of long range dependence, some results on overdifferenced processes being given in earlier work (Phillips, 1995, lemma 8.1). We first develop a general form of asymptotic expansion for the autocovariance matrix of such a multivariate process. The approximation induced by this expansion is of independent interest. It has a simple form which gives the power law decay structure of the elements of the autocovariance matrix and reveals an interesting asymmetric structure for the cross autocovariances. In the scalar case, the result reduces to a formula obtained recently in Lieberman and Phillips (2006). The expansion is particularly useful in developing a limiting form of a standardized sum of the autocovariance matrices, allowing us to define the long run variance matrix of a multivariate fractional process. In contrast to the autocovariance function, the long run variance matrix has a simple symmetric form that depends on the long memory parameters of the constituent processes and the long run variance matrix of the short memory components.

2. Results

Let X_t be a real-valued covariance stationary m-vector time series generated by the system

$$\begin{pmatrix} (1-L)^{d_1} & & 0 \\ & \ddots & \\ 0 & & (1-L)^{d_m} \end{pmatrix} \begin{pmatrix} X_{1t} - EX_{1t} \\ \vdots \\ X_{mt} - EX_{mt} \end{pmatrix} = \begin{pmatrix} u_{1t} \\ \vdots \\ u_{mt} \end{pmatrix}, \quad 0 \leq d_1, \ldots, d_m < \frac{1}{2}, \tag{1}$$

where $u_t = (u_{1t}, \ldots, u_{mt})'$ is a covariance stationary process whose spectral density matrix $f_{uu}(\lambda)$ is assumed to be continuously differentiable on $[-\pi, \pi]$ and bounded away from zero (in the sense of positive definite matrices) at the zero frequency $\lambda = 0$. The smoothness condition on $f_{uu}(\lambda)$ is needed to develop an asymptotic expansion of the autocovariance function defined by a Fourier integral inversion of $f_{uu}(\lambda)$.

X_t is a multivariate extension of a scalar fractionally integrated process (the so-called $I(d)$ process) and each component X_{at} exhibits long-range dependence whenever $d_a > 0$. X_t reduces to a multivariate ARFIMA process when u_t is a vector ARMA process, but the specification (1) does not require u_t to be of this or any other parametric form.

Let $f_{xx}(\lambda)$ denote the spectral density of X_t, so that the autocovariance matrix is given by

$$\Gamma_{xx}(k) = E(X_t - EX_t)(X_{t+k} - EX_t)' = \int_{-\pi}^{\pi} e^{ik\lambda} f_{xx}(\lambda) d\lambda.$$

Define

$$\Phi(\lambda) = \operatorname{diag}\left((1-e^{i\lambda})^{-d_1}, \cdots, (1-e^{i\lambda})^{-d_m}\right) = \operatorname{diag}\left((1-e^{i\lambda})^{-d_a}\right),$$

and then the spectral density of X_t satisfies (e.g., Hannan, 1970, p.61)

$$f_{xx}(\lambda) = \Phi(\lambda) f_{uu}(\lambda) \Phi^*(\lambda), \tag{2}$$

where the affix $*$ signifies complex conjugate transpose. As is well known, the memory parameters, d_a, govern the long-run dynamics of X_t and the behavior of its spectrum $f_{xx}(\lambda)$ around the origin. Often, when attention is focused on long-run dynamics, it is useful to specify the spectral density only locally in the vicinity of the origin and to avoid short-run dynamic specifications concerning u_t altogether. In the multivariate case, we also are interested in the behavior of cross spectra at the origin and the corresponding cross autocovariances at long lags, as well as the individual spectra and autocovariance functions.

A first order approximation to the behavior of $f_{xx}(\lambda)$ at the origin is easily seen to be given by

$$f_{xx}(\lambda) \sim \operatorname{diag}(\lambda^{-d_a} e^{i\pi d_a/2}) f_{uu}(0) \operatorname{diag}(\lambda^{-d_a} e^{-i\pi d_a/2}), \quad \lambda \to 0+, \tag{3}$$

and higher order approximations may be similarly obtained (e.g., Phillips and Shimotsu, 2004; Shimotsu, 2006). The factors involving the complex exponentials $e^{i\pi d_a/2}$ turn out to be important in the off diagonal elements of $f_{xx}(\lambda)$ and these figure in the analysis below. When $0 < d_a < 1/2$, the individual time series X_{at} have long memory and the j-lag autocovariances decrease slowly, according to the power law j^{2d_a-1} as $j \to \infty$. In this case, the autocovariances are not summable and the usual formula for the long run variance of X_{at} is undefined. However, as shown below, upon suitable standardization, we may define the long run variances and covariances of the elements of X_t.

We start with the following result, which gives an asymptotic approximation to the autocovariance matrix function for long lags. Lieberman and Phillips (2006) gave a complete asymptotic series expansion of the autocovariance function of a scalar long memory time series. Under stronger smoothness conditions on the spectrum $f_{uu}(\lambda)$, similar asymptotic series may be developed here. The theorem below gives the leading term in the corresponding expansion for the multivariate case, which is sufficient for the present purpose of developing a formula for the long run variance matrix. The proof is given in the Appendix.

Theorem 1 *If* $d_a \in [0, 1/2)$ for all $a = 1, .., m$, *and the spectral density matrix* $f_{uu}(\lambda)$ *of* u_t *is continuously differentiable, then*

$$[\Gamma_{xx}(k)]_{ab} = \frac{2 f_{u_a u_b}(0) \Gamma(1-d_a-d_b) \sin(\pi d_b)}{k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \tag{4}$$

Remark 1. Note that the asymptotic approximation (4) is asymmetric. Suppose for example that $d_a < d_b$ and $f_{u_a u_b}(0) > 0$. Then, for large k

$$\begin{aligned}\gamma_{ab}(k) &= \frac{2 f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} \sin\{\pi d_b\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\ &> \frac{2 f_{u_b u_a}(0)\,\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} \sin\{\pi d_a\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\ &= \gamma_{ba}(k),\end{aligned}$$

since $\sin\{\pi d_a\} < \sin\{\pi d_b\}$. In particular, when $d_a = 0 < d_b < 1/2$, $\sin\{\pi d_a\} = 0$ and we have the interesting phenomena that $\gamma_{ba}(k)$ decays faster than the power law $k^{d_b - 1}$, corresponding to the short memory property of X_{at}, whereas $\gamma_{ab}(k)$ decays according to the power law k^{d_b-1} as $k \to \infty$, corresponding to the long memory property of X_{bt}. This asymmetry is explained by the fact that $\gamma_{ab}(k)$ is dominated by the slow decay in the impulse responses affecting X_{bt+k}, whereas the impulse responses and autocovariances of X_{at+k} decay faster than any power rate[1], thereby determining the different behavior of $\gamma_{ba}(k)$ when $0 = d_a < d_b < 1/2$.

Remark 2. When $d_a = d_b$, we have

$$\gamma_{aa}(k) = \frac{2 f_{u_a u_a}(0)\,\Gamma(1-2d_a)}{k^{1-2d_a}} \sin\{\pi d_a\} + O\left(\frac{1}{k^{2-2d_a}}\right), \tag{5}$$

corresponding to the leading term given in the asymptotic expansion of the autocovariance function for the scalar case in Lieberman and Phillips (2006).

We now define the standardization matrix $D_n = \mathrm{diag}\left(n^{d_1}, ..., n^{d_m}\right)$, the partial sum $S_t = \sum_{s=1}^{t} X_s$, and let $d = \min_{a \le m} d_a$ for the following theorem, whose proof is in the Appendix.

Theorem 2 *If $d_a \in (0, 1/2)$ for all $a = 1, .., m$, and the spectral density matrix $f_{uu}(\lambda)$ of u_t is continuously differentiable, then as $n \to \infty$*

$$\left[\frac{1}{n} D_n^{-1} E\left\{S_n S_n'\right\} D_n^{-1}\right]_{ab} \to \frac{2\pi f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\left\{\sin(\pi d_b) + \sin(\pi d_a)\right\}}{\pi(d_a+d_b)(1+d_a+d_b)}. \tag{6}$$

Remark 3. The diagonal elements of (6) are

$$\begin{aligned}\left[\frac{1}{n} D_n^{-1} E\left\{S_n S_n'\right\} D_n^{-1}\right]_{aa} &\to \frac{2\pi f_{u_a u_a}(0)\,\Gamma(1-2d_a)\sin(\pi d_a)}{\pi d_a(1+2d_a)} \\ &= \mathrm{lrvar}(u_{at}) \frac{\Gamma(1-2d_a)\sin(\pi d_a)}{\pi d_a(1+2d_a)},\end{aligned} \tag{7}$$

and, as $d_a \to 0$, this formula tends to $2\pi f_{uu}(0) = \mathrm{lrvar}(u_t)$, the limiting variance of the standardized partial sum $n^{-1/2} S_n = n^{-1/2} \sum_{t=1}^{n} u_t$.

[1] This may be proved directly using a Fourier integral asymptotic expansion when the spectrum of the short memory component is analytic.

Remark 4. In the scalar case with $u_t \sim iid\left(0, \sigma_u^2\right)$, Sowell (1990) showed that

$$\lim_{n\to\infty} \frac{\operatorname{var}(S_n)}{n^{1+2d}} = \sigma_u^2 \frac{\Gamma(1-2d)}{(1+2d)\Gamma(1+d)\Gamma(1-d)}. \tag{8}$$

We may compare this formula with (7). By the reflection formula for the gamma function we have $\Gamma(d)\Gamma(1-d) = \frac{\pi}{\sin(\pi d)}$, so that (7) may be written in the alternate form

$$\begin{aligned}\frac{2\pi f_{uu}(0)\sin\{\pi d\}\Gamma(1-2d)}{\pi d(1+2d)} &= \frac{2\pi f_{uu}(0)\Gamma(1-2d)}{d(1+2d)\Gamma(d)\Gamma(1-d)} \\ &= \frac{2\pi f_{uu}(0)\Gamma(1-2d)}{(1+2d)\Gamma(d+1)\Gamma(1-d)},\end{aligned} \tag{9}$$

which clearly reduces to (8) in the case of $iid\ u_t$.

Remark 5. Formula (7) for the asymptotic variance corresponds to that delivered by the covariance kernel of the limiting fractional Brownian motion. In particular, it is well known (e.g. Chan and Terrin, 1995; Marinucci and Robinson, 2000) that under certain regularity conditions we have the weak convergence

$$\frac{1}{n^{\frac{1}{2}+d_a}} \sum_{t=1}^{[nr]} X_{at} \xrightarrow{d} B_{d_a}(r).$$

It is often convenient to define the limiting fractional Brownian motions $B_{d_a}(r)$ in terms of their harmonizable representations (see Samorodnitsky and Taqqu, 1994; Chan and Terrin, 1995; Davidson and Hashimzade, 2007) as follows

$$B_{d_a}(r) = \frac{1}{\sqrt{2\pi}} \int_R \frac{e^{i\lambda r}-1}{i\lambda} (i\lambda)^{-d_a}\, dW_a(\lambda), \tag{10}$$

where $\{W_a(\lambda) : a = 1, ..., m\}$ are complex-valued Gaussian random measures satisfying

$$\begin{aligned} dW_a(\lambda) &= d\overline{W_a(-\lambda)}, \\ \mathbf{E}[dW_a(\lambda)] &= 0, \\ \mathbf{E}\left[dW_a(\lambda)d\overline{W_b(\mu)}\right] &= \begin{cases} \omega_{ab} d\lambda & , \ \lambda = \mu \\ 0 & , \ \lambda \neq \mu \end{cases} \ , \ a, b = 1, .., m, \end{aligned}$$

for $\lambda \in [-\pi, \pi]$, where $\omega_{ab} = 2\pi f_{u_a u_a}(0)$ and where the bar denotes complex conjugation. Observe that

$$
\begin{aligned}
E\left\{B_{d_a}(1)^2\right\} &= \frac{\omega_{aa}}{2\pi}\int_R \left|\frac{e^{i\lambda}-1}{i\lambda}\right|^2 |\lambda|^{-2d_a}\, d\lambda \\
&= \frac{\omega_{aa}}{2\pi}\int_R \frac{2-2\cos\lambda}{\lambda^{2+2d_a}} d\lambda = \frac{2\omega_{aa}}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^{2+2d_a}} d\lambda \\
&= \frac{4\omega_{aa}}{\pi}\int_0^\infty \frac{\sin^2\frac{\lambda}{2}}{\lambda^{2+2d_a}} d\lambda = -\frac{2\omega_{aa}}{\pi}\Gamma(-1-2d_a)\cos\left(\frac{(1+2d_a)\,\pi}{2}\right) \qquad (11) \\
&= -\frac{2\omega_{aa}}{\pi}\Gamma(-1-2d_a)\cos\left(\frac{(1+2d_a)\,\pi}{2}\right) \\
&= \frac{2\omega_{aa}}{\pi}\Gamma(-1-2d_a)\sin\left(d_a\pi\right) \\
&= \frac{2\omega_{aa}}{\pi}\frac{\pi\sin\left(d_a\pi\right)}{\sin\left(\pi\left(2+2d_a\right)\right)\Gamma\left(2+2d_a\right)} \\
&= \frac{2\omega_{aa}\sin\left(d_a\pi\right)}{\sin\left(2\pi d_a\right)\Gamma\left(2+2d_a\right)} = \frac{\omega_{aa}}{\cos\left(\pi d_a\right)\Gamma\left(2+2d_a\right)} \qquad (12)
\end{aligned}
$$

using Gradshteyn and Ryzhik (2000, formula 3.823) in (11), c.f. Davidson and Hashimzade (2007). Applying the reflection formula $\Gamma\left(1-z\right)\Gamma\left(z\right)=\frac{\pi}{\sin(\pi z)}$ with $z=2+2d_a$ so that $1-z=-1-2d_a$, we have

$$
\begin{aligned}
&\Gamma(2+2d_a)\cos\pi d_a \\
&= \frac{\pi}{\sin\left(2\pi d_a+2\pi\right)\Gamma\left(-1-2d_a\right)}\cos\pi d_a \\
&= \frac{\pi}{\sin\left(2\pi d_a\right)\Gamma\left(-1-2d_a\right)}\cos\pi d_a \\
&= \frac{\pi\left(-2d_a\right)\left(-1-2d_a\right)}{2\sin\left(\pi d_a\right)\Gamma\left(1-2d_a\right)} = \frac{\pi d_a\left(1+2d_a\right)}{\sin\left(\pi d_a\right)\Gamma\left(1-2d_a\right)}, \qquad (13)
\end{aligned}
$$

using $\sin\left(2\pi d_a\right)=2\sin\pi d_a\cos\pi d_a$. It now follows from (12) and (13) that

$$
\begin{aligned}
E\left\{B_{d_a}(1)^2\right\} &= \frac{\omega_{aa}}{\Gamma(2+2d_a)\cos\left(\pi d_a\right)} = \frac{\omega_{aa}\Gamma\left(1-2d_a\right)\sin\left(\pi d_a\right)}{\pi d_a\left(1+2d_a\right)} \\
&= \frac{\omega_{aa}\Gamma\left(1-2d_a\right)}{d_a\left(1+2d_a\right)\Gamma\left(1-d_a\right)\Gamma\left(d_a\right)} = \frac{\omega_{aa}\Gamma\left(1-2d_a\right)}{\left(1+2d_a\right)\Gamma\left(1-d_a\right)\Gamma\left(1+d_a\right)},
\end{aligned}
$$

where we use the reflection formula again in the form $\Gamma\left(1-d_a\right)\Gamma\left(d_a\right)=\frac{\pi}{\sin(\pi d_a)}$, leading to the stated correspondence with (7) and (9). Expression (12) was also obtained in Davidson and Hashimzade (2007, formula 2.6 with $\kappa=1$). Similar arguments show that

$$
E\left\{B_{d_a}(1)B_{d_b}(1)\right\} = \frac{\omega_{ab}\Gamma\left(1-d_a-d_b\right)\left\{\sin\left(\pi d_b\right)+\sin\left(\pi d_a\right)\right\}}{\pi\left(d_a+d_b\right)\left(1+d_a+d_b\right)},
$$

corresponding to (6).

3. Discussion and Application

Some estimation procedures like fully modified estimation in a fractional cointegration model (Kim and Phillips, 1999; Davidson, 2004) involve unknown long run variance and covariance matrices for fractional processes such as those given in Theorem 2, which need to be estimated consistently for these procedures to be implemented. Consistent estimation of these long run covariances can be accomplished by a stepwise procedure that involves separate consistent estimation of the memory parameters and the long run variance matrix, $\Omega_{uu} = 2\pi f_{uu}(0)$ of u_t. The memory parameters $\{d_a : a = 1, ..., m\}$ can be estimated by any consistent semiparametric method, such as the local Whittle (Robinson, 1995) or exact local Whittle (Shimotsu and Phillips, 2005) procedures. Using estimates $\hat{d}_a$ obtained in this way, estimates of the residuals u_{at} can be constructed by the truncated filtering operation

$$\sum_{k=0}^{t} \widehat{\pi}_k \left(X_{at-k} - \bar{X}_a \right) = \widehat{u}_{at}, \tag{14}$$

where $\widehat{\pi}_k = \Gamma(k - \widehat{d}_a)/\{\Gamma(k+1)\Gamma(-\widehat{d}_a)\}$. Using $\widehat{u}_{at}$, the long run variance matrix Ω_{uu} of u_t may then be consistently estimated by any conventional HAC procedure[2]. These estimates of Ω_{uu} and d_a may then be plugged into formulae such as those given in Theorem 2 to produce consistent estimates of the required long run covariances of the fractional processes.

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[2] Demonstration of consistency requires attention to the effect of the finite length filtering operation (14), as in Velasco (2003) and Robinson and Hualde(2003), and the use of residuals $\hat{u}_{at}$ in the estimation of Ω_{uu}, as in conventional HAC estimation.

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Appendix : Proofs

Proof of Theorem 1. The derivation of (4) uses an asymptotic expansion of the Fourier inversion formula for $\Gamma_{xx}(k)$, which can be written as

$$\Gamma_{xx}(k) = \int_{-\pi}^{\pi} e^{iwk} L\left(e^{iw}\right) f_{uu}(w) L\left(e^{-iw}\right) dw, \tag{15}$$

where

$$L\left(e^{iw}\right)=\operatorname{diag}\left\{\left(1-e^{iw}\right)^{-d_1},...,\left(1-e^{iw}\right)^{-d_m}\right\}.$$

In what follows, we will work with the ab'th element of $\Gamma_{xx}(k)$ denoted by

$$\gamma_{ab}(k)=E\left\{\left(X_{at}-EX_{at}\right)\left(X_{bt+k}-EX_{bt+k}\right)\right\}.$$

Observe that $|1-e^{iw}|=|2\sin(w/2)|$ and

$$\arg(1-e^{-iw})=\begin{cases}(w-\pi)/2 & \text{for } 0\le w<\pi\\ (\pi-w)/2 & \text{for } -\pi\le w<0\end{cases},$$

so that

$$1-e^{iw}=\begin{cases}|2\sin(w/2)|e^{i(w-\pi)/2} & 0\le w<\pi\\ |2\sin(w/2)|e^{i(\pi-w)/2} & -\pi\le w<0\end{cases},$$

and then

$$\begin{aligned}(1-e^{iw})^{\theta}&=\begin{cases}|2\sin(w/2)|^{\theta}e^{i(w-\pi)\theta/2} & 0\le w<\pi\\ |2\sin(w/2)|^{\theta}e^{i(\pi-w)\theta/2} & -\pi\le w<0\end{cases}\\ &=\begin{cases}|w|^{\theta}\left|\frac{2\sin(w/2)}{w}\right|^{\theta}e^{i(w-\pi)\theta/2} & 0\le w<\pi\\ |w|^{\theta}\left|\frac{2\sin(w/2)}{w}\right|^{\theta}e^{i(\pi-w)\theta/2} & -\pi\le w<0\end{cases}.\end{aligned}$$

It follows that for $0\le w<\pi$

$$\begin{aligned}&\left(1-e^{iw}\right)^{-d_a}\left(1-e^{-iw}\right)^{-d_b}f_{u_au_b}(w)\\ &=w^{-d_a}\left(\frac{2\sin(w/2)}{w}\right)^{-d_a}e^{-\frac{i(w-\pi)d_a}{2}}\\ &\times w^{-d_b}\left(\frac{2\sin(w/2)}{w}\right)^{-d_b}e^{-\frac{i(\pi-w)d_b}{2}}f_{u_au_b}(w)\\ &=e^{\frac{i\pi(d_a-d_b)}{2}}w^{-d_a-d_b}\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b}e^{-\frac{iw(d_a-d_b)}{2}}f_{u_au_b}(w),\end{aligned}$$

and for $-\pi<w\le 0$

$$\begin{aligned}&\left(1-e^{iw}\right)^{-d_a}\left(1-e^{-iw}\right)^{-d_b}f_{u_au_b}(w)\\ &=|w|^{-d_a}\left|\frac{2\sin(w/2)}{w}\right|^{-d_a}e^{-\frac{i(w+\pi)d_a}{2}}\\ &\times|w|^{-d_b}\left|\frac{2\sin(w/2)}{w}\right|^{-d_b}e^{-\frac{i(-w-\pi)d_b}{2}}\\ &=e^{-\frac{i\pi(d_a-d_b)}{2}}|w|^{-d_a-d_b}\left|\frac{2\sin(w/2)}{w}\right|^{-d_a-d_b}e^{-\frac{iw(d_a-d_b)}{2}}f_{u_au_b}(w).\end{aligned}$$

Hence,

$$\begin{aligned}\gamma_{ab}(k) &= \int_{-\pi}^{\pi} e^{iwk}\left(1-e^{iw}\right)^{-d_a}\left(1-e^{-iw}\right)^{-d_b} f_{u_a u_b}(w)\,dw \\ &= \left\{\int_0^{\pi} + \int_{-\pi}^{0}\right\} e^{iwk}\left(1-e^{iw}\right)^{-d_a}\left(1-e^{-iw}\right)^{-d_b} f_{u_a u_b}(w)\,dw \\ &= e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{iwk}\left[\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} f_{u_a u_b}(w)\, e^{-\frac{iw(d_a-d_b)}{2}}\right] w^{-d_a-d_b}dw \\ &+ e^{-\frac{i\pi(d_a-d_b)}{2}} \int_{-\pi}^{0} e^{iwk}\left[\left|\frac{2\sin(w/2)}{w}\right|^{-d_a-d_b} f_{u_a u_b}(w)\, e^{-\frac{iw(d_a-d_b)}{2}}\right] |w|^{-d_a-d_b}\,dw \\ &= e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{iwk}\left[\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} f_{u_a u_b}(w)\, e^{-\frac{iw(d_a-d_b)}{2}}\right] w^{-d_a-d_b}dw \\ &+ e^{-\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{-iwk}\left[\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} f_{u_a u_b}(-w)\, e^{\frac{iw(d_a-d_b)}{2}}\right] w^{-d_a-d_b}dw\end{aligned}$$

$$= e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{iwk} F_{u_a u_b}(w)\, w^{-d_a-d_b}dw + e^{-\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{-iwk} F_{u_a u_b}(-w)\, w^{-d_a-d_b}dw. \tag{16}$$

where $F_{u_a u_b}(w) = \left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} f_{u_a u_b}(w)\, e^{-\frac{iw(d_a-d_b)}{2}} \in C^1[-\pi,\pi]$.

When at least one of d_a or $d_b > 0$, the two integrals in (16) have critical points (singularities in the integrand) at the origin $w = 0$. For Fourier integrals of this type asymptotic expansions for large k were originally developed by Erdélyi (1956), a convenient reference being Bleistein and Handelsman (1986). In particular, if $F(w) \in C^\infty[a,b]$, and α and β are not integers, then Erdélyi's result implies that the integral

$$I(k) = \int_a^b e^{ikw}(w-a)^{\alpha-1}(b-w)^{\beta-1} F(w)\,dw \tag{17}$$

has the following complete asymptotic series representation as $k \to \infty$

$$I(k) = I_a(k) + I_b(k),$$

where

$$I_a(k) \sim \sum_{n=0}^{\infty} \frac{d^n}{da^n}\left\{(b-a)^{\beta-1} F(a)\right\} \frac{\Gamma(n+\alpha)}{n!k^{n+\alpha}} e^{\frac{\pi i}{2}(n+\alpha)+ika}, \tag{18}$$

and

$$I_b(k) \sim \sum_{n=0}^{\infty} \frac{d^n}{db^n}\left\{(b-a)^{\alpha-1} F(b)\right\} \frac{\Gamma(n+\beta)}{n!k^{n+\beta}} e^{\frac{\pi i}{2}(n-\beta)+ikb}. \tag{19}$$

These expansions hold to the first term $n = 0$ provided $F(w) \in C^1[a,b]$, and this degree of smoothness is all that is required for the present application.

Specializing the expansion of (17) to the present case, we first consider the integral

$$\int_0^\pi e^{iwk} F_{u_a u_b}(w)\, w^{-d_a-d_b} dw,$$

and set $a = 0$, $b = \pi$, $\beta = 1$, and $\alpha = 1 - d_a - d_b$ in formula (18). We deduce that

$$\begin{aligned}
&\int_0^\pi e^{ikw} w^{\alpha-1} F_{u_a u_b}(w)\, dw \\
&= \sum_{n=0}^{\infty} \frac{d^n}{da^n} \{F_{u_a u_b}(a)\}_{a=0} \frac{\Gamma(n+\alpha)}{n! k^{n+\alpha}} e^{\frac{\pi i}{2}(n+\alpha)+ika} \\
&= \frac{\Gamma(1-d_a-d_b)\, e^{\frac{\pi i}{2}(1-d_a-d_b)}}{k^{1-d_a-d_b}} F_{u_a u_b}(0) + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \qquad (20)
\end{aligned}$$

Using the same settings $a = 0$, $b = \pi$, $\beta = 1$, and $\alpha = 1 - d_a - d_b$ again, we next find that

$$\begin{aligned}
&\int_0^\pi e^{-iwk} F_{u_a u_b}(-w)\, w^{-d_a-d_b} dw \\
&= \sum_{n=0}^{\infty} \frac{d^n}{da^n} \{F_{u_a u_b}(a)\}_{a=0} \frac{\Gamma(n+\alpha)}{n!(-k)^{n+\alpha}} e^{\frac{\pi i}{2}(n+\alpha)+ika} \\
&= \frac{\Gamma(1-d_a-d_b)\, e^{\frac{\pi i}{2}(1-d_a-d_b)}}{k^{1-d_a-d_b} e^{\pi i(1-d_a-d_b)}} F_{u_a u_b}(0) + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\Gamma(1-d_a-d_b)\, e^{-\frac{\pi i}{2}(1-d_a-d_b)}}{k^{1-d_a-d_b}} F_{u_a u_b}(0) + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \qquad (21)
\end{aligned}$$

Combining results (20) and (21) in (16) we obtain

$$\begin{aligned}
\gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^\pi e^{iwk} F_{u_a u_b}(w)\, w^{-d_a-d_b} dw + e^{-\frac{i\pi(d_a-d_b)}{2}} \int_0^\pi e^{-iwk} F_{u_a u_b}(-w)\, w^{-d_a-d_b} dw \\
&= e^{\frac{i\pi(d_a-d_b)}{2}} \frac{\Gamma(1-d_a-d_b)\, e^{\frac{\pi i}{2}(1-d_a-d_b)}}{k^{1-d_a-d_b}} F_{u_a u_b}(0) \\
&+ e^{-\frac{i\pi(d_a-d_b)}{2}} \frac{\Gamma(1-d_a-d_b)\, e^{-\frac{\pi i}{2}(1-d_a-d_b)}}{k^{1-d_a-d_b}} F_{u_a u_b}(0) + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} F_{u_a u_b}(0) \left\{ e^{\frac{\pi i}{2}(1-2d_b)} + e^{-\frac{\pi i}{2}(1-2d_b)} \right\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{2\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} F_{u_a u_b}(0) \cos\left\{\frac{\pi}{2}(1-2d_b)\right\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{2\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} F_{u_a u_b}(0) \sin\{\pi d_b\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right),
\end{aligned}$$

which gives the stated result since $F_{u_a u_b}(0) = f_{u_a u_b}(0)$. ■

Proof of Theorem 2. From theorem 1, as $k \to \infty$

$$\gamma_{ab}(k) = \frac{2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\sin(\pi d_b)}{k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \tag{22}$$

Correspondingly, as $k \to -\infty$, we have, since $\gamma_{ab}(k) = \gamma_{ba}(-k)$

$$\gamma_{ab}(k) = \frac{2f_{u_b u_a}(0)\,\Gamma(1-d_a-d_b)\sin(\pi d_a)}{|k|^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \tag{23}$$

Then,

$$\begin{aligned}
&\frac{1}{n}D_n^{-1}E\left\{S_nS_n'\right\}D_n^{-1} \\
&= \frac{1}{n}D_n^{-1}E\left(\sum_{t=1}^{n}X_t\right)\left(\sum_{t=1}^{n}X_t\right)' D_n^{-1} = \frac{1}{n}D_n^{-1}\sum_{t,s=1}^{n}\left(EX_tX_s'\right)D_n^{-1} \\
&= \frac{1}{n}D_n^{-1}\sum_{t,s=1}^{n}\Gamma_{xx}(s-t)\,D_n^{-1} = D_n^{-1}\sum_{h=-n+1}^{n-1}\left(1-\frac{|h|}{n}\right)\Gamma_{xx}(h)\,D_n^{-1} \\
&= D_n^{-1}\sum_{h=-n+1}^{n-1}\Gamma_{xx}(h)\,D_n^{-1} - n^{-1}D_n^{-1}\sum_{h=-n+1}^{n-1}|h|\,\Gamma_{xx}(h)\,D_n^{-1}.
\end{aligned} \tag{24}$$

For some L such that $\frac{1}{L} + \frac{L}{n^{2d}} \to 0$ as $n \to \infty$, we decompose the first sum in (24) as follows

$$\begin{aligned}
D_n^{-1}\sum_{h=-n+1}^{n-1}\Gamma_{xx}(h)\,D_n^{-1} &= D_n^{-1}\left[\sum_{-L+1}^{L-1}\Gamma_{xx}(h) + \sum_{L}^{n-1}\Gamma_{xx}(h) + \sum_{-n+1}^{-L}\Gamma_{xx}(h)\right]D_n^{-1} \\
&= D_n^{-1}\left[\sum_{L}^{n-1}\Gamma_{xx}(h) + \sum_{-n+1}^{-L}\Gamma_{xx}(h)\right]D_n^{-1} + o(1).
\end{aligned}$$

The ab'th element of this matrix is

$$\begin{aligned}
&\frac{1}{n^{d_a+d_b}}\left[\sum_{L}^{n-1}\gamma_{ab}(h) + \sum_{-n+1}^{-L}\gamma_{ab}(h)\right] + o(1) = \frac{1}{n^{d_a+d_b}}\left[\sum_{L}^{n-1}\gamma_{ab}(h) + \sum_{-n+1}^{-L}\gamma_{ba}(-h)\right] + o(1) \\
&= \frac{1}{n^{d_a+d_b}}\sum_{L}^{n-1}\left\{\gamma_{ab}(h) + \gamma_{ba}(h)\right\} + o(1) \\
&= \frac{1}{n^{d_a+d_b}}\sum_{L}^{n-1}\left[\frac{2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\left\{\sin(\pi d_b)+\sin(\pi d_a)\right\}}{h^{1-d_a-d_b}} + O\left(\frac{1}{h^{2-d_a-d_b}}\right)\right] + o(1)
\end{aligned}$$

.

$$= 2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}\frac{1}{n^{d_a+d_b}}\sum_{1}^{n}\frac{1}{h^{1-d_a-d_b}}+o(1)$$
$$= 2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}\frac{1}{n^{d_a+d_b}}\int_{1}^{n}\frac{dh}{h^{1-d_a-d_b}}+o(1)$$
$$= 2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}\frac{1}{n^{d_a+d_b}}\left[\frac{h^{d_a+d_b}}{d_a+d_b}\right]_1^n+o(1)$$
$$= \frac{2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{d_a+d_b}+o(1), \tag{25}$$

by Euler summation. It follows that

$$\left[D_n^{-1}\sum_{h=-n+1}^{n-1}\Gamma_{xx}(h)\,D_n^{-1}\right]_{ab}\to\frac{2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{d_a+d_b}.$$

Next consider the second sum in (24). The ab'th element is

$$\frac{1}{n^{1+d_a+d_b}}\sum_{h=-n+1}^{n-1}|h|\,\gamma_{ab}(h)$$
$$= \frac{1}{n^{1+d_a+d_b}}\sum_{h=-L+1}^{L-1}|h|\,\gamma_{ab}(h)+\frac{1}{n^{1+d_a+d_b}}\sum_{h=L}^{n-1}h\gamma_{ab}(h)+\frac{1}{n^{1+d_a+d_b}}\sum_{-n+1}^{-L}|h|\,\gamma_{ab}(h)$$
$$= \frac{1}{n^{1+d_a+d_b}}\sum_{h=-L+1}^{L-1}|h|\,\gamma_{ab}(h)+\frac{1}{n^{1+d_a+d_b}}\sum_{h=L}^{n-1}h\{\gamma_{ab}(h)+\gamma_{ba}(h)\}$$
$$= \frac{1}{n^{1+d_a+d_b}}\sum_{h=L}^{n-1}h\left\{\frac{2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{h^{1-d_a-d_b}}+O\left(\frac{1}{h^{2-d_a-d_b}}\right)\right\}+o(1)$$

$$= \frac{1}{n^{1+d_a+d_b}}\sum_{h=1}^{n-1}\left\{\frac{2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{h^{-d_a-d_b}}\right\}+o(1)$$
$$= 2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}\frac{1}{n^{1+d_a+d_b}}\int_1^n h^{d_a+d_b}dh+o(1)$$
$$= \frac{2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{1+d_a+d_b}+o(1), \tag{26}$$

by Euler summation again.

We now combine (25) and (26) in (24) giving, as $n \to \infty$,

$$\begin{aligned}
&\left[\frac{1}{n} D_n^{-1} E\left\{S_n S_n'\right\} D_n^{-1}\right]_{ab} \\
&= \frac{2 f_{u_a u_b}(0)\, \Gamma(1-d_a-d_b)\left\{\sin(\pi d_b)+\sin(\pi d_a)\right\}}{d_a+d_b} \\
&- \frac{2 f_{u_a u_b}(0)\, \Gamma(1-d_a-d_b)\left\{\sin(\pi d_b)+\sin(\pi d_a)\right\}}{1+d_a+d_b} + o(1) \\
&= \frac{2 f_{u_a u_b}(0)\, \Gamma(1-d_a-d_b)\left\{\sin(\pi d_b)+\sin(\pi d_a)\right\}}{(d_a+d_b)(1+d_a+d_b)} + o(1) \\
&\to \frac{2 f_{u_a u_b}(0)\, \Gamma(1-d_a-d_b)\left\{\sin(\pi d_b)+\sin(\pi d_a)\right\}}{(d_a+d_b)(1+d_a+d_b)},
\end{aligned}$$

which corresponds to the stated result. ■