WEAKLY DEMOCRATIC REGULAR TAX EQUILIBRIA

IN A LOCAL PUBLIC GOODS ECONOMY WITH PERFECT CONSUMER MOBILITY

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1. Introduction

In this paper we study the existence of equilibria in a local public
goods economy which has the following structure. There are a finite number
of geographical regions. Each region has a government which provides
public goods locally by buying private goods inputs on competitive markets
and transforming these inputs into outputs of public goods by using a con-
vex technology. The local public goods are pure in the sense that all
the local residents consume the total produced, with no spillovers to other
regions. Each regional government raises revenue from the residents of
its region to cover the costs of the public goods it provides. The local
governments are assumed to be weakly democratic in the sense that no public
sector proposal (that is, a vector of public good provisions and a tax
scheme to pay for it) will be enacted for which there exists an alternative
proposal which is unanimously preferred by the local residents. (Note
that it is <u>not</u> being assumed that unanimity is a necessary condition for
a public sector proposal to be enacted.)

Private goods are produced by price-taking, profit maximizing pro-
ducers using convex technologies, and are traded across regions. A
continuum of price-taking, utility maximizing consumers, who are perfectly

mobile across regions, treat parametrically the public sector proposals of the various local governments in deciding where to live. There are a finite number of different types of consumers, with type defined by a consumer's preferences, initial endowments, and share of private sector profits.

A feasible allocation is a specification of a partition of the consumers among the regions, a vector of technically feasible public good provisions (and an associated vector of private good inputs used in its production) for each region, a technically feasible aggregate production vector for the private sector, and a vector of private goods consumption for each consumer such that excess demand for each private good is zero. An equilibrium is defined by a vector of private good prices, a feasible allocation, and a vector of regional tax rates (on wealth), which satisfy the following conditions: (i) the private sector production vector is profit maximizing at the given price vector; (ii) each consumer's private goods bundle is utility maximizing subject to the after-tax budget constraint faced in the region in which he lives, and taking as given the vector of public good provisions in the region; (iii) no consumer desires to move to another region; (iv) the local public sectors are in equilibrium in the following sense: (a) each local government has a balanced budget (i.e., the revenue raised equals the cost of the private goods used to produce the region's public goods vector); (b) a regular tax scheme (specifically a proportional wealth tax) is used in each region (with the tax rate allowed to vary across regions); (c) in each region there is no alternative public sector proposal (involving any scheme of taxation) which the residents would unanimously prefer to the given public sector proposal.

This characterization of public sector equilibrium dates back to Foley [5], who dealt with a single region model in which mobility considerations played no role. The basic idea is to avoid a detailed examination of the (quite possibly very complex) political mechanisms of the local governments while still requiring that the equilibrium public sector proposals have certain realistic features. Whatever the local political mechanism is, it seems reasonable to assume that it is weakly democratic and that the tax scheme which is used to balance the budget reflects some sort of regular incidence of taxation over the residents of the region.

We say an allocation is locally Pareto efficient if there is no other feasible allocation involving the same partition as the given allocation which can make everyone better off. The assumption that the local governments are weakly democratic, along with the profit maximization equilibrium condition, insure that an equilibrium allocation as defined above is efficient in this restricted sense.

In light of this property, asking whether an equilibrium exists amounts to asking the following question. Among the locally Pareto efficient allocations corresponding to all possible partitions, are there any which are decentralizable (in the sense described by conditions (i)-(iii) of equilibrium) for a given initial distribution of endowments, and correspond to regular regional tax structures in which all regions have balanced budgets?

An allocation is said to be globally Pareto efficient if there is no other feasible allocation which can make everyone better off. Unlike the definition of local Pareto efficiency, there is no requirement that the partition involved in the alternative feasible allocation be the same as the one in the given allocation.

We have motivated interest in locally Pareto efficient allocations
on positive grounds. But normative considerations would seem to motivate
interest in the set of globally Pareto efficient allocations.[1] In particu-
lar we can ask the analogous decentralization question for them--will there
always be a globally Pareto efficient allocation which is decentralizable
and corresponds to a regular regional tax structure in which all regions
have balanced budgets. Under conventional economic assumptions in economies
where the local public goods are pure, the answer is no.[2]

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[1]See, for example, Bewley [2] and Wooders [14]. The possibility of decen-
tralizing globally Pareto efficient allocations is generally referred to
as the Tiebout hypothesis.

[2]Suppose we impose a very weak regularity requirement on the local tax sys-
tems, namely that any two individuals that are identical except for tastes
are taxed the same. Thus we rule out Lindahl taxation. Then consider an
economy having 2 regions, 1 type of private good and 1 type of public good,
and a continuum of consumers represented by the interval $[0,1]$. Half
the consumers (type 1) have the utility function (applicable to either
region) $u^1(g,y) = g + (y/8)$, where g is the amount of the local public
good provided in the region in which the consumer lives, and y denotes
the consumption of the private good. The other half of the consumers
(type 2) have the utility function $u^2(g,y) = g + 2y$. There is an aggre-
gate endowment of 1 unit of the private good, divided evenly among all
consumers. Each local government uses a constant returns to scale technology
in producing the public good in its region, with 1 unit of input of the
private good yielding 1 unit of output of the public good. There is no
private good production.
    Clearly we can confine our search for a decentralizable globally
Pareto efficient (GPE) allocation to GPE allocations which assign all con-
sumers of the same type the same positive utility level. Given a GPE
allocation with this property, at least half of the consumers living in
one of the regions must be of type 1 (where these type 1 consumers repre-
sent at least $r \geq 1/4$ of all the consumers in the economy), and each
consumer of type 1 must be consuming the same bundle $(\bar{g}, y^1)$. Suppose
$y^1 > 0$. Then by reducing y^1 to 0 , the resulting ry^1 units of the
private good, raising the utility level of type 1 consumers by $ry^1 - (y^1/8) > 0$,
could be used to produce ry^1 additional units of the public good;
and increasing the utility level of any type 2 consumers living in this
region. Hence the given allocation could not have been GPE. Thus $y^1 = 0$,
and hence $\bar{g} > 0$ (since the given allocation yields a positive utility
level to each type). But since both utility functions are monotonic in
g , $\bar{g} > 0$ implies everyone must live in this region if the allocation is
GPE. (There are no congestion effects nor any characteristics specific

Richter [11] and Greenberg [6] have studied equilibria of the type described earlier, with one important exception. They both assumed that the partition of consumers was exogenous, thus ignoring probably the most crucial aspect of local public goods economies--the mobility of consumers. Another way of phrasing the question we are asking in this paper is: as we sift through the proportional wealth tax equilibria of Richter and Greenberg corresponding to all possible fixed partitions, will we always be able to find a fixed partition equilibrium which satisfies one additional condition--nobody wants to move?

In [12] an attempt was made to explicitly treat consumer mobility in the context of a regional model with pure local public goods. That model was only a partial success. While public goods were consumed locally, it was assumed that a weakly democratic central government levied a proportional wealth tax on all the consumers in the economy in order to raise revenue to pay for all the local public goods provided in all the regions. Furthermore, local governments did not actually produce their public goods. Instead it was assumed that public goods were produced by an aggregated, profit maximizing production sector. Thus local governments played no real role. In the present paper we have given the local governments a real role to play, requiring them to produce their own public goods, making

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to a particular region which affect utility or production in this example.) But since $y^1 = 0$, all income of type 1 consumers must be taxed away for the given allocation to qualify as a decentralized equilibrium. But then from our regularity assumption on the tax system, all income of type 2 consumers must also be taxed away. Hence the economy's total endowment of 1 unit of the private good must be used to produce 1 unit of the public good, yielding a utility level of 1 for type 2 consumers. But by moving to the empty region (where the tax rate would be zero), type 2 consumers could achieve a utility level of 2 by consuming their endowments of the private good. Thus no GPE allocations in this economy can be decentralized if a regular tax system is used.

them fiscally autonomous, and permitting them to compete for residents on the basis of tax rates as well as public goods menus. These changes have led us to a radically different approach to the existence question from that taken in [12].

The paper is organized as follows. In Section 2 the basic assumptions of the model are presented and the equilibrium notion is rigorously defined. Section 3 contains a heuristic overview of the existence proof and Section 4 contains the rigorous existence proof.

2. The Model

R_+^n (R_-^n) denotes the nonnegative (nonpositive) orthant of n-dimensional Euclidean space R^n. The inequality convention used for vectors is $>>$, $>$, $\geq$.

Superscripts are primarily used to refer to consumer type, while subscripts are generally reserved for regions or goods. The index i or ℓ will mainly be used to refer to consumer type, and the index j for regions. If $y_j^i \in R^n$, where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, \gamma\}$, then the vector $y = (y_1^1, y_1^2, \ldots, y_1^m, y_2^1, y_2^2, \ldots, y_2^m, \ldots, y_\gamma^1, y_\gamma^2, \ldots, y_\gamma^m) \in R^{nm\gamma}$ will be denoted by (y_j^i). $\sum_{ij} y_j^i$ denotes the summation $\sum_{i=1}^m \sum_{j=1}^\gamma y_j^i$.

A. Regions, Public Goods, and Private Goods

There are $\gamma \geq 2$ non-overlapping geographical regions. $g_j \in R_+^{\sigma_j}$ denotes the vector of local public goods provided in region j.

There are n private goods. Some of these goods, such as land, may be associated with a particular region. The total endowment of private goods in the economy is denoted by the n-vector w.

B. Production

The government of region j produces public goods for its residents using the production set $Y_j \subset R_+^{\sigma_j} \times R_-^n$, where $(g_j, x_j) \epsilon Y_j$ means the vector of public goods g_j is produced from the vector of private good inputs x_j . The aggregate public goods production set $Y_G \equiv \underset{1}{\overset{\gamma}{\times}} Y_j$. We write $(g,x) \epsilon Y_G$ when $g = (g_1, \ldots, g_\gamma)$, $x = (x_1, \ldots, x_\gamma)$, and $(g_j, x_j) \epsilon Y_j \ \forall j$.

For simplicity of exposition we treat only an aggregated private production sector. $Y \subset R^n$ summarizes the production possibilities for the private sector, with outputs measured positively and inputs negatively. No public goods are used as inputs or produced as outputs by the private sector.

(B.1) Y and $Y_j \ \forall j$ are closed and convex.

(B.2) $Y \cap R_+^n = \{0\}$ and $Y_j \cap R_+^{\sigma_j + n} = \{0\} \ \forall j$.

(B.3) If $(g_j, x_j) \epsilon Y_j$ and $\tilde{x}_j < x_j$, $\exists \tilde{g}_j > g_j \ni (\tilde{g}_j, \tilde{x}_j) \epsilon Y_j$.[3]

C. Consumption

There are a continuum of consumers of m different types. All consumers of the same type have the same tastes and initial endowments, and receive the same share of total profits. $r^i \epsilon (0,1]$ denotes the fraction of all consumers who are of type i , with $\sum_1^m r^i = 1$. The preferences of a consumer of type i $(i = 1, \ldots, m)$, if he resides

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[3] (B.3) is a very strong assumption. We will assume later that preferences are monotonic in public goods, but for technical reasons we cannot assume monotonicity in private goods over all of R_+^n . (B.3) essentially substitutes for the latter assumption, because it says that each private good can be used to increase the output of some public good.

in region j $(j = 1, \ldots, \gamma)$, are summarized by a utility function $u_j^i(g_j, y_j^i)$, defined on the consumption set $R_+^{\sigma_j + n}$, where y_j^i represents the private goods consumption bundle. We summarize all these bundles with the $nm\gamma$-dimensional vector $y \equiv (y_j^i)$.

Each consumer of type i has a semi-positive initial endowment n-vector w^i of private goods, with $w = \sum_1^m r^i w^i$. There are no initial endowments of public goods. Each consumer of type i receives a non-negative share θ^i of the profits of the private sector, where $\sum_1^m r^i \theta^i = 1$.

A partition of the consumers among the γ regions is denoted by a $m\gamma$-vector $r = (r_j^i)$, where $r_j^i \in [0, r^i]$ denotes the fraction of all consumers who are of type i and live in region j . The set of all possible partitions is $P \equiv \{r \in R_+^{m\gamma} | \sum_{j=1}^{\gamma} r_j^i = r^i \; \forall i = 1, \ldots, m\}$.

(C.1) u_j^i is continuous on $R_+^{\sigma_j + n}$

(C.2) u_j^i is concave on $R_+^{\sigma_j + n}$.

(C.3) u_j^i is increasing in every component of g_j .

(C.4) There exists a parameter $\bar{\zeta} > 0$ such that u_j^i is non-increasing in each private good coordinate above the level $\bar{\zeta}$.[4]

(C.5) $w^i > 0$ and $w \gg 0$.

Let $y^i \equiv (y_1^i, \ldots, y_\gamma^i)$. For fixed i and j and any given $\rho^i \in (0,1)$, define $\phi_j^i(\rho^i) \equiv \sup\{\bar{u}|\exists\, (r_1^i, \ldots, r_\gamma^i) \geq 0$ with $r_j^i > 0$ and $\sum_j r_j^i = \rho^i r^i$, and $(g, x, y^i, s) \in Y_G \times R_+^{n\gamma} \times Y \ni u_j^i(g_j, y_j^i) = \bar{u}$ if

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[4] This satiation assumption is motivated by technical convenience. Since ζ can be chosen arbitrarily large, it is not very restrictive.

$r_j^i > 0$ and $\sum_j r_j^i y_j^i - \sum_j x_j - s - w \leq 0\}$. (Since (B.2) implies the set

$\{(g,x,s) \in Y_G \times Y | s + \sum_j x_j + w \geq 0\}$ is bounded, $\phi_j^i(\rho^1)$ clearly exists.)

If all the resources in the economy are devoted to pleasing a fraction

ρ^i of the type i consumers, $\phi_j^i(\rho^1)$ is the largest common utility

level they could obtain, provided that some of them live in the given

region j . (Obviously ϕ_j^i is non-increasing in ρ^1 .)

(C.6) $\exists | \rho^1 \in (0,1)$ and $K_j^i > 0$ such that

    (a) $\displaystyle \max_{y_j^i \in R_+^n} u_j^i(0, y_j^i) > \max_{\hat{j}} \phi_{\hat{j}}^i(\rho^1)$

    (b)   If $(g_j, x_j) \in Y_j$ with $x_j < 0$, and $u_j^i(g_j, y_j^i) \leq \phi_j^i(\rho^1)$,

         $\exists | \bar{y}_j^i \leq y_j^i - K_j^i x_j \ni u_j^i(0, \bar{y}_j^i) > u_j^i(g_j, y_j^i)$.

The left-hand side maximum in (a) exists because of (C.4). (C.6a) could

be satisfied if, for example, utilities were increasing over a sufficiently

large range in some private good whose marginal utility diminished suffi-

ciently slowly. If this good was a necessary input in the production

of public goods, (b) could also be satisfied. (The closer ρ^i is chosen

to 1, the lower will be the values of ϕ_j^i , and hence the more likely

the conditions are to be satisfied.)

D. <u>Allocations and the Equilibrium Concept</u>

<u>Definition 2.1</u>. An <u>allocation</u> (r,g,x,y,s) is an element of the set

$A = P \times Y_G \times R_+^{mn\gamma} \times Y$.

     An allocation specifies a partition r , public good provisions

in all regions $g = (g_1, \ldots, g_\gamma)$, private good inputs to public goods

production $x = (x_1, \ldots, x_\gamma)$, private good consumption bundles for all consumer types in all regions $y = (y_j^i)$, and the private sector production vector s . (Implicit in this definition is the property that all consumers of the same type living in the same region receive the same private goods bundle.)

Definition 2.2. The _excess_ _demand_ _function_ $z : A \rightarrow R^n$ is defined by

$$z(r,g,x,y,s) = \sum_{ij} r_j^i y_j^i - \sum_1^\gamma x_j - s - w .$$

Definition 2.3. An allocation (r,g,x,y,s) is _feasible_ if $z(r,g,x,y,s) = 0$.

Definition 2.4. A feasible allocation (r,g,x,y,s) is _locally_ Pareto _efficient_ if there does not exist a feasible allocation $(r,\bar{g},\bar{x},\bar{y},\bar{s})$ with $u_j^i(\bar{g}_j, \bar{y}_j^i) > u_j^i(g_j, y_j^i)$ $\forall ij \ni r_j^i > 0$.

This efficiency notion was first introduced by Ellickson [4], who called it Pareto efficiency with respect to a partition. As mentioned in the Introduction, the important feature is that only alternative feasible allocations involving the given partition r are considered.

Definition 2.5. A _proportional_ _wealth_ _tax_ _local_ _public_ _competitive_ _equilibrium_ is a price vector $p^* >> 0$, a feasible allocation $(r^*, g^*, x^*, y^*, s^*)$, and a vector of tax rates $(t_1^*, \ldots, t_\gamma^*) \in \underset{1}{\overset{\gamma}{\times}}[0,1]$ such that

    (a) (private sector profit maximization) $p^*s^* \geq p^*s$ $\forall s \in Y$;

    (b) (utility maximization subject to after-tax budget constraint)
if $r_j^{*i} > 0$, y_j^{*i} maximizes $u_j^i(g_j^*, y_j^i)$ on
$\{y_j^i \in R_+^n | p^* y_j^i \leq (p^* w^i + \theta^i p^* s^*)(1 - t_j^*)\}$;

(c) (mobility equilibrium) if $r_j^{*i} > 0$, $u_j^i(g_j^*, y_j^{*i}) \geq u_j^i(g_j^*, \tilde{y}_j^i)$

$\forall j = 1, \ldots, \gamma$, where $\tilde{y}_j^i$ maximizes $u_j^i(g_j^*, y_j^i)$ on

$\{y_j^i \in R_+^n | p^*y_j^i \leq (p^*w^i + \theta^i p^*s^*)(1 - t_j^*)\}$;

(d) (weak democracy) if $\sum_{\ell=1}^{m} r_j^{*\ell} > 0$, $(g_j, x_j) \in Y_j$ and

$u_j^i(g_j, y_j^i) > u_j^i(g_j^*, y_j^{*i})$ $\forall i \ni r_j^{*i} > 0$, then

$-p^*x_j + p^* \sum_{i \ni} r_j^{*i}y_j^i > p^*\sum_i r_j^{*i}w^i + p^*s^*\sum_i r_j^{*i}\theta^i$;
$\qquad\qquad r_j^{*i}>0$

(e) (balanced budgets) $t_j^*p^*\sum_i r_j^{*i}(w^i + \theta^i s^*) = -p^*x_j^*$ and $t_j^* = 0$

if $\sum_i r_j^{*i} = 0$.

Condition (c) requires that the equilibrium level of utility of

a consumer of type i who lives in region $\hat{j}$ is not less than the level

he could obtain by moving to another region and maximizing his utility,

given the public good provisions and the tax rate prevailing in that region.

Condition (d) states that no nonempty region can afford an alternative

public sector proposal that would be unanimously preferred by its residents.

The right-hand side of the inequality in (d) represents the total before

tax income or wealth of region j , and the left-hand side represents

the total cost of the preferred alternative bundles. After subtracting

the total expenditure on the preferred bundles of private goods from before

tax wealth, there would not be enough wealth left to tax away (using any

tax scheme, including ones involving subsidies to some consumers) to cover the

cost $-p^*x_j$ of the alternative bundle of public goods. ($-p^*x_j$ is the cost of

the private good inputs purchased by the local government in order to produce g_j .)

Condition (e) requires that each local government set its tax rate

so that the revenue raised (i.e., the tax rate times the region's before

tax wealth) equals the cost of producing the public goods. If region j

is empty, then its before tax wealth is 0 and the balanced budget condition

implies $x_j^* = 0$, and hence the region provides no public goods. However,

this still leaves t_j^* indeterminate, and hence we explicitly require a

zero tax rate if a region is empty.

The definition of an equilibrium allocation allows for a trivial

level of non-uniqueness in the sense that if $r_j^{*i} = 0$, then y_j^{*i} can

be chosen arbitrarily.

Theorem 2.1. If $(r^*, g^*, x^*, y^*, s^*)$ is an equilibrium allocation as

described in Definition 2.5, then it is locally Pareto efficient.

Proof. Suppose not. Then $\exists$ a feasible allocation (r^*, g, x, y, s) with

$u_j^i(g_j, y_j^i) > u_j^i(g_j^*, y_j^{*i})$ $\forall ij \ni r_j^{*i} > 0$. Then (d) of Definition 2.5 implies

$-p^* \sum_{\substack{j \ni \\ \sum_\ell r_j^{*\ell} > 0}} x_j + p^* \sum_{\substack{ij \ni \\ r_j^{*i} > 0}} r_j^{*i} y_j^i > p^* \sum_{ij} r_j^{*i} w^i + p^* s^* \sum_{ij} r_j^{*i} \theta^i$. (We have taken

the summations on the right-hand side over all ij , since terms involving

$r_j^{*i} = 0$ obviously do not affect the sum.) Since $x_j \leq 0$ $\forall j$ and $p^* \gg 0$,

we can let the first summation on the left-hand side of the inequality

run over all j and maintain the inequality. The second summation can

trivially be extended to run over all ij . Incorporating these changes,

using $\sum_{ij} r_j^{*i} w^i = w$ and $\sum_{ij} r_j^{*i} \theta^i = 1$ on the right-hand side, and rearrang-

ing yields $p^*(\sum_{ij} r_j^{*i} y_j^i - \sum_j x_j - w) > p^* s^*$. Since (r^*, g, x, y, s) is feasible,

the term in parentheses equals s , and hence we obtain $p^* s > p^* s^*$,

contradicting condition (a) of equilibrium.

$\hspace{10cm}$ Q.E.D.

Since our equilibrium allocations are efficient in the restricted sense of Theorem 2.1, perhaps they can be found by maximizing some sort of social welfare function. This is the motivation for the following definition. Let $S_{m\gamma} \equiv \{(k_j^i) \varepsilon R_+^{m\gamma} \mid \sum_{ij} k_j^i = 1\}$.

Definition 2.6. The <u>social</u> <u>welfare</u> <u>function</u> $U : S_{m\gamma} \times Y_G \times R_+^{mn\gamma} \times Y \rightarrow R$ is defined by $U(k,a) \equiv \sum_{ij} k_j^i u_j^i(g_j, y_j^i)$, where $a \equiv (g,x,y,s)$.

Even though U is independent of x and s , we use the larger domain for notational convenience.

3. Existence Proof--Heuristic Discussion

In the next section we rigorously demonstrate the existence of a proportional wealth tax local public competitive equilibrium. In this section an overview of the underlying strategy of the existence proof is provided.

The proof is a generalization of an idea of Negishi's [10]. In proving the existence of an equilibrium in a competitive economy with only private goods, Negishi confines his attention to the set of feasible allocations which maximize nonnegative linear combinations of the individual utilities, and hence are Pareto efficient. For any given vector of utility weights for this social welfare function a maximizing allocation, and a vector of shadow prices having many of the properties of an equilibrium price vector, are generated. In particular, profits will be maximized at these prices, and any bundle of goods preferred by a consumer to the one he receives in the maximizing allocation will cost more at these prices. However, for an arbitrary vector of weights, there is no guarantee that

at these prices, the value of the consumer's assigned bundle will be less than or equal to his endowment plus profit income.

In order to find a vector of weights such that all consumers will satisfy their budget constraints when the dual prices are used, Negishi constructs a "penalty" mapping. Given a vector of weights and a corresponding efficient allocation, the basic idea is to lower the weights of consumers whose budget constraints are violated when evaluated at the efficient allocation and corresponding dual prices, and raise the weights of those whose budget constraints are not binding. At a fixed point of such a mapping, the weights will not be adjusted, implying everyone's budget constraint is just satisfied.

Foley's [5] proof of existence in the context of a single region public goods model embodied such a welfare maximization approach, and Matsuda [8] suggested a similar approach for demonstrating existence of equilibrium in the local public goods model studied by Richter [11] and Greenberg [6], in which consumer mobility was explicitly prohibited.

In the next section we present a proof which is very much in the spirit of Negishi's original work. By Theorem 2.1, we can confine our search for an equilibrium to the set of locally efficient allocations corresponding to all the various partitions. For given vectors k (the social welfare function weights) and r (a partition), a feasible allocation which maximizes the social welfare function of Definition 2.6 will (in most cases) be locally Pareto efficient. (Unlike Negishi, our social welfare maximization process is parameterized by two vectors rather than just the vector of utility weights, because r enters the characterization of feasible allocations.) Furthermore, we can find a vector of dual prices which, together with the efficient allocation, will satisfy all the require-

ments of the equilibrium of Definiiton 2.5 except possibly two--the budget

constraint in condition (b) may be violated (where the tax rates are defined

by (e)), and the mobility requirement (condition (c)) may also not hold.

Then we construct two penalty mappings. One of these mappings is very

similar to Negishi's penalty mapping, with the welfare weights being

adjusted with reference to the after-tax budget constraints evaluated at

the dual prices. The other penalty mapping adjusts the partition in order

to fix up the mobility condition. For each type of consumer, it penalizes

a region which doesn't "measure up" in utility terms to the "best" regions

for that type by assigning the region zero consumers of that type. We

then show that at a fixed point of a correspondence involving these penalty

mappings, both the budget constraints and the mobility condition will be

satisfied. Having built all the other requirements of Definition 2.5 into

the maximization process, an equilibrium will have been determined.

4. The Existence Proof

As noted above, assumption (B.2) implies the set

$\{(g,x,s) \in Y_G \times Y \mid s + \sum_1^Y x_j + w \geq 0\}$ is bounded. Hence there exist positive

vectors $\overline{b}_g$, $\overline{b}_x$, $\overline{b}_s$ such that for all (g,x,s) in this set, $g << \overline{b}_g$,

$x >> -\overline{b}_x$ and $-\overline{b}_s << s << \overline{b}_s$. We define $\hat{Y}_G \equiv \{(g,x) \in Y_G \mid g \leq \overline{b}_g, x \geq -\overline{b}_x\}$

and $\hat{Y} \equiv \{s \in Y \mid -\overline{b}_s \leq s \leq \overline{b}_s\}$.

Let $\zeta \equiv (\overline{\zeta}, \ldots, \overline{\zeta})$, where $\overline{\zeta}$ is the satiation parameter of

assumption (C.4). (The dimensionality of ζ will be implied by its context.)

Then we define $W \equiv Y_G \times \{y \in R_+^{mnY} \mid y \leq \zeta\} \times Y$ and $\hat{W} \equiv \hat{Y}_G \times \{y \in R_+^{mnY} \mid y \leq \zeta\} \times \hat{Y}$.

If $a \in W$ and $z(r,a) \leq 0$ for some $r \in P$, then $a \in \hat{W}$.

The set $P \times \hat{W}$ is a convex, compact subset of the set of allocations

A (see Definition 2.1), and includes all allocations (r,g,x,y,s) in which $y \leq \zeta$ and $z(r,g,x,y,s) \leq 0$. We will confine our search for an equilibrium allocation to the set $P \times \hat{W}$.[5]

Given $(k,r) \varepsilon S_{m\gamma} \times P$, consider the problem

(1) $\max_{a \varepsilon \hat{W}} U(k,a)$ subject to $z(r,a) \leq 0$

where U is defined in Definition 2.6. Since $\hat{W}$ is compact and U is continuous in a (by (C.1)), (1) has a non-empty solution set for each $(k,r) \varepsilon S_{m\gamma} \times P$.

Definition 4.1. $N : S_{m\gamma} \times P \to \hat{W}$ is defined by $N(k,r) = \{a \varepsilon \hat{W} | a$ is an optimal solution to (1) for the given $(k,r)\}$.

Given $(k,r) \varepsilon S_{m\gamma} \times P$, let $\bar{a} \varepsilon N(k,r)$. Let $C_{kr}(\bar{a}) \equiv \{z \varepsilon R^n | z = z(r,a)$ for some $a \varepsilon W$ with $U(k,a) \geq U(k,\bar{a})\}$ and $D \equiv \{z \varepsilon R^n | z << 0\}$. Since $\bar{a} \varepsilon N(k,r)$, $C_{kr}(\bar{a}) \cap D = \phi$. For suppose not. Then $\exists a \varepsilon W \ni z(r,a) << 0$ (hence $a \varepsilon \hat{W}$) and $U(k,a) \geq U(k,\bar{a})$. Hence from (C.3) and (B.3) $\exists \tilde{a} \varepsilon \hat{W}$ with $z(r,\tilde{a}) \leq 0$ and $U(k,\tilde{a}) > U(k,\bar{a})$,[6] contradicting $\bar{a} \varepsilon N(k,r)$.

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[5] Even though all allocations (r,g,x,y,s) with $z(r,g,x,y,s) \leq 0$ have the property that $\sum_{ij} r_j^i y_j^i \leq s + \sum_j x_j + w << \bar{b}_s + w$, this inequality is not sufficient to bound the y_j^i because the r_j^i do not have a positive lower bound. Hence we have invoked (C.4) in order to confine our search to a compact subset of the consumption set.

[6] Since $k \varepsilon S_{m\gamma}$, $\exists ij \ni k_j^i > 0$. Let $a = (g,x,y,s)$. Form $\tilde{a}$ by changing only the components g_j and x_j of a , in the following manner. Since $z(r,a) << 0$, we can choose $\tilde{x}_j < x_j$ so that $z(r,\tilde{a}) \leq 0$ and obtain (by (B.3)) $\tilde{g}_j > g_j$. Then $\tilde{a} \varepsilon \hat{W}$, and by (C.3), $u_j^i(\tilde{g}_j, y_j^i) > u_j^i(g_j, y_j^i)$ and hence $U(k,\tilde{a}) > U(k,a) \geq U(k,\bar{a})$.

W is convex, and U is concave in a because the u_j^i are concave.

Hence $C_{kr}(\bar{a})$ is convex, and by the separating hyperplane theorem $\exists\, p \neq 0$,

with $\sum_1^n p_\ell = 1 \ni pz \geq 0 \;\; \forall z \in C_{kr}(\bar{a})$. Since D contains points with

arbitrarily large (in absolute value) negative coordinates, it follows that

$p > 0$. Furthermore, (C.3) and (B.3) imply $p \gg 0$.[7]

Summarizing, we have shown that if $\bar{a} \in N(k,r)$, $\exists\, p \gg 0$ with

$\sum_\ell p_\ell = 1$ such that $pz(r,a) \geq 0 \;\; \forall a \in W$ with $U(k,a) \geq U(k,\bar{a})$. Hence

the following correspondence is well-defined, where $\overset{o}{S}_n$ denotes the

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[7] Since $z(r,\bar{a}) \in C_{kr}(\bar{a})$, and $z(r,\bar{a}) \leq 0$ and $p > 0$, it follows that

$pz(r,\bar{a}) = 0$. Thus if $p_k > 0$ it follows that the kth coordinate of

$z(r,\bar{a})$ equals 0, i.e., the kth coordinate of $\sum_{ij} r_j^i \bar{y}_j^i - \sum_j \bar{x}_j - \bar{s} - w$

is 0. Since $w \gg 0$, it follows that if $p_k > 0$, then either

(a) $\exists\, ij \ni r_j^i > 0$ and $(\bar{y}_j^i)_k$ (the kth coordinate of $\bar{y}_j^i$) is positive

or (b) $\exists\, j \ni (\bar{x}_j)_k < 0$ or (c) $\bar{s}_k < 0$. (Of course these cases are not

mutually exclusive.) Suppose $p_\ell = 0$ for some ℓ and (a) holds for some

k with $p_k > 0$. Then by decreasing $(\bar{y}_j^i)_k$ slightly and increasing some

component of $\bar{g}_j$ by (algebraically) decreasing $(\bar{x}_j)_\ell$ (we appeal here

to (B.3)), we can obtain an allocation $\tilde{a} \in W \ni U(k,\tilde{a}) \geq U(k,\bar{a})$ (by (C.1)
and (C.3)) and $pz(r,\tilde{a}) < 0$. This contradiction rules out case (a) hold-
ing for any k for which $p_k > 0$ under the assumption that $p_\ell = 0$.
Thus suppose $p_\ell = 0$ and (b) holds for some k with $p_k > 0$. Then by

(algebraically) increasing $(\bar{x}_j)_k$ slightly and decreasing (algebraically)

$(\bar{x}_j)_\ell$ sufficiently (and hence by (B.3) increasing some component of $\bar{g}_j$)

we can (by (C.1) and (C.3)) obtain an allocation $\tilde{a} \in W$ with $U(k,\tilde{a}) \geq U(k,\bar{a})$
and $pz(r,\tilde{a}) < 0$. This contradiction rules out (b) holding for any k
with $p_k > 0$. Thus neither (a) nor (b) can hold for any k with $p_k > 0$
under the assumption that $p_\ell = 0$. Thus (c) must hold for all k with

$p_k > 0$. But then $p\bar{s} < 0$, and by changing $\bar{s}$ to 0 and leaving the

other components of $\bar{a}$ the same, we would obtain an allocation $\tilde{a} \in W$

with $U(k,\tilde{a}) = U(k,\bar{a})$ and $pz(r,\tilde{a}) < 0$. Thus the supposition that $p_\ell = 0$

has led to a contradiction in all possible cases. Hence $p \gg 0$.

interior of the (n-1)-dimensional simplex $S_n \equiv \{p \in R_+^n \mid \sum_\ell p_\ell = 1\}$.

<u>Definition 4.2.</u> $M : S_{m\gamma} \times P \to \hat{W} \times \overset{\circ}{S}_n$ is defined by

$M(k,r) = \{(\bar{a},\bar{p}) \in \hat{W} \times \overset{\circ}{S}_n \mid \bar{a} \in N(k,r)$ and $\bar{p}z(r,a) \geq 0$ $\forall a \in W \ni U(k,a)$

$\geq U(k,\bar{a})\}$.

(Note that $\bar{a} \in \hat{W}$, but that the a need only belong to W .)

As we vary (k,r) over $S_{m\gamma} \times P$, most allocations (r,a) , where
$a \in N(k,r)$, will be locally Pareto efficient, and hence M essentially
represents the locus of locally Pareto efficient allocations together with
the associated dual price vectors. We formalize these remarks in the
following lemma.

<u>Lemma 4.1.</u> If $(a,p) \in M(k,r)$ and $r_j^i > 0$ whenever $k_j^i > 0$, the allo-
cation (r,a) is locally Pareto efficient.

<u>Remark</u>. The hypothesis of Lemma 4.1 rules out the placing of any positive
weight in the social welfare function on nobody. The correspondence M
will be part of the mapping whose fixed points will correspond to equilibria.
We will show that the vectors k and r determined by such a fixed point
will satisfy the property that $r_j^i > 0$ whenever $k_j^i > 0$.

<u>Proof</u>. For (r,a) to be locally Pareto efficient, we must first show
it is feasible in the sense of Definition 2.3. Since $a \in N(k,r)$,
$z(r,a) \leq 0$. But $p \gg 0$ and $pz(r,a) \geq 0$, and hence $z(r,a) = 0$.
Suppose (r,a) is not locally Pareto efficient. Let $a = (g,x,y,s)$.
Then $\exists$ an allocation $(r,\tilde{g},\tilde{x},\tilde{y},\tilde{s})$ with $z(r,\tilde{g},\tilde{x},\tilde{y},\tilde{s}) = 0$ such that

$$(2) \qquad u_j^i(\tilde{g}_j, \tilde{y}_j^i) > u_j^i(g_j, y_j^i) \quad \forall ij \ni r_j^i > 0 .$$

If $\tilde{y} \not\leq \zeta$, form $\hat{y}$ by lowering all coordinates of $\tilde{y}$ which exceed $\bar{\zeta}$ to the level $\bar{\zeta}$. Then $z(r,\tilde{g},\tilde{x},\hat{y},\tilde{s}) \leq 0$ and hence $\hat{a} \equiv (\tilde{g},\tilde{x},\hat{y},\tilde{s}) \in \hat{W}$. By (C.4), $u_j^i(\tilde{g}_j, \hat{y}_j^i) \geq u_j^i(\tilde{g}_j, \tilde{y}_j^i)$ $\forall ij$ and hence (2) holds with $\tilde{y}_j^i$ replaced by $\hat{y}_j^i$. Then $U(k,\hat{a}) - U(k,a) = \sum_{\substack{ij \ni \\ k_j^i > 0}} k_j^i (u_j^i(\tilde{g}_j, \hat{y}_j^i) - u_j^i(g_j, y_j^i))$.

Since $r_j^i > 0$ if $k_j^i > 0$, each term of this summation is positive. Hence $U(k,\hat{a}) > U(k,a)$. Since $\hat{a} \in W$ with $z(r,\hat{a}) \leq 0$, this contradicts $a \in N(k,r)$.

$$Q.E.D.$$

__Lemma 4.2.__ M is upper semicontinuous on $S_{m\gamma} \times P$, and for each $(k,r) \in S_{m\gamma} \times P$, the image set $M(k,r)$ is convex.

__Proof.__ See the Appendix.

The next task is to construct a mapping which penalizes consumers who violate their budget constraints. Let δ be an arbitrary positive number, and let

$$(3) \qquad I \equiv \{t = (t_1, \ldots, t_\gamma) | t_j \in [0, 1+\delta] \; \forall j\} .$$

__Definition 4.3.__ The __budgetary deficits function__ $\eta : \hat{W} \times S_n \times I \to R^{m\gamma}$ where $\eta(a,p,t) = (\eta_j^i(a,p,t))$ and $a = (g,x,y,s)$, is defined by $\eta_j^i(a,p,t) = (pw^i + \theta^i ps)(1 - t_j) - py_j^i$.

Interpreting t_j as a tax rate (even though for technical reasons we allow it to exceed 1) applied to wealth (which equals endowment income plus profit income), $\eta_j^i(a,p,t)$ represents after-tax income minus private goods expenditure by a type i consumer residing in region j .

η is a continuous function defined on a compact set. Hence $\exists \ \varepsilon > 0 \ni \sum\limits_{ij} \eta_j^i(a,p,t) > -\frac{1}{\varepsilon} \ \forall (a,p,t) \ \varepsilon \ \hat{W} \times S_n \times I$. We use ε in the following definition.

Definition 4.4. The <u>budget penalty mapping</u> $K : S_{m\gamma} \times \hat{W} \times S_n \times I \rightarrow S_{m\gamma}$ is defined by $K(k,a,p,t) = \hat{k}$ where

$$\hat{k}_j^i = [\max(0, \ k_j^i + \varepsilon\eta_j^i(a,p,t))]/[\sum\limits_{ij}\max(0, \ k_j^i + \varepsilon\eta_j^i(a,p,t))] \ .$$

Let $D \equiv \sum\limits_{ij}\max(0, \ k_j^i + \varepsilon\eta_j^i(a,p,t))$ for arbitrary $(a,p,t) \ \varepsilon \ \hat{W} \times S_n \times I$. Then $D \geq \sum\limits_{ij} k_j^i + \varepsilon\sum\limits_{ij}\eta_j^i(a,p,t) = 1 + \varepsilon\sum\limits_{ij}\eta_j^i(a,p,t)$. Since $\varepsilon\sum\limits_{ij}\eta_j^i(a,p,t) > -1$ from the definition of ε, we have $D > 0$. Hence the mapping K is well-defined, and obviously maps into $S_{m\gamma}$. Since η is continuous, so is K.

The numerator of this mapping raises a consumer's weight in the social welfare function if his after-tax budget constraint is not binding for the given vector (a,p,t), and lowers his weight if his budget constraint is violated. (Note that violation of the budget constraint can result even if private expenditure is zero, either because the tax rate exceeds 1 or because ps is sufficiently negative.)

The next task is to define a regional penalty mapping, which distributes consumers among the regions so that they will have no incentive to move. The following two definitions will be used in constructing this regional penalty mapping. Recall the definitions of ρ^i and K_j^i from (C.6).

Definition 4.5. $\lambda : \overset{\circ}{S}_n \rightarrow R_+$ is defined by $\lambda(p) = \overline{\lambda} \min\limits_i(r^i p w^i)$ where $\overline{\lambda} > 0$ is any constant satisfying $\overline{\lambda} \leq \min(1/[\max\limits_{ij} K_j^i], \ \min\limits_i[(1-\rho^i)r^i]/\gamma)$.

Since $w^i > 0 \ \forall i$, $\lambda(p) > 0 \ \forall p \ \varepsilon \ \overset{\circ}{S}_n$.

<u>Definition 4.6</u>. $v : P \times \hat{W} \times \overset{\circ}{S}_n \to R^{mY}$, where $v(r,a,p) = (v_j^i(r,a,p))$ and $a = (g,x,y,s)$, is defined by $v_j^i(r,a,p) = u_j^i(g_j, \bar{y}_j^i)$ where $\bar{y}_j^i$ is an optimal solution to the problem:

(4)
$$\max_{\hat{y}_j^i \geq 0} u_j^i(g_j, \hat{y}_j^i)$$

$$\text{subject to } p\hat{y}_j^i \leq \max\left[0, (pw^i + \theta^i ps)\left[1 + \frac{px_j}{\max(p\sum_\ell r_j^\ell(w^\ell + \theta^\ell s), \lambda(p))}\right]\right] .$$

Suppose that $p\sum_\ell r_j^\ell(w^\ell + \theta^\ell s) \geq \lambda(p)$ and $-px_j/[p\sum_\ell r_j^\ell(w^\ell + \theta^\ell s)] \leq 1$. Then $v_j^i(r,a,p)$ is the maximum utility level a type i consumer could achieve by moving to region j , taking the region's public goods vector g_j and tax rate $-px_j/[p\sum_\ell r_j^\ell(w^\ell + \theta^\ell s)]$ as given. This tax rate would balance the budget in region j for the given allocation (r,a) . The max operator in the denominator of the tax rate rules out undefined tax rates stemming, for example, from empty regions. (The specific choice of the function λ is motivated by technical reasons.) The other max operator insures "after-tax income" is nonnegative.

For each (r,a,p) in the domain of v , the constraint set in (4) is compact (since $p \gg 0$), and hence an optimal solution always exists. Furthermore $p \gg 0$ rules out the "exceptional case" and hence v is continuous on its domain.

<u>Definition 4.7</u>. The <u>regional penalty mapping</u> $R : P \times \hat{W} \times \overset{\circ}{S}_n \to P$, where $a = (g,x,y,s)$, is defined by $R(r,a,p) = \{\hat{r} \in P | \forall ij, \hat{r}_j^i = 0$ if

$\max(v_j^i(r,a,p), u_j^i(g_j, y_j^i)) < \max_{\hat{j} \in \{1,\ldots,\gamma\}} [\max(v_{\hat{j}}^i(r,a,p), u_{\hat{j}}^i(g_{\hat{j}}, y_{\hat{j}}^i))]\}$.

This mapping is well-defined because for some value of j , the left-hand side of the inequality equals the right-hand side, thus insuring that $\hat{r}^i_j$ is not restricted to zero for all j . To make transparent the intuition underlying this mapping, suppose for the moment that $v^i_j \geq u^i_j$ $\forall j$. Then the mapping penalizes a region j by assigning it no type i consumers if the region does not measure up, in the budget constrained utility sense underlying the definition of the v^i_j , to some of the other regions (i.e., $v^i_j < \max_j v^i_j$) for this consumer type. Thus it should be clear that this mapping is designed to insure satisfaction of the mobility condition of equilibrium. The presence of the additional terms u^i_j will insure, as we shall see later, that some type i consumers are assigned to region j if the social welfare function assigns a positive weight to type i consumers residing in region j .

Lemma 4.3. R is upper semicontinuous on $P \times \hat{W} \times \overset{\circ}{S}_n$, and for each $(r,a,p) \in P \times \hat{W} \times \overset{\circ}{S}_n$, $R(r,a,p)$ is a convex set.

Proof. Convexity is obvious, and upper semicontinuity follows straightforwardly from the continuity of v and the u^i_j .

In order to insure continuity of v , we restricted our attention to positive price vectors in Definition 4.6. Hence R , which inherits its domain from v , is not defined on a compact set. However, we need a correspondence defined on a compact set for our fixed point arguments. Hence we extend R to an upper semicontinuous correspondence on the closure of its domain. Let $G_R \equiv \{(r,a,p,\hat{r}) \in P \times \hat{W} \times \overset{\circ}{S}_n \times P | \hat{r} \in R(r,a,p)\}$, the graph of R . Let $\overline{G}_R$ denote the closure of G_R in $P \times \hat{W} \times S_n \times P$.

We then extend R to an upper semicontinuous correspondence $\overset{\curvearrowright}{R}$ on the domain $P \times \hat{W} \times S_n$ as follows.

<u>Definition 4.8.</u> $\tilde{R} : P \times \hat{W} \times S_n \to P$ is defined by

$$\tilde{R}(r,a,p) = \{\hat{r} \ \epsilon \ P \,|\, (r,a,p,\hat{r}) \ \epsilon \ \overline{G}_R\} \ . $$

Of course on the set $P \times \hat{W} \times \overset{\circ}{S}_n$, $\tilde{R}$ agrees with R .

We have now defined 3 mappings-- M , K , and $\overset{\curvearrowright}{R}$ --which will be components of our fixed point mapping. The final component mapping is the content of the next definition. Recall that $I \equiv \overset{\gamma}{\underset{1}{X}}[0, \ 1+\delta]$.

<u>Definition 4.9.</u> The <u>tax mapping</u> $T : P \times \hat{W} \times S_n \to I$, where $T(r,a,p) = (T_j(r,a,p))$ and $a = (g,x,y,s)$, is defined by

(5) $\qquad T_j(r,a,p) = \min\left[1+\delta, \ \dfrac{-px_j}{E_j(r,a,p)}\right]$ if $E_j(r,a,p) \equiv \sum_\ell r_j^\ell (pw^\ell + \theta^\ell \max(0,ps)) >$

(6) $\qquad T_j(r,a,p) = 1+\delta$ if $-px_j > 0$ and $E_j(r,a,p) = 0$

(7) $\qquad T_j(r,a,p) = [0, \ 1+\delta]$ if $px_j = 0$ and $E_j(r,a,p) = 0$

Suppose profits ps are nonnegative. Then $E_j(r,a,p)$ is region j's before tax income, and if this income is positive and region j's expenditure on public goods does not exceed it, then $T_j(r,a,p)$ equals the proportional wealth tax rate which will balance the region's budget. Technical reasons motivate the other complications in the definition.

<u>Lemma 4.4.</u> T is upper semicontinuous, and for each $(r,a,p) \ \epsilon \ P \times \hat{W} \times S_n$, $T_j(r,a,p)$ is a convex set.

<u>Proof</u>. Convexity is obvious. T_j is obviously upper semicontinuous at any point (r,a,p) with $E_j(r,a,p) > 0$. Thus suppose $E_j(r,a,p) = 0$ and $-px_j > 0$. Let $(r^v, a^v, p^v) \to (r,a,p)$ (where $a^v = (g^v, x^v, y^v, s^v)$), $t_j^v \in T_j(r^v, a^v, p^v)$ $\forall v$, and $t_j^v \to t_j$. According to (6), we must show $t_j = 1+\delta$. Clearly $\exists v_0 \ni \forall v \geq v_0$, $-p^v x_j^v > 0$ and either $E_j(r^v, a^v, p^v) = 0$ or $E_j(r^v, a^v, p^v) > 0$ and $(-p^v x_j^v / E_j(r^v, a^v, p^v)) > 1+\delta$. If $E_j(r^v, a^v, p^v) = 0$, then by (6), $t_j^v = 1+\delta$. If $E_j(r^v, a^v, p^v) > 0$, then by (5), $t_j^v = \min(1+\delta, [-p^v x_j^v / E_j(r^v, a^v, p^v)]) = 1+\delta$. Thus $\forall v \geq v_0$, $t_j^v = 1+\delta$, and hence $t_j = 1+\delta$. Thus T_j is upper semicontinuous at all points (r,a,p) with $E_j(r,a,p) = 0$ and $-px_j > 0$. Finally it is obvious that T_j is upper semicontinuous at the remaining points (r,a,p) where $px_j = 0$ and $E_j(r,a,p) = 0$. Thus T_j is upper semicontinuous everywhere, and hence so is T.

Q.E.D.

Definitions 4.2, 4.4, 4.8, and 4.9 define 4 upper semicontinuous mappings M, K, $\tilde{R}$, and T. M, K, and T have convex image sets at all points of their respective domains, and $\tilde{R}$ has convex image sets (at least) at all points in the relative interior of its domain. Hence the mapping $M \times K \times \tilde{R} \times T : S_{m\gamma} \times P \times \hat{W} \times S_n \times I \to S_{m\gamma} \times P \times \hat{W} \times S_n \times I$ is upper semicontinuous, and has convex image sets on the relative interior of its domain. Thus by a slight extension of Kakutani's Fixed Point Theorem,[8] $\exists (k^*, r^*, a^*, p^*, t^*) \in S_{m\gamma} \times P \times \hat{W} \times S_n \times I$ such that

$$(8) \qquad (a^*, p^*) \in M(k^*, r^*)$$

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[8] The only change from the standard version that we are exploiting is that the image sets need not be convex at boundary points of the domain. See [9] for a proof.

(9) $k^* \in K(k^*, a^*, p^*, t^*)$

(10) $r^* \in \tilde{R}(r^*, a^*, p^*)$

(11) $t^* \in T(r^*, a^*, p^*)$

where $a^* = (g^*, x^*, y^*, s^*)$.

The price vector p^* , the allocation (r^*, a^*) , and a vector of tax
rates τ^* , where $\tau_j^* = t_j^*$ if region j is nonempty and $\tau_j^* = 0$ if region j
is empty in the partition r^* , correspond to an equilibrium (see Theorem 4.1 below).
Essentially (8) insures (r^*, a^*) is feasible and that the profit maximization
condition of Definition 2.5 is satisfied. (8) also does the lion's share of the
work in guaranteeing the weak democracy condition is satisfied. (8) and
(9) insure satisfaction of the utility maximization condition, (10) bears
most of the burden in guaranteeing the mobility condition, and (11) aims
primarily at balancing the regional budgets. However, there are some
subtle interactions among the mappings. Thus before stating and proving
the major theorem, we will prove a series of lemmas.

We first derive some implications which stem primarily from
property (8).

Lemma 4.5.

(12) $z(r^*, a^*) = 0$

(13) $p^* s^* \geq p^* s \quad \forall s \in Y$

(14)   If $r_j^{*i} > 0$, $u_j^i(g_j^*, y_j^i) > u_j^i(g_j^*, y_j^{*i}) \implies p^* y_j^i > p^* y_j^{*i}$

(15)   If $\sum_{\ell} r_j^{*\ell} > 0$, $(\hat{g}_j, \hat{y}_j) \in Y_j$, $u_j^i(\hat{g}_j, \hat{y}_j^i) > u_j^i(g_j^*, y_j^{*i})$ $\forall i \ni r_j^{*i} > 0$,

and $\forall i$ $r_j^{*i} > 0$ whenever $k_j^{*i} > 0$, then

$$-p^*\hat{x}_j + p^* \sum_{\substack{i \ni \\ r_j^{*i} > 0}} r_j^{*i}\hat{y}_j^i \geq -p^*x_j^* + p^* \sum_{\substack{i \ni \\ r_j^{*i} > 0}} r_j^{*i}y_j^{*i}$$

(16) $\sum_{ij} r_j^{*i}\eta_j^i(a^*, p^*, t^*) \geq 0$

(17)   If $k_j^{*i} = 0$ and $r_j^{*i} > 0$, then $y_j^{*i} = 0$

(18)   If $\sum_i k_j^{*i} = 0$, then $x_j^* = 0$.

Proof. Since $(a^*, p^*) \in M(k^*, r^*)$ it follows from Definition 4.2 that

(19) $p^*z(r^*, \hat{a}) \geq 0$ $\forall \hat{a} \in W \equiv Y_G \times \{y \in R_+^{mn\gamma} | y \leq \zeta\} \times Y$ such that

$$U(k^*, \hat{a}) \geq U(k^*, a^*) .$$

Since $z(r^*, a^*) \leq 0$, $p^* >> 0$, and $\hat{a} = a^*$ satisfies (19), we have

(20) $p^*z(r^*, a^*) = 0$

and hence $z(r^*, a^*) = 0$, verifying (12). Combining (19) and (20) yields

(21) $p^*z(r^*, \hat{a}) \geq p^*z(r^*, a^*)$ $\forall \hat{a} \in W \ni U(k^*, \hat{a}) \geq U(k^*, a^*) .$

Let s be any element of Y . Then $\hat{a} = (g^*, x^*, y^*, s) \in W$ and
it follows from (21) that $-p^*s \geq -p^*s^*$, verifying (13).

Next suppose $\hat{a} = (g^*, x^*, \hat{y}, s^*)$ where $\hat{y}$ is identical to y^*
except possibly in the component vector $\hat{y}_j^i$ corresponding to some fixed

pair ij for which $r_j^{*i} > 0$. If $u_j^i(g_j^*, \hat{y}_j^i) \geq u_j^i(g_j^*, y_j^{*i})$, then

$U(k^*, \hat{a}) \geq U(k^*, a^*)$. If $\hat{y}_j^i \leq \zeta$, then $\hat{a} \, \epsilon \, W$ and (21) implies

$p^*\hat{y}_j^i \geq p^*y_j^{*i}$. Suppose $u_j^i(g_j^*, \hat{y}_j^i) > u_j^i(g_j^*, y_j^{*i})$ and $p^*\hat{y}_j^i = p^*y_j^{*i}$.

Then by reducing some positive component of $\hat{y}_j^i$ a little (since $y_j^{*i} \geq 0$,

clearly $\hat{y}_j^i > 0$), we could obtain a vector $\tilde{y}_j^i \leq \zeta$ for which

$u_j^i(g_j^*, \tilde{y}_j^i) \geq u_j^i(g_j^*, y_j^{*i})$ (by continuity of u_j^i) and $p^*\tilde{y}_j^i < p^*y_j^{*i}$ (since

$p^* >> 0$), leading to a contradiction of (21). Thus $p^*\hat{y}_j^i > p^*y_j^{*i}$ if

$u_j^i(g_j^*, \hat{y}_j^i) > u_j^i(g_j^*, y_j^{*i})$ and $\hat{y}_j^i \leq \zeta$. If $\hat{y}_j^i \not\leq \zeta$, then by (C.4) $\exists$

a vector $y_j^i < \hat{y}_j^i$ with $y_j^i \leq \zeta$ and $u_j^i(g_j^*, y_j^i) \geq u_j^i(g_j^*, \hat{y}_j^i)$. Thus

$u_j^i(g_j^*, y_j^i) > u_j^i(g_j^*, y_j^{*i})$ and since $y_j^i \leq \zeta$, $p^*y_j^i > p^*y_j^{*i}$. Since

$y_j^i < \hat{y}_j^i$, $p^*\hat{y}_j^i > p^*y_j^i$, and hence $p^*\hat{y}_j^i > p^*y_j^{*i}$. Thus $u_j^i(g_j^*, \hat{y}_j^i) > u_j^i(g_j^*, y_j^{*i})$

implies $p^*\hat{y}_j^i > p^*y_j^{*i}$ even if $\hat{y}_j^i \not\leq \zeta$, verifying (14).

Next suppose $\sum_\ell r_j^{*\ell} > 0$ for some fixed j and define $\hat{a} = (\hat{g}, \hat{x}, \hat{y}, s^*)$,

where $(\hat{g}, \hat{x})$ is identical to (g^*, x^*) except possibly for the component

vectors $\hat{g}_j$ and $\hat{x}_j$ (where $(\hat{g}_j, \hat{x}_j) \, \epsilon \, Y_j$), and $\hat{y} \geq 0$ is identical

to y^* , except possibly in the component vectors $\hat{y}_j^i$ for $i \ni r_j^{*i} > 0$.

Then $U(k^*, \hat{a}) - U(k^*, a^*) = \sum_{\substack{i \ni \\ k_j^{*i} > 0}} k_j^{*i}(u_j^i(\hat{g}_j, \hat{y}_j^i) - u_j^i(g_j^*, y_j^{*i}))$. Suppose

the hypotheses of (15) hold. Then either this summation is vacuous (i.e.,

$k_j^{*i} = 0 \,\forall i$) or each of its terms is positive. (The latter assertion

follows from the fact that if $k_j^{*i} > 0$, then $r_j^{*i} > 0$ and hence

$u_j^i(\hat{g}_j, \hat{y}_j^i) > u_j^i(g_j^*, y_j^{*i})$.) Thus $U(k^*, \hat{a}) \geq U(k^*, a^*)$ and if $\hat{y} \leq \zeta$,

(21) directly implies the conclusion of (15). The argument can easily be

extended to allow for $\hat{y} \not\leq \zeta$, as we did in proving (14), and hence (15)

is verified.

It follows from (13) and $0 \, \epsilon \, Y$ that

(22) $p^*s^* \geq 0$.

Hence $E_j(r^*, a^*, p^*)$ (see Definition 4.9) equals before-tax wealth in region j , and we have

$$(23) \qquad E_j^* \equiv E_j(r^*, a^*, p^*) = \sum_{\ell} r_j^{*\ell}(p^*w^{\ell} + \theta^{\ell}p^*s^*) \geq 0 \quad \forall j .$$

If $E_j^* > 0$, then since $t^* \in T(r^*, a^*, p^*)$, (5) implies

$$(24) \qquad t_j^*E_j^* \leq -p^*x_j^* .$$

If $E_j^* = 0$, then since $x_j^* \leq 0$, (24) also holds. Thus (24) holds $\forall j$, implying

$$(25) \qquad \sum_j t_j^*E_j^* \leq -p^*\sum_j x_j^* .$$

Using Definition 4.3, $\sum_{ij} r_j^{*i}n_j^i(a^*, p^*, t^*) = \sum_{ij} r_j^{*i}(p^*w^i + \theta^i p^*s^*)(1 - t_j^*) -$

$p^* \sum_{ij} r_j^{*i}y_j^{*i} = p^*w + p^*s^* - \sum_j t_j^*E_j^* - p^* \sum_{ij} r_j^{*i}y_j^{*i} \geq p^*w + p^*s^* + p^*\sum_j x_j^* - p^*\sum_{ij} r_j^{*i}y_j^{*i}$,

the inequality following from (25). But the right-hand side of the inequality equals $-p^*z(r^*, a^*)$, which equals 0 by (12). Thus (16) is verified.

With respect to (17), suppose $k_j^{*i} = 0$, $r_j^{*i} > 0$ and $y_j^{*i} > 0$.
$\exists \hat{j}$ (possibly $\hat{j} = j$) $\ni \sum_{\ell} k_{\hat{j}}^{*\ell} > 0$. Reduce y_j^{*i} to 0 (which doesn't alter the value of U since $k_j^{*i} = 0$) and let $x_{\hat{j}} = x_{\hat{j}}^* - r_j^{*i}y_j^{*i}$. This oper-
ation leaves the excess demand vector $z(r^*, a^*)$ unchanged. But by (B.3), as a result we obtain a vector $g_{\hat{j}} > g_{\hat{j}}^*$, which raises the value of U (by (C.3) and the fact that $\sum_{\ell} k_{\hat{j}}^{*\ell} > 0$), contradicting the maximality of a^* . Thus (17) is verified.

Finally, with respect to (18), suppose $\sum_i k^{*i}_j = 0$ and $x^*_j < 0$.

$\exists \; \hat{j} \neq j \ni \sum_i k^{*i}_{\hat{j}} > 0$. Then by reducing x^*_j to 0 (and consequently reducing

g^*_j to 0, which does not alter the value of U since $\sum_i k^{*i}_j = 0$) and

algebraically decreasing $x^*_{\hat{j}}$ so as to maintain a nonpositive excess

demand vector (and hence by (B.3) increasing $g^*_{\hat{j}}$), we can increase the

value of U (by (C.3) and $\sum_i k^{*i}_{\hat{j}} > 0$), thus contradicting the maximality

of a* . Thus (18) is verified.

Q.E.D.

Lemma 4.6. If $r^{*i}_j > 0$, $(p^*w^i + \theta^i p^*s^*)(1 - t^*_j) - p^*y^{*i}_j = 0$.

Proof. Since $k^* \varepsilon K(k^*, a^*, p^*, t^*)$, it follows from Definition 4.4

that

$$(26) \qquad D^*k^{*i}_j = \max(0, \; k^{*i}_j + \varepsilon \eta^{*i}_j)$$

where $D^* \equiv \sum_{ij} \max(0, \; k^{*i}_j + \varepsilon \eta^{*i}_j) > 0$ and $\eta^{*i}_j \equiv \eta^i_j(a^*, p^*, t^*)$. It follows

directly from (26) that:

$$(27) \qquad \forall ij \ni k^{*i}_j > 0 \text{ , all the corresponding } \eta^{*i}_j \text{ have the same sign;}$$

$$(28) \qquad \text{if } k^{*i}_j = 0 \text{ , then } \eta^{*i}_j \leq 0 \text{ .}$$

Furthermore, we can refine (28) so that

$$(29) \qquad \text{if } k^{*i}_j = 0 \text{ and } r^{*i}_j > 0 \text{ , then } \eta^{*i}_j = 0 \text{ .}$$

To establish (29), suppose for some fixed pair $\hat{i}\hat{j}$, $k^{*\hat{i}}_j = 0$, $r^{*\hat{i}}_j > 0$,

and $\eta^{*\hat{i}}_j \neq 0$. Then by (28), $\eta^{*\hat{i}}_j < 0$. By (17), $y^{*\hat{i}}_j = 0$. Since

$n_{\hat{j}}^{*\hat{i}} < 0$ and $y_{\hat{j}}^{*\hat{i}} = 0$, Definition 4.3 implies $t_{\hat{j}}^* > 1$. Hence

$t_{\hat{j}}^* \in T_j(r^*, a^*, p^*)$ and (5) imply $x_{\hat{j}}^* < 0$. ($E_{\hat{j}}^* > 0$ because $r_{\hat{j}}^{*\hat{i}} > 0$.)

But $x_{\hat{j}}^* < 0$ implies, by (18), that $\sum_i k_{\hat{j}}^{*i} > 0$. Hence $\exists \, \bar{i} \ni k_{\hat{j}}^{*\bar{i}} > 0$.

But since $t_{\hat{j}}^* > 1$, $n_{\hat{j}}^{*\bar{i}} < 0$. Hence by (27), $\displaystyle\sum_{\substack{ij\ni \\ k_j^{*i}>0}} r_j^{*i} n_j^{*i} \le 0$. Then

by (16), $\displaystyle 0 \le \sum_{ij} r_j^{*i} n_j^{*i} = \sum_{\substack{ij\ni \\ k_j^{*i}>0 \\ r_j^{*i}>0}} r_j^{*i} n_j^{*i} + \sum_{\substack{ij\ni \\ k_j^{*i}=0 \\ r_j^{*i}>0}} r_j^{*i} n_j^{*i}$. The first summation

is nonpositive, and hence the second summation is nonnegative. But using

(28), every term in this second summation is nonpositive, and by assump-

tion $r_{\hat{j}}^{*\hat{i}} n_{\hat{j}}^{*\hat{i}} < 0$. This contradiction establishes (29).[9]

Given (29), if $k^* r^* = 0$, the proof is complete, since if $r_j^{*i} > 0$,

then $k_j^{*i} = 0$ and hence $n_j^{*i} = 0$. Thus suppose $k^* r^* > 0$. We first

need to show that (16) holds with equality.

$$0 \le \sum_{ij} r_j^{*i} n_j^{*i} = \sum_{\substack{ij\ni \\ k_j^{*i}>0 \\ r_j^{*i}>0}} r_j^{*i} n_j^{*i} + \sum_{\substack{ij\ni \\ k_j^{*i}=0 \\ r_j^{*i}>0}} r_j^{*i} n_j^{*i} .$$

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[9] Since the logic underlying the verification of (29) is somewhat intricate, it may be useful to review what has been done. The key is to show there exists some consumer type in region $\hat{j}$ which receives positive weight in the social welfare function (SWF) and violates its budget constraint. (For then (27) and the weak form of the Walras law (16) can be used straightforwardly to establish a contradiction.) Essentially we assumed there were consumers in region $\hat{j}$ who receive no weight in the SWF who violate their budget constraints. Since they receive no weight, they are assigned 0 private goods, and hence violation of their budget constraints must stem from a tax rate larger than 1. But a tax rate larger than 1 implies public goods are being provided in their region, and hence some consumer type in the region must be receiving positive weight in the SWF (even though none of this type may be living in the region). But then this type would be violating its budget constraint, since the tax rate exceeds 1.

Either the last summation is vacuous, or (29) implies it is 0, and hence

$$(30) \qquad \sum_{\substack{ij \ni \\ k_j^{*i}>0 \\ r_j^{*i}>0}} r_j^{*i} \eta_j^{*i} \geq 0 .$$

This summation is not vacuous, since we are assuming $k^*r^* > 0$. (27) and (30) imply

$$(31) \qquad \eta_j^{*i} \geq 0 \quad \forall ij \ni k_j^{*i} > 0 .$$

If $\eta_j^{*i} \geq 0$, then Definition 4.3 implies $t_j^* \leq 1$. Thus (31) implies

$$(32) \qquad t_j^* \leq 1 \quad \text{if} \quad \sum_\ell k_j^{*\ell} > 0 .$$

We also claim

$$(33) \qquad t_j^* \leq 1 \quad \text{if} \quad \sum_\ell r_j^{*\ell} > 0 .$$

There are 2 subcases. If $\sum_\ell k_j^{*\ell} > 0$, (33) follows from (32). If $\sum_\ell k_j^{*\ell} = 0$, then $\exists i \ni k_j^{*i} = 0$ and $r_j^{*i} > 0$, and hence by (29), $\eta_j^{*i} = 0$, which in turn implies $t_j^* \leq 1$. Thus (33) is verified.

It follows from (33), $t^* \varepsilon T(r^*, a^*, p^*)$ and (5) that

$$(34) \qquad t_j^* = \frac{-p^* x_j^*}{E_j^*} \quad \forall j \ni \sum_\ell r_j^{*\ell} > 0 .$$

(Note that $\sum_\ell r_j^{*\ell} > 0 \iff E_j^* > 0$.) Then using Definition 4.3,

$$\sum_{ij} r_j^{*i} \eta_j^{*i} = p^*w + p^*s^* - \sum_j t_j^* E_j^* - p^* \sum_{ij} r_j^{*i} y_j^{*i} \ , \quad \text{which equals, since (34) holds}$$

$\forall j$ with $E_j^* > 0$ (or equivalently $\sum_\ell r_j^{*\ell} > 0$),

$$(35) \qquad p^*w + p^*s^* - \sum_{\substack{j \ni \\ \sum_\ell r_j^{*\ell} > 0}} (-p^*x_j^*) - p^* \sum_{ij} r_j^{*i} y_j^{*i} \ .$$

Furthermore,

$$(36) \qquad x_j^* = 0 \quad \text{if} \quad \sum_\ell r_j^{*\ell} = 0 \ .$$

If $\sum_\ell k_j^{*\ell} = 0$, (36) follows directly from (18). Thus suppose $\sum_\ell r_j^{*\ell} = 0$,

$\sum_\ell k_j^{*\ell} > 0$ and $x_j^* < 0$. (32) implies $t_j^* \le 1$. But $t^* \ \varepsilon \ T(r^*, a^*, p^*)$,

$x_j^* < 0$, $\sum_\ell r_j^{*\ell} = 0$ and (6) imply $t_j^* = 1 + \delta$. This contradiction establishes

(36). Thus we can add $\displaystyle \sum_{\substack{j \ni \\ \sum_\ell r_j^{*\ell} = 0}} p^*x_j^*$ to (35) without altering its value.

But then (35) becomes $-p^*z(r^*, a^*)$, which equals 0 since $z(r^*, a^*) = 0$.

Thus we have established that (16) holds with equality and hence

$$(37) \qquad 0 = \sum_{ij} r_j^{*i} \eta_j^{*i} = \sum_{\substack{ij \ni \\ k_j^{*i} > 0 \\ r_j^{*i} > 0}} r_j^{*i} \eta_j^{*i} + \sum_{\substack{ij \ni \\ k_j^{*i} = 0 \\ r_j^{*i} > 0}} r_j^{*i} \eta_j^{*i} \ .$$

(29) implies the second summation is 0, and then (31) implies $\eta_j^{*i} = 0$

if $r_j^{*i} > 0$ and $k_j^{*i} > 0$. This result, coupled with (29), proves the

lemma for the case where $k^*r^* > 0$. Since (29) proves the lemma by itself

if $k^*r^* = 0$, the proof is complete.[10]

$$\text{Q.E.D.}$$

Next we state and prove the final lemma before getting to the main theorem of the paper. Recall the definition of v (Definition 4.6).

<u>Lemma 4.7.</u>

(38)   If $r_j^{*i} > 0$, y_j^{*i} maximizes $u_j^i(g_j^*, y_j^i)$ on

$$\{y_j^i \in R_+^n \mid p^* y_j^i \leq (p^* w^i + \theta^i p^* s^*)(1 - t_j^*)\} ;$$

(39) $r_j^{*i} > 0$ whenever $k_j^{*i} > 0$;

(40)   If $r_j^{*i} > 0$, then $u_j^i(g_j^*, y_j^{*i}) \geq v_j^i(r^*, a^*, p^*)$ $\forall j$.

<u>Remark</u>. Since $p^* \gg 0$, v_j^i in (40) is defined at the point (r^*, a^*, p^*). In the proof we shall use the notation $v_j^{*i} \equiv v_j^i(r^*, a^*, p^*)$.

<u>Proof</u>. Suppose $r_j^{*i} > 0$. Then (14), Lemma 4.6, and $p^* \gg 0$ directly imply (38). Since $p^* y_j^{*i} = (p^* w^i + \theta^i p^* s^*)(1 - t_j^*)$, $t_j^* \leq 1$. Then $t_j^* \in T_j(r^*, a^*, p^*)$ and (5) imply $t_j^* = -p^* x_j^*/E_j^*$. ($E_j^* > 0$ since $r_j^{*i} > 0$.) Thus y_j^{*i} maximizes $u_j^i(g_j^*, y_j^i)$ subject to the budget constraint $p^* y_j^i \leq (p^* w^i + \theta^i p^* s^*)(1 + [p^* x_j^*/E_j^*])$. If we now add the assumption that $E_j^* \geq \lambda(p^*)$ (where λ is defined in Definition 4.5), inspection reveals that this budget constraint is identical to the one underlying the defini-tion of v_j^i (see (4)). Hence

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[10] It may be useful to review the overall logic of the proof. Starting from (29), the penalty mapping K together with the weak form of the Walras law (16), tell us that if $r_j^{*i} > 0$, then $\eta_j^{*i} \geq 0$. (This is the con-tent of (29) and (31).) The fact that these budget constraints are not violated tells us that we are on the part of the tax mapping (see (34)) where the strong Walras law holds. Then the strong Walras law allows us to strengthen the claim that $r_j^{*i} > 0 \implies \eta_j^{*i} \geq 0$ to the claim that $\eta_j^{*i} = 0$.

$$(41) \qquad u_j^i(g_j^*, y_j^{*i}) = v_j^{*i} \quad \text{if} \quad r_j^{*i} > 0 \quad \text{and} \quad E_j^* \geq \lambda(p^*) \ .$$

Since $r^* \in \tilde{R}(r^*, a^*, p^*)$, $p^* \gg 0$, and $\tilde{R}$ agrees with R on $P \times \hat{W} \times \overset{o}{S}_n$, $r^* \in R(r^*, a^*, p^*)$. Hence by (41) and Definition 4.7,

$$(42) \qquad u_j^i(g_j^*, y_j^{*i}) = \max_{\hat{j}}[\max(v_{\hat{j}}^{*i}, u_{\hat{j}}^i(g_{\hat{j}}^*, y_{\hat{j}}^{*i}))] \quad \text{if} \quad r_{\hat{j}}^{*i} > 0 \quad \text{and} \quad E_{\hat{j}}^* \geq \lambda(p^*) \ .$$

To verify (39), suppose $k_j^{*i} > 0$ but $r_j^{*i} = 0$. Then since $a^* \in M(k^*, r^*)$, $u_{\hat{j}}^i(g_{\hat{j}}^*, y_{\hat{j}}^{*i}) = \max_{y_{\hat{j}}^i \in R_+^n} u_{\hat{j}}^i(g_{\hat{j}}^*, y_{\hat{j}}^i)$ (i.e., since $r_{\hat{j}}^{*i} y_{\hat{j}}^i = 0$ $\forall y_{\hat{j}}^i \in R_+^n$ with $y_{\hat{j}}^i \leq \zeta$, $y_{\hat{j}}^{*i}$ must correspond to a maximum of $u_{\hat{j}}^i(g_{\hat{j}}^*, y_{\hat{j}}^i)$ on $y_{\hat{j}}^i \in R_+^n \ni y_{\hat{j}}^i \leq \zeta$, and hence by (C.4) to a maximum on R_+^n as well). Since $\gamma\bar{\lambda} \leq (1-\rho^i)r^i$ (see Definition 4.5) and there are only γ regions, at least $\rho^i r^i$ consumers of type i live in regions j in which $\sum_\ell r_j^{*\ell} \geq \bar{\lambda}$. Since $\sum_\ell r_j^{*\ell} \geq \bar{\lambda} \Longrightarrow E_j^* \geq \lambda(p^*)$,[11] these consumers must obtain the utility level, call it $\bar{u}^i$, equal to the right-hand side of (42). From $z(r^*,a^*) = 0$ and the definition of $\phi_j^i(\rho^i)$ (see the paragraph following (C.5)), it follows that $\bar{u}^i \leq \max_j \phi_j^i(\rho^i)$. Thus

$$\max_j \phi_j^i \geq \bar{u}^i \geq u_{\hat{j}}^i(g_{\hat{j}}^*, y_{\hat{j}}^{*i}) = \max_{y_{\hat{j}}^i \in R_+^n} u_{\hat{j}}^i(g_{\hat{j}}^*, y_{\hat{j}}^i) \geq \max_{y_{\hat{j}}^i \in R_+^n} u_{\hat{j}}^i(0, y_{\hat{j}}^i) \ .$$

The second inequality follows trivially from the definition of $\bar{u}^i$ ($\hat{j}$ is now fixed), and the last inequality from (C.3). But this result contradicts (C.6a). This contradiction establishes that $k_{\hat{j}}^{*i} > 0 \Longrightarrow r_{\hat{j}}^{*i} > 0$, and thus (39) is verified.

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[11] If $\sum_\ell r_j^{*\ell} \geq \bar{\lambda}$, then $E_j^* = \sum_\ell r_j^{*\ell}(p^*w^\ell + \theta^\ell p^*s^*) \geq \sum_\ell r_j^{*\ell} p^*w^\ell \geq (\sum_\ell r_j^{*\ell})(\min_\ell p^*w^\ell) \geq \bar{\lambda} \min_\ell p^*w^\ell \geq \bar{\lambda} \min_\ell r^\ell p^*w^\ell = \lambda(p^*)$.

Next we wish to show that the restriction that $E_j^* \geq \lambda(p^*)$ can be

dropped from (42). If $\sum_{\ell} r_j^{*\ell} \geq \bar{\lambda}$, then $E_j^* \geq \lambda(p^*)$ (see footnote 11),

and hence if $r_j^{*i} > 0$, (42) would be satisfied. Using (C.6b) it can be

shown that

(43) $x_j^* = 0$ if $0 < \sum_{\ell} r_j^{*\ell} < \bar{\lambda}$.[12]

Hence if $0 < \sum_{\ell} r_j^{*\ell} < \bar{\lambda}$ and $r_j^{*i} > 0$, $t_j^* = -p^* x_j^*/E_j^* = 0$ and the budget

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[12]Since $z(r^*, a^*) = 0$, it follows from the definition of $\phi_j^i(\rho^i)$ that
$\bar{u}^i \leq \phi_j^i(\rho^i)$ if $r_j^{*i} > 0$. Suppose $0 < \sum_{\ell} r_j^{*\ell} < \bar{\lambda}$. Hence $\exists i \ni r_j^{*i} > 0$.
Since the budget constraint underlying v_j^{*i} is no more restrictive than
the budget constraint in (38), $u_j^i(g_j^*, y_j^{*i}) \leq v_j^{*i}$. Hence from Definition
4.7, $u_j^i(g_j^*, y_j^{*i}) \leq \bar{u}^i$ ($\leq \phi_j^i(\rho^i)$). Suppose $x_j^* < 0$. Then the hypotheses
of (C.6b) are satisfied. By reducing g_j^* to 0 we can redistribute the
vector of private goods $-x_j^*$ evenly among the $\sum_{\ell} r_j^{*\ell}$ consumers of the
region without altering the value of $z(r^*, a^*)$. Let $\bar{a} \equiv (\bar{g}, \bar{x}, \bar{y}, s^*)$,
where $(\bar{g}_{\hat{j}}, \bar{x}_{\hat{j}}) = (g_{\hat{j}}^*, x_{\hat{j}}^*) \; \forall \hat{j} \neq j$, $(g_j, x_j) = 0$, $\bar{y}_{\hat{j}}^i = y_{\hat{j}}^{*i} \; \forall i$ and $\forall \hat{j} \neq j$,
and for j and $\forall i \ni r_j^{*i} = 0$. For $\forall i \ni r_j^{*i} > 0$, suppose
(I) $\bar{y}_j^i \leq y_j^{*i} - (x_j^*/\sum_{\ell} r_j^{*\ell})$. Then $\bar{a} \varepsilon \hat{W}$ and $z(r^*, \bar{a}) \leq z(r^*, a^*) = 0$.
Since $0 < \sum_{\ell} r_j^{*\ell} < \bar{\lambda} \leq 1/(\max_{ij} K_j^i)$ (the last inequality following from Defini-
tion 4.5), if we choose (II) $\bar{y}_j^i \leq y_j^{*i} - K_j^i x_j^*$ if $r_j^{*i} > 0$, then (I) will
also be satisfied. Furthermore, (C.6b) tells us that we can choose the
$\bar{y}_j^i$ in (II) so that $u_j^i(0, \bar{y}_j^i) > u_j^i(g_j^*, y_j^{*i})$. Summarizing, we have shown
that there exists $\bar{a} \varepsilon \hat{W}$ with $z(r^*, \bar{a}) \leq 0 \ni u_{\hat{j}}^i(\bar{g}_{\hat{j}}, \bar{y}_{\hat{j}}^i) = u_{\hat{j}}^i(g_{\hat{j}}^*, y_{\hat{j}}^{*i}) \; \forall i$
and $\forall \hat{j} \neq j$, and (III) $u_j^i(\bar{g}_j, \bar{y}_j^i) > u_j^i(g_j^*, y_j^{*i})$ if $r_j^{*i} > 0$. Since
we are assuming $x_j^* < 0$, it follows from (18) that $\sum_{\ell} k_j^{*\ell} > 0$. Hence
by (39) $\exists \hat{i} \ni k_j^{*\hat{i}} > 0$ and $r_j^{*\hat{i}} > 0$, with $k_j^{*i} = 0$ if $r_j^{*i} = 0$. Then
(III) implies $U(k^*, \bar{a}) > U(k^*, a^*)$, contradicting the maximality of a^* .
This contradiction establishes that $x_j^* = 0$.

constraint underlying $v_j^{*^1}$ also involves a zero tax rate, and thus by (38), $u_j^i(g_j^*, y_j^{*^1}) = v_j^{*^1}$. Thus (41) holds without the restriction that $E_j^* \geq \lambda(p^*)$, and hence (42) also holds without this restriction. Since the right-hand side of (42) is trivially greater than or equal to $v_j^{*^1} \; \forall j$, (40) is verified.

<div align="right">Q.E.D.</div>

We are now ready to prove the main theorem. (12) implies that $(r^*, g^*, x^*, y^*, s^*)$ is a feasible allocation. Lemma 4.6 implies $t_j^* \leq 1$ if $r_j^{*^1} > 0$ and hence

$$(44) \qquad t_j^* \; \epsilon \; [0,1] \quad \text{if} \quad \sum_\ell r_j^{*^\ell} > 0 \; .$$

Let

$$(45) \qquad \tau^* \equiv (\tau_1^*, \ldots, \tau_\gamma^*) \; , \quad \text{where} \quad \tau_j^* = t_j^* \quad \text{if} \quad \sum_\ell r_j^{*^\ell} > 0 \quad \text{and}$$

$$\tau_j^* = 0 \quad \text{if} \quad \sum_\ell r_j^{*^\ell} = 0 \; .^{13}$$

<u>Theorem 4.1.</u> The price vector $p^* >> 0$, the feasible allocation $(r^*, g^*, x^*, y^*, s^*)$, and the vector of tax rates $(\tau_1^*, \ldots, \tau_\gamma^*) \; \epsilon \; \underset{1}{\overset{\gamma}{X}}[0,1]$ constitute an equilibrium as described by Definition 2.5.

<u>Proof.</u> Conditions (a) and (b) have already been verified (see (13) and (38)). (In (38), $t_j^* = \tau_j^*$ since $\sum_\ell r_j^{*^\ell} > 0$.)

Since $\sum_\ell r_j^{*^\ell} > 0$ implies $t_j^* \leq 1$ (by (44)), (5) implies

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[13]If $\sum_\ell r_j^{*^\ell} = 0$, it is not necessarily true that $t_j^* = 0$.

$$(46) \qquad \tau_j^* = t_j^* = \frac{-p^* x_j^*}{E_j^*} \quad \text{if} \quad \sum_\ell r_j^{*\ell} > 0 .$$

Next note that

$$(47) \qquad x_j^* = 0 \quad \text{if} \quad 0 \leq \sum_\ell r_j^{*\ell} < \bar\lambda .$$

(If $\sum_\ell r_j^{*\ell} = 0$, then (39) implies $\sum_\ell k_j^{*\ell} = 0$, which in turn implies $x_j^* = 0$ by (18). The rest of (47) has already been established in (43).) If $\sum_\ell r_j^{*\ell} \geq \bar\lambda$, then $E_j^* \geq \lambda(p^*)$ (see footnote 11). Hence using (46) and the fact that $\tau_j^* = 0$ if $\sum_\ell r_j^{*\ell} = 0$, inspection reveals that for all ij , the budget constraints underlying v_j^{*i} (Definition 4.6) and the $\tilde y_j^i$ of condition (c) are identical. Thus

$$(48) \qquad v_j^{*i} = u_j^i(g_j^*, \tilde y_j^i) \quad \forall ij .$$

Then (40) and (48) directly imply condition (c).

Under the hypotheses of condition (d), (15) and (39) imply

$$(49) \qquad -p^* x_j + p^* \sum_{\substack{i \ni \\ r_j^{*i} > 0}} r_j^{*i} y_j^i \geq -p^* x_j^* + p^* \sum_{\substack{i \ni \\ r_j^{*i} > 0}} r_j^{*i} y_j^{*i} .$$

Using (46) and Lemma 4.6, the right-hand side of (49) can be seen to equal $p^* \sum_{\substack{i \ni \\ r_j^{*i} > 0}} r_j^{*i} w^i + p^* s^* \sum_{\substack{i \ni \\ r_j^{*i} > 0}} r_j^{*i} \theta^i$. Thus under the hypotheses of (d),

$$(50) \qquad -p^* x_j + p^* \sum_{\substack{i \ni \\ r_j^{*i} > 0}} r_j^{*i} y_j^i \geq p^* \sum_i r_j^{*i} w^i + p^* s^* \sum_i r_j^{*i} \theta^i$$

where we have trivially extended the summation indices on the right-hand

side to include all i . The right-hand side of (50) is positive, since

$p^* \gg 0$, $w^i > 0$ $\forall i$, $p^*s^* \geq 0$, and by assumption, $\sum_i r_j^{*i} > 0$. Hence

some term on the left-hand side must be positive. Suppose equality held

in (50). Then by a straightforward argument, appealing primarily to

$p^* \gg 0$ and the continuity of preferences, we can obtain a contradiction.[14]

Thus the strict inequality holds in (50), and hence condition (d) is verified.

If $\sum_\ell r_j^{*\ell} > 0$, condition (e) follows immediately from (46). If

$\sum_\ell r_j^{*\ell} = 0$, then (47) implies $p^*x_j^* = 0$ and $\tau_j^* = 0$ by definition, and

hence (e) is also satisfied.

<div align="right">Q.E.D.</div>

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[14] If $-p^*x_j > 0$, then by convexity of Y_j and continuity of the u_j^i , $\exists \, (\tilde{g}_j, \tilde{x}_j) \in Y_j$ with $\tilde{x}_j > x_j$ and $u_j^i(\tilde{g}_j, y_j^i) > u_j^i(g_j^*, y_j^{*i}) \; \forall i \ni r_j^{*i} > 0$. But since $p^* \gg 0$, $-p^*\tilde{x}_j + p^* \sum_{\substack{i \ni \\ r_j^{*i} > 0}} r_j^{*i}y_j^i$ is less than the right-hand

side of (50)--contradiction. Thus suppose $\exists \, \hat{i} \ni r_j^{*\hat{i}}p^*y_j^{\hat{i}} > 0$. Then by reducing some positive component of $y_j^{\hat{i}}$ slightly we can obtain a vector $\tilde{y}_j^{\hat{i}} \ni u_j^{\hat{i}}(g_j, \tilde{y}_j^{\hat{i}}) > u_j^{\hat{i}}(g_j^*, y_j^{\hat{i}})$. Then the left-hand side of (50), with $y_j^{\hat{i}}$ replaced by $\tilde{y}_j^{\hat{i}}$, will be less than the right-hand side--contradiction.

APPENDIX

In this appendix we prove Lemma 4.2 of the text. We first prove a preliminary lemma. Recall the definition of N (Definition 4.1).

<u>Lemma A.1</u>. N is upper semicontinuous on $S_{m\gamma} \times P$.

<u>Proof</u>. Let $(k^v, r^v) \to (k,r)$, $a^v \to a$, and $a^v \in N(k^v, r^v)$ $\forall v$. Since $z(r^v, a^v) \leq 0$ and $a^v \in \hat{W}$ $\forall v$, it follows that $z(r,a) \leq 0$ and $a \in \hat{W}$ (since $\hat{W}$ is closed). Suppose $a \notin N(k,r)$. Then for $\hat{a} \in N(k,r)$,

(i) $U(k,\hat{a}) > U(k,a)$. Given $\hat{a}$, suppose we can exhibit a sequence $\hat{a}^v \to \hat{a} \ni \hat{a}^v \in \hat{W}$ and $z(r^v, \hat{a}^v) \leq 0$ $\forall v$. Then $U(k^v, a^v) \geq U(k^v, \hat{a}^v)$ (since $a^v \in N(k^v, r^v)$), and it would follow from the continuity of U that $U(k,a) \geq U(k,\hat{a})$, contradicting (i). Thus the supposition that $a \notin N(k,r)$ would be ruled out, and hence N would be upper semicontinuous at the arbitrary point (k,r) .

Thus it remains to define a sequence $\hat{a}^v$ with the desired properties. If $z(r,\hat{a}) << 0$, then $\exists v_0 \ni \forall v \geq v_0$, $z(r^v, \hat{a}) \leq 0$. Hence define $\hat{a}^v = 0$ if $v < v_0$ and $\hat{a}^v = \hat{a}$ $\forall v \geq v_0$. This sequence clearly has the desired qualities.

Thus suppose $z(r,\hat{a}) \leq 0$ but the equality holds for at least one component. For any v one of the following two cases must hold: (a) $z(r^v, \hat{a}) \leq 0$; (b) $z(r^v, \hat{a})$ has at least one positive coordinate. If (a) holds, define $\alpha^v = 1$. If (b) holds, choose $\alpha^v \in (0,1) \ni z(r^v, \alpha^v \hat{a}) \leq 0$, with equality holding in at least one coordinate. (Clearly this can be done, since $z(r^v, 0) << 0$ because $w >> 0$, and z is continuous.) Then define $\hat{a}^v \equiv \alpha^v \hat{a}$ $\forall v$. Since 0 and $\hat{a} \in \hat{W}$ and $\hat{W}$ is convex, $\hat{a}^v \in \hat{W}$.

By definition of α^v , $z(r^v, \hat{a}^v) \leq 0$. Thus it remains to show $\hat{a}^v \to \hat{a}$. Since the sequence α^v is bounded, it has a limit point. Let $\alpha*$ be any such limit point. Then to show $\hat{a}^v \to \hat{a}$ it is sufficient to show $\alpha* = 1$. If case (a) holds for an infinite number of v , trivially such a subsequence of the α^v has a limit point of 1. Thus suppose (b) holds for an infinite number of v and $\alpha*$ is a limit point of this subsequence. From the continuity of z it follows that $z(r, \alpha*\hat{a}) \leq 0$, with equality holding in at least one coordinate. By assumption, $z(r,\hat{a}) \leq 0$ with equality holding in at least one coordinate. If $\alpha* < 1$, then $z(r, \alpha*\hat{a}) << 0$ (since $w >> 0$). This contradiction establishes that $\alpha* = 1$, and hence $\hat{a}^v \to \hat{a}$.

Q.E.D.

__Lemma 4.2__. M is upper semicontinuous on $S_{m\gamma} \times P$, and for each $(k,r) \in S_{m\gamma} \times P$, the image set $M(k,r)$ is convex.

__Proof__. Let $(k^v, r^v) \to (k,r)$, $(a^v, p^v) \to (\overline{a},\overline{p})$, and $(a^v, p^v) \in M(k^v, r^v)$ $\forall v$. To show $(\overline{a},\overline{p}) \in M(k,r)$ we must show (i) $\overline{a} \in N(k,r)$; (ii) $\overline{p}z(r,\hat{a}) \geq 0$ $\forall \hat{a} \in W \ni U(k,\hat{a}) \geq U(k,\overline{a})$. (i) follows immediately from the upper semi-continuity of N . Given $\hat{a} \in W$ with $U(k,\hat{a}) \geq U(k,\overline{a})$, for each v either (I) $U(k^v, \hat{a}) \geq U(k^v, a^v)$ or (II) $U(k^v, a^v) > U(k^v, \hat{a})$. For each v satisfying (I), $p^v z(r^v, \hat{a}) \geq 0$ since $(a^v, p^v) \in M(k^v, r^v)$. If there are an infinite number of v satisfying (I), then in the limit of this subsequence we obtain $\overline{p}z(r,\hat{a}) \geq 0$, verifying (ii) for this case.

By (C.3) and (B.3) $\exists \tilde{a} \in W \ni U(k,\tilde{a}) > U(k,\hat{a})$, and hence $U(k,\tilde{a}) > U(k,\overline{a})$. Since $(k^v, a^v) \to (k,\overline{a})$, it follows from the continuity of U that $\exists v_0 \ni \forall v \geq v_0$, $U(k^v, \tilde{a}) > U(k^v, a^v)$. Suppose (II) holds for an infinite number of v . Then $\forall v \geq v_0$ satisfying (II), $U(k^v, \tilde{a}) > U(k^v, a^v) > U(k^v, \hat{a})$. For __these__ v define

$\hat{a}^V \equiv \alpha_V \tilde{a} + (1 - \alpha_V)\hat{a}$ where $\alpha_V \varepsilon (0,1)$ is chosen so that $U(k^V, \hat{a}^V) = U(k^V, a^V)$. Such a choice is possible because U is continuous. Then $p^V z(r^V, \hat{a}^V) \geq 0$ since $\hat{a}^V \varepsilon W$ and $(a^V, p^V) \varepsilon M(k^V, r^V)$. The bounded sequence α_V has a convergent subsequence. Suppose $\alpha_V \to \alpha_*$ along such a subsequence of v satisfying (II). If we can show $\alpha_* = 0$, then it will follow that $\hat{a}^V \to \hat{a}$ and hence $\bar{p} z(r, \hat{a}) \geq 0$, verifying (ii) for this case. Suppose $\alpha_* > 0$. Then $\hat{a}^V \to \alpha_* \tilde{a} + (1 - \alpha_*)\hat{a}$ and $U(k, \alpha_* \tilde{a} + (1 - \alpha_*)\hat{a}) \geq \alpha_* U(k, \tilde{a}) + (1 - \alpha_*)U(k, \hat{a}) > U(k, \hat{a})$, the first inequality following from the concavity of U and the second from the definition of $\tilde{a}$. But $U(k, \alpha_* \tilde{a} + (1 - \alpha_*)\hat{a}) \equiv \lim U(k^V, \hat{a}^V) = \lim U(k^V, a^V) = U(k, \bar{a}) \leq U(k, \hat{a})$, the second equality following from the definition of $\hat{a}^V$, the third from continuity of U and $(k^V, a^V) \to (k, \bar{a})$, and the inequality from the definition of $\hat{a}$. This contradiction establishes that $\alpha_* = 0$.

The proof of convexity is straightforward.

<div align="right">Q.E.D.</div>

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